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New conditions for (non)uniform behaviour of linear cocycles over flows

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We give a characterization of tempered exponential dichotomies for linear cocycles over flows in terms of the spectral properties of certain linear operators. We consider noninvertible linear cocycles acting on infinite-dimensional spaces and our approach avoids the use of Lyapunov norms. Finally, we apply obtained results to give new conditions for uniform exponential stability of linear cocycles.

1. Introduction

The notion of a linear cocycle (or a linear skew-product system) plays a fundamental role in the modern dynamical systems theory. This notion was essentially introduced by Oseledets [26] and systematically studied in the pioneering works of Sacker and Sell [32–36]. Sacker and Sell also showed that the notion of a linear cocycle can be used to model many nonautonomous problems. We stress that this line of the research can also be considered as a natural extension of the smooth dynamical systems theory since the derivative cocycle associated to a map or a flow is a particular example of a linear cocycle (see [4]). More recently, Arnold [2] proposed an abstract framework for the study of random dynamical systems that also relies on the notion of a linear cocycle. We refer to [10] for more details and further references.

Following the seminal works of Perron [27] and Smale [40], Sacker and Sell [32–34] introduced the concept of a uniform exponential dichotomy for linear cocycles and formulated sufficient conditions for its existence. We in particular mention the paper [36] which deals with cocycles that take values in the space of bounded operators acting on an arbitrary Banach space. Since then, many authors dealt with the problem of finding necessary and sufficient conditions for the existence of a uniform exponential dichotomy for linear cocycles.

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Relevant to our work are those characterizations of exponential dichotomy which can be phrased in terms of spectral properties of certain linear operators associated to a linear cocycle. This line of the research in the context of ordinary differential equations can be traced to the work of Perron [27] and was considerably developed by Massera and Schäffer [20]. In the context of derivative cocycles, first results in this direction are due to Mather [21] and Chicone and Swanson [11]. More precisely, to a diffeomorphism f acting on a compact Riemannian manifold M, Mather associated a bounded linear operator Γ acting on a suitably constructed Banach space consisting of vector fields on M and proved that f is Anosov (i.e. Df admits a uniform exponential dichotomy) if and only if spectrum of Γ doesn't intersect the unit circle (see [11] for corresponding results in the case of flows).

For general linear cocycles over maps, in their seminar paper [12], Chow and Leiva proposed a slightly different approach. More precisely, for each base point q of a linear cocycle they construct a bounded linear operator Γ_q acting on a certain Banach space and they prove that a linear cocycle admits a uniform exponential dichotomy if and only if for each q, $\mathrm{Id} - \Gamma_q$ is an invertible operator and $\sup_q \|(\mathrm{Id} - \Gamma_q)^{-1}\| < \infty$. Furthermore, they apply this result to the problem of stability of exponential dichotomies under small linear perturbations. We stress that their approach was inspired by the related results of Henry [16] who considered linear nonautonomous dynamics with discrete time generated by a sequence of linear operators.

Since then, many authors obtain further functional-theoretic characterization of uniform exponential behaviour for linear cocycles. We in particular mention the works of Latushkin and Schnaubelt [18], Megan, Sasu and Sasu [22–25], C. Preda and his collaborators [29,31,30], Huy [17] and more recently Sasu and Sasu [37–39]. We stress that all of those works consider *uniform* exponential behaviour.

On the other hand, uniform behaviour is rather restrictive and it is of great importance to look for weaker forms of exponential behaviour. Among many different approaches, arguably the most important is the concept of *nonuniform hyperbolicity*. The study of nonuniformly hyperbolic dynamical systems was initiated by Oseledets [26] and in particular Pesin [28] (see [4] for the overview of this theory). Recently, many of the elements of this theory have been extended to the infinite-dimensional case [8,19]. We emphasize that nonuniform hyperbolicity (or the existence of tempered exponential dichotomy) is ensured by nonvanishing of Lyapunov exponents.

In their paper [41] devoted to the study of the roughness property of tempered dichotomies, Zhou, Lu and Zhang raised a question whether the notion of a tempered dichotomy can be expressed in terms of spectral properties of certain linear operators. In [5], the authors gave a positive answer to this question and used it to give a simplified proof of the main result from [41]. We refer to [6] for similar results dealing with the derivative cocycle associated to a smooth flow. However, the results in [5,6] have an important disadvantage that the spaces (so-called admissible spaces) on which linear operators act are constructed in terms of the so-called Lyapunov norms. Since there are no tools for constructing Lyapunov norms a priori (i.e. without knowing whether the dynamics is nonuniformly hyperbolic), these results are somewhat unsatisfactory.

In order to overcome this problem, we have proposed a different approach in [1] (partially based on ideas from [14]) that avoids the use of Lyapunov norms, i.e. admissible spaces are carefully constructed in terms of the original norm. Furthermore, this type of spaces haven't appeared earlier in the literature and they don't share many standard properties of admissible space for uniformly hyperbolic dynamics (for example, the norm is not invariant under translations). We stress that the results in [1] deal with cocycles over maps.

