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Collateral and growth cycles with heterogeneous agents☆

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ABSTRACT

We investigate the effects of collateral and monetary policy on economic growth within a Ramsey equilibrium model where agents have different discount factors. Introducing liquidity constraints in segmented markets where (poor) impatient agents without collateral have limited access to credit, we study their implications in terms of welfare and business cycles (based on deterministic cycles through bifurcations and self-fulfilling prophecies). We find that an accommodative monetary policy may be growth-enhancing and welfare-improving (through the inequality reduction) while making unpleasant fluctuations more likely. Conversely, a regulation reinforcing the role of collateral and tempering the financial market imperfections may stimulate the economic growth while pursuing the goal of stabilization.

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1. Introduction

The financial crisis of 2007–2008, better known as subprime (mortgage) crisis, led at the very end to credit tightening and economic slowdown in the U.S. and Europe. The drop in the value of collateral, the securitization of bad credits and the spread of toxic assets promoted a liquidity crisis together with financial and real markets instability. Two main ingredients are at the origin of such a global crisis: a strong development of subprime loans distributed to households with insufficient collateral and new banking practices with respect to risk based on insufficient banks' own funds. American and European governments and central banks have then decided to increase the collateral constraints to both households and banks to avoid bad credits, and to supply money to offset the lack of liquidity. However, besides the problems related to sovereign debts, the effects of these policies on the growth rate have been contrasted. While the U.S. recovers today its pre-2007 growth rate, most European countries have experienced a lasting stagnation or even recession.

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In particular, the quantitative easing policy first implemented by the US FED and then by the European Central Bank since September 2012, which has significantly increased over the last months, has clearly provided liquidities in the economy in order to support investment and consumption but it is claimed to generate some instability in the financial markets. Maintaining the interest rate close to the zero lower bound, this policy provides strong incentives to agents to invest in assets and may generate bubbles disconnected from the real side of the economy. Over the last year, the FED has decided to weaken its quantitative easing policy and the volatility has indeed significantly increased in the financial markets, agents expecting some possible increase if the interest rate. An additional regulation policy aiming at affecting consumption and investment decisions could then be appropriate in order to rule out fluctuations.

To address these questions, we consider an endogenous growth model. We study the stability issue of growth rate dynamics taking in account the role of collateral in a monetary economy with credit market imperfections and agents with heterogeneous discount rates. Agents pay cash a part of their consumption purchases and on credit the remainder whose extent depends on the amount of collateral understood in a broad sense as depending on both physical and human capital. Such an assumption captures the increased restrictions recently faced by households when accessing credit as financial intermediaries require collateral guarantees based at the same time on the stock of physical capital owned by households and on their potential wage rate which depends on their stock of human capital. Depending on their discount rate, agents are differently affected by these constraints. [1] We aim at understanding the role of collateral in the occurrence of economic crises (viewed as self-fulfilling prophecies) and making policy recommendations to stabilize the economy and possibly reduce social inequalities. As the quantitative easing increases the liquidity, we will focus on the consequences of money creation by the central bank.

We consider a credit constraint, formulated as an extended cash-in-advance constraint, similar in spirit to the one considered by Kiyotaki and Moore (1997). We assume that consumers may borrow to consume more than the cash they hold but they are constrained by the amount of collateral they own. Similarly to Kiyotaki and Moore (1997), we show that the credit constraint promotes the emergence of endogenous fluctuations. Depending on whether this constraint is binding for all agents or only a part of them, we get two different regimes: the one with expectation-driven fluctuations based on equilibrium multiplicity (sunspots); the other without transitional dynamics based on a unique equilibrium, the Balanced Growth Path (henceforth, BGP). Although we do not consider a micro-founded formulation based on monetary search models *à la* Lagos and Wright (2005), our results look similar to the main conclusions by Ferraris and Watanabe (2008); 2011). Embedding a credit model *à la* (Kiyotaki and Moore, 1997) into a search model *à la* (Lagos and Wright, 2005), these authors show that larger fluctuations arise when the credit constraint is binding and agents are at their borrowing limit.[2]

We focus on agents' heterogeneity. They have different discount rates, cannot borrow against future labor income, but are allowed to pay a part of current consumption on credit if the collateral is positive. We know that heterogeneous discounting catalyses the accumulation of capital in the hands of the most patient agents (henceforth, the capitalists). If individuals are permitted to borrow against future income, the impatient agents borrow from the patient ones and spend the rest of their life to work in order to refund the debt. In the spirit of Becker (1980), Becker and Foias (1987); 1994), but in contrast to Le Van and Vailakis (2003), we also introduce a borrowing constraint: we rule out the possibility of negative capital (of debt). Through the credit constraint, a positive capital works as collateral to reduce the amount of balances needed to finance the current consumption. Thus markets are incomplete and the collateral weakens the cash-in-advance.

