Algebras and Representation Theory (2005) 8: 275–295
DOI: 10.1007/s10468-005-3605-4

Dual Entwining Structures and Dual Entwined Modules

JAWAD Y. ABUHLAIL⋆
Mathematics Department, Birzeit University, P.O. Box 14, Birzeit, Palestine.
e-mail: abuhlail@kfupm.edu.sa

(Received: April 2003; accepted: March 2004)

Presented by A. Verschoren

Abstract. In this note we introduce and investigate the concepts of dual entwining structures and dual entwined modules. This generalizes the concepts of dual Doi–Koppinen structures and dual Doi–Koppinen modules introduced (in the infinite case over rings) by the author in his dissertation.

Mathematics Subject Classifications (2000): 16W30, 18E15.

Key words: entwining structures, entwined modules, Doi–Koppinen structures, Doi–Koppinen modules, Hopf–Galois (co)extensions, duality, (co)module (co)algebras.

Introduction

This note deals with the following problem: let (A, C, ψ) be a given entwining structure over a commutative base ring R. Find an R-subalgebra $\widetilde{A} \subseteq C^*$ and an R-coalgebra $\widetilde{C} \subseteq A^*$, such that $(\widetilde{A}, \widetilde{C}, \psi^*)$ is an entwining structure.

For general entwining structures, it is not clear if such a *dual* entwining structure exists. However, once it is found, we have the *expected* duality relations between the corresponding categories of entwined modules. For the special case of Doi–Koppinen structures over Noetherian rings, the problem was solved by the author in his dissertation. Our results are formulated for right–right entwining structures. Corresponding versions for left–left, right–left and left–right entwining structures can be derived easily using the left–right dictionary (e.g., [9]).

The paper consists of three sections. In the first section, we give the necessary definitions and results from the theory of Hopf algebras and entwining structures. In the second section we present and investigate the concepts of dual entwining structures and dual entwined modules. The third section is an extended version of [2, § 3.4] formulated for *right–right* Doi–Koppinen structures. Our results in the third section generalize also those *achieved independently* by L. Zhang [29] on *dual relative Hopf modules* in the case of a commutative base field.

⋆ Current address: Department of Mathematical Sciences, P.O. Box 5046, King Fahd University of Petroleum & Minerals, 31261 Dhahran, Saudi Arabia.

Throughout this paper R denotes a commutative ring with $1_R \neq 0_R$. The category of R-(bi)modules will be denoted by $\mathcal{M}_R$. For an R-coalgebra $(C, \Delta_C, \varepsilon_C)$ and an R-algebra (A, μ_A, η_A) we consider $\mathrm{Hom}_R(C, A)$ as an R-algebra with the so called *convolution product* $(f \star g)(c) := \sum f(c_1)g(c_2)$ and unity $\eta_A \circ \varepsilon_C$. For an R-algebra A and an A-module M, an A-submodule $N \subset M$ will be called *R-cofinite*, if M/N is f.g. in $\mathcal{M}_R$. We call an R-submodule $K \subseteq M$ *pure* (in the sense of Cohn), if the canonical map $\iota_K \otimes \mathrm{id}_N\colon\ K \otimes_R N \to M \otimes_R N$ is injective for every R-module N.

1. Preliminaries

In this section we give some definitions and lemmata from the theory of Hopf algebras and entwining structures.

1.1. MEASURING R-PAIRINGS

If C is an R-coalgebra and A is an R-algebra with a morphism of R-algebras $\kappa\colon\ A \to C^*$, $a \mapsto [c \mapsto \langle a, c\rangle]$, then we call $P := (A, C)$ a *measuring R-pairing*. In this case C is an A-bimodule through

$$a \rightharpoonup c := \sum c_1\langle a, c_2\rangle \quad \text{and} \quad c \leftharpoonup a := \sum \langle a, c_1\rangle c_2 \quad \text{for all } a \in A,\ c \in C. \tag{1}$$

Let (A, C) and (B, D) be measuring R-pairings, $\xi\colon\ A \to B$ an R-algebra morphism and $\theta\colon\ D \to C$ an R-coalgebra morphism. Then we say $(\xi, \theta)\colon\ (B, D) \to (A, C)$ is a *morphism of measuring R-pairings*, if

$$\langle \xi(a), d\rangle = \langle a, \theta(d)\rangle \quad \text{for all } a \in A \text{ and } d \in D.$$

The category of measuring R-pairings and morphisms described above will be denoted by $\mathcal{P}_m$.

1.2. THE α-CONDITION

Let $P = (A, C)$ be a measuring R-pairing. We say P satisfies the *α-condition* (or P is a *measuring α-pairing*), if for every R-module M the following map is injective:

$$\alpha_M^P\colon\ M \otimes_R C \to \mathrm{Hom}_R(A, M),\ \sum m_i \otimes c_i \mapsto \Big[a \mapsto \sum m_i\langle a, c_i\rangle\Big]. \tag{2}$$

With $\mathcal{P}_m^\alpha \subset \mathcal{P}_m$ we denote the *full* subcategory of measuring α-pairings.

We say an R-coalgebra C satisfies the *α-condition,* if (C^*, C) satisfies the α-condition (equivalently, if ${}_RC$ is *locally projective* in the sense of B. Zimmermann-Huignes [30, Theorem 2.1], [16, Theorem 3.2]).

1.3. SUBGENERATORS

Let A be an R-algebra and K an A-module. We say an A-module N is *K-subgenerated*, if N is isomorphic to a submodule of a K-generated A-module (equivalently, if N is Kernel of K-generated A-modules). The *full* subcategory of A-modules, whose objects are the K-subgenerated A-modules is denoted by $\sigma[K]$. Moreover $\sigma[K]$ is the *smallest* Grothendieck full subcategory of the category of A-modules that contains K. The reader is referred to [25] for the well developed theory of categories of this type.

Rational Modules

1.4. Let $P = (A, C)$ be a measuring α-pairing. Let M be a left (a right) A-module, $\rho_M\colon M \to \mathrm{Hom}_R(A, M)$ the canonical A-linear map and put $\mathrm{Rat}^C({}_AM) := \rho_M^{-1}(M \otimes_R C)$ (resp. ${}^C\mathrm{Rat}(M_A) := \rho_M^{-1}(C \otimes_R M)$). We call ${}_AM$ (resp. M_A) *C-rational,* if $\mathrm{Rat}^C({}_AM) = M$ (resp. ${}^C\mathrm{Rat}(M_A) = M$). If M is an A-bimodule, then we set ${}^C\mathrm{Rat}^C({}_AM_A) = \mathrm{Rat}^C({}_AM) \cap {}^C\mathrm{Rat}(M_A)$ and call M C-birational, if ${}^C\mathrm{Rat}^C({}_AM_A) = M$.

LEMMA 1.5 ([2, Lemma 2.2.7]). *Let $P = (A, C)$ be a measuring α-pairing. For every left (resp. right) A-module M we have*:

(1) $\mathrm{Rat}^C({}_AM) \subset M$ (*resp.* ${}^C\mathrm{Rat}(M_A) \subset M$) *is an A-submodule.*
(2) *For every A-submodule $N \subset M$, it follows that* $\mathrm{Rat}^C({}_AN) = N \cap \mathrm{Rat}^C({}_AM)$ (*resp.* ${}^C\mathrm{Rat}(N_A) = N \cap {}^C\mathrm{Rat}(M_A)$).
(3) $\mathrm{Rat}^C(\mathrm{Rat}^C({}_AM)) = \mathrm{Rat}^C({}_AM)$ (*resp.* ${}^C\mathrm{Rat}({}^C\mathrm{Rat}(M_A)) = {}^C\mathrm{Rat}(M_A)$).
(4) *For a left* (*resp. a right*) *A-module L and an A-linear map $f\colon M \to L$, we have* $f(\mathrm{Rat}^C({}_AM)) \subseteq \mathrm{Rat}^C({}_AL)$ (*resp.* $f({}^C\mathrm{Rat}(M_A)) \subseteq {}^C\mathrm{Rat}(L_A)$).

NOTATION. For a measuring α-pairing (A, C) we denote with $\mathrm{Rat}^C({}_A\mathcal{M}) \subseteq {}_A\mathcal{M}$ (resp. ${}^C\mathrm{Rat}(\mathcal{M}_A) \subseteq \mathcal{M}_A$, ${}^C\mathrm{Rat}^C({}_A\mathcal{M}_A) \subseteq {}_A\mathcal{M}_A$) the *full* subcategory of C-rational left A-modules (resp. C-rational right A-modules, C-birational A-bimodules).

THEOREM 1.6 ([2, Satz 2.2.16, Folgerung 2.2.22]). *For a measuring R-pairing $P = (A, C)$ the following are equivalent*:

(1) *P satisfies the α-condition;*
(2) *${}_RC$ is locally projective and $\kappa_P(A) \subseteq C^*$ is dense* (*w.r.t. the finite topology*).