The main objective of the present paper is to extend the results from [1] to the case of cocycles over flows. We emphasize that the passage from discrete and to continuous time is nontrivial and that our arguments require substantial changes when compared to [1]. Finally, in the present paper we also apply our results dealing with the existence of nonuniform behaviour to obtain new conditions for the existence of uniform exponential stability.

2. Preliminaries

2.1. Basic notions

Let M be a locally compact metric space and $X = (X, \|\cdot\|)$ a separable Banach space. Furthermore, by $B(X)$ we will denote the space of all bounded linear operators on X. We recall the notion of a flow on M.

Definition 1. A family $\Phi = (\phi_t)_{t\in\mathbb{R}}$ is said to be a *flow* on M if:

- $\phi_t \colon M \to M$ is a homeomorphism for each $t \in \mathbb{R}$;
- $\phi_{t+s} = \phi_t \circ \phi_s$ for every $t, s \in \mathbb{R}$;
- $(q, t) \mapsto \phi_t(q)$ is a continuous map on $M \times \mathbb{R}$.

Furthermore, we recall the notions of an invariant measure for a flow and ergodicity.

Definition 2. Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a flow on M and assume that μ is a Borel probability measure on M. We say that μ is *invariant* for Φ if

$$\mu(\phi_t(E)) = \mu(E) \quad \text{for each } t \geq 0 \text{ and a Borel subset } E \subset M.$$

For an Φ-invariant measure μ is said to be *ergodic* if $\mu(E) \in \{0, 1\}$ for each Borel $E \subset M$ such that $\phi_t(E) = E$ for every $t \geq 0$.

By $\mathcal{E}(\Phi)$ we will denote the space of all Φ-invariant measures on M which are ergodic. We recall that $\mathcal{E}(\Phi) \neq \emptyset$ if M is compact.

Definition 3. Assume that $\Phi = (\phi_t)_{t\in\mathbb{R}}$ is a flow on M. A map $\mathcal{A} \colon M \times [0, \infty) \to B(X)$ is said to be a *linear cocycle* over Φ if:

- $\mathcal{A}(q, 0) = \mathrm{Id}$ for every $q \in M$;
- $\mathcal{A}(q, t + s) = \mathcal{A}(\phi_t(q), s)\mathcal{A}(q, t)$ for every $q \in M$ and $t, s \geq 0$;
- for each $x \in X$, $(q, t) \mapsto \mathcal{A}(q, t)x$ is continuous map on $M \times [0, \infty)$.

Finally, we recall the notion of a tempered random variable.

Definition 4. We say that a Borel measurable map $K \colon M \to (0, \infty)$ is *tempered* with respect to $\mu \in \mathcal{E}(\Phi)$ if:

$$\lim_{t\to\pm\infty} \frac{1}{t} \log K(\phi_t(q)) = 0 \quad \text{for } \mu\text{-a.e. } q \in M.$$

The following well-known result (see [2] for example) will be useful in our arguments.

Proposition 1. *Assume that $K \colon M \to (0, \infty)$ is tempered with respect to $\mu \in \mathcal{E}(\Phi)$. Then, for $\varepsilon > 0$ there exists a Borel measurable map $C \colon M \to (0, \infty)$ such that:*

1. *for μ-a.e. $q \in M$,*

$$K(q) \leq C(q); \tag{1}$$

2. *for μ-a.e. $q \in M$ and every $t \in \mathbb{R}$, we have that*

$$C(\phi_t(q)) \le C(q)e^{\varepsilon|t|}. \tag{2}$$

Finally, we introduce the notion of a tempered (exponential) dichotomy.

Definition 5. Let $\mathcal{A}$ be a linear cocycle over $\Phi = (\phi_t)_{t\in\mathbb{R}}$. We say that $\mathcal{A}$ admits a *tempered dichotomy* with respect to $\mu \in \mathcal{E}(\Phi)$ if there exist $\lambda > 0$, $K\colon M \to (0,\infty)$ tempered with respect to μ, Φ-invariant Borel subset $E \subset M$ satisfying $\mu(E) = 1$ and a family of projections $P(q) \in B(X)$, $q \in M$ such that:

1. $q \mapsto P(q)x$ is a Borel measurable map for each $x \in X$;
2. for $q \in E$ and $t \ge 0$,

$$\mathcal{A}(q,t)P(q) = P(\phi_t(q))\mathcal{A}(q,t) \tag{3}$$

   and the map

$$\mathcal{A}(q,t)|\operatorname{Ker} P(q)\colon\ \operatorname{Ker} P(q) \to \operatorname{Ker} P(\phi_t(q))$$

   is invertible;
3. for $q \in E$ and every $t \ge 0$,

$$\|\mathcal{A}(q,t)P(q)\| \le K(q)e^{-\lambda t} \tag{4}$$

   and

$$\|\mathcal{A}(q,-t)(\mathrm{Id} - P(q))\| \le K(q)e^{-\lambda t}, \tag{5}$$

   where

$$\mathcal{A}(q,-t) := (\mathcal{A}(\phi_{-t}(q),t)|\operatorname{Ker} P(\phi_{-t}(q)))^{-1}$$

   is a well-defined linear map from $\operatorname{Ker} P(q)$ to $\operatorname{Ker} P(\phi_{-t}(q))$.