We consider an endogenous growth model with a Romer-type (1986) learning-by-doing externality leading to constant returns to scale in capital accumulation. Endogenous growth rates are suitable to capture the effects of financial frictions (borrowing constraints) on the emergence of growth cycles (endogenous fluctuations of the growth rates). Business-cycle fluctuations are here represented as deterministic cycles arising out of bifurcations, and self-fulfilling prophecies.[3]

Under heterogeneous discounting, the population splits into two classes: patient capitalists and impatient workers.[4] There is room for two regimes: the one where the capitalist holds capital and no balances, the other where money and capital enter her portfolio.

In the first, the credit constraint is not binding and the liquidity constraint does not affect the patient agents' capital accumulation. The economy jumps on the BGP. This conclusion goes along the regime with agents below their borrowing limits in Ferraris and Watanabe (2011). In this case, they show that fluctuations never occur in capital.

Alternatively, the credit constraint of patient agents binds and equilibrium transitions take place: endogenous fluctuations (two-period cycles and indeterminacy) may arise around the BGP. A similar conclusion is reached by Ferraris and Watanabe (2011) when agents are at their borrowing limits and fluctuations in capital occur.[5] Eventually, we prove that

[1] Barbar and Bosi (2010) consider a similar model but with homogeneous agents.

[2] See also (Fostel and Geanakoplos, 2008) and (Geanakoplos and Zame, 2007), where the existence of collateralized assets affects market prices and allocations and may generate fluctuations.

[3] See also Matsuyama (1999) and (2001) for the emergence of growth cycles from the interplay between investment and innovation.

[4] Woodford (1986) also studies an economy with patient capitalists and impatient workers financially constrained. Even if these agents are infinite-lived, the reduced forms of his model look like an overlapping-generations model because of the nature of borrowing constraint. Our model is more standard and closer to Ramsey equilibrium models with cash-in-advance.

[5] See also Matsuyama (2007) for the existence of credit cycles in a model with credit market imperfections and endogenous changes in investment technologies.

an accommodative monetary policy may be growth-enhancing and possibly welfare-improving, while involving unpleasant fluctuations.

More precisely, when capitalists hold no balances, money growth enhances their welfare, while worsening the situation of impatient agents, who bear the opportunity cost of holding money (nominal interest rate). It affects the initial consumption without effects on the growth rate. Focusing on the welfare differential between the two classes of agents along the BGP, we find an increase in social inequalities: the patient agents become richer ending up to own the entire stock of capital, and experience a welfare gain through an increase in consumption; the impatient ones become poorer and experience a welfare loss.

In the second case (capitalists hold also money), money growth has a positive impact on the growth rate but an ambiguous effect on initial consumptions and welfare. This regime differs significantly from the first one: to show the possibility of opposite effects, we perform a numerical exercise with plausible parameter values where money growth worsens the welfare of capitalists (sharing now the opportunity cost of holding balances) and improves the welfare of workers. We conclude that an accommodative monetary policy may reduce the inequalities viewed as welfare differential between agents along the BGP.

It is valuable to complement and supplement a welfare analysis with a bifurcation analysis to grasp the risk of destabilization. Money growth promotes crises through the occurrence of endogenous fluctuations and self-fulfilling prophecies. Indeed, it raises the inflation rate and the nominal interest rate, that is the opportunity cost of holding money. The imperfection (liquidity constraint) has a larger impact on the consumption smoothing of patient agents and makes indeterminacy more likely. Conversely, collateralization tempers the effects of financial frictions and renders the endogenous fluctuations less likely. We conclude that, if the government goal is to prevent growth cycles, any increase in money supply should go with an increase in the sensitivity of credit grants to collateral. In the light of the recent crisis, our results can be reinterpreted. The Basel III agreements raising the weight of collateral of banks that have to be extended up to 2019, together with the quantitative easing policy implemented by the European Central Bank since September 2012 and recently increased, appear as two facets of a global policy intended to promote growth without expectation-driven fluctuations and to fight inequalities.

This rest of the paper is organized as follows. In Section 2 and 3, we present the model and we derive the demand for goods and balances. In Section 4, we study the equilibrium properties when capitalists hold no money. Section 5 treats the case where they are also liquidity-constrained. Section 6 concludes. All the proofs and mathematical details are provided in a technical appendix available online.