If these equivalent conditions are satisfied, then we have isomorphisms of categories

$$\begin{aligned} \mathcal{M}^C &\simeq \mathrm{Rat}^C({}_A\mathcal{M}) = \sigma[{}_AC] \\ &\simeq \mathrm{Rat}^C({}_{C^*}\mathcal{M}) = \sigma[{}_{C^*}C]; \\ {}^C\mathcal{M} &\simeq {}^C\mathrm{Rat}(\mathcal{M}_A) = \sigma[C_A] \end{aligned}$$

$$\simeq {}^{C}\mathrm{Rat}(\mathcal{M}_{C^*}) = \sigma[C_{C^*}];$$
$$^{C}\mathcal{M}^{C} \simeq {}^{C}\mathrm{Rat}^{C}({}_{A}\mathcal{M}_{A}) = \sigma[{}_{A}(C \otimes_R C)_{A}]$$
$$\simeq {}^{C}\mathrm{Rat}^{C}({}_{C^*}\mathcal{M}_{C^*}) = \sigma[{}_{C^*}(C \otimes_R C)_{C^*}].$$

LEMMA 1.7 ([2, Lemma 2.1.23]). *Let $P = (A, C)$, $Q = (B, D)$ be measuring R-pairings and $\xi\colon A \to B$, $\theta\colon D \to C$ be R-linear maps with*

$$\langle \xi(a), d\rangle = \langle a, \theta(d)\rangle \quad \textit{for all } a \in A \textit{ and } d \in D.$$

(1) *Set $P \otimes P := (A \otimes_R A, C \otimes_R C)$ and assume that $C \otimes_R C \overset{\chi_{P\otimes P}}{\hookrightarrow} (A \otimes_R A)^*$ is an embedding. If ξ is an R-algebra morphism, then θ is an R-coalgebra morphism. Moreover, if A is commutative, then C is cocommutative.*

(2) *Assume $B \overset{\kappa_Q}{\hookrightarrow} D^*$ to be an embedding. If θ is an R-coalgebra morphism, then ξ is an R-algebra morphism. Moreover, if C is cocommutative and $A \subseteq C^*$, then A is commutative.*

1.8 ([4, Theorem 2.8], [5, Remark 2.14, Proposition 2.15]). Assume R to be Noetherian. Let A be an R-algebra and consider A^* as an A-bimodule through the left and the right regular A-action $(af)(b) = f(ba)$ and $(fa)(b) = f(ab)$. We define the *finite dual* of A as the R-module

$$A^\circ := \{f \in A^* \mid AfA \text{ is f.g. in } \mathcal{M}_R\}$$
$$= \{f \in A^* \mid f(I) = 0 \text{ for some } R\text{-cofinite ideal } I \lhd A\}.$$

An R-algebra (resp. an R-bialgebra, a Hopf R-algebra) A with $A^\circ \subset R^A$ pure will be called an *α-algebra* (an *α-bialgebra*, a *Hopf α-algebra*). For every α-algebra (resp. α-bialgebra, Hopf α-algebra) A, the finite dual A° becomes a locally projective R-coalgebra (resp. R-bialgebra, Hopf R-algebra). If A is an α-algebra and $\widetilde{C} \subseteq A^\circ$ is an R-subcoalgebra, then $(A, \widetilde{C})$ is a measuring α-pairing. For α-algebras (resp. α-bialgebra, Hopf α-algebras) A, B and a morphism of R-algebras (resp. R-bialgebras, Hopf R-algebras) $f\colon A \to B$, it follows directly from Lemma 1.7 that the restriction of $f^*\colon B^* \to A^*$ to B° induces a morphism of R-coalgebras (resp. R-bialgebras, Hopf R-algebras) $f^\circ\colon B^\circ \to A^\circ$.

Remark 1.9 ([2, Folgerung 2.1.10(1)]). Let V, W be R-modules and $X \subseteq V^*$, $Y \subseteq W^*$ be R-submodules. If R is Noetherian and $X \subset R^V$ is Y-pure (or $Y \subset R^W$ is X-pure), then the following induced canonical map is injective:

$$\varpi\colon X \otimes_R Y \to (V \otimes_R W)^*,\ f \otimes g \mapsto f \underline{\otimes} g,$$

where

$$(f \underline{\otimes} g)(v \otimes w) := f(v)g(w).$$

Entwined Modules

1.10. A *right–right entwining structure* (A, C, ψ) consists of an R-algebra (A, μ_A, η_A), an R-coalgebra $(C, \Delta_C, \varepsilon_C)$ and an R-linear map

$$\psi\colon C \otimes_R A \to A \otimes_R C,\ c \otimes a \mapsto \sum a_\psi \otimes c^\psi,$$

such that

$$\begin{array}{ll} \sum (ab)_\psi \otimes c^\psi = \sum a_\psi b_\Psi \otimes c^{\psi\Psi}, & \sum (1_A)_\psi \otimes c^\psi = 1_A \otimes c, \\ \sum a_\psi \otimes \Delta_C(c^\psi) = \sum a_{\psi\Psi} \otimes c_1^\Psi \otimes c_2^\psi, & \sum \varepsilon_C(c^\psi) a_\psi = \varepsilon_C(c) a. \end{array} \tag{3}$$

Let (A, C, ψ) and (B, D, Ψ) be right–right entwining structures. A morphism $(\gamma, \delta)\colon (A, C, \psi) \to (B, D, \Psi)$ consists of right entwining structures an R-algebra morphism $\gamma\colon A \to B$ and an R-coalgebra morphism $\delta\colon C \to D$, such that

$$\sum \gamma(a_\psi) \otimes \delta(c^\psi) = \sum \gamma(a)_\Psi \otimes \delta(c)^\Psi.$$

With $\mathbb{E}^{\bullet}_{\bullet}$ we denote the category of right–right entwining structures. For definitions of the categories of left–left, right–left and left–right entwining structures the interested reader may refer to [9].

1.11. Let (A, C, ψ) be a right–right entwining structure. An *entwined module* corresponding to (A, C, ψ) is a right A-module M, which is also a right C-comodule, such that

$$\varrho_M(ma) = \sum m_{\langle 0\rangle} a_\psi \otimes m_{\langle 1\rangle}^\psi \quad \text{for all } m \in M,\ a \in A.$$

For entwined modules M, N corresponding to (A, C, ψ) we denote with $\mathrm{Hom}_A^C(M, N)$ the set of A-linear C-colinear morphisms from M to N. The category of right–right entwined modules and A-linear C-colinear morphisms is denoted by $\mathcal{M}_A^C(\psi)$. Entwined modules were introduced by T. Brzeziński and S. Majid in [8] as a generalization of Doi–Koppinen modules presented in [13] and [17].

LEMMA 1.12. *Let (A, C, ψ) be a right–right entwining structure over R and set $\mathfrak{C} := A \otimes_R C$.*

(1) *$\mathfrak{C}$ is an A-coring with A-bimodules structure given by*

$$a(\widetilde{a} \otimes c) := a\widetilde{a} \otimes c,\ (\widetilde{a} \otimes c)a := \sum \widetilde{a} a_\psi \otimes c^\psi, \tag{4}$$

comultiplication

$$\Delta_{\mathfrak{C}}\colon A \otimes_R C \to (A \otimes_R C) \otimes_A (A \otimes_R C),$$
$$a \otimes c \mapsto \sum (a \otimes c_1) \otimes_A (1_A \otimes c_2)$$

and counity $\varepsilon_{\mathfrak{C}} := \vartheta_A^r \circ (\mathrm{id}_A \otimes \varepsilon_C)$. Moreover $\mathcal{M}_A^C(\psi) \simeq \mathcal{M}^{\mathfrak{C}}$.

(2) $\#_{\psi}^{\mathrm{op}}(C, A) := \mathrm{Hom}_R(C, A)$ *is an A-ring with A-bimodule structure given by* $(af)(c) := \sum a_\psi f(c^\psi)$, $(fa)(c) := f(c)a$, *multiplication*

$$(f \cdot g)(c) = \sum f(c_2)_\psi g(c_1^\psi) \tag{5}$$

and unity $\eta_A \circ \varepsilon_C$.

(3) *Consider* ${}^*\mathfrak{C} := \mathrm{Hom}_{A-}(\mathfrak{C}, A)$ *as an A-ring with the canonical A-bimodule structure, multiplication*

$$(f \star_l g)(c) = \sum g(c_1 f(c_2)) \quad \textit{for all } f, g \in {}^*\mathfrak{C} \textit{ and } c \in \mathfrak{C}$$

and unity $\varepsilon_{\mathfrak{C}}$. *Then* $\#_{\psi}^{\mathrm{op}}(C, A) \simeq {}^*\mathfrak{C}$ *as A-rings via*

$$\nu\colon \mathrm{Hom}_R(C, A) \longrightarrow \mathrm{Hom}_{A-}(A \otimes_R C, A), \;\; f \mapsto [a \otimes c \mapsto af(c)] \tag{6}$$

with inverse $h \mapsto [c \mapsto h(1_A \otimes c)]$.

Proof. (1) This was noticed first by M. Takeuchi and can be found in several references (e.g., [7, Proposition 2.2]).

(2) For all $a, b \in A$, $f \in \#_{\psi}^{\mathrm{op}}(C, A)$ and $c \in C$ we have

$$\begin{aligned}((ab)f)(c) &= \sum (ab)_\psi f(c^\psi) = \sum a_\psi b_\Psi f(c^{\psi\Psi}) \\ &= \sum a_\psi (bf)(c^\psi) = (a(bf))(c).\end{aligned}$$

It is clear then that the left and the right A-actions given above define on $\#_{\psi}^{\mathrm{op}}(C, A)$ a structure of an A-bimodule. Moreover we have for all $f, g, h \in \mathrm{Hom}_R(C, A)$ and $c \in C$:

$$\begin{aligned}((f \cdot g) \cdot h)(c) &= \sum [(f \cdot g)(c_2)]_{\widehat{\psi}} h(c_1^{\widehat{\psi}}) \\ &= \sum [f(c_{22})_\psi g(c_{21}^\psi)]_{\widehat{\psi}} h(c_1^{\widehat{\psi}}) \\ &= \sum [f(c_{22})_{\psi\widehat{\psi}} g(c_{21}^\psi)_{\psi'}] h((c_1^{\widehat{\psi}})^{\psi'}) \\ &= \sum [f(c_2)_{\psi\widehat{\psi}} g(c_{12}^\psi)_{\psi'}] h((c_{11}^{\widehat{\psi}})^{\psi'}) \\ &= \sum f(c_2)_\psi g((c_1^\psi)_2)_{\psi'} h((c_1^\psi)_1^{\psi'}) \\ &= \sum f(c_2)_\psi (g \cdot h)(c_1^\psi) \\ &= (f \cdot (g \cdot h))(c).\end{aligned}$$

It is clear that $\eta_A \circ \varepsilon_C$ is a unity for $\#_{\psi}^{\mathrm{op}}(C, A)$.