2.2. Important function spaces and admissibility

We now introduce a class of function spaces that will play a crucial role in our results. For $k \in \mathbb{N}$, set

$$Y_k := \left\{ f\colon \mathbb{R} \to X : f \text{ continuous and } \|f\|_k := \sup_{t\in\mathbb{R}} \|f(t)\| e^{-\frac{|t|}{k}} < \infty \right\}.$$

Observe that

$$Y_{k+1} \subset Y_k \quad \text{for each } k \in \mathbb{N}.$$

The proof of the following result is straightforward.

Proposition 2. *For each $k \in \mathbb{N}$, $(Y_k, \|\cdot\|_k)$ is a Banach space.*

Finally, we introduce the central concept of admissibility.

Definition 6. Assume that $\mathcal{A}$ is a linear cocycle over a flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$. Furthermore, suppose that $k \in \mathbb{N}$ and that $E \subset M$ is a Borel set. We say that $\mathcal{A}$ is (k, E)-*admissible* if for each $q \in E$ and every $g \in Y_{k+1}$ there exists a unique $f \in Y_k$ such that

$$f(t) = \mathcal{A}(\phi_s(q), t-s)f(s) + \int_s^t \mathcal{A}(\phi_r(q), t-r)g(r)\, dr, \tag{6}$$

for $t \geq s$.

We shall also use the following weaker concept of admissibility.

Definition 7. Assume that $\mathcal{A}$ is a linear cocycle over a flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$. Furthermore, suppose that $k \in \mathbb{N}$ and that $E \subset M$ is a Borel set. We say that $\mathcal{A}$ is (k, E)-*weakly admissible* if for each $q \in E$ and every $g \in Y_{k+1}$ there exists at most one $f \in Y_k$ such that (6) holds for $t \geq s$.

Remark 1. We observe that the concept of (k, E)-weak admissibility doesn't require that (6) has a (unique) solution $f \in Y_k$ for $g \in Y_{k+1}$. Indeed, it only requires that if (6) has a solution $f \in Y_k$ that then this solution is unique. Hence, if $\mathcal{A}$ is (k, E)-admissible then it is also (k, E)-weakly admissible.

2.3. Multiplicative ergodic theorem

We begin by recalling some terminology together with the most general version of the multiplicative ergodic theorem.

For $A \in B(X)$, let $\|A\|_{\text{ic}}$ denote the infimum over all $\varepsilon > 0$ with the property that $A(B(0,1))$ can be covered by finitely many open balls of radius ε. Here, $B(0,1)$ denotes the open ball in X of radius 1 centred at 0.

Let $\mathcal{A}$ be a linear cocycle over a flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ and assume that $\mu \in \mathcal{E}(\Phi)$. Furthermore, suppose that

$$\sup_{0\leq t\leq 1} \int_M \max\{\log\|\mathcal{A}(q,t)\|, 0\}\, d\mu(q) < \infty. \tag{7}$$

It follows from the Kingman's subadditive ergodic theorem that there exist $\kappa(\mu), \lambda(\mu) \in [-\infty, \infty)$ such that

$$\lambda(\mu) = \lim_{t\to\infty} \frac{1}{t} \log\|\mathcal{A}(q,t)\|$$

and

$$\kappa(\mu) = \lim_{t\to\infty} \frac{1}{t} \log\|\mathcal{A}(q,t)\|_{\text{ic}},$$

for μ-a.e. $q \in M$. Furthermore, since $\|A\|_{\text{ic}} \leq \|A\|$ for $A \in B(X)$, we have that $\kappa(\mu) \leq \lambda(\mu)$. We say that $\mathcal{A}$ is *quasicompact* with respect to μ if $\kappa(\mu) < \lambda(\mu)$.

The following result for linear cocycles over maps can be found in [7,15]. The extension to linear cocycles over flows is straightforward since it can be obtained by applying the usual trick of passing to discrete-time linear cocycle over a time-one map of a flow (see [4] for example).