2. Fundamentals

Firms produce a single good, households work, consume and save through capital and balances provided by a monetary authority. We specify the fundamentals (endowments, preferences and technology) as follows.

2.1. Heterogenous households

We assume that individuals differ in terms of initial capital $k_{i0} \geq 0$ and preferences. More precisely, they have the same utility function

$$u(c_i) \equiv \frac{c_i^{1-1/\sigma}}{1-1/\sigma} \quad \text{for } \sigma \neq 1$$
$$u(c_i) \equiv \ln c_i \quad \text{for } \sigma = 1 \tag{1}$$

but different discount factors. $\sigma > 0$ is the constant elasticity of intertemporal substitution. Without loss of generality, we will consider only two types of agents with discount factors β_i such that $\beta_1 < \beta_0$. We call 0 the patient type and 1 the impatient one. There are n_i individuals of type i with $n_0 + n_1 \equiv n$. Each consumer of type i maximizes an additively separable intertemporal utility function

$$\sum_{t=0}^{\infty} \beta_i^t u(c_{it}) \tag{2}$$

with respect to a sequence $(k_{it+1}, c_{it})_{t=0}^{\infty}$, and faces a sequence of budget constraints:

$$M_{it+1} - M_{it} + p_t(k_{it+1} - \Delta k_{it}) + p_t c_{it} \leq p_t r_t k_{it} + p_t w_t l_{it} + p_t \tau_{it} \tag{3}$$

and borrowing constraints

$$p_t c_{it} - \gamma p_t k_{it} \leq M_{it} \tag{4}$$

with r_t the real interest rate, w_t the wage rate, $k_{it} \geq 0$ and $\delta = 1 - \Delta \in (0, 1)$, the depreciation rate of capital. Consumption (c_i) and capital (k_i) are the same good: they are produced with the same technology and have the same price (p).

The RHS of constraint (3) is the disposable monetary income. In real terms, it is the sum of capital income $(r_t k_{it})$, labor income $(w_t l_{it})$ and transfers from the monetary authority (τ_{it}). The revenue is partly saved in form of money $(M_{it+1} - M_{it})$

and capital ($k_{it+1} - \Delta k_{it}$), and partly consumed. In the spirit of Grandmont and Younès (1972), we assume that a share of purchases is paid cash, while the rest on credit. The cash-in-advance constraint (4) is a possible interpretation of Lucas and Stokey (1987) with a cash and a credit good.

The individual's wealth matters to buy on credit. We consider an extended cash-in-advance to introduce a credit constraint in the spirit of Kiyotaki and Moore (1997). The amount of credit depends on the household's collateral. For simplicity, we assume the credit grant to be proportional to the nominal value of collateral ($\gamma p_t k_{it}$). γ represents either the common credit sensitivity to collateral or the market rules. More precisely, γ captures the lenders' or sellers' prudential attitude towards borrowers in presence of asymmetric informations, but also credit market regulation policies, that is legal constraints to credit grants to ensure borrowers' solvability. To buy a constant amount of consumption on credit: $d = \gamma k_{it}$, the household faces a given amount of collateral: $k_{it} = d/\gamma$. Thereby, the higher the parameter γ, the lower the needs of collateral to buy on credit an amount d. If γ is sufficiently high, the cash-in-advance is no longer binding and the capitalist no longer needs balances to finance her consumption purchases. Put differently, a larger γ can be interpreted as a consumption-oriented policy based on larger agents' solvability.

Note also that the introduction of collateral in a model of capital accumulation implies an endogenous velocity of money (with respect to consumption). Indeed, in the case of a representative agent, we get from (4)

$$v_t = \frac{p_t c_t}{M_t} = 1 + \gamma \frac{p_t k_t}{M_t}$$

This formulation takes in account the criticism addressed to the cash-in-advance literature, that is the implausibility of a constant unitary velocity of money.[6] However, this property is of little interest in our model as we will show that our main conclusions hold under low values of γ.

