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# Characterization of best Chebychev approximation using the frequency response of IIR digital filters with convex stability

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ABSTRACT

This paper deals with the application of Chebychev's approximation theory to IIR digital filter frequency response (FR) approximation. It explores the properties of the frequency response of IIR digital filters as a nonlinear complex approximating function; IIR digital filter frequency response is used to approximate a prescribed magnitude and phase responses. The approximation problem is closely related to optimization. If the set of approximating functions is non-convex, the optimization problem is difficult and may converge to a local minimum. The main results presented in the paper are proposing a convex stability domain by introducing a condition termed "sign condition" and characterization of the best approximation by the Global Kolmogorov's Criterion (GKC). The Global Kolmogorov's Criteria is shown to be also a necessary condition for the approximation problem. Finally, it is proved that the best approximation is a global minimum. The sign condition can be incorporated as a constraint in an optimization algorithm.



## 1. Introduction

The transfer function of an IIR digital filter is

$$H(z)=\frac{N(z^{-1})}{D(z^{-1})}=\frac{\sum_{k=0}^{m}b_k z^{-k}}{1+\sum_{l=1}^{n}a_l z^{-l}},$$
$$z\in\mathbf{U},\ U \text{ is the unit disc.} \tag{1}$$

Its frequency response is $H(\omega)=H(z)|_{z=e^{j\omega}}$. This function is used to approximate a prescribed frequency response on a compact interval, $\Omega$.

In many applications of digital signal processing filter design with arbitrary magnitude and phase responses is required.

One design trend of IIR digital filters is to meet frequency response magnitude specifications that minimize a specific error norm ($L_\infty$ norm). The designed filter may have a nonlinear phase. An all-pass filter is cascaded with the filter as an equalizer [1,2]. The equalizer is a real nonlinear phase function of the all-pass filter [3]. The minimum error is often characterized by the alternation theorem (equiripple of the error on a frequency interval).

One of the drawbacks associated with the use of equalizer is that the number of independent coefficients in an all-pass section is less than the number of the filter coefficients. Moreover, based on approximation theory, the original coefficients of the IIR digital filter, $a$ and $b$, are no longer the independent coefficients for the magnitude approximation problem. The valid independent approximation parameters are, in this case, the coefficients of the magnitude which are functions of $a$ and $b$ that are probably not easily solved.

Another trend is to approximate both magnitude and phase simultaneously using the complex FR functions. The major challenges in any approximation problem are: existence, uniqueness, characterization of best approximation and designing an algorithm. The Chebychev approximation with general continuous complex valued rational functions is tackled in [34–36]. As it was established by Walsh [34], the existence of best approximation is guaranteed provided the domain of approximation is compact and has no singularity points. In addition, the best approximation is known to be non-unique [36].

In the real approximation the alternation theorem is the tool for characterizing the best approximation. This theorem no longer holds in the complex case. The main tools for characterization of an optimal solution in the complex case are the Global Kolmogorov Criterion (GKC) and the Local Kolmogorov Criterion (LKC) [30,31]. GKC is generally a sufficient condition while LKC is a necessary condition. The intimate connection between approximation and optimization is well recognized [32,33]. The optimization algorithm is used to determine the coefficients of a stable IIR digital filter that minimizes the max-norm error ($L_\infty$). Various design methods are proposed to compute an optimal solution [11–22]. The optimization problem is difficult if the set of approximating functions are non-convex. In such cases, the algorithm may converge to a local minimum. Another major problem in the design of IIR

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digital filters is stability. Some design methods start with a point corresponding to a stable IIR digital filter, i.e., the roots of $D(A,z)$ lie inside the unit disc (Schur Polynomial), and monitor the stability iteratively; other methods follow stabilization steps. Numerical optimization algorithms incorporate the stability requirements in a constrained optimization setting. A linear programming algorithm was proposed for the optimization utilizing the positive realness of $D(A,z)$, ($Re\{D(A,\omega)\} > 0$), to ensure stability [13]. The convex stability of $(N(z), D(z))$ and the positive realness of $H(z)$ are utilized to obtain a convex set of IIR digital filters [21]. Rouche's condition on the denominator perturbation is incorporated to preserve stability [20]. Stability margin approach was proposed in [18]. Iterative Lyaponov inequality constraint is incorporated for the filter stability [17].

The objective of this paper is to investigate the approximation properties of the rational complex FR functions using non-linear Chebychev approximation theory. The concepts of "functions with betweeness property" [25] and theory of "regular systems" [27] play an important role in this paper. The results of this study are three folds: it proposes a convex stability domain of FR functions by introducing a condition termed "*sign condition*". In addition, the proof that GKC (Theorem 2) is also a necessary condition in order to characterize a best approximation, is provided. Finally, the best approximation is shown to be a global minimum (Theorem 3). This sign condition has to be incorporated as a constraint in the optimization algorithm.

The interested reader about convex stability is invited to review the references [4, Chapter 7] [5–8] (see also Appendix A).

This paper is organized as follows: some definitions are provided in Section 2. Section 3 states the complex approximation problem of FR functions. The results of the paper are included in Section 4. Section 5 presents examples. The conclusion is presented in Section 6. Appendix A is about convex stability.

## 2. Preliminaries

### 2.1. IIR digital filters

The transfer function of an IIR digital filter is defined in Eq. (1) where $N(z)$ and $D(z)$ are relatively prime of fixed degrees $m$ and $n$ with cardinality $m \leqslant n$. The sets of parameters $\{A = (a_1 \dots a_n)$, $a_0 = 1\}$ and $\{B = (b_0 \dots b_m)\}$ are real.