(3) Note that ν is given by the canonical isomorphisms

$$\mathrm{Hom}_R(C, A) \simeq \mathrm{Hom}_R(C, \mathrm{Hom}_{A-}(A, A)) \simeq \mathrm{Hom}_{A-}(A \otimes_R C, A).$$

For all $a \in A$, $f \in \#_{\psi}^{\mathrm{op}}(C, A)$ and $c \in C$ we have

$$\begin{aligned}\nu(af)(b \otimes c) &= b((af)(c)) = b\Big(\sum a_{\psi} f(c^{\psi})\Big)\\ &= \sum b a_{\psi} f(c^{\psi}) = \nu(f)\Big(\sum b a_{\psi} \otimes c^{\psi}\Big)\\ &= \nu(f)((b \otimes c)a) = (a\nu(f))(b \otimes c).\end{aligned}$$

It is obvious that ν is right A-linear. For all $f, g \in \#_{\psi}^{\mathrm{op}}(C, A)$, $a \in A$ and $c \in C$ we have

$$\begin{aligned}\nu(f \cdot g)(a \otimes c) &= a((f \cdot g)(c)) = a \sum f(c_2)_{\psi} g(c_1^{\psi})\\ &= \sum a f(c_2)_{\psi} g(c_1^{\psi}) = \nu(g)\Big(\sum a f(c_2)_{\psi} \otimes c_1^{\psi}\Big)\\ &= \nu(g)\Big(\sum (a \otimes c_1) f(c_2)\Big) = \nu(g)\Big(\sum (a \otimes c_1) 1_A f(c_2)\Big)\\ &= \nu(g)\Big(\sum (a \otimes c_1)\nu(f)(1_A \otimes c_2)\Big)\\ &= (\nu(f) \star_l \nu(g))(a \otimes c).\end{aligned}$$

Consequently, ν is an isomorphism of A-rings. □

1.13. Let (A, C, ψ) be a right–right entwining structure over R and consider the corresponding A-coring $\mathcal{C} := A \otimes_R C$. We say that (A, C, ψ) *satisfies the α-condition* (or is an *α-entwining structure*) if for every right A-module M, the following map is injective

$$\alpha_M^{\psi}\colon M \otimes_R C \to \mathrm{Hom}_R(\#_{\psi}^{\mathrm{op}}(C, A), M),\ m \otimes c \mapsto [f \mapsto mf(c)]$$

(equivalently if ${}_A\mathcal{C}$ is locally projective).

Inspired by [14, 3.1] we introduce

DEFINITION 1.14. Let (A, C, ψ) be a right–right entwining structure that satisfies the α-condition. Let M be a right $\#_{\psi}^{\mathrm{op}}(C, A)$-module, $\rho_M\colon M \to \mathrm{Hom}_{-\#_{\psi}^{\mathrm{op}}(C,A)}(\#_{\psi}^{\mathrm{op}}(C, A), M)$ the canonical map and put $\mathrm{Rat}^C(M_{\#_{\psi}^{\mathrm{op}}(C,A)}) := \rho_M^{-1}(M \otimes_R C)$. Then M will be called *#-rational*, if $\mathrm{Rat}^C(M_{\#_{\psi}^{\mathrm{op}}(C,A)}) = M$. For a #-rational right $\#_{\psi}^{\mathrm{op}}(C, A)$-module M we set $\varrho_M := (\alpha_M^{\psi})^{-1} \circ \rho_M\colon M \to M \otimes_R C$. The category of #-rational right $\#_{\psi}^{\mathrm{op}}(C, A)$-modules and $\#_{\psi}^{\mathrm{op}}(C, A)$-linear maps will be denoted with $\mathrm{Rat}^C(\mathcal{M}_{\#_{\psi}^{\mathrm{op}}(C,A)})$.

THEOREM 1.15 ([3, Lemma 3.8, Theorem 3.10]). *Let (A, C, ψ) be a right–right entwining structure and consider the corresponding A-coring $\mathcal{C} := A \otimes_R C$.*

(1) *If ${}_RC$ is flat, then ${}_A\mathcal{C}$ is flat and $\mathcal{M}_A^C(\psi)$ is a Grothendieck category with enough injective objects.*

(2) *If* ${}_RC$ *is locally projective* (*resp. f.g. projective*), *then* ${}_A\mathfrak{C}$ *is locally projective* (*resp. f.g. projective*) *and*

$$\begin{aligned}&\mathcal{M}_A^C(\psi) \simeq \mathrm{Rat}^C(\mathcal{M}_{\#_\psi^{\mathrm{op}}(C,A)}) \simeq \sigma[(A \otimes_R C)_{\#_\psi^{\mathrm{op}}(C,A)}]\\ &\quad (\textit{resp. } \mathcal{M}_A^C(\psi) \simeq \mathcal{M}_{\#_\psi^{\mathrm{op}}(C,A)}). \end{aligned} \tag{7}$$

2. Dual Entwined Modules

In this section we fix the following: R is Noetherian, (A, C, ψ) is a right–right entwining structure with A an α-algebra and $\widetilde{A} \subseteq C^*$ is an R-subalgebra with $\varepsilon_C \in \widetilde{A}$, $\widetilde{C} \subseteq A^\circ$ is an R-subcoalgebra. So we have a measuring R-pairing $(\widetilde{A}, C)$ and a measuring α-pairing $(A, \widetilde{C})$. Besides the above technical assumptions we assume moreover that $\psi^*(\widetilde{C} \otimes_R \widetilde{A}) \subseteq \widetilde{A} \otimes_R \widetilde{C}$, i.e. the following diagram

$$\begin{array}{ccc} (A \otimes_R C)^* & \xrightarrow{\ \psi^*\ } & (C \otimes_R A)^* \\ \uparrow & & \uparrow \\ \widetilde{C} \otimes_R \widetilde{A} & \xrightarrow[\varphi]{\cdots\cdots} & \widetilde{A} \otimes_R \widetilde{C} \end{array}$$

can be completed commutatively with an R-linear morphism

$$\begin{aligned}&\varphi\colon\ \widetilde{C} \otimes_R \widetilde{A} \to \widetilde{A} \otimes_R \widetilde{C},\ \ \widetilde{f} \otimes \widetilde{g} \mapsto \sum \widetilde{g}_\varphi \underline{\otimes} \widetilde{f}^\varphi,\\ &\quad \text{where } \Big(\sum \widetilde{g}_\varphi \underline{\otimes} \widetilde{f}^\varphi\Big)(c \otimes a) := \sum \widetilde{f}(a_\psi)\widetilde{g}(c^\psi). \end{aligned} \tag{8}$$

THEOREM 2.1. $(\widetilde{A}, \widetilde{C}, \varphi)$ *is a right–right entwining structure and we have isomorphisms of categories*

$$\mathcal{M}_{\widetilde{A}}^{\widetilde{C}}(\varphi) \simeq \mathrm{Rat}^{\widetilde{C}}(\mathcal{M}_{\#_\varphi^{\mathrm{op}}(\widetilde{C},\widetilde{A})}) = \sigma[(\widetilde{A} \otimes_R \widetilde{C})_{\#_\varphi^{\mathrm{op}}(\widetilde{C},\widetilde{A})}]. \tag{9}$$

If moreover ${}_R\widetilde{C}$ *is f.g. projective, then*

$$\mathcal{M}_{\widetilde{A}}^{\widetilde{C}}(\varphi) \simeq \mathcal{M}_{\#_\varphi^{\mathrm{op}}(\widetilde{C},\widetilde{A})}. \tag{10}$$

Proof. Let $\widetilde{f} \in \widetilde{C}$, $\widetilde{g}, \widetilde{h} \in \widetilde{A}$, $c \in C$ and $a, b \in A$ be arbitrary. Then we have

$$\begin{aligned} \sum((\widetilde{g} \star \widetilde{h})_\varphi \underline{\otimes} \widetilde{f}^\varphi)(c \otimes a) &= \sum \widetilde{f}(a_\psi)(\widetilde{g} \star \widetilde{h})(c^\psi)\\ &= \sum \widetilde{f}(a_\psi)\widetilde{g}((c^\psi)_1)\widetilde{h}((c^\psi)_2)\\ &= \sum \widetilde{f}(a_{\psi\Psi})\widetilde{g}(c_1^\Psi)\widetilde{h}(c_2^\psi)\\ &= \sum(\widetilde{g}_\varphi \underline{\otimes} \widetilde{h}_\Phi \underline{\otimes} \widetilde{f}^{\varphi\Phi})(c_1 \otimes c_2 \otimes a)\\ &= \sum((\widetilde{g}_\varphi \star \widetilde{h}_\Phi) \underline{\otimes} \widetilde{f}^{\varphi\Phi})(c \otimes a) \end{aligned}$$

and

$$\Big(\sum(\varepsilon_C)_\varphi \underline{\otimes} \widetilde{f}^\varphi\Big)(c\otimes a) = \sum \widetilde{f}(a_\psi)\varepsilon_C(c^\psi) = \widetilde{f}(\varepsilon_C(c)a) = (1_{\widetilde{A}} \underline{\otimes} \widetilde{f})(c\otimes a).$$