For the sake of simplicity, the patient agents supply no labor while the impatient ones supply one unit: $l_{0t} = 0$ and $l_{1t} = 1$. This assumption is not restrictive. Similar models with endogenous labor supply exhibit a regime where patient agents supply no labor. Our main results are robust to extensions with endogenous labor for impatient agents.[7]

2.2. Homogeneous firms

Firms are represented by a technology with constant private returns and increasing social returns. There is a large number (J) of identical competitive firms. The representative firm rents capital and labor to produce the good under constant (private) returns to scale. Technology is rationalized by an aggregate Romer-type (1986) production function allowing for endogenous growth:

$$Y = F(K, L) \equiv \bar{A} K^s L^{1-s} \tag{5}$$

where Y is GDP, $K \equiv \sum_{j=1}^{J} K_j$ and $L \equiv \sum_{j=1}^{J} L_j$ are the aggregate capital and labor force with K_j and L_j the jth firm's inputs, and $s \in (0, 1)$ represents the share of capital in total income.

As in Romer (1986), the TFP $\bar{A} = A\bar{K}^{1-s}$ is affected by a capital externality interpreted, for instance, as technological knowledge spillover, with A a scaling parameter. At the aggregate level, we then get an AK-type technology $Y = AKL^{1-s}$ with increasing (social) returns. Recalling that we have assumed $l_0 = 0$ and $l_1 = 1$, total labor is $L = n_0 l_0 + n_1 l_1 = n_1$ and from now on we consider the normalization $n_0 = n_1 = 1$. Denoting $k = K/L$ the capital-labor ratio, the real interest rate r_t and the wage bill w_t at period t are given by

$$r_t = sA \equiv r \text{ and } w_t = (1 - s)Ak_t \tag{6}$$

The equilibrium interest rate turns out to be constant over time. Seminal AK-type endogenous growth models such as Romer (1986) share this feature. Although our results will not depend on a specific value of the share of capital s, we will mainly consider in the sequel a value $s \in (1/2, 1)$, quite usual in the endogenous growth literature (Mankiw et al., 1992), implying that capital includes both physical and human capital. This assumption is introduced for two reasons. First, it has been shown by Basu and Fernald (1997) that the degree of increasing returns to scale in the durable manufacturing sector in the US economy is 0.33, with standard deviation 0.11. Assuming for instance $s = 2/3$ as in Mankiw et al. (1992) allows to match this estimate. Second, when they contract a credit, beside a collateral based on physical capital, households may also be constrained on the basis of their human wealth, i.e. their discounted wage income, which highly depends on their productivity, i.e. their human capital. Our formulation allows to take into account these two effects.

2.3. Monetary authority

Let M_t denotes the aggregate supply of money at time t. Each period, an amount $M_{t+1} - M_t$ of money is "helicoptered" to agents. More explicitly, an exogenous share θ_i is given to agents of type i. For mathematical tractability, we assume the

[6] Money-In-the-Utility (MIU) models are immunized against this criticism: the functional equivalence highlighted by Feenstra (1986) between CIA and MIU, no longer holds with a Cash-When-I'm-Done (CWID) timing or a CIA timing and strictly positive elasticity of substitution between consumption and real balances (see Carlstrom and Fuerst, 2003).

[7] Bosi and Seegmuller (2010) among others. In Woodford (1986), the capitalists do not work but hold capital while workers own no capital.

same shares as initial conditions: $M_{i0} = \theta_i M_0$ for $i = 0, 1$. In addition, money supply grows at a constant factor $\eta \equiv M_{t+1}/M_t$. The monetary policy consists in a pair of instruments (η, θ_0) because the other share is determined: $\theta_1 = 1 - \theta_0$.[8] Normally, real economies experience a monetary expansion ($\eta \geq 1$).

3. Demand for goods and balances

The individual i maximizes the intertemporal utility (2) under the constraints (3) and (4). Her demand for money and for goods rests on the first-order conditions:

$$\beta_i^t c_{it}^{-1/\sigma} = p_t(\lambda_{it} + \mu_{it}) \tag{7}$$

$$p_t \lambda_{it} \geq p_{t+1}[\lambda_{it+1}(r_{t+1} + \Delta) + \gamma \mu_{it+1}] \tag{8}$$

$$\lambda_{it} \geq \lambda_{it+1} + \mu_{it+1} \tag{9}$$

where λ_{it} and μ_{it} are the non-negative multipliers associated to constraints (3) and (4) respectively.