The set of FR functions of stable IIR digital filters is denoted by $\mathcal{H}$.

A digital IIR digital filter is stable if the denominator $D(A,z)$ has all its zeros inside the unit disc., i.e. $D(A,z)$, is a Schur polynomial. $D(A,\omega) \neq 0$ on the boundary of the unit circle and $Re[D(A,\omega)] > 0$ [13].

The convex combination of two polynomials, $D_0(\omega)$ and $D_1(\omega)$, is

$$D_\lambda = D_0 + \lambda(D_1 - D0),$$

where $\lambda$ belongs to [0, 1].

The real and imaginary parts of $D(A,\omega)$ are respectively,

$$g(A,\omega) = 1 + \sum_{l=1}^{n} a_l \cos(l\omega), \tag{2}$$

$$u(A,\omega) = \sum_{l=1}^{n} a_l \sin(l\omega). \tag{3}$$

## 3. Statement of the Chebychev approximation problem

The following brief review considers the general problem of approximation of a continuous function, $f(\omega)$, by an approximating function depending on a finite number of parameters. Thus, the problem under consideration is to approximate, $f(\omega)$, by an approximating function, $F(A,\omega)$, which may depend on the parameter, $A$, in a linear or non linear way. The problem is to determine those parameters, $A^*$, which make, $F(A^*,\omega)$, closest to $f(\omega)$ with respect to some norm, i.e., $\|f(\omega) - F(A^*,\omega)|$ is a minimum. The functions, $f(\omega)$, and, $F(A,\omega)$, can be real or complex functions. The function, $F(A^*,\omega)$, may be termed best approximation, optimal approximation and minimal solution. Once the problem is formulated in a mathematical form, there are four main issues related to its solution after the choice of $F(A,\omega)$: existence, uniqueness, characterization and computation of $F(A^*,\omega)$. A norm is defined by $\|f\|_p = (\int |(f(\omega)|^p\, d\omega))^{\frac{1}{p}}$ and denoted $L_p$ norm. The norms $L_1$, $L_2$, and $L_\infty$ are often used in the approximation theory.

### 3.1. Mathematical formulation of the approximation problem

Let $C(\Omega)$ be the space of continuous complex valued function on a real compact interval, $\Omega$, endowed with the max-norm, $L_\infty$

$$\big\|H(A,B,.)\big\| = \max_{\omega}\big|H(A,B,\omega)\big|. \tag{4}$$

Let $H_d(\omega) \in C(\Omega) \setminus \mathcal{H}$ be a prescribed frequency response and $H(A,B,\omega) \in \mathcal{H}$ be the approximating function. For example, $H_d(\omega)$ may be $\Gamma e^{-j\phi(\omega)} \in C(\Omega)\setminus\mathcal{H}$, where $\Gamma$ is a constant and $\phi(\omega)$ is a linear function of $\omega$. The error function of approximation is defined as:

$$e = H_d(\omega) - H(A,B,\omega). \tag{5}$$

This function attains its norm on a discrete point set $M^* \subset \Omega$ with cardinality $\geqslant m+n+2$ points. The minimum solution of the Chebychev approximation problem $H_0(a^*,b^*,\omega)$ is the solution of:

$$\begin{aligned} E^* &= \big\|H_d(.) - H_0\big(A^*,B^*,.\big)\big\| \\ &= \min_{(A,B)}\max_{\omega}\big|H_d(\omega) - H(A,B,\omega)\big|, \end{aligned} \tag{6}$$

where $E^*$ is the max-norm of $e$. This solution is characterized by the GKC and the LKC [30].

## 4. Results

The keys of this study are the concept of "betweeness property" [24,25] and the more general concept "regular systems" [27,28]. These two concepts were introduced in the context of nonlinear approximation theory as a generalization of convexity. A variant concept termed "weak betweeness property" has been applied for characterization and uniqueness of best approximation [29]. The three concepts have considered complex rational functions provided that the denominator is a real positive function.

The convex stability and the line homotopy [37] are additional concepts playing an important role in the study.

In the approximation problem at hand, the concepts of betweeness property and regular systems have been applied to the complex FR functions provided the denominator, $D(A,\omega)$, is a complex function and non-zero on the unit circle.

### 4.1. The main result of the paper is Theorem 1

Its consequences are proposing a convex stability domain in FR functions by introducing a condition, i.e., "sign condition". Recalling, from introduction that GKC is generally a sufficient condition in order to characterize the best approximation; the investigation of this work shows that GKC is also a necessary condition (Theorem 2) under the existence of the monotone sequence denoted $h_\lambda$