On the other hand we have

$$\begin{aligned}
&\Big(\sum \widetilde{g}_\varphi \underline{\otimes} ((\widetilde{f}^\varphi)_1 \underline{\otimes} (\widetilde{f}^\varphi)_2)\Big)(c\otimes a\otimes b)\\
&= \Big(\sum \widetilde{g}_\varphi \underline{\otimes} \widetilde{f}^\varphi\Big)(c\otimes ab)\\
&= \sum \widetilde{f}((ab)_\psi)\widetilde{g}(c^\psi)\\
&= \sum \widetilde{f}(a_\psi b_\Psi)\widetilde{g}(c^{\psi\Psi})\\
&= \sum \widetilde{f}_1(a_\psi)\widetilde{f}_2(b_\Psi)\widetilde{g}(c^{\psi\Psi})\\
&= \sum(\widetilde{g}_{\varphi\Phi} \underline{\otimes} \widetilde{f}_1^\Phi \underline{\otimes} \widetilde{f}_2^\varphi)(c\otimes a\otimes b)
\end{aligned}$$

and

$$\Big(\sum \varepsilon_{\widetilde{C}}(\widetilde{f}^\varphi)\widetilde{g}_\varphi\Big)(c) = \sum(\widetilde{g}_\varphi \underline{\otimes} \widetilde{f}^\varphi)(c\otimes 1) = \sum \widetilde{f}(1_\psi)\widetilde{g}(c^\psi) = \widetilde{f}(1_A)\widetilde{g}(c) = (\varepsilon_{\widetilde{C}}(\widetilde{f})\widetilde{g})(c).$$

Hence $(\widetilde{A}, \widetilde{C}, \varphi)$ is a right–right entwining structure. Since $(A, \widetilde{C})$ is a measuring α-pairing, it follows by Theorem 1.6 that ${}_R\widetilde{C}$ is locally projective. The isomorphisms of categories 9 and 10 follow then by Theorem 1.15. □

LEMMA 2.2. *Consider the entwining structure* $(\widetilde{A}, \widetilde{C}, \varphi)$.

(1) *Consider the measuring* α*-pairing* $(A, \widetilde{C})$. *Let* $M \in \mathcal{M}_A^C(\psi)$ *and consider* M^* *with the induced right* $\widetilde{A}$*-module and left* A*-module structures. Then* $M_r := \mathrm{Rat}^{\widetilde{C}}({}_AM^*) \in \mathcal{M}_{\widetilde{A}}^{\widetilde{C}}(\varphi)$.
If $M, N \in \mathcal{M}_A^C(\psi)$ *and* $f\colon M \to N$ *is* A*-linear* C*-colinear, then* $f^*\colon N_r \to M_r$ *is* $\widetilde{A}$*-linear* $\widetilde{C}$*-colinear.*

(2) *Assume the measuring* R*-pairing* $(\widetilde{A}, C)$ *to satisfy the* α*-condition (equivalently,* ${}_RC$ *is locally projective and* $\widetilde{A} \subseteq C^*$ *is dense). Let* $K \in \mathcal{M}_{\widetilde{A}}^{\widetilde{C}}(\varphi)$ *and consider* K^* *with the induced left* $\widetilde{A}$*-module and right* A*-module structures. Then* $K^r := \mathrm{Rat}^C({}_{\widetilde{A}}K^*) \in \mathcal{M}_A^C(\psi)$.
If $K, L \in \mathcal{M}_{\widetilde{A}}^{\widetilde{C}}(\varphi)$ *and* $g\colon K \to L$ *is* $\widetilde{A}$*-linear* $\widetilde{C}$*-colinear, then* $g^*\colon L^r \to K^r$ *is* A*-linear* C*-colinear.*

Proof. (1) Let $M \in \mathcal{M}_A^C(\psi)$. Since $(A, \widetilde{C})$ is a measuring α-pairing, $M_r := \mathrm{Rat}^{\widetilde{C}}({}_AM^*)$ is by Theorem 1.6 a right $\widetilde{C}$-comodule. Moreover we have for all $a \in$

A, $\widetilde{g} \in \widetilde{A}$, $m \in M$ and $h \in M_r$:

$$\begin{aligned}
[a(h\widetilde{g})](m) &= (h\widetilde{g})(ma) \\
&= h(\widetilde{g}[ma]) \\
&= \sum h((ma)_{\langle 0\rangle}\widetilde{g}((ma)_{\langle 1\rangle})) \\
&= \sum h(m_{\langle 0\rangle}a_\psi \widetilde{g}(m^\psi_{\langle 1\rangle})) \\
&= \sum (a_\psi h)(m_{\langle 0\rangle})\widetilde{g}(m^\psi_{\langle 1\rangle}) \\
&= \sum h_{\langle 0\rangle}(m_{\langle 0\rangle})h_{\langle 1\rangle}(a_\psi)\widetilde{g}(m^\psi_{\langle 1\rangle}) \\
&= \sum h_{\langle 0\rangle}(m_{\langle 0\rangle})\widetilde{g}_\varphi(m_{\langle 1\rangle})h^\varphi_{\langle 1\rangle}(a) \\
&= \sum h_{\langle 0\rangle}(\widetilde{g}_\varphi m)h^\varphi_{\langle 1\rangle}(a) \\
&= \sum (h_{\langle 0\rangle}\widetilde{g}_\varphi)(m)h^\varphi_{\langle 1\rangle}(a) \\
&= \Big(\sum (h_{\langle 0\rangle}\widetilde{g}_\varphi)h^\varphi_{\langle 1\rangle}(a)\Big)(m),
\end{aligned}$$

i.e. $h\widetilde{g} \in M_r$ with $\varrho(h\widetilde{g}) = \sum h_{\langle 0\rangle}\widetilde{g}_\varphi \otimes h^\varphi_{\langle 1\rangle}$. Hence $M_r \in \mathcal{M}^{\widetilde{C}}_{\widetilde{A}}(\varphi)$.

The second statement follows now by Lemma 1.5(4) and Theorem 1.6.

(2) Let $K \in \mathcal{M}^{\widetilde{C}}_{\widetilde{A}}(\varphi)$. By assumption $(\widetilde{A}, C)$ satisfies the α-condition, hence $K^r := \mathrm{Rat}^C({}_{\widetilde{A}}K^*)$ is by Theorem 1.6 a right C-comodule. Moreover we have for all $a \in A$, $\widetilde{g} \in \widetilde{A}$, $k \in K$ and $f \in K^r$:

$$\begin{aligned}
[\widetilde{g}(fa)](k) &= (fa)(k\widetilde{g}) \\
&= f(a(k\widetilde{g})) \\
&= \sum f((k\widetilde{g})_{\langle 0\rangle}(k\widetilde{g})_{\langle 1\rangle}(a)) \\
&= \sum f(n_{\langle 0\rangle}\widetilde{g}_\varphi(k^\varphi_{\langle 1\rangle})(a)) \\
&= \sum (\widetilde{g}_\varphi f)(k_{\langle 0\rangle})k^\varphi_{\langle 1\rangle}(a) \\
&= \sum f_{\langle 0\rangle}(k_{\langle 0\rangle})\widetilde{g}_\varphi(f_{\langle 1\rangle})k^\varphi_{\langle 1\rangle}(a) \\
&= \sum f_{\langle 0\rangle}(k_{\langle 0\rangle})\langle a_\psi, k_{\langle 1\rangle}\rangle\widetilde{g}(f^\psi_{\langle 1\rangle}) \\
&= \sum f_{\langle 0\rangle}(a_\psi k)\widetilde{g}(f^\psi_{\langle 1\rangle}) \\
&= \sum (f_{\langle 0\rangle}a_\psi)(k)\widetilde{g}(f^\psi_{\langle 1\rangle}) \\
&= \Big(\sum (f_{\langle 0\rangle}a_\psi)\widetilde{g}(f^\psi_{\langle 1\rangle})\Big)(k),
\end{aligned}$$

i.e. $fa \in \mathrm{Rat}^C({}_{\widetilde{A}}K^*)$ with $\varrho(fa) = \sum f_{\langle 0\rangle}a_\psi \otimes f^\psi_{\langle 1\rangle}$. Hence $K^r \in \mathcal{M}^C_A(\psi)$. As in (1), the second statement follows by Lemma 1.5 (4) and Theorem 1.6. □

DEFINITION 2.3. With the same notation and assumptions as above, we call the right–right entwining structure $(\widetilde{A}, \widetilde{C}, \varphi)$ a *dual entwining structure* of (A, C, ψ). We also call M_r (resp. K^r) a *dual entwined module* of M (resp. of K).