Theorem 3. *Assume that $\mathcal{A}$ is a linear cocycle over a flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ and suppose that $\mathcal{A}$ satisfies* (7) *and is quasicompact with respect to $\mu \in \mathcal{E}(\Phi)$. Then there exists a Borel set $\mathcal{R}^\mu \subset M$ such that $\mu(\mathcal{R}^\mu) = 1$ and either:*

1. *There is a finite sequence of numbers*

$$\lambda(\mu) = \lambda_1(\mu) > \lambda_2(\mu) > \cdots > \lambda_k(\mu) > \kappa(\mu)$$

and a measurable decomposition

$$X = E_1(q) \oplus \cdots \oplus E_k(q) \oplus F_\infty(q)$$

such that for $q \in \mathcal{R}^\mu$,

$$\mathcal{A}(q,t)E_i(q) = E_i(\phi_t(q)), \quad i = 1, \ldots, k, \ t \geq 0,$$
$$\mathcal{A}(q,t)E_\infty(q) \subset E_\infty(\phi_t(q)), \quad t \geq 0,$$
$$\lim_{|t|\to\infty} \frac{1}{t} \log\|\mathcal{A}(q,t)x\| = \lambda_i(\mu),$$

for $x \in E_i(q) \setminus \{0\}$, $i \in \{1, \ldots k\}$ *and*

$$\limsup_{t\to\infty} \frac{1}{t} \log\|\mathcal{A}(q,t)x\| \leq \kappa(\mu) \quad \text{for } x \in F_\infty(q).$$

Moreover, each $E_i(q)$, $i = 1, \ldots, k$ *is a finite-dimensional subspace of* X.

2. *There exists an infinite sequence of numbers*

$$\lambda(\mu) = \lambda_1(\mu) > \lambda_2(\mu) > \cdots > \lambda_k(\mu) > \ldots > \kappa(\mu),$$
$$\lim_{k\to\infty} \lambda_k(\mu) = \kappa(\mu),$$

and for each $k \in \mathbb{N}$ *a measurable decomposition*

$$X = E_1(q) \oplus \cdots \oplus E_k(q) \oplus F_k(q)$$

such that for $q \in \mathcal{R}^\mu$,

$$\mathcal{A}(q,t)E_i(q) = E_i(\phi_t(q)), \quad i = 1, \ldots, k, \ t \geq 0,$$
$$\mathcal{A}(q,t)F_k(q) \subset F_k(\phi_t(q)), \quad t \geq 0,$$
$$\lim_{|t|\to\infty} \frac{1}{t} \log\|\mathcal{A}(q,t)x\| = \lambda_i(\mu),$$

for $x \in E_i(q) \setminus \{0\}$, $i = 1, \ldots, k$ *and*

$$\limsup_{t\to\infty} \frac{1}{t} \log\|\mathcal{A}(q,t)x\| \leq \lambda_{k+1}(\mu), \quad \text{for } x \in F_k(q).$$

Moreover, each $E_i(q), i \in \mathbb{N}$ *is a finite-dimensional subspace of* X.

We note that the numbers $\lambda_i(\mu)$ are called *Lyapunov exponents* of the cocycle $\mathcal{A}$ with respect to μ. In addition, subspaces $E_i(q)$ are called *Oseledets subspaces*.

The following results show that nonvanishing of Lyapunov exponents implies the existence of a tempered dichotomy.

Proposition 4. *Assume that $\mathcal{A}$ is a linear cocycle over a flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ and suppose that $\mathcal{A}$ satisfies (7) and is quasicompact with respect to $\mu \in \mathcal{E}(\Phi)$. Furthermore, suppose that all Lyapunov exponents of $\mathcal{A}$ with respect to μ are nonzero. Then, $\mathcal{A}$ admits a tempered dichotomy.*

Proof. The proof can be obtained by repeating the arguments in the proof of [3, Proposition 3.2.] simply by passing from discrete to continuous time without any changes. □

Remark 2. We emphasize that prior to [3], the version of Proposition 4 has appeared in [8,19], where the authors deal with injective cocycles and in [13], where the authors considered cocycles on Hilbert spaces.

3. Main results

The following section contains the main results of our paper. We will split it into two parts.

3.1. From tempered dichotomy to admissibility

Theorem 5. *Assume that a cocycle $\mathcal{A}$ over $\Phi = (\phi_t)_{t\in\mathbb{R}}$ admits a tempered dichotomy with respect to $\mu \in \mathcal{E}(\Phi)$. Then, there exists an Φ-invariant Borel $E \subset M$ satisfying $\mu(E) = 1$ and such that $\mathcal{A}$ is (k, E)-admissible for sufficiently large $k \in \mathbb{N}$.*

Proof. Let $\lambda > 0, K : M \to (0, \infty)$ and $E \subset M$ be as in Definition 5. Furthermore, let us fix an arbitrary $k \in \mathbb{N}$ such that $\frac{1}{k-1} < \lambda$.