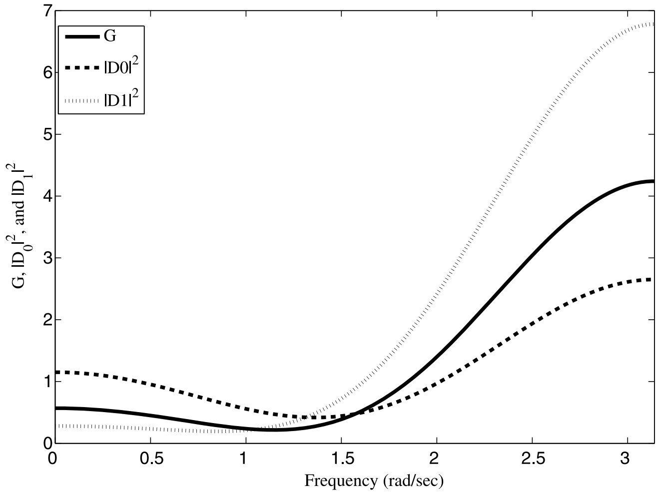


**Fig. 1.** Sign condition $G$, $|D_0|^2$, and $|D_1|^2$ of Eq. (8) as a function of frequency for LP filter.

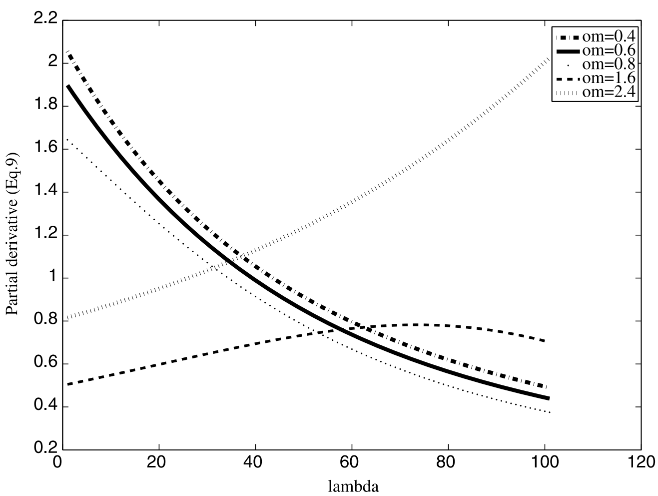


**Fig. 2.** Partial derivative of $|h_\lambda - H_0|$ for variable frequencies – LP filter.

in Theorem 1. In addition, it is established, relying on the sequence $h_\lambda$, that a best approximation is a global minimum (Theorem 3). LKC (Theorem 4) is only a necessary criterion in order to characterize the best approximation [30] if the partial derivatives are continuous.

Summarizing, Corollaries 2 and 3 are immediate conclusion of Theorem 2. Theorem 1 and Theorem 3 are proved with the help of the existence of the monotonic sequence $h_\lambda$. Discussion of the interrelationship among the theorems is presented in the sequel of this section.

While Fig. 1 and Fig. 3 demonstrate the concepts of sign condition, Fig. 2 and Fig. 4 demonstrate the monotonicity of the sequence of $h_\lambda$ of Theorem 1.

The convexity domain is computed numerically. The results of the examples were obtained by MATLAB.

### 4.2. Approximation properties of frequency response

For brevity, the arguments $(A, B, \omega)$ are often suppressed in the denominator and numerator.

Let $H_i = \frac{N_i}{D_i}$, stands for $H(A^{(i)}, B^{(i)}, \omega) = \frac{N(B^{(i)}, \omega)}{D(A^{(i)}, \omega)}$, and let $g_i, u_i$ be the real and imaginary parts of the denominator $D_i$, respectively, where $i = 0, 1$.

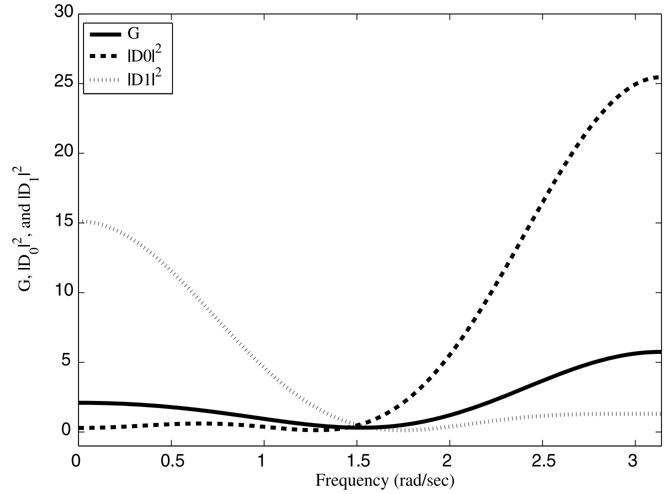


**Fig. 3.** Sign condition $G$, $|D_0|^2$, and $|D_1|^2$ of Eq. (9) as a function of frequency for notch filter.

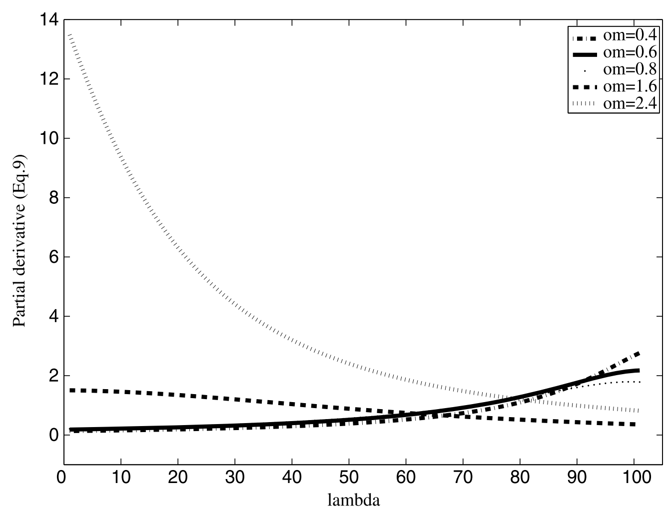


**Fig. 4.** Partial derivative of $|h_\lambda - H_0|$ for variable frequencies – notch filter.

**Definition 1** *(Homotopy).* (See [37].) If $f, g$ are continuous maps of the space $X$ into the space $Y$, we say that $f$ is homotopic to $g$ if there is a continuous map $F : [0, 1] \times X \to Y$ such that $F(0, \omega) = f(\omega)$ and $F(1, \omega) = g(\omega)$ for each $\omega \in X$. The map $F$ is called a homotopy between $f$ and $g$.