THEOREM 2.4. *Assume that $(\widetilde{A}, C)$ satisfies the α-condition. Then we have right adjoint contravariant functors*

$$(-)_r\colon \mathcal{M}_A^C(\psi) \to \mathcal{M}_{\widetilde{A}}^{\widetilde{C}}(\varphi) \quad \textit{and} \quad (-)^r\colon \mathcal{M}_{\widetilde{A}}^{\widetilde{C}}(\varphi) \to \mathcal{M}_A^C(\psi).$$

Proof. Let $M \in \mathcal{M}_A^C(\psi)$, $K \in \mathcal{M}_{\widetilde{A}}^{\widetilde{C}}(\varphi)$ and consider the canonical R-linear maps

$$\lambda_M\colon M \to (M_r)^* \quad \text{and} \quad \lambda_K\colon K \to (K^r)^*.$$

Clearly λ_M is $\widetilde{A}$-linear and λ_K is A-linear, hence $\lambda_M(M) \subseteq (M_r)^r$ and $\lambda_K(K) \subseteq (K^r)_r$ by Lemma 1.5(4). It is easy then to see that the right-adjointness is given by the functorial inverse isomorphisms

$$\Lambda_{M,K}\colon \mathrm{Hom}_A^C(M, K^r) \to \mathrm{Hom}_{\widetilde{A}}^{\widetilde{C}}(K, M_r),\ f \mapsto f^* \circ \lambda_K,$$

$$\Gamma_{M,K}\colon \mathrm{Hom}_{\widetilde{A}}^{\widetilde{C}}(K, M_r) \to \mathrm{Hom}_A^C(M, K^r),\ g \mapsto g^* \circ \lambda_M. \qquad \square$$

2.5. Let (A, C, ψ), (B, D, Ψ) be right–right entwining structures and assume that A, B are α-algebras. Let $(\gamma, \delta)\colon (A, C, \psi) \to (B, D, \Psi)$ be a morphism in $\mathbb{E}_\bullet^\bullet$, $\gamma^\circ\colon B^\circ \to A^\circ$ the induced R-coalgebra morphism and $\delta^*\colon D^* \to C^*$ the induced R-algebra morphism. Let $\widetilde{C} \subseteq A^\circ$, $\widetilde{D} \subseteq B^\circ$ be R-subcoalgebras and $\widetilde{A} \subseteq C^*$, $\widetilde{B} \subseteq D^*$ be R-subalgebras with $\varepsilon_C \in \widetilde{A}$, $\varepsilon_D \in \widetilde{B}$ and assume that $\gamma^\circ(\widetilde{D}) \subseteq \widetilde{C}$ and $\delta^*(\widetilde{B}) \subseteq \widetilde{A}$. Assume moreover that $\psi^*(\widetilde{C} \otimes_R \widetilde{A}) \subseteq \widetilde{A} \otimes_R \widetilde{C}$, $\Psi^*(\widetilde{D} \otimes_R \widetilde{B}) \subseteq \widetilde{B} \otimes_R \widetilde{D}$ and let $(\widetilde{A}, \widetilde{C}, \varphi)$ and $(\widetilde{B}, \widetilde{D}, \Phi)$ be the induced dual entwining structures of (A, C, ψ) and (B, D, Ψ) respectively. Then we have for all $\widetilde{g} \in \widetilde{B}$, $\widetilde{f} \in \widetilde{D}$, $d \in D$ and $b \in B$:

$$\begin{aligned}\Big(\sum \delta^*(\widetilde{g}_\Phi) \underline{\otimes} \gamma^\circ(\widetilde{f}^\Phi)\Big)(c \otimes a) &= \sum (\widetilde{g}_\Phi \underline{\otimes} \widetilde{f}^\Phi)(\delta(c) \otimes \gamma(a)) \\ &= \sum (\widetilde{f} \underline{\otimes} \widetilde{g})(\gamma(a)_\Psi \otimes \delta(c)^\Psi) \\ &= \sum (\widetilde{f} \underline{\otimes} \widetilde{g})(\gamma(a_\psi) \otimes \delta(c^\psi)) \\ &= \sum (\gamma^\circ(\widetilde{f}) \underline{\otimes} \delta^*(\widetilde{g}))(a_\psi \otimes c^\psi) \\ &= \sum (\delta^*(\widetilde{g})_\varphi \underline{\otimes} \gamma^\circ(\widetilde{f})^\varphi)(c \otimes a),\end{aligned}$$

i.e. $(\delta^*, \gamma^\circ)\colon (\widetilde{B}, \widetilde{D}, \Phi) \to (\widetilde{A}, \widetilde{C}, \varphi)$ is a morphism in $\mathbb{E}_\bullet^\bullet$.

3. Dual Doi–Koppinen Modules

Doi–Koppinen structures were presented independently by Y. Doi [13] and M. Koppinen [17] and provide a fundamental example of entwining structures. The corresponding categories of Doi–Koppinen modules unify themselves many categories of modules well studied by Hopf-algebraists such as the categories of *Hopf modules* [24, 4.1], *relative Hopf modules* [11], *Doi's* $[C, H]$*-modules* [11], *Dimodules, Yetter–Drinfeld modules* and *modules graded by* G*-sets* [9].

Dual Module (Co)algebras & Comodule (Co)algebras

Before we present our dual Doi–Koppinen modules we introduce some definitions and results concerning duality of (co)module (co)algebras.

DEFINITION 3.1. Let H be an R-bialgebra.

(1) A *right* H*-module algebra* is an R-algebra (A, μ_A, η_A) with a right H-module structure through $\phi_A\colon A \otimes_R H \to A$, such that μ_A and η_A are H*-linear*, i.e.

$$(ab)h = \sum (ah_1)(bh_2) \quad \text{and} \quad 1_A h = \varepsilon_H(h) 1_A \quad \text{for all } a, b \in A \text{ and } h \in H. \tag{11}$$

In a similar way we define a *left* H*-module algebra*. An H*-bimodule algebra,* is a left and a right H-module algebra, such that A is an H-bimodule with the given left and right H-actions.

(2) A *right* H*-module coalgebra* is an R-coalgebra $(C, \Delta_C, \varepsilon_C)$ with a right H-module structure through $\phi_C\colon C \otimes_R H \to C$, such that Δ_C and ε_C are H*-linear* (equivalently, ϕ_C is an R-coalgebra morphism), i.e.

$$\Delta_C(ch) = \sum c_1 h_1 \otimes c_2 h_2 \quad \text{and} \quad \varepsilon_C(ch) = \varepsilon_C(c)\varepsilon_H(h) \quad \text{for all } c \in C \text{ and } h \in H. \tag{12}$$

In a similar way we define a *left* H*-module coalgebra*. An H*-bimodule coalgebra*, is a left and a right H-module coalgebra, which is an H-bimodule with the given left and right H-actions.

(3) A *right* H*-comodule algebra* is an R-algebra (A, μ_A, η_A) with a right H-comodule structure through $\varrho_A\colon A \to A \otimes_R H$, such that μ_A and η_A are H*-colinear* (equivalently, ϱ_A is an R-algebra morphism), i.e.

$$\varrho_A(ab) = \sum a_{\langle 0\rangle} b_{\langle 0\rangle} \otimes a_{\langle 1\rangle} b_{\langle 1\rangle} \quad \text{and} \quad \varrho_A(1_A) = 1_A \otimes 1_H. \tag{13}$$

In a similar way we define a *left* H*-comodule algebra*. An H*-bicomodule algebra* is a left and right H-comodule algebra, which is an H-bicomodule under the given left and right H-coactions.

(4) A *right* H*-comodule coalgebra* is an R-coalgebra $(C, \Delta_C, \varepsilon_C)$ with a right H-comodule structure through $\varrho_C\colon C \to C \otimes_R H$, such that Δ_C and ε_C are H*-colinear*, i.e.

$$\sum c_{\langle 0\rangle 1} \otimes c_{\langle 0\rangle 2} \otimes c_{\langle 1\rangle} = \sum c_{1\langle 0\rangle} \otimes c_{2\langle 0\rangle} \otimes c_{1\langle 1\rangle} c_{2\langle 1\rangle}, \sum \varepsilon_C(c_{\langle 0\rangle}) c_{\langle 1\rangle} = \varepsilon_C(c) 1_H. \tag{14}$$

In a similar way we define a *left H-comodule coalgebra*. An *H-bicomodule coalgebra* is a left and a right H-comodule coalgebra, which is an H-bicomodule with the given left and right H-coactions.

LEMMA 3.2. *Let R be Noetherian and H an R-bialgebra. If A is a right (resp. a left) H-module algebra, then $A^\circ \subset A^*$ is a left (resp. a right) H-submodule. If A is an H-bimodule algebra, then $A^\circ \subseteq H^*$ is an H-subbimodule.*

Proof. Let A be a right H-module algebra. If $f \in A^\circ$, then we have for all $h \in H$ and $a, b \in A$:

$$\begin{aligned}(b(hf))(a) &= (hf)(ab)\\ &= f((ab)h)\\ &= f\Big(\sum(ah_1)(bh_2)\Big)\\ &= \sum f_1(ah_1)f_2(bh_2)\\ &= \Big[\sum(h_2f_2)(b)(h_1f_1)\Big](a).\end{aligned}$$

So $hf \in A^\circ$ for every $h \in H$, i.e. $A^\circ \subset A^*$ is a left H-submodule.

If A is a left H-module algebra, then a similar argument shows that $A^\circ \subset A^*$ is a right H-submodule. The last statement becomes then obvious. □

PROPOSITION 3.3. *Let R be Noetherian, H an α-bialgebra and $U \subseteq H^\circ$ an R-subbialgebra.*

(1) *Consider the measuring R-pairing (U, H). If A is a right (a left) H-comodule algebra, then A is a left (a right) U-module algebra and A° is a right (a left) U-module coalgebra. If A is an H-bicomodule algebra, then A is a U-bimodule algebra and A° is a U-bimodule coalgebra.*
(2) *Consider the measuring α-pairing (H, U).*
    (a) *If A is a right (a left) U-comodule algebra, then A is a left (a right) H-module algebra. If A is a U-bicomodule algebra, then A is an H-bimodule algebra.*
    (b) *If A is a left (a right) H-module algebra, then $\mathrm{Rat}^U({}_HA)$ (resp. ${}^U\mathrm{Rat}(A_H)$) is a right (a left) U-comodule algebra. If A is an H-bimodule algebra, then ${}^U\mathrm{Rat}^U({}_HA_H)$ is a U-bicomodule algebra.*

Proof. (1) Without loss of generality assume that A is a right H-comodule algebra through an R-algebra morphism $\varrho\colon A \to A \otimes_R H$. For all $a, b \in A$ and $f \in U$ we have

$$\begin{aligned}f \rightharpoonup ab &= \sum(ab)_{\langle 0\rangle} f((ab)_{\langle 1\rangle})\\ &= \sum a_{\langle 0\rangle}b_{\langle 0\rangle} f(a_{\langle 1\rangle}b_{\langle 1\rangle}) \sum a_{\langle 0\rangle}b_{\langle 0\rangle} f_1(a_{\langle 1\rangle})f_2(b_{\langle 1\rangle})\\ &= \sum(f_1 \rightharpoonup a)(f_2 \rightharpoonup b),\end{aligned}$$

and moreover

$$f \rightharpoonup 1_A = \sum 1_{\langle 0\rangle} f(1_{\langle 1\rangle}) = 1_A f(1_H) = 1_A \varepsilon_U(f).$$

Hence A is a left U-module algebra.

