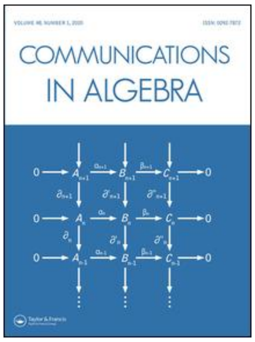

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On divided commutative rings

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ON DIVIDED COMMUTATIVE RINGS

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ABSTRACT. Let R be a commutative ring with identity having total quotient ring T. A prime ideal P of R is called divided if P is comparable to every principal ideal of R. If every prime ideal of R is divided, then R is called a divided ring. If P is a nonprincipal divided prime, then $P^{-1} = \{ x \in T : xP \subset P \}$ is a ring. We show that if R is an atomic domain and divided, then the Krull dimension of $R \leq 1$. Also, we show that if a finitely generated prime ideal containing a nonzerodivisor of a ring R is divided, then it is maximal and R is quasilocal.

INTRODUCTION

Through out this paper, R denotes a commutative ring with 1 and T denotes the total quotient ring of R. Given a ring R, then $Z(R)$ denotes the set of zerodivisors of R, and N denotes the set of nonunits of R. D. Dobbs in [8] studied divided domains. Our main purpose is to generalize the study of divided domains to the context of arbitrary rings where possibly $Z(R)$ is nonzero. Our definition of divided rings is the same as that one given in [8] for integral domains.

We start with the following definitions :

Definition. A prime ideal P of a ring R is called divided if P is comparable to every principal ideal of R. If every prime ideal of R is divided, then R is called a divided ring.

Definition. Recall from [6], a prime ideal P of R is called strongly prime if aP and bR are comparable for every $a,b \in R$. If every prime ideal of a ring R is strongly prime, then R is called a pseudo-valuation ring (PVR).

The first part of the following result is clear by the definition of divided rings, and the second part is also clear by [6, Lemma 1(a)] where it was shown that if a prime ideal P of R is strongly prime, then it is comparable to every principal ideal of R and therefore it is divided.

Proposition 1. (a). If R is a divided ring, then the prime ideals of R are linearly ordered and therefore R is quasi-local. **(b).** If R is a PVR, then R is a divided ring.

In [5, Proposition 2] we gave several characterizations of divided domains. In view of the proof of [5, Proposition 2], we see that these characterizations still valid for an arbitrary ring R. Thus, we state them here without proof.

Proposition 2. The following statements are equivalent for a ring R.

(1) R is a divided ring.

(2) For every pair of proper ideals I, J of R, I and $\mathrm{Rad}(J)$ are comparable, where $\mathrm{Rad}(J)$ denotes the radical of J.

(3) For every $a,b \in R$, the ideals (a) and $\mathrm{Rad}((b))$ are comparable.

(4) For every $a,b \in R$, either $a|b$ or $b|a^n$ for some $n \geq 1$.

In light of Proposition 2(4), we have the following result.

Corollary 3. Any homomorphic image of a divided ring is divided. In particular, if R is divided and I is an ideal of R, then R/I is divided.

The following result is a generalization of [8, Lemma 2.2 (a)]. Our proof is different than that given in [8].

Proposition 4. Any localization of a divided ring is divided.

Proof. Let S be a nonzero multiplicatively closed subset of R and $x,y \in R_S$. Then $x = a/s$ and $y = b/s$ for some $s \in S$ and $a,b \in R$. Since R is divided, $a|b$ or $b|a^n$ for some $n \geq 1$ by Proposition 2(4). Hence, $b = ca$ or $a^n = gb$ for some $c,g \in R$. Thus, $b/s = c(a/s)$ or $a^n/s^n = (g/s^{n-1})(b/s)$. Thus, $x|y$ or $y|x^n$. Therefore, R is divided by Proposition 2(4). ■

In [7, Theorem 1], we proved that R is a PVR if and only if for every $a,b \in R$, bR and aN are comparable. The following result is an analog of this fact.

Proposition 5. The following statements are equivalent for a ring R.

(1) R is divided.

(2) For every $a,b \in R$, there is an $n \geq 1$ such that bR and a^nN are comparable.

Proof. **(1)$\Longrightarrow$(2).** Suppose that $bR \not\subset a^mN$ for every $m \geq 1$. Then either a and b are associative in R or a does not divide b in R. If a and b are associative, then

$b|a$ and therefore $aN \subset bR$. If a does not divide b in R, then $b|a^n$ for some $n \geq 1$ by Proposition 2(4) and hence $a^nN \subset bR$. **(2) $\Longrightarrow$ (1).** Let $a,b \in R$. By proposition 2(4) we need show that either $a|b$ or $b|a^n$ for some $n \geq 1$. Now, if $bR \subset a^nN$ for some $n \geq 1$, then $a|b$. If $a^mN \subset bR$ for some $m \geq 1$, then $b|a^{m+1}$. Thus, R is a divided ringt. ■

A consequence of the above result is the following corollary.

Corollary 6. Let R be a quasilocal ring with the maximal ideal M. The following statements are equivalent.