Take an arbitrary $q \in E$ and $g \in Y_{k+1}$. Set

$$f(t) = f_1(t) - f_2(t),$$

where

$$f_1(t) = \int_{-\infty}^{t} \mathcal{A}(\phi_r(q), t - r)P(\phi_r(q))g(r)\, dr \tag{8}$$

and

$$f_2(t) = \int_{t}^{\infty} \mathcal{A}(\phi_r(q), t - r)(\mathrm{Id} - P(\phi_r(q)))g(r)\, dr. \tag{9}$$

Lemma 1. *For μ-a.e. $q \in E$, we have that $f_1 \in Y_k$.*

Proof of the lemma. By (4), we have

$$\|f_1(t)\|e^{-\frac{|t|}{k}} \le e^{-\frac{|t|}{k}} \int_{-\infty}^{t} K(\phi_r(q))e^{-\lambda(t-r)}\|g(r)\|\, dr, \tag{10}$$

for each $t \in \mathbb{R}$ and for μ-a.e. $q \in M$. Take $\varepsilon > 0$ such that $\varepsilon < \frac{1}{k} - \frac{1}{k+1}$ and let $C : M \to (0, \infty)$ be the corresponding map given by Proposition 1. It follows from (1), (2) and (10) that

$$
\begin{aligned}
\|f_1(t)\|e^{-\frac{|t|}{k}} &\le \int_{-\infty}^{t} C(\phi_r(q))e^{-\lambda(t-r)}\|g(r)\|e^{-\frac{|r|}{k}+\frac{t-r}{k}}\,dr \\
&\le C(q)\int_{-\infty}^{t} e^{\varepsilon|r|}e^{-(\lambda-1/k)(t-r)}\|g(r)\|e^{-\frac{|r|}{k}}\,dr \\
&\le C(q)\int_{-\infty}^{t} e^{-(\lambda-1/k)(t-r)}\|g(r)\|e^{-\frac{|r|}{k+1}}\,dr \\
&\le C(q)\|g\|_{k+1}\int_{-\infty}^{t} e^{-(\lambda-1/k)(t-r)}\,dr \\
&\le \frac{C(q)}{\lambda-1/k}\|g\|_{k+1},
\end{aligned}
$$

for each $t \in \mathbb{R}$ and μ-a.e. $q \in E$. Hence,

$$\sup_{t\in\mathbb{R}}\|f_1(t)\|e^{-\frac{|t|}{k}} < \infty,$$

for μ-a.e. $q \in E$. Thus, (since f_1 is obviously continuous) we conclude that $f_1 \in Y_k$. □

Lemma 2. *For μ-a.e. $q \in E$, we have that $f_2 \in Y_k$.*

Proof of the lemma. Let $\varepsilon > 0$ and $C\colon M \to (0,\infty)$ be as in the proof of the previous lemma. It follows from (1), (2) and (5) that

$$
\begin{aligned}
\|f_2(t)\|e^{-\frac{|t|}{k}} &\le e^{-\frac{|t|}{k}}\int_{t}^{\infty} K(\phi_r(q))e^{-\lambda(r-t)}\|g(r)\|\,dr \\
&\le \int_{t}^{\infty} C(\phi_r(q))e^{-\lambda(r-t)}\|g(r)\|e^{-\frac{|r|}{k}+\frac{r-t}{k}}\,dr \\
&\le C(q)\int_{t}^{\infty} e^{\varepsilon|r|}e^{-(\lambda-1/k)(r-t)}\|g(r)\|e^{-\frac{|r|}{k}}\,dr \\
&\le C(q)\int_{t}^{\infty} e^{-(\lambda-1/k)(r-t)}\|g(r)\|e^{-\frac{|r|}{k+1}}\,dr \\
&\le \frac{C(q)}{\lambda-1/k}\|g\|_{k+1},
\end{aligned}
$$

for each $t \in \mathbb{R}$ and μ-a.e. $q \in E$. As in the proof of the previous lemma, we conclude that $f_2 \in Y_k$ for μ-a.e. $q \in E$. □

It follows from Lemmas 1 and 2 that $f \in Y_k$ for μ-a.e. $q \in E$. We now show that (6) holds. This is well-known but we give proof for the sake of completeness.

Lemma 3. *We have that (6) holds for μ-a.e. $q \in M$ and $t \ge s$.*

Proof of the lemma. We have that

$$\begin{aligned}
&f(t) - \mathcal{A}(\phi_s(q), t-s)f(s) \\
&= \int_{-\infty}^{t} \mathcal{A}(\phi_r(q), t-r)P(\phi_r(q))g(r)\, dr \\
&\quad - \mathcal{A}(\phi_s(q), t-s) \int_{-\infty}^{s} \mathcal{A}(\phi_r(q), s-r)P(\phi_r(q))g(r)\, dr \\
&\quad - \int_{t}^{\infty} \mathcal{A}(\phi_r(q), t-r)(\mathrm{Id} - P(\phi_r(q)))g(r)\, dr \\
&\quad + \mathcal{A}(\phi_s(q), t-s) \int_{s}^{\infty} \mathcal{A}(\phi_r(q), s-r)(\mathrm{Id} - P(\phi_r(q)))g(r)\, dr \\
&= \int_{s}^{t} \mathcal{A}(\phi_r(q), t-r)P(\phi_r(q))g(r)\, dr \\
&\quad + \int_{s}^{t} \mathcal{A}(\phi_r(q), t-r)(\mathrm{Id} - P(\phi_r(q)))g(r)\, dr \\
&= \int_{s}^{t} \mathcal{A}(\phi_r(q), t-r)g(r)\, dr,
\end{aligned}$$