Consider now the canonical R-linear map $\varpi\colon A^\circ \otimes_R U \to (A \otimes_R H)^\circ$. Then (A, A°), $(A \otimes_R H, A^\circ \otimes_R U)$ are measuring α-pairings and we have a morphism of R-pairings

$$(\varrho, \varrho^\circ \circ \varpi)\colon (A \otimes_R H, A^\circ \otimes_R U) \to (A, A^\circ).$$

Moreover $A^\circ \otimes_R A^\circ \hookrightarrow (A \otimes_R A)^*$ and it follows from the assumption and Lemma 1.7(1) that $\varrho^\circ \circ \varpi\colon A^\circ \otimes_R U \to A^\circ$ is an R-coalgebra morphism, i.e. A° is a right U-module coalgebra. If A is an H-bicomodule, then A is a U-bimodule by Theorem 1.6 and $A^\circ \subseteq A^*$ is a U-subbimodule by Lemma 3.2, hence A is a U-bimodule algebra and A° is a U-bimodule algebra.

(2) Consider the measuring α-pairing (H, U).

(a) Without loss of generality, let A be a right U-comodule algebra. Then we have for all $h \in H$ and $a, b \in A$:

$$\begin{aligned} h \rightharpoonup (ab) &= \sum (ab)_{\langle 0\rangle} \langle h, (ab)_{\langle 1\rangle}\rangle \\ &= \sum a_{\langle 0\rangle} b_{\langle 0\rangle} \langle h, a_{\langle 1\rangle} \star b_{\langle 1\rangle}\rangle \\ &= \sum a_{\langle 0\rangle} b_{\langle 0\rangle} \langle h_1, a_{\langle 1\rangle}\rangle \langle h_2, b_{\langle 1\rangle}\rangle \\ &= \sum (h_1 \rightharpoonup a)(h_2 \rightharpoonup b) \end{aligned}$$

and

$$h \rightharpoonup 1_A = \varepsilon(h) 1_A,$$

i.e. A is a left H-module algebra. If A is a U-bicomodule algebra, then A is by Theorem 1.6 an H-bimodule, hence an H-bimodule algebra.

(b) Assume now that A is a left H-module algebra. Then we have for all $a, b \in \mathrm{Rat}^U({}_H A)$ and $h \in H$:

$$\begin{aligned} (h \rightharpoonup ab) &= \sum (h_1 \rightharpoonup a)(h_2 \rightharpoonup b) \\ &= \sum a_{\langle 0\rangle} \langle h_1, a_{\langle 1\rangle}\rangle b_{\langle 0\rangle} \langle h_2, b_{\langle 1\rangle}\rangle \\ &= \sum a_{\langle 0\rangle} b_{\langle 0\rangle} \langle h, a_{\langle 1\rangle} \star b_{\langle 1\rangle}\rangle, \end{aligned}$$

i.e. $ab \in \mathrm{Rat}^U({}_H A)$ with $\varrho(ab) = \sum a_{\langle 0\rangle} b_{\langle 0\rangle} \otimes a_{\langle 1\rangle} \star b_{\langle 1\rangle}$. Note also that $h \rightharpoonup 1_A = \varepsilon_H(h) 1_A$, i.e. $1_A \in \mathrm{Rat}^U({}_H A)$, with $\varrho(1_A) = 1_A \otimes \varepsilon_H = 1_A \otimes 1_U$. Hence, $\mathrm{Rat}^U({}_H A)$ is a right U-comodule algebra. If A is an H-bimodule algebra, then ${}^U\mathrm{Rat}^U({}_H A_H)$ is by Theorem 1.6 a U-bicomodule, hence a U-bicomodule algebra. □

PROPOSITION 3.4. *Let R be Noetherian, H an α-bialgebra and $U \subseteq H^\circ$ an R-subbialgebra.*

(1) *Consider the measuring α-pairing (H, U). If C is a right (a left) H-module coalgebra, then C^* is a left (a right) H-module algebra and $\mathrm{Rat}^U({}_HC^*)$ is a right (a left) U-comodule algebra. If C is an H-bimodule coalgebra, then C^* is an H-bimodule algebra and ${}^U\mathrm{Rat}^U({}_HC^*_H)$ is a U-bicomodule algebra.*
(2) *Consider the measuring R-pairing (U, H). If C is a right (a left) H-comodule coalgebra, then C is a left (a right) U-module coalgebra and C^* is a right (a left) U-module algebra. If C is an H-bicomodule coalgebra, then C is a U-bimodule coalgebra and C^* is a U-bimodule algebra.*

Proof. (1) Let C be a right H-module coalgebra. Then we have for all $f, g \in C^*$, $h \in H$, $c \in C$:

$$\begin{aligned}(h \rightharpoonup (f \star g))(c) &= (f \star g)(ch) = \sum f((ch)_1)g((ch)_2) \\ &= \sum f(c_1h_1)g(c_2h_2) = \sum(h_1f)(c_1)(h_2g)(c_2) \\ &= \Big(\sum(h_1f) \star (h_2g)\Big)(c)\end{aligned}$$

and

$$(h\varepsilon_C)(c) = \varepsilon_C(ch) = \varepsilon_C(c)\varepsilon_H(h) = (\varepsilon_H(h)\varepsilon_C)(c),$$

i.e. C^* is a left H-module algebra. By Proposition 3.3(2-b), $\mathrm{Rat}^U({}_HC^*)$ is a right U-comodule algebra. If C is an H-bimodule coalgebra, then C^* is an H-bimodule algebra and ${}^U\mathrm{Rat}^U({}_HC^*_H)$ is a U-bicomodule by Theorem 1.6, hence a U-bicomodule algebra.

(2) Without loss of generality, assume that C is a right H-comodule coalgebra. For all $c \in C$, $f \in U$ we have

$$\begin{aligned}\sum(f \rightharpoonup c)_1 \otimes (f \rightharpoonup c)_2 &= \sum c_{\langle 0\rangle 1} \otimes c_{\langle 0\rangle 2} f(c_{\langle 1\rangle}) \\ &= \sum c_{1\langle 0\rangle} \otimes c_{2\langle 0\rangle} f(c_{1\langle 1\rangle}c_{2\langle 1\rangle}) \\ &= \sum c_{1\langle 0\rangle} \otimes c_{2\langle 0\rangle} f_1(c_{1\langle 1\rangle}) f_2(c_{2\langle 1\rangle}) \\ &= \sum c_{1\langle 0\rangle} f_1(c_{1\langle 1\rangle}) \otimes c_{2\langle 0\rangle} f_2(c_{2\langle 1\rangle}) \\ &= \sum f_1 \rightharpoonup c_1 \otimes f_2 \rightharpoonup c_2\end{aligned}$$

and

$$\varepsilon_C(f \rightharpoonup c) = \sum \varepsilon_C(c_{\langle 0\rangle}) f(c_{\langle 1\rangle}) = f(\varepsilon_C(c)1_H) = \varepsilon_C(c)\varepsilon_U(f),$$

i.e. C is a left U-module coalgebra. Analogous to (1) one can show that C^* is a right U-module algebra. If C is an H-bicomodule coalgebra, then C is a U-bimodule by Theorem 1.6. Hence, C is a U-bimodule coalgebra and C^* is a U-bimodule algebra. □

The following result generalizes ([22, Example 4.1.10]):

COROLLARY 3.5. *Let H be an R-bialgebra and consider H^* as an H-bimodule with the regular left and right H-actions.*

(1) *Since H is an H-bimodule coalgebra, it follows* (*by Proposition* 3.4(1)) *that H^* is an H-bimodule algebra. If moreover R is Noetherian and H is an α-algebra, then $H^\circ \subset H^*$ is an H-subbimodule algebra.*
(2) *Let R be Noetherian, H an α-algebra and $U \subseteq H^\circ$ an R-subbialgebra. Since H is an H-bicomodule algebra, it follows* (*by Proposition* 3.3(1))*, that H is a U-bimodule algebra. In particular H is an H°-bimodule algebra.*

Doi–Koppinen Modules

3.6. A *right–right Doi–Koppinen structure* over R is a triple (H, A, C) consisting of an R-bialgebra H, a right H-comodule algebra A and a right H-module coalgebra C. Let (H, A, C), (K, B, D) be right–right Doi–Koppinen structures. Then a morphism $(\beta, \gamma, \delta)\colon (H, A, C) \to (K, B, D)$ of Doi–Koppinen structures, consists of an R-bialgebra morphism $\beta\colon H \to K$, an R-algebra morphism $\gamma\colon A \to B$ and an R-coalgebra morphism $\delta\colon C \to D$, such that

$$\sum \gamma(a_{\langle 0\rangle}) \otimes \delta(ca_{\langle 1\rangle}) = \sum \gamma(a)_{\langle 0\rangle} \otimes \delta(c)\gamma(a)_{\langle 1\rangle} \quad \text{for all } a \in A \text{ and } c \in C.$$

The category of right–right Doi–Koppinen modules is denoted by $\mathbb{DK}^{\bullet}_{\bullet}$. For definitions of the categories of left–left, right–left and left–right Doi–Koppinen structures the reader may refer to [9].