(1) R is divided.

(2) For every $a,b \in R$, there is an $n \geq 1$ such that bR and a^nM are comparable.

Recall that if I is an ideal of R, then $I^{-1} = \{ x \in T : xI \subset R \}$ and $I:I = \{ x \in T : xI \subset I \}$. We leave the proof of the following lemma to the reader.

Lemma 7. Let I be a nonprincipal ideal of R. Then $xI \subset N$ for every $x \in I^{-1}$.

The following lemma is needed in the next result.

Lemma 8. Let P be a divided prime ideal of R containing a nonzerodivisor of R. Then $Z(R) \subset P$.

Proof. Let $s \in P$ be a nonzerodivisor of R. Suppose that there is a $z \in Z(R) \setminus P$. Since P is divided, $P \subset (z)$ and in particular $z|s$ which is impossible. Thus, $Z(R) \subset P$. ■

The following result is a generalization of the first part in [4, Proposition 6].

Proposition 9. Let P be a nonprincipal divided prime ideal of R. Then $P^{-1} = P:P$ is a ring.

Proof. Suppose there is an $x \in P^{-1}\backslash R$. Write $x = a/b$ for some $a \in R$ and a nonzerodivisor $b \in R$. Suppose that for some $p \in P$, $(a/b)p = c \in R\backslash P$. Then $ap = bc$ in R. Hence, $(a/b)(p/c) = 1$ in T. Since P is prime and $c \in R\backslash P$, $b \in P$. Hence, $Z(R) \subset P$ by Lemma 8. Thus, c is a nonzerodivisor of R. Since P is divided and $c \in R\backslash P$, $p/c \in P$. But $(a/b)(p/c) = 1$ which is a contradiction by Lemma 7. Thus, $P^{-1} = P:P$ is a ring. ∎

The following is a generalization of [4, Proposition 7].

Proposition 10. Let I be a proper ideal of R containing a nonzerodivisor of R. The following statements are equivalent.

(1) I is a nonprincipal divided prime ideal.

(2) I^{-1} is a ring and I is comparable to every principal ideal of R.

Proof. **(1) $\Longrightarrow$ (2).** This is clear by Proposition 9 and the definition of divided prime. **(2) $\Longrightarrow$ (1).** Since I Contains a nonzerodivisor of R and it is comparable to every principal ideal of R, we see that $Z(R) \subset I$. Since I^{-1} is a ring and I contains a nonzerodivisor of R, I is nonprincipal. For, if I is principal, then $I = (s)$ for some nonzerodivisor $s \in R$. Hence, $1/s \in I^{-1}$. Since I^{-1} is a ring, $1/s^2 \in I^{-1}$. But $(1/s^2)s = 1/s \notin R$, a contradiction. Now, we show that I is prime. Let $S = R\backslash I$ and $x,y \in S$. Since $Z(R) \subset I$, neither x nor y is a zerodivisor of R. Since I is comparable to every principal ideal of R, $1/x$ and $1/y$ are in I^{-1}. Since I^{-1} is a ring, $(1/x)(1/y) = 1/xy \in I^{-1}$. Since I is nonprincipal and $1/xy \in I^{-1}$, $xy \in S$. Thus, S is a multiplicatively closed subset of R and therefore I is prime ∎

The following example shows that the hypothesis that I contains a nonzerodivisor of R is crucial.

Example 11. Let $R = Z_8$ and $I = (2)$. Then $I^{-1} = R$ is a ring and I is divided but I is principal.

In view of Example 11, we have the following result.

Proposition 12. Let I be a proper ideal of R such that $Z(R) \subset I$. If I^{-1} is a ring and I is comparable to every principal ideal of R, then I is prime.

Proof. To show that I is prime, see the argument given in the proof of Proposition 10. ∎

The following example shows that the hypothesis $Z(R) \subset I$ is crucial in the above Proposition.

Example 13. Let $R = Z_8$ and $I = (4)$. Then $I^{-1} = R$ is a ring and I is comparable to every principal ideal of R but I is not prime.

The first part of the following lemma is taken from [5, Theorem 1].

Lemma 14. (a). The prime ideals of a ring R are linearly ordered if and only if the radical of every proper principal ideal of R is prime if and only if for every $a,b \in R$, either $a|b^n$ or $b|a^m$ for every $n,m \geq 1$. **(b).** If $a,b \in R$, then $Rad((a)) = Rad((b))$ if and only if there are $n,m \geq 1$ such that $a|b^n$ and $b|a^m$.

Proof. (b). Just observe that $Rad((a)) = Rad((b))$ iff $a \in$

$\mathrm{Rad}((b))$ and $b \in \mathrm{Rad}((a))$ iff there are $n, m \geq 1$ such that $a | b^n$ and $b | a^m$. ■

Recall that a ring B is called an overring of R if $R \subset B \subset T$. A prime ideal P of R contains a nonzerodivisor element of R is called a minimal regular prime ideal of R if whenever $Q \subsetneq P$ for some prime ideal Q of R, then $Q \subset Z(R)$.