for μ-a.e. $q \in M$ and $t \geq s$. The proof of the lemma is completed. □

It remains to establish the uniqueness of solutions of (6). Assume that for $g \in Y_{k+1}$, there exist $f_i \in Y_k$, $i = 1, 2$ such that

$$f_i(t) = \mathcal{A}(\phi_s(q), t-s)f_i(s) + \int_{t}^{s} \mathcal{A}(\phi_r(q), t-r)g(r)\, dr,$$

for $t \geq s$ and $i \in \{1, 2\}$. Hence,

$$f(t) = \mathcal{A}(\phi_s(q), t-s)f(s),$$

for $t \geq s$ where $f := f_1 - f_2 \in Y_k$. Set

$$h_1(t) = P(\phi_t(q))f(t) \quad \text{and} \quad h_2(t) = (\mathrm{Id} - P(\phi_t(q)))f(t),$$

for $t \in \mathbb{R}$. Choose $\varepsilon > 0$ such that $\varepsilon < \frac{1}{k-1} - \frac{1}{k}$ and let C be the corresponding map given by Proposition 1. We have that

$$\begin{aligned}\|h_1(t)\|e^{-\frac{|t|}{k-1}} &= \|\mathcal{A}(\phi_s(q), t-s)P(\phi_s(q))f(s)\|e^{-\frac{|t|}{k-1}} \\ &\le K(\phi_s(q))e^{-\lambda(t-s)}\|f(s)\|e^{-\frac{|t|}{k-1}} \\ &\le C(\phi_s(q))e^{-\lambda(t-s)}\|f(s)\|e^{-\frac{|t|}{k-1}} \\ &\le C(q)e^{\varepsilon|s|}e^{-\lambda(t-s)}\|f(s)\|e^{\frac{t-s}{k-1}-\frac{|s|}{k-1}} \\ &\le C(q)e^{-(\lambda-\frac{1}{k-1})(t-s)}\|f(s)\|e^{-\frac{|s|}{k}} \\ &\le C(q)e^{-(\lambda-\frac{1}{k-1})(t-s)}\|f\|_k,\end{aligned}$$

for $t \ge s$. Letting $s \to -\infty$, we conclude that $h_1(t) = 0$ for each $t \in \mathbb{R}$. Similarly, one can show that $h_2(t) = 0$ for each $t \in \mathbb{R}$ and thus $f = 0$. The proof of the theorem is completed. □

Our next aim is to formulate the conclusion of Theorem 5 in a slightly different manner. Set

$$Y = \bigcap_{k\in\mathbb{N}} Y_k.$$

Proposition 6. *We have that*

$$Y = \left\{ f : \mathbb{R} \to X : f \text{ continuous and } \limsup_{t\to\pm\infty} \frac{1}{|t|}\log\|f(t)\| \le 0 \right\}.$$

Proof. Take an arbitrary $f \in Y$ and fix $l \in \mathbb{N}$. Since $f \in Y_l$, we have that $\|f\|_l < \infty$. Consequently,

$$\|f(t)\| \le e^{\frac{|t|}{l}}\|f\|_l,$$

for every $t \in \mathbb{R}$ and hence

$$\limsup_{t\to\pm\infty} \frac{1}{|t|}\log\|f(t)\| \le \limsup_{t\to\pm\infty} \frac{1}{|t|}\log\left(e^{\frac{|t|}{l}}\|f\|_l\right) = \frac{1}{l}.$$

Since $l \in \mathbb{N}$ was arbitrary, we conclude that

$$\limsup_{t\to\pm\infty} \frac{1}{|t|}\log\|f(t)\| \le 0.$$

The converse inclusion can be obtained similarly. □

Proposition 7. *Assume that $(f_n)_n \subset Y$ is a Cauchy sequence in Y_k for each $k \in \mathbb{N}$. Then, there exists $f \in Y$ such that f_n converges to f in Y_k for every $k \in \mathbb{N}$.*

Proof. Let $C(\mathbb{R}, X)$ be a space of all continuous functions $f : \mathbb{R} \to X$ such that $\|f\|_\infty := \sup_{t\in\mathbb{R}}\|f(t)\| < \infty$. We recall that $(C(\mathbb{R}, X), \|\cdot\|_\infty)$ is a Banach space.