3.7. Let (A, H, C) be a right–right Doi–Koppinen structure. A *right–right Doi–Koppinen module* corresponding to (H, A, C) is a right A-module M, which is also a right C-comodule, such that

$$\varrho_M(ma) = \sum m_{\langle 0\rangle}a_{\langle 0\rangle} \otimes m_{\langle 1\rangle}a_{\langle 1\rangle} \quad \text{for all } m \in M \text{ and } a \in A.$$

For Doi–Koppinen modules M, N corresponding to (A, H, C) we denote with $\mathrm{Hom}_A^C(M, N)$ the set of all A-linear C-colinear maps from M to N. By $\mathcal{M}(H)_A^C$ we denote the category of right–right Doi–Koppinen modules corresponding to (A, H, C) and with A-linear C-colinear morphisms. Setting

$$\psi\colon C \otimes_R A \to A \otimes_R C,\ c \otimes a \mapsto \sum a_{\langle 0\rangle} \otimes ca_{\langle 1\rangle}, \tag{15}$$

it follows by [6, page 295] that (A, C, ψ) is a right–right entwining structure and $\mathcal{M}(H)_A^C \simeq \mathcal{M}_A^C(\psi)$. Moreover $\#^{\mathrm{op}}(C, A) := \mathrm{Hom}_R(C, A)$, introduced first in [17, 2.2], is an R-algebra with multiplication

$$(f \cdot g)(c) = \sum f(c_2)_{\langle 0\rangle} g(c_1 f(c_2)_{\langle 1\rangle}) \tag{16}$$

and unity $\eta_A \circ \varepsilon_C$.

Duality Theorems

In what follows we present for every α-bialgebra H over a Noetherian ground ring R and every right H-module coalgebra (left H-module coalgebra) C a right H°-comodule R-algebra (a left H°-comodule algebra) C^0, that plays an important role by the dualization process in the rest of this note. In our *infinite versions* of duality theorems, C^0 will play the role of C^* in the finite versions (e.g., [28]).

DEFINITION 3.8. Let R be Noetherian, H an α-bialgebra and consider the measuring α-pairing (H, H°). For every right (resp. left) H-module coalgebra C we have by Proposition 3.4(1) the right (resp. the left) H°-comodule algebra

$$C^0 := \mathrm{Rat}^{H^\circ}({}_H C^*) \quad (\text{resp. } C^0 := {}^{H^\circ}\mathrm{Rat}(C^*_H)).$$

In view of 1.3 and Propositions 3.3, 3.4 we get

THEOREM 3.9. *Let R be Noetherian.*

(1) *Let (H, A, C) be a right–right Doi–Koppinen structure and assume that H, A are α-algebras. Then (H°, C^0, A°) is a dual right–right Doi–Koppinen structure of (H, A, C) and we have isomorphism of categories*

$$\mathcal{M}(H^\circ)^{A^\circ}_{C^0} \simeq \mathrm{Rat}^{A^\circ}(\mathcal{M}_{\#^{\mathrm{op}}(A^\circ, C^0)}) = \sigma[(C^0 \otimes_R A^\circ)_{\#^{\mathrm{op}}(A^\circ, C^0)}].$$

If, moreover, ${}_R A$ is f.g. projective, then

$$\mathcal{M}(H^\circ)^{A^*}_{C^0} \simeq \mathcal{M}_{\#^{\mathrm{op}}(A^*, C^0)}.$$

(2) *Let $(\beta, \gamma, \delta)\colon (H, A, C) \to (K, B, D)$ be a morphism in $\mathbb{DK}^{\bullet}_{\bullet}$. If H, K, A, B are α-algebras and $\delta^*(D^0) \subseteq C^0$ (e.g., δ is H-linear, or $C^0 = C^*$), then $(\beta^\circ, \delta^0, \gamma^\circ)\colon (K^\circ, D^0, B^\circ) \to (H^\circ, C^0, A^\circ)$ is a morphism in $\mathbb{DK}^{\bullet}_{\bullet}$.*

As a corollary of Theorem 2.4 we get the following theorem:

THEOREM 3.10. *Let R be Noetherian, (H, A, C) a right–right Doi–Koppinen structure with H, A α-algebras and consider the dual right–right Doi–Koppinen structure (H°, C^0, A°).*

(1) *There is a contravariant functor*

$$(-)^0\colon \mathcal{M}(H)^C_A \to \mathcal{M}(H^\circ)^{A^\circ}_{C^0},\ M \longmapsto M^0 := \mathrm{Rat}^{A^\circ}({}_A M^*).$$

(2) *If $P := (C^0, C)$ satisfies the α-condition (equivalently ${}_RC$ is locally projective and $C^0 \subset C^*$ is dense), then there is a contravariant functor*

$$(-)^{\diamond}\colon\ \mathcal{M}(H^\circ)_{C^0}^{A^\circ} \to \mathcal{M}(H)_A^C,\ K \longmapsto K^{\diamond} := \mathrm{Rat}^C({}_{C^0}K^*). \tag{17}$$

Moreover the contravariant functors $(-)^0$ and $(-)^{\diamond}$ are right adjoint.

DEFINITION 3.11. We say an R-algebra A is *R-c-cogenerated*, if for every R-cofinite ideal $I \lhd A$, the R-module A/I is R-cogenerated.

Remark 3.12. Let R be Noetherian and A an α-algebra. For every right (resp. left) A-module M, set $M^0 := \mathrm{Rat}^{A^\circ}({}_AM^*)$ (resp. $M^0 := {}^{A^\circ}\mathrm{Rat}(M_A^*)$). If A is R-c-cogenerated, then we have by [2, Proposition 3.3.15]

$$\begin{aligned} M^0 := \{f \in M^* \mid\ & f(MI) = 0 \text{ (resp. } f(IM) = 0) \\ & \text{for some } R\text{-cofinite ideal } I \lhd A\}. \end{aligned} \tag{18}$$

EXAMPLE 3.13. Let A be an R-algebra, C an R-coalgebra and consider the category of right A-modules and right C-comodules satisfying the compatibility relation

$$\varrho_M(ma) = \sum m_{\langle 0\rangle}a \otimes m_{\langle 1\rangle} \quad \text{for all } m \in M \text{ and } a \in A.$$

The category of such modules and A-linear C-colinear morphisms is called the category of *Long dimodules,* denoted by $\mathcal{L}_A^C$, and was introduced by F. Long in [21]. Considering A as a trivial R-comodule algebra and C as a trivial right R-module coalgebra we get a right–right Doi–Koppinen structure (R, A, C) and it follows that $\mathcal{L}_A^C \simeq \mathcal{M}(R)_A^C$. If A is an α-algebra, then (R, C^*, A°) is a dual right–right Doi–Koppinen structure of (R, A, C) and the contravariant functors $(-)^0\colon\ \mathcal{L}_A^C \to \mathcal{L}_{C^*}^{A^\circ}$ and $(-)^{\diamond}\colon\ \mathcal{L}_{C^*}^{A^\circ} \to \mathcal{L}_A^C$ are right adjoint.

Inspired by [20] and in contradiction to [1, page 138], the following example shows that for a Hopf R-algebra H and an H-module algebra A over a field, the dual R-coalgebra A° need *not* be an H°-comodule coalgebra.

COUNTEREXAMPLE 3.14. Let R be a field and H a *coreflexive* Hopf R-algebra with $\dim(H) = \infty$ (e.g., the Hopf R-algebra of [19, Example 5]). By Lemma 3.5 H^* is a right H-module algebra. If $H^{*\circ} \simeq H$ were a right H°-comodule coalgebra, then we would have an R-cofinite ideal $J \lhd H$ with

$$0 = \langle 1_H, H^* \leftharpoondown J\rangle = \langle J, H^*\rangle.$$

But we would get then $J = 0$ (which contradicts the assumption $\dim(H) = \infty$).

Remark 3.15. Let H be an R-bialgebra, A a right H-module algebra and C a right H-comodule coalgebra. Then (H, A, C) is called a *right–right alternative Doi–Koppinen structure*. Such structures were introduced by P. Schauenburg in [23], who showed that with

$$\psi\colon\ C \otimes_R A \to A \otimes_R C,\ c \otimes a \mapsto \sum ac_{\langle 1\rangle} \otimes c_{\langle 0\rangle},$$

(A, C, ψ) is a right–right entwining structure. Moreover he gave an example of such an entwining structure that *cannot* be derived form a Doi–Koppinen structure. The previous counterexample shows that, even over base fields, (H°, C^0, A°) *may not* be a dual alternative Doi–Koppinen structure of (H, A, C).

Cleft *H*-Extensions

Hopf–Galois extensions were presented by S. Chase and M. Sweedler [10] for a *commutative* R-algebra acting on a Hopf R-Hopf and are considered as generalization of the classical Galois extensions over fields (e.g., [22, 8.1.2]). In [18] H. Kreimer and M. Takeuchi extended these to the *noncommutative* case.

3.16. *H*-EXTENSIONS ([12])

Let H be an R-bialgebra, B a right H-comodule algebra and $A := B^{\mathrm{co}H} = \{a \in B \mid \varrho(a) = a \otimes 1_H\}$. Then A is an R-algebra and the algebra extension $A \hookrightarrow B$ is called a *right H-extension.*

A (*total*) *integral* for B is an H*-colinear* map $\gamma\colon\ H \to B$ (with $\gamma(1_H) = 1_B$). If B admits an integral, that is invertible in $(\mathrm{Hom}_R(H, B), \star)$, then $A \hookrightarrow B$ is called a *cleft* right H-extension.

EXAMPLE 3.17. Let H be an R-bialgebra. By [15, Corollary 6] H/R is a *cleft* H-extension, iff H is a Hopf R-algebra. In this case $\mathrm{id}_H\colon\ H \to H$ is an invertible total integral with inverse the antipode S_H.