Proposition 15. Suppose that the prime ideals of a ring R are linearly ordered, and B is an overring of R containing an element of the form $1/s$ for some nonunit nonzerodivisor $s \in R$. Furthermore, suppose that $\mathrm{Rad}((s))$ is a minimal regular prime ideal of R, then $B = T$. In particular, if R is divided, then $B = T$ is divided.

Proof. To show that $B = T$, it suffices to show that $1/d \in B$ for every nonzerodivisor $d \in R$. Let d be a nonzerodivisor of R. We consider two cases : case 1. Suppose that $d \in R \backslash \mathrm{Rad}((s))$. Then $d | s^n$ for some $n \geq 1$ by Lemma 14 (a). Hence, $s^n = dk$ for some $k \in R$. Thus, $k/s^n = 1/d$ in T. Since $1/s \in B$, $k/s^n = 1/b \in B$. Case 2. Suppose that $d \in \mathrm{Rad}((s))$. Since $\mathrm{Rad}((s))$ is a minimal prime ideal of R and $\mathrm{Rad}((d))$ is prime by Lemma 14 (a), $\mathrm{Rad}((s)) = \mathrm{Rad}((d))$. Hence, $d | s^n$ for some $n \geq 1$ by Lemma 14 (b). Now, a similar argument as in case 1, we conclude that $1/d \in B$. Thus, $B = T$. The remaining part is clear by Proposition 4. ■

In light of the above Proposition, we have the following.

Corollary 16. Let R be a quasilocal ring of a Krull dimension 1 containing a nonunit nonzerodivisor element. Then T is the only overring of R containing an element of the form $1/s$ for some nonunit nonzerodivisor $s \in R$.

D. Dobbs in [9, Proposition 2.2 (a)] proved that if P is a divided prime ideal of a domain R, then P^n is a P-primary ideal of R, for every $n \geq 1$. The following is a generalization of this fact.

Proposition 17. Let P be a divided prime ideal of R such that $Z(R) \subset P$. Then P^n is P-primary, for every $n \geq 1$.

Proof. We show that if $a, b \in R$ satisfy $ab \in P^n$ and $a \notin \mathrm{Rad}(P^n) = P$, then $b \in P^n$. Consider an element of the form $y = p_1p_2\ldots p_n$ in P^n where the p_i's are in P. To show $b \in P^n$, it suffices to show that $y/a \in P^n$, since b is a finite sum of element of the form of y. Since $Z(R) \subset P$ and $a \notin P$ and P is divided, a is a nonzerodivisor of R and $p/a \in P$ for every $p \in P$. Thus, $y/a = (p_1/a)p_2\ldots p_n \in P^n$. Hence, $b \in P^n$. Therefore, P^n is P-primary. ∎

The following example shows that the hypothesis $Z(R) \subset P$ in the above Proposition is crucial.

Example 18. Let $V = Z_{(2)} + XQ[[X]]$, a two dimensional valuation domain with prime ideals $(0) \subset P = XQ[[X]] \subset M = 2Z_{(2)} + XQ[[X]]$. Let $R = V/X^2V$. Then R is a PVR (see [6, Example 10(b)]) with prime ideals $G = P/X^2V$ and $Z(R) = N = M/X^2V$. Then G is divided. Now, $(2 + X^2V)(X^2/2 + X^2V) \in G^3 = [P^3 + X^2V] / X^2V = 0$ in R. But neither $2 + X^2V \in \mathrm{Rad}(G^3) = G$ nor $X^2/2 + X^2V \in G^3$ since $1/2 \notin X^2V$. Hence, G^3 is not G-primary of R.

Proposition 19. Let R be an atomic domain. Then R is divided if and only if R is quasilocal of Krull dimension 1.

Proof. Suppose that R is divided with maximal ideal M. Suppose that there is a nonzero prime ideal P of R such that $P \not\subset M$. Then there are atoms a, b of R such that $a \in P$ and $b \in M \setminus P$. Since P is divided, $b \mid a$ which is a contradiction. The converse is clear. ■

The following lemma is well-known. See for example [10, Theorem 15]. We state it here without proof.

Lemma 20. Let s be a nonunit nonzerodivisor element of R. Then $1/s$ is never integral over R.

It is easy to see that if P is a divided prime ideal of R, then $P \subset J(R)$ where $J(R)$ is the Jacobson radical of R. In the following result, we show that if a finitely generated prime ideal containing a nonzerodivisor of R is divided, then P is maximal and therefore R is quasilocal.

Proposition 21. Let P be a finitely generated prime ideal containing a nonzerodivisor of R. If P is divided, then P is maximal and therefore R is quasilocal.

Proof. Deny. Then there is an $s \in N \setminus P$. Since P contains a nonzerodivisor of R, $Z(R) \subset P$ by Lemma 8. Hence, s is a nonzerodivisor of R. Since P is prime and divided, $(1/s)P \subset P$. Since P contains a nonzerodivisor of R, the annihlator of P in T is 0. Hence, by [10, Theorem 12], $1/s$ is integral over R. A contradiction by Lemma 20. ■

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