We think of a homotopy as a continuous one-parameter family of maps from $X$ to $Y$. If we imagine the parameter $\lambda \in [0, 1]$ as representing time, then the homotopy $F$ represents a continuous "deforming" of the map $f$ to the map $g$, as $\lambda$ goes from 0 to 1.

**Definition 2** *(The betweeness property).* (See [25].) A subset $G$ of $C(X, Y)$ is said to have the *betweeness property* if for any two elements $g_0$, $g_1 \in G$, there exists a $\lambda$-set $\{h_\lambda\} \subset G$ such that

1. $H_0 = g_0$, $H_1 = g_1$.
2. For $\lambda \in [0, 1]$ and $x \in X$, $h_\lambda(x)$ is on the joining line $g_0(x)$ and $g_1(x)$.
3. If $g_0 = g_1$ then $h_\lambda(x) = g_0$, $0 < \lambda < 1$ and if $g_0 \neq g_1$, $|g_0 - h_\lambda(x)|$ is strictly monotonic continuous function of $\lambda \in [0, 1]$.

**Definition 3.** The condition $G = g_0 g_1 + u_0 u_1 > 0$ is termed "*sign condition*".

The following lemma establishes an equivalent condition to that proposed by B. Dumitrescu [4] for the characterization of the convex stability domain.

**Lemma 1.** *The sign condition G is equivalent to the convex stability condition* $Re(\frac{D1}{D_0}) > 0$ [4].

**Proof of Lemma 1.**

$$Re\left(\frac{D1}{D_0}\right) = Re\left(\frac{g_1 + ju_1}{g_0 + ju_0}\right) = Re\left(\frac{(g_1 + ju_1)(g_0 + ju_0)}{|D_0|^2}\right)$$
$$= Re\left(\frac{(g_1 g_0 + u_1 u_0) + j(u_1 g_0 - u_0 g_1)}{|D_0|^2}\right) > 0$$

$|D_0|^2$ is positive, hence $g_1 g_0 + u_1 u_0$ is positive. □

**Corollary 1** *(Convex stability). If G holds, then the polynomial* $D_\lambda$ *of any two stable polynomials* $D_0$ *and* $D_1$ *is a Schur polynomial.*

**Proof of Corollary 1.**

$$Re\left(\frac{D_\lambda}{D_0}\right) = Re\left(\frac{g_0 + \lambda(g_1 - g_0) + j[u_0 + \lambda(u_1 - u_0)]}{D_0}\right)$$

must be positive. By simple mathematical manipulations, similar to the proof of Lemma 1, the real part of

$$Re\left(\frac{D_\lambda}{D_0}\right) = \left(\frac{g_0^2 + u_0^2 - \lambda(g_0^2 + u_0^2) + \lambda(g_0 g_1 + u_0 u_1)}{|D_0|^2}\right).$$

Observe that $(\frac{g_0^2+u_0^2-\lambda(g_0^2+u_0^2)}{|D_0|2})$ is $\geqslant 0$ for $\lambda \in [0,1]$.

Thus for $Re(\frac{D_\lambda}{D_0})$ to be positive the term $u_0 u_1 + g_0 g_1$ must be positive. □

The next theorem is applied to FR functions as an application to the notion of homotopy and betweeness property. Its consequences will be discussed in the sequel.

**Theorem 1.** *Let* $H_0$ *and* $H_1$ *be frequency responses of stable IIR digital filters of the same order and same type defined on* $\Omega$, $H_1 \neq H_0$.

*Under the sign condition G*

1. *there exists a continuous line homotopy* $h_\lambda \in \mathcal{H}$, *between* $H_0 = h_\lambda(0,\omega)$, $H_1 = h_\lambda(1,\omega)$, *for any* $\omega \in \Omega$, $\lambda \in [0,1]$.
2. *The function* $\|h_\lambda - H_0\| \to 0$ *is a continuous strictly monotonic function of* $\lambda \in [0,1]$, *i.e.,* $\{h_\lambda\}$ *converges uniformly to* $H_0$. *where,*

$$h_\lambda = \frac{N_0(1-\lambda) + \lambda N_1}{D_0(1-\lambda) + \lambda D_1}, \tag{7}$$

*and*

$$(h_\lambda - H_0) = \frac{\lambda D_1}{\lambda D_1 + (1-\lambda) D_0}(H_1 - H_0), \tag{8}$$

$$\frac{\partial |h_\lambda - H_0|}{\partial \lambda} = \frac{|D_1|}{|D_\lambda|^3}\left[|D_0|^2 + \lambda\left(u_0 u_1 + g_0 g_1 - |D_0|^2\right)\right].|H_1 - H_0|. \tag{9}$$

**Remark 1.** By the "same type" in Theorem 1 we mean that the FR functions $H_0$ and $H_1$ are both LPs or HPs or Notch Filter, etc., as illustrated in Examples 1 and 2.

**Remark 2.** $G$ is a necessary and sufficient condition for $|h_\lambda - H_0|$ to be a monotonic sequence, i.e., $\frac{\partial |h_\lambda - H_0|}{\partial \lambda}$ is positive on $\lambda \in [0,1]$.

In fact, if the derivative 9, $\frac{\partial |h_\lambda - H_0|}{\partial \lambda}$, is positive then $G$ holds. The converse is true.