Take now a sequence $(f_n)_n \subset Y$ as in the statement of the proposition and fix an arbitrary $k \in \mathbb{N}$. For each $n \in \mathbb{N}$, we define a map $g_n : \mathbb{R} \to X$ by

$$g_n(t) = e^{-\frac{|t|}{k}} f_n(t), \quad t \in \mathbb{R}.$$

Since $(f_n)_n$ is a Cauchy sequence in Y_k, we have that $(g_n)_n$ is a Cauchy sequence in $C(\mathbb{R}, X)$ and therefore it converges to some $g \in C(\mathbb{R}, X)$. Hence, f_n converges to f^k in Y_k, where $f^k : \mathbb{R} \to X$ is given by

$$f^k(t) = e^{\frac{|t|}{k}} g(t), \quad t \in \mathbb{R}.$$

Finally, we note that

$$\|f_n - f^k\|_1 \le \|f_n - f^k\|_k,$$

for each $n \in \mathbb{N}$, we conclude that $f^k = f^1$. The proof of the proposition is completed. □

As a consequence of Proposition 7, we have the following.

Proposition 8. *Y is a Frechét space.*

The following result is a simple consequence of Theorem 5.

Corollary 1. *Assume that the cocycle $\mathcal{A}$ over $\Phi = (\phi_t)_{t\in\mathbb{R}}$ admits a tempered dichotomy with respect to $\mu \in \mathcal{E}(\Phi)$. Then, for μ-a.e. $q \in M$ and every $g \in Y$ there exists a unique $f \in Y$ such that (6) holds for $t \ge s$.*

Proof. Take $g \in Y$. Then, $g \in Y_k$ for every $k \in \mathbb{N}$. Let $f = f_1 - f_2$, where f_1 and f_2 are defined as in (8) and (9) respectively. It follows from Lemmas 1 and 2 that for μ-a.e. $q \in M$, $f \in Y$. Moreover, Lemma 3 implies that (6) holds for μ-a.e. $q \in M$ and $t \ge s$. The uniqueness of f follows directly from Theorem 5. □

3.2. Converse result

We now formulate a partial converse of Theorem 5.

Theorem 9. *Assume that $\mathcal{A}$ is a linear cocycle over a flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ and suppose that $\mathcal{A}$ satisfies (7) and is quasicompact with respect to $\mu \in \mathcal{E}(\Phi)$. Furthermore, suppose that there exists $k \in \mathbb{N}$ and a Borel $E \subset M$ satisfying $\mu(E) > 0$ and such that $\mathcal{A}$ is (k, E)-weakly admissible. Then, $\mathcal{A}$ admits a tempered dichotomy with respect to μ.*

Proof. In a view of Proposition 4, it is sufficient to show that all Lyapunov exponents of $\mathcal{A}$ are nonzero. Suppose that there $i \in \mathbb{N}$ such that $\lambda_i(\mu) = 0$ and take $q \in E \cap \mathcal{R}^\mu$, where $\mathcal{R}^\mu$ is as in the statement of Theorem 3. Finally choose $v \in E_i(q) \setminus \{0\}$ and define $x \colon \mathbb{R} \to X$ by

$$x(t) = \mathcal{A}(q,t)v, \quad t \in \mathbb{R}.$$

Since

$$\lim_{|t|\to\infty} \frac{1}{t} \log\|x(t)\| = 0,$$

we have that $x \in Y_k$. Furthermore, (6) holds with $y = 0$. Hence, it follows from the assumptions of the theorem that $x = 0$ and thus $v = 0$, which yields a contradiction. □

4. Applications

In this section, we use the main results to establish new conditions for the existence of a uniform exponential stability. We begin by recalling the notion of a uniform exponential stability.

Definition 8. We say that $\mathcal{A}$ is *uniformly exponentially stable* if there exist $D, \lambda > 0$ such that

$$\|\mathcal{A}(q,t)\| \le De^{-\lambda t} \quad \text{for } q \in M \text{ and } t \ge 0. \tag{11}$$

Theorem 10. *Assume that:*

- M *is compact and* $\mathcal{A}\colon M \times [0,\infty) \to B(X)$ *is a continuous map;*
- $\mathcal{A}$ *is quasicompact with respect to every* $\mu \in \mathcal{E}(\Phi)$*;*
- *for each* $\mu \in \mathcal{E}(\Phi)$ *there exist* $k \in \mathbb{N}$ *and a Borel subset* $E \subset M$ *satisfying* $\mu(E) > 0$ *such that* $\mathcal{A}$ *is* (k,E)*-weakly admissible;*
- *for each* $\mu \in \mathcal{E}(\Phi)$ *there exist a Borel subset* $F \subset M$ *satisfying* $\mu(F) > 0$ *such that for each* $q \in F$ *and continuous* $y\colon [0,\infty) \to X$ *satisfying* $\int_0^\infty \|y(t)\|\,dt < 0$*, we have that* $x_q\colon [0,\infty) \to X$ *defined by*

$$x_q(t) = \int_0^t \mathcal{A}(\phi_r(q), t-r)y(r)\,dr \quad t \ge 0,$$

satisfies $\sup_{t\ge 0}\|x_q(t)\| < \infty$.