3.18. *H*-COEXTENSIONS

Let H be an R-bialgebra and D a right H-module coalgebra. Then $H^+ := \mathrm{Ker}(\varepsilon_H)$ is an H-coideal, DH^+ is a D-coideal and $C := D/DH^+$ is a right H-module coalgebra with the induced H-module structure. The canonical coalgebra epimorphism $\pi\colon\ D \to C$ is called a right H*-coextension* of D.

A (*total*) *cointegral* for D is an H-linear map $\omega\colon\ D \to H$ (with $\varepsilon_H \circ \omega = \varepsilon_D$). A right H-coextension $\pi\colon\ D \to C$ is called *cocleft*, if D admits cointegral, that is invertible in $(\mathrm{Hom}_R(D, H), \star)$.

As a corollary of our results in this section we get

PROPOSITION 3.19. *Let R be Noetherian, H a Hopf α-algebra with bijective antipode, D a right H-module coalgebra and $C := D/DH^{+}$. If $\pi\colon D \to C$ is a* (cocleft) *right H-coextension, then $\pi^{\circ}\colon C^{0} \hookrightarrow D^{0}$ is a* (cleft) *right H°-extension.*

Proof. Let D be a right H-module coalgebra through $\phi_D\colon D \otimes_R H \to D$. By Proposition 3.4(1) D^0 is a right H°-comodule algebra through $\phi_D^{\circ}\colon D^0 \to D^0 \otimes_R H^{\circ}$. Moreover we have $C^* = (D^*)^H := \{g \in D^* \mid hg = \varepsilon(h)g$ for every $h \in H\}$. Hence $(D^0)^{\mathrm{co}H^{\circ}} = (D^0)^H = C^0$ (by [2, Lemma 2.5.15]), i.e. $\pi^{\circ}\colon C^0 \hookrightarrow D^0$ is a right H°-extension.

If $\omega\colon D \to H$ is a cointegral for D, then ω is by definition H-linear and so $\omega^{\circ} \in \mathrm{Hom}_{H-}(H^{\circ}, D^0) = \mathrm{Hom}^{H^{\circ}}(H^{\circ}, D^0)$, i.e. ω° is an integral for D^0.

Let ω be invertible in $(\mathrm{Hom}_R(D, H), \star)$ with inverse $\omega^{-1}\colon D \to H$. In a similar way to [29] we get

$$\omega^{-1}(dh) = S_H(h)\omega^{-1}(d) \quad \text{for all } h \in H \text{ and } d \in D.$$

If $f \in H^{\circ}$, then we have for all $d \in D$ and $h \in H$:

$$\begin{aligned}
(h(\omega^{-1})^{\circ}(f))(d) &= ((\omega^{-1})^{\circ}(f))(dh) = f(\omega^{-1}(dh))\\
&= f(S(h)\omega^{-1}(d)) = \sum f_1(S(h))f_2(\omega^{-1}(d))\\
&= \sum (S^{\circ}(f_1))(h)((\omega^{-1})^{\circ}(f_2))(d)\\
&= \Big(\sum S^{\circ}(f_1)(h)(\omega^{-1})^{\circ}(f_2)\Big)(d),
\end{aligned}$$

i.e. $(\omega^{-1})^{\circ} \in D^0$ with $\varrho((\omega^{-1})^{\circ}) = \sum (\omega^{-1})^{\circ}(f_2) \otimes S^{\circ}(f_1)$. Moreover we have for all $f \in H^{\circ}$ and $d \in D$:

$$\begin{aligned}
((\omega^{\circ} \star (\omega^{-1})^{\circ})(f))(d) &= ((\omega^{\circ} \underline{\otimes} (\omega^{-1})^{\circ})(\Delta(f)))(d)\\
&= \Big(\sum \omega^{\circ}(f_1) \star (\omega^{-1})^{\circ}(f_2)\Big)(d)\\
&= \Big(\sum \omega^{\circ}(f_1) \underline{\otimes} (\omega^{-1})^{\circ}(f_2)\Big)(d_1 \otimes d_2)\\
&= \sum \omega^{\circ}(f_1)(d_1)(\omega^{-1})^{\circ}(f_2)(d_2)\\
&= \sum f_1(\omega(d_1))f_2(\omega^{-1}(d_2))\\
&= \sum f(\omega(d_1)\omega^{-1}(d_2))\\
&= f((\omega \star \omega^{-1})(d))\\
&= f(\varepsilon_D(d)1_H)\\
&= \varepsilon_{H^{\circ}}(f)\varepsilon_D(d),
\end{aligned}$$

i.e. $\omega^{\circ} \star (\omega^{-1})^{\circ} = \mathrm{id}_{\mathrm{Hom}_R(H^{\circ}, D^0)}$. In a similar way, one can prove that $(\omega^{-1})^{\circ} \star \omega^{\circ} = \mathrm{id}_{\mathrm{Hom}_R(H^{\circ}, D^0)}$. So ω° is $\star$-invertible with inverse $(\omega^{-1})^{\circ}$ and $\pi^{\circ}\colon C^0 \hookrightarrow D^0$ is a *cleft* right H°-extension. □

References

1. Abe, E.: *Hopf Algebras*, Cambridge Tracts in Math. 74, Cambridge University Press, 1980.
2. Abuhlail, J. Y.: *Dualitätstheoreme für Hopf-Algebren über Ringen*, Ph.D. Dissertation, Heinrich-Heine Universität, Düsseldorf-Germany, 2001. http://ww.ulb.uniduesseldorf.de/diss/mathnat/2001/abuhlail.html.
3. Abuhlail, J. Y.: Rational modules for corings, *Comm. Algebra* **31**(12) (2003), 5793–5840.
4. Abuhlail, J. Y., Gómez-Torrecillas, J. and Wisbauer, R.: Dual coalgebras of algebras over commutative rings, *J. Pure Appl. Algebra* **153**(2) (2000), 107–120.
5. Abuhlail, J. Y., Gómez-Torrecillas, J. and Lobillo, F.: Duality and rational modules in Hopf algebras over commutative rings, *J. Algebra* **240** (2001), 165–184.
6. Brzeziński, T.: On modules associated to coalgebra Galois extensions, *J. Algebra* **215** (1999), 290–317.
7. Brzeziński, T.: The structure of corings. Induction functors, Maschke-type theorem, and Frobenius and Galois-type properties, *Algebr. Represent. Theory* **5** (2002), 389–410.
8. Brzeziński, T. and Majid, S.: Coalgebra bundles, *Comm. Math. Phys.* **191** (1998), 467–492.
9. Caenepeel, S., Militaru, G. and Zhu, S.: *Frobenius and Separable Functors for Generalized Module Categories and Nonlinear Equations*, Lecture Notes in Math. 1787, Springer-Verlag, Berlin, 2002.
10. Chase, S. and Sweedler, M.: *Hopf Algebras and Galois Theory*, Lecture Notes in Math. 97, Springer-Verlag, Berlin 1969.
11. Doi, Y.: On the structure of relative Hopf modules, *Comm. Algebra* **11** (1983), 243–255.
12. Doi, Y.: Algebras with total integrals, *Comm. Algebra* **13** (1985), 2137–2159.
13. Doi, Y.: Unifying Hopf modules, *J. Algebra* **153** (1992), 373–385.
14. Doi, Y.: Generalized smash products and Morita contexts for arbitrary Hopf algebras, In: Bergen-Montgomery (ed.), *Advances in Hopf Algebras*, Lecture Notes in Pure Appl. Math. 158, Marcel Dekker, New York, 1994, pp. 39–53.
15. Doi, Y. and Takeuchi, M.: Cleft comodule alegrbas for a bialgebra, *Comm. Algebra* **14** (1986), 801–817.
16. Garfinkel, G.: Universally torsionless and trace modules, *J. Amer. Math. Soc.* **215** (1976), 119–144.
17. Koppinen, M.: Variations on the smash product with applications to group-graded rings, *J. Pure Appl. Algebra* **104** (1995), 61–80.
18. Kreimer, H. and Takeuchi, M.: Hopf algebras and Galois extensions of an algebra, *Indiana Univ. Math. J.* **30** (1981), 675–692.
19. Lin, B: Semiperfect coalgebras, *J. Algebra* **49** (1977), 357–373.
20. Liu, G.: The duality between modules and comodules, *Acta Math. Sinica* **37** (1994), 150–154.
21. Long, F.: The Brauer group of dimodule algebras, *J. Algebra* **30** (1974), 559–601.
22. Montogomery, S.: *Hopf Algebras and Their Coactions on Rings*, Reg. Conf. Ser. Math. 82, Amer. Math. Soc., Providence, R.I., 1993.
23. Schauenburg, P.: Doi–Koppinen Hopf modules versus entwined modules, *New York J. Math.* **6** (2000), 325–329.
24. Sweedler, M.: *Hopf Algebras*, Benjamin, New York, 1969.
25. Wisbauer, R.: *Grundlagen der Modul- und Ringtheorie*, Verlag Reinhard Fischer, Munich, 1988; *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, 1991.
26. Wisbauer, R.: Weak corings, *J. Algebra* **245** (2001), 123–160.
27. Wisbauer, R.: Introduction to coalgebras and comodules, Lecture Notes, 2000.
28. Yokogawa, K.: On dual Hopf Galois extension, In: *Proc. 15th Sympos. Ring Theory* (*Takarazuka/Jap. 1982*), 1982, pp. 84–92.
29. Zhang, L.: The duality of relative Hopf modules, *Acta Math. Sinica* **40** (1998), 73–79.
30. Zimmermann-Huignes, B.: Pure submodules of direct products of free modules, *Math. Ann.* **224** (1976), 233–245.