**Proof of Theorem 1.** $h_\lambda : [0,1] \times \Omega \to C(\Omega)$ is a continuous homotopy between $H_0$ and $H_1$ where $h_\lambda(0,\omega) = H_0, h_\lambda(1,\omega) = H_1$ for $\lambda \in [0,1]$. The sign of the derivative depends on the term $X(\lambda) = |D_0|^2 + \lambda(u_0 u_1 + g_0 g_1 - |D_0|^2)$. $X(\lambda)$ is positive for both cases $G > |D_0|^2$ or $G < |D_0|^2$ $\forall \lambda \in [0,1]$. $X(0) = |D_0^2|$ and $X(1) = (u_0 u_1 + g_0 g_1) = G$. The terms $|D_1|, |D_\lambda|^3$ are positive and have no effect on the sign of the derivative. Consequently, the function $\|h_\lambda - H_0\|$ is a strictly monotonic function under the assumption of the sign condition, $G$, concluding that the derivative is positive $\forall \lambda \in [0,1]$. By Dini's theorem [38] $h_\lambda$ converges uniformly to $H_0$. The condition $G$ implies that $D_\lambda$ is a Schur polynomial by Corollary 1 and $h_\lambda(z)$ is a stable IIR digital filter $\forall \lambda \in [0,1]$, resulting in $\{h_\lambda(\omega)\} \subset \mathcal{H}$. □

**Corollary 2.** *The set of FR functions has the betweeness property.*

The best approximation of functions with betweeness property is characterized by corollary of [25]. Similarly, Theorem 1 and Corollary 2 establish the characterization of best approximation in FR functions by the following corollary.

**Corollary 3.** $H_0(A^*, B^*, \omega) \in \mathcal{H}$ *is the best approximation for* $H_d(\omega)$ *if and only if there is no* $H(A, B, \omega)$ *such that*

$$Re\left[\overline{e\left(A^*, B^*, \omega\right)}\left(H(A,B,\omega) - H_0\left(A^*, B^*, \omega\right)\right)\right] > 0, \omega \in M^*. \tag{10}$$

The bar over the expression $Re[\overline{e(A^*, B^*, \omega)}]$ in Corollary 3 and in the remainder of the paper denotes a complex conjugate.

**Sketch of the Proof to Corollary 3.** Suppose there exists $H(A, B, \omega) \in \mathcal{H}$ such that $|H_d(\omega) - H(A,B,\omega)| < |H_d(\omega) - H_0(A^*, B^*, \omega)|$, $\omega \in M^*$

$$\begin{aligned} &\left|H_d(\omega) - H(A,B,\omega) - H_0\left(A^*, B^*, \omega\right) + H_0\left(A^*, B^*, \omega\right)\right|^2 \\ &\quad = \left|H_d(\omega) - H_0\left(A^*, B^*, \omega\right)\right|^2 \\ &\qquad - 2\,Re\left(\overline{\left(H_d(\omega) - H_0\left(A^*, B^*, \omega\right)\right)}\left(H(A,B,\omega)\right.\right. \\ &\qquad \left.\left. - H_0\left(A^*, B^*, \omega\right)\right)\right) \\ &\qquad + \left|H(A,B,\omega) - H_0\left(A^*, B^*, \omega\right)\right|^2, \quad \omega \in M^*. \end{aligned}$$

Simple manipulations yield $Re[(\overline{(H_d(\omega) - H_0(A^*, B^*, \omega))}(H(A,B,\omega) - H_0(A^*, B^*, \omega)))] > 0$.

The proof of necessity is similar to that of the next theorem (GKC). □

A more general system of functions is the concept of regular systems. The abstract definition of such systems is often intractable. Alternatively, regular systems are defined if the necessity of GKC is satisfied [27]. Paraphrased in still another way, any set of functions satisfying the GKC to be a necessary condition for characterizing the best approximation is a subset of a regular system and the best approximation is a global minimum. Recalling, that GKC is generally a sufficient condition. Relying on the existence of a sequence, $h_\lambda$, from Theorem 1, the necessity of GKC for frequency response functions is shown by the proof of the following theorem. Moreover, Theorem 3 specifies the best approximation to be a global minimum for the FR functions.

**Theorem 2** *(GKC). $H_0 \in \mathcal{H}$ is a best approximation for $H_d$ if and only if*

$$\min_{\omega \in M^*} Re\{\overline{(H_d - H_0)}(H_1 - H_0)\} \leqslant 0 \tag{11}$$

*for any frequency response $H_1 \in \mathcal{H}$.*

**Proof of Theorem 2.** Sufficiency is a direct result from [30].

Necessity: Let $H_0$ be not a best approximation. Suppose there exists $H_1$ such that $\|H_d - H_1\| < \|H_d - H_0\|$ $\forall \omega \in M^*$. Since the set $M^* \subset \Omega$ is a compact set, there exists $\alpha > 0$ such that $Y_d(\omega) = Re[\overline{(H_d - H_0)}(H_1 - H_0)] \geqslant \alpha$ for all $\omega \in M^*$. By the continuity of $(H_d - H_0)$ and $(H_1 - H_0)$ there exists an open set $W$ that covers $M^*$, $W = \{\omega \in \Omega | Y_d(\omega) > \frac{\alpha}{2}\}$, i.e., $\overline{W}$ includes $M^*$. By Theorem 1 there exists an element $h_\lambda$ between $H_1$ and $H_0$ such that $\|H_d - h_\lambda\| < \|H_d - H_0\|$. It follows

$$2\,Re\big[\overline{(H_d - H_0)}(h_\lambda - H_0)\big] > |h_\lambda - H_0|^2, \omega \in W \tag{12}$$