Then, $\mathcal{A}$ *is uniformly exponentially stable.*

Proof. Let us fix $\mu \in \mathcal{E}(\Phi)$. By $\lambda(\mu)$ we will denote the largest Lyapunov exponent of $\mathcal{A}$ with respect to μ. Furthermore, let F be as in statement of the theorem. Take $q \in F \subset \mathcal{R}^\mu$, $v \in X$ and choose a continuous map $\alpha\colon [0,1] \to [0,2]$ with support in $(0,1)$ and such that $\int_0^1 \alpha(r)\,dr = 1$. We define $y\colon [0,\infty) \to X$ by

$$y(t) = \begin{cases} \alpha(t)\mathcal{A}(q,t)v & t \in [0,1) \\ 0 & t \ge 1. \end{cases}$$

Obviously, y is continuous and $\int_0^\infty \|y(t)\|\,dt < \infty$. Moreover,

$$\begin{aligned} x_q(t) &= \int_0^t \mathcal{A}(\phi_r(q), t-r)y(r)\,dr \\ &= \int_0^1 \mathcal{A}(\phi_r(q), t-r)\alpha(r)\mathcal{A}(q,r)v\,dr \\ &= \mathcal{A}(q,t)v\int_0^1 \alpha(r)\,dr \\ &= \mathcal{A}(q,t)v, \end{aligned}$$

for $t \ge 1$, and therefore

$$\sup_{t\ge 1}\|\mathcal{A}(q,t)v\| \le \sup_{t\ge 0}\|x_q(t)\| < \infty.$$

We conclude that $\lambda(\mu) \le 0$. On the other hand, it follows from the assumptions of the theorem and Theorem 9 that $\lambda(\mu) \ne 0$. Thus,

$$\lambda(\mu) < 0 \quad \text{for } \mu \in \mathcal{E}(\Phi). \tag{12}$$

For $t \geq 0$, we define $F_t \colon M \to \mathbb{R} \cup \{-\infty\}$ by

$$F_t(q) = \|\mathcal{A}(q,t)\| \quad \text{for } q \in M.$$

Observe that F_t is upper semicontinuous for each $t \geq 0$. Then, it follows from [9, Theorem 4.3.] that there exists $\nu \in \mathcal{E}(\Phi)$ such that

$$\lim_{t\to\infty} \frac{1}{t} \sup_{q\in M} F_t(q) = \lambda(\nu).$$

The above equality together with (12) yields

$$\lim_{t\to\infty} \frac{1}{t} \sup_{q\in M} F_t(q) < 0,$$

which easily implies (11). □

Remark 3. Observe that in the statement of Theorem 10 we have essentially required for two admissibility conditions to hold (see the last two requirements in the statement). We observe that the conclusion of the theorem would not hold if one of those assumptions is eliminated. Indeed, the last requirement ensures that $\lambda(\mu) \leq 0$ for any $\mu \in \mathcal{E}(\Phi)$ but it doesn't imply any type of exponential decay since the identity cocycle $\mathcal{A}(q,t) = \mathrm{Id}$, $q \in M$, $t \in [0,\infty)$ trivially satisfies this condition. Hence, we have imposed third condition to ensure that $\lambda(\mu) < 0$ for each $\mu \in \mathcal{E}(\Phi)$.

On the other hand, the last assumption also cannot be eliminated since without it we would not be able to eliminate the existence of positive Lyapunov exponents and of the unstable component of our cocycle.

Remark 4. We would also like to stress that our admissibility assumptions in the statement of Theorem 10 are optimal. Indeed, if $\mathcal{A}$ is uniformly exponentially stable, it is easy to verify that the last two requirements in the statement of Theorem 10 hold with $E = F = M$.

Remark 5. Finally, we would like to compare Theorem 10 with the existing results in the literature. We first observe that in the statement of Theorem 10, we don't require that admissibility conditions hold for each $q \in M$ (like in all previous work devoted to spectral characterization of hyperbolic behaviour for linear cocycles) but rather for $q \in M$ which belong to a set which is essentially as large as M from the measure-theoretic point of view but not necessarily equal to M.

However, we stress that Theorem 10 only works under the assumption that $\mathcal{A} \colon M \times [0,\infty) \to B(X)$ is continuous. Hence, when compared with the results in the literature, we have stronger regularity assumptions for $\mathcal{A}$.

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