and then by Theorem 1

$$\|h_\lambda - H_0\| \to 0. \tag{13}$$

The following estimation on $W$

$$\begin{aligned}
&|H_d - h_\lambda|^2 \\
&\quad = |H_d - H_0|^2 - 2\,Re\big[\overline{(H_d - H_0)}(h_\lambda - H_0| + |h_\lambda - H_0|^2\big], \\
&\qquad \omega \in W \\
&\quad = |H_d - H_0|^2 \\
&\qquad - 2\,Re\left(\overline{(H_d - H_0)}(H_1 - H_0).\frac{\lambda D_1}{\lambda D_1 + (1-\lambda)D_0}\right) \\
&\qquad + \left|\frac{\lambda D_1}{\lambda D_1 + (1-\lambda)D_0)}\right|^2 .|H_1 - H_0|^2 \\
&\quad \leqslant \|H_d - H_0\|^2 \\
&\qquad - 2\,Re\left((H_d - H_0)(H_1 - H_0)\frac{\lambda D_1}{\lambda D_1 + (1-\lambda)D_0}\right) \\
&\qquad + \left|\frac{\lambda D_1}{\lambda D_1 + (1-\lambda)D_0}\right|^2 .\|H_1 - H_0\|^2 \\
&\quad < \|H_d - H_0\|^2.
\end{aligned} \tag{14}$$

The next estimation is performed on $V$, $V = \Omega \setminus W$. Consider the expression $\max_V |H_d - H_0|$. As $V$ is a compact set and $M^* \cap V = \phi$, then $\max_V |H_d - H_0|$ and $\|H_d - H_0\|$ cannot be on $M^* \cap V$. Let

$$\max_{M^*} |H_d - H_0| - \Big\{\max_V |H_d - H_0|\Big\} = \mu > 0.$$

By Theorem 1, $\|h_\lambda - H_0\|$ converges uniformly to 0. Choosing a $\lambda$ such that

$$\begin{aligned}
&\|H_0 - h_\lambda\| < \mu > 0, \\
&|H_d - h_\lambda| = \big|H_d - H_0 - (h_\lambda - H_0)\big| \leqslant |H_d - H_0| + |H_0 - h_\lambda| \\
&\qquad\qquad \leqslant \|H_d - H_0\| - \mu + \mu < \big\|(H_d - H_0)\big\|.
\end{aligned} \tag{15}$$

The two estimations (14), (15) on $W \cup V$ contradict the optimality of $H_0$. $\square$

In computing Chebychev's approximation problems, it is important to characterize the best approximation as a local or global minimum. It is recognized that the approximation problem is closely related to optimization. It is difficult to compute the optimal solution if the set of approximating function is non-convex and the algorithm may converge to a local minimum. Theorem 3 ensures that the best approximation in the set of frequency response is a global minimum. Its proof relies on the next lemma and existence of the sequence $h_\lambda$ of Theorem 1.

**Lemma 2.** *(See [26,27, Lemma 3.1].) Given $H_0$, $H_1 \in \mathcal{H}$ such that $\|H_d - H_1\| < \|H_d - H_0\|$ implies $Re\{\overline{(H_d - H_0)}(H_1 - H_0)\} > 0$, $\omega \in M^*$. Let the neighborhood $ON_\lambda(H_0)$, $ON_\lambda(H_0) = \{H\colon \|H - H_0\| < \lambda\} \cap \mathcal{H}$ for every $0 < \lambda \leqslant 1$. Then there exists $h_\lambda \in ON_\lambda$ such that $\|H_d - h_\lambda\| < \|H_d - H_0\|$, i.e., $H_0$ is not a local minimum.*

**Proof of Lemma 2.** The proof of $\|H_d - h_\lambda\| < \|H_d - H_0\|$ is similar to the estimations (14), (15). $\square$

**Theorem 3.** *The minimum of the functional $\|H_d - H_0\|$ in $\mathcal{H}$ is a global minimum.*

**Proof of Theorem 3.** Let $H_0$ be not a global minimum. Then there exists $H_1$ such that $\|H_d - H_1\| < \|H_d - H_0\|$, implying $Re\{\overline{(H_d - H_0)}(H_1 - H_0)\} > 0$ for every $\omega \in M^*$. By Theorem 1 there exists a sequence, $\{h_\lambda\} \subset \mathcal{H}$ which converges uniformly to $H_0$ such that $\|H_d - h_\lambda\| < \|H_d - H_0\|$, implying $Re\{\overline{(H_d - H_0)}(h_\lambda - H_0)\} > 0$ for $\omega \in M^*$. Hence by Lemma 2 and the expression (10) of Corollary 3, $H_0$ is not a local minimum. $\square$

The following are the partial derivatives with respect to the parameters $A$ and $B$.

$$\begin{aligned}
&\frac{\partial H(A,B,\omega)}{\partial b_k} = \frac{\cos(k\omega) - j\sin(k\omega)}{D(A,\omega)}, \quad k = 0,\dots,m, \\
&\frac{\partial H(A,B,\omega)}{\partial a_l} = -\frac{\sum_{k=0}^n b_k(\cos(l+k)\omega - j\sin(l+k)\omega}{[D(A,\omega)]^2}, \\
&\quad l = 1,\dots,n, \\
&\big[D(A,\omega)\big]^2 = 1 + \sum_{l=1}^n a_l^2\big[\cos 2(l\omega) - j\sin 2(l\omega)\big] \\
&\qquad\qquad + \sum_{l=1}^n 2a_l\big[\cos(l\omega) - j\sin(l\omega)\big] \\
&\qquad\qquad + \sum_{l=0}^n \sum_{p=l+1}^n 2a_l a_p\big[\cos(l+p)\omega - j\sin(l+p)\omega\big],
\end{aligned} \tag{16}$$

where the expressions of the derivatives are continuous with respect to $A, \omega$ and $B$. We define the linear function $L_{(A,B)}(\delta a_l, \delta b_i, \omega)$,

$$\begin{aligned}
&L_{(A,B)}(\delta a_l, \delta b_i, \omega) \\
&\quad = \sum_{l=1}^n \frac{\partial H(A,B,\omega)}{\partial a_l}\delta a_l + \sum_{i=0}^m \frac{\partial H(A,B,\omega)}{\partial b_i})\delta b_i.
\end{aligned} \tag{17}$$

The following theorem is the previously declared LKC. It is generally a necessary condition for the characterization of best approximation. In fact, it applies for those functions having partial derivatives on the parameter domain $A$ and $B$ and the derivatives are continuous on the frequency domain. Obviously, these requirements are satisfied in the FR functions where the partial derivatives (16) are continuous with respect to the coefficients and frequency. This is obvious from the fact that $D(A,\omega) \neq 0$. The linear function (17) is important for the LKC.

**Theorem 4** *(LKC). (See [30–32].) Let $H_0$ be a best approximation of $H_d$. The partial derivatives (16) are continuous then,*

$$\min_{\omega\in M^*} Re\left\{\overline{H_d - H\left(A^*, B^*, \omega\right)}\right\} L_{(A^*,B^*)}(\delta A, \delta B, \omega) \leqslant 0 \qquad (18)$$

*for all $\delta A$ and $\delta B$.*

**Sketch for the proof.** LKC is known as a standard criterion in [30, 31]. The continuity of the partial derivatives is evident. The denominator $D(A,\omega)$ in both expressions (16) is never equal to zero. This ensures that the partial derivatives (16) are continuous with respect to $\omega$, $A$ and $B$. □

## 5. Examples

The purpose of the following examples is to illustrate the monotonicity of the homotopic sequence of frequency response function and the existence of convexity domain under the sign condition assumption. The filters of both examples were selected so that they are stable and satisfy the sign condition in a prescribed band of frequency.

**Example 1.** Two 3rd-order LP filters of bandwidths $0.25\pi$ and $0.35\pi$ [23]:

$$H_0(z) = \frac{0.0662272(1+z^{-1})^3}{(1.0000 - 0.9356z^{-1} + 0.5671z^{-2} - 0.1016z^{-3})},$$

$$H_1(z) = \frac{0.13402309(1+z^{-1})^3}{(1 - 0.2543z^{-1} + 0.3504z^{-2} - 0.0234z^{-3})},$$

where the denominators of $H_0, H_1$ are $D_0 = 1.0000 - 0.9356z^{-1} + 0.5671z^{-2} - 0.1016z^{-3}$ and $D_1 = 1 - 0.2543z^{-1} + 0.3504z^{-2} - 0.0234z^{-3}$.

The poles of the first filter are $0.0925 \pm 0.5736i$ and $0.6993$. The poles of the second filter are $0.3381 \pm 0.5267i$ and $0.2594$. Both filters have their poles inside the unit disk (Schur polynomials).

Fig. 1 shows the partial terms, $|D_0|^2$, $|D_1|^2$ and $G$, of Eq. (9). The plot shows the positive behavior of the sign condition with respect to the frequency.

Fig. 2 describes the behavior of the derivative of the sequence $|h_\lambda - H_0|$ with respect to $\lambda \in [0,1]$ for different values of frequencies. The plot shows that the derivative is always positive and so $|h_\lambda - H_0|$ is a strict monotonic sequence.

Range of convex stability: Applying the theory of convex stability in Appendix A we compute the convex stability domain of the two filters.

The matrices $S(D_0)$ and $S((D_1)-(D_0))$ are used to construct the block matrix $F$. The eigenvalues of $F$ are: $-0.0764, 0.0801, -0.0160 + 0.2160i, -0.0160 - 0.2160i, 0.7842, -0.4900, -0.0160 + 0.2160i, -0.0160 - 0.2160i$.

The eigenvalues are out of the range $[1,\infty)$. The range of convexity domain is $(1/(-0.49), 1/(0.7842))$ and $\lambda \in [0,1]$ lies between $(-2.0408, 1.2752]$ or $\{[-1.66, 1.2752]\}$, i.e., the convex combination of the matrix $S(D_0)$ and the matrix $S(D_1)$ is stable.

Consequently the convex combination of $D_0$ and $D_1$ is a Schur polynomial.

In addition, by Appendix A.2 the eigenvalues of $S^{-1}(D_0)S(D_1)$, 4.3419, 3.6120, are out of the range $(-\infty, 0)$ which emphasize the convex stability of $D_0, D_1$.

**Example 2.** Two fourth order stable notch filters. The bandwidth of the first filter is $0.25\pi$ and that of the second filter is $0.25\pi$. The center frequency of the first filter $0.375\pi$ and that of the second filter is $0.325\pi$

$$H_0(z) = \frac{0.569z^4 - 0.9428z^3 + 1.5286z^2 - 0.9428z + 0.5690}{z^4 - 1.219z^3 + 1.333z^2 - 0.6667z + 0.3333},$$

$$H_1(z) = \frac{0.569z^4 - 1.287z^3 + 1.866z^2 - 1.287z + 0.569}{z^4 - 1.664z^3 + 1.671z^2 - 0.9102z + 0.0333}.$$

The poles of the first filter are $0.5264 + 0.5882i, 0.5264 - 0.5882i, 0.0831 + 0.7266i, 0.0831 - 0.7266i$, and of the second filter $0.3680 + 0.9042i, 0.3680 - 0.9042i, 0.8887, 0.0393$.

Fig. 3 shows the partial terms, $|D_0|^2$, $|D_1|^2$ and $G$, of Eq. (9). The plot shows the positive behavior of the sign condition with respect to the frequency (3).

Fig. 4 describes the behavior of the derivative of the sequence $|h_\lambda - H_0|$ with respect to $\lambda \in [0,1]$ for different values of frequencies. The plot shows that the derivative is always positive and so $|h_\lambda - H_0|$ is a strictly monotonic sequence.

Using the same procedure of computation of Example 1, the eigenvalues of the matrix $F$ are: $0.2250, -0.4500, -0.4500, 0.0072 + 0.3377i, 0.0072 - 0.3377i, 0.2250, -0.4500 + 0.0000i, -0.4500 - 0.0000i, 0.2250 + 0.0000i, 0.2250 - 0.0000i, 0.5067, -0.2536, 0.0072 + 0.3377i, 0.0072 - 0.3377i, 0.0072 + 0.3377i, 0.0072 - 0.3377i, 0.0072 + 0.3377i, 0.0072 - 0.3377i$.

The range of eigenvalues of the matrix $F$ are out of the range $[1,\infty)$. The convex combination of the polynomials $D_0, D_1, D_\lambda$, is stable.

According to Appendix A.2 the eigenvalues of, $S^{-1}(D_0)S(D_1)$, 0.0999, 0.0999, 2.0150, are out of the range $(-\infty, 0)$. Hence the polynomial $D_\lambda$ is a Schur polynomial.

## 6. Conclusion

The main contribution of this paper is the application of the approximation theory in IIR digital filter design. The FR functions are commonly used to approximate a desired frequency response (magnitude and phase) on a defined frequency band according to the max-norm. This study has been based on the nonlinear Chebychev approximation theory. The results of the investigations conclude: the best approximation is completely characterized by GKC (necessary and sufficient), LKC as a necessary condition, a convexity domain is proposed by introducing the sign condition (convex stability condition) and the optimal solution is shown to be a global minimum. The sign condition has to be incorporated as optimization constraint in an algorithm to preserve stability of the designed filter.

The examples presented in the paper demonstrate Theorem 1. Furthermore, numerical values in the provided examples asses the convex stability domain. A future work can be conducted to design an algorithm to verify the results of this work as a fundamental theory.

## Appendix A

*A.1.* Computing the convexity domain: The coefficients of a polynomial $D(A,z)$ are the entries of the following matrix [5,10]

$$S(D) = \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & a_3 & a_2 - a_0 \\ 0 & a_n & a_{n-1} & a_3 & a_3 - a_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_0 & -a_1 & a_n - a_{n-4} & a_{n-1} - a_{n-3} \\ a_0 & -a_1 & a_2 & -a_{n-3} & a_n - a_2 \end{bmatrix}.$$

Define the polynomial $D_\lambda(z)$,

$$D_\lambda(z) = D_0(z) + \lambda(D_1 - D_0) \qquad (A.1)$$

and denote the related matrices by $S(D_0), S(D_1 - D_0)$. Find $\lambda$ such that

$$S(D_\lambda) = S(D_0) + \lambda S(D_1 - D_0) \qquad (A.2)$$

be convex stable. The following matrices are constructed to determine the range of convex stability so that $\lambda \in [0,1]$ falls in this range:

$$F_0 == S(D_0) \otimes S(D_0) - I \otimes I,$$
$$F_1 = S(D_0) \otimes S(D_1 - D_0) + S(D_1 - D_0) \otimes S(D_0),$$
$$F_2 = S(D_1 - D_0) \otimes S(D_1 - D_0). \tag{A.3}$$

The symbol $\otimes$ denotes the Kronecker product,

$$\begin{aligned} &F_0 + \lambda F_1 + \lambda^2 F_2 \\ &\quad = \big(S(D_0) \otimes S(D_0) - I \otimes I\big) + \lambda\big(S(D_0) \otimes S(\delta D) \\ &\qquad + S(\delta D) \otimes S(D_0)\big) + \lambda^2\big(S(\delta D) \otimes S(\delta D)\big). \end{aligned}$$

The following block matrix $F$

$$F = \begin{bmatrix} \mathbf{0} & I \\ -F_0{}^{-1} F_2 & -F_0{}^{-1} F_1 \end{bmatrix}$$

is convex stable if and only if it has no eigenvalues $\rho_i$ in the interval $[1, \infty)$.

The matrix $F$ may have real eigenvalues. Denote the minimum negative real eigenvalue with $\rho^-_{min}$ and the maximum positive real eigenvalue with $\rho^+_{max}$. $\lambda \in [0, 1]$ falls in the range of convex stability, $(\frac{1}{\rho^-_{min}}), \frac{1}{\rho^+_{max}}$ [9].

The polynomial $D_\lambda(a, z)$ and the matrix $S(D_\lambda)$ belongs to the convexity domain.

*A.2.* [5,10] The convex combination of two Schur polynomials, $D_0$ and $D_1$, is a stable polynomial if and only if the matrix $S^{-1}(D_0)S(D_1)$ has no eigenvalues in $(-\infty, 0)$.

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