Let $\pi_{t+1} \equiv p_{t+1}/p_t$ denote the inflation factor. Eqs. (8) and (9) hold with equality if $k_{it+1} > 0$ and $M_{it+1} > 0$ respectively. The complementary slackness conditions hold (Karush–Kuhn–Tucker):

$$p_t \lambda_{it}(r_t k_{it} + w_t + \tau_{it} - \pi_{t+1} m_{it+1} + m_{it} - k_{it+1} + \Delta k_{it} - c_{it}) = 0$$
$$p_t \mu_{it}(m_{it} - c_{it} + \gamma k_{it}) = 0$$

where $m_{it} = M_{it}/p_t$ is the individual demand for real balances. If $\mu_{it} > 0$, the cash-in-advance constraint is binding, while, if $\mu_{it} = 0$, the constraint becomes superfluous. In such a case, under the standard assumption of positive nominal interest rate, the agent holds no money because of the opportunity cost of holding money ($M_{it} = 0$). [9]

The initial and the final conditions are given by the initial endowments: M_{i0}, $k_{i0} \geq 0$ and the transversality conditions:

$$\lim_{t \to +\infty} \beta_i^t c_{it}^{-1/\sigma}(k_{it+1} + \pi_{t+1} m_{it+1}) = 0 \tag{10}$$

for each individual $i = 0, 1$.

In the spirit of Becker (1980), we will show that impatient agents hold no capital in the long run, along the balanced growth path (BGP), but hold money because of the (binding) liquidity constraint. However, the patient agent's liquidity constraint may be binding or not along the BGP. Indeed, owning the entire stock of capital, patient agents may hold money or not through the borrowing constraint depending on their level of wealth. Of course, in the case of transition, the equilibrium may switch from a regime to another along the dynamics. Nevertheless, in a neighborhood of the BGP, by continuity, the constraint will remain binding (or not). These cases will be treated separately in the following.

4. Patient agents buy on credit

A wealthy patient agent does not need money: she holds enough collateral to buy on credit (the liquidity constraint (4) becomes ineffective). In this section, we consider the regime where money is demanded only by impatient agents and we characterize the long-run equilibrium. The economy jumps from the very beginning on the BGP and experiences no transition.

4.1. Patient agent

In the first regime, the liquidity constraint of a patient agent becomes superfluous. Thus, in the case of positive interest rate, she holds no money ($M_{0t} = 0$) and $c_{0t} < \gamma k_{0t}$. From (6) and first-order conditions (7)–(9) with $\mu_{0t} = 0$, we derive the Euler inequality

$$\left(\frac{c_{0t+1}}{c_{0t}}\right)^{1/\sigma} \geq \beta_0(\Delta + sA) \tag{11}$$

which holds with equality if $k_{0t+1} > 0$, and the budget constraint:

$$k_{0t+1} - \Delta k_{0t} + c_{0t} = sAk_{0t} + \tau_{0t} \tag{12}$$

The transversality condition simplifies as well:

$$\lim_{t \to +\infty} \beta_0^t c_{0t}^{-1/\sigma} k_{0t+1} = 0 \tag{13}$$

[8] As most of our results do not depend on a specific value of the shares θ_i, we could simplify the formulation assuming $\theta_0 = \theta_1 = 1/2$. However, in the case where patient agents hold money and pay cash part of their consumption, we will use conditions on these shares to derive the impact of the policy parameters of η and γ on welfare (see Proposition 6).

[9] Indeed, when $\mu_{it+1} = 0$, the inequality (8) becomes $\lambda_{it} \geq \lambda_{it+1}\pi_{t+1}(\Delta + r_{t+1})$. If, at the same time, $M_{it+1} > 0$, then $\lambda_{it} = \lambda_{it+1}$ and $1 \geq \pi_{t+1}(\Delta + r_{t+1})$ which implies in turn a negative nominal interest rate.

4.2. Impatient agent

Focus now on the impatient agent who needs money to finance her future consumption. From (6) and first-order conditions (7)–(9) with $\mu_{it} > 0$ and $\lambda_{it} = \lambda_{it+1} + \mu_{it+1}$ (the liquidity constraint is binding), we derive the Euler inequality:

$$\left(\frac{c_{1t+1}}{c_{1t}}\right)^{1/\sigma} \geq \beta_1 \frac{\pi_t}{\pi_{t+1}} \frac{\Delta + sA - \gamma}{1 - \gamma \pi_t} \tag{14}$$

which holds with equality if $k_{1t+1} > 0$, and the constraints:

$$m_{1t+1}\pi_{t+1} - m_{1t} + k_{1t+1} - \Delta k_{1t} + c_{1t} = sAk_{1t} + w_t + \tau_{1t} \tag{15}$$

$$c_{1t} - \gamma k_{1t} = m_{1t} \tag{16}$$

The transversality condition (10) still holds.

4.3. Balanced growth path

In the *AK*-like endogenous growth models, the long-run equilibrium is given by an endogenous (stationary) growth rate and the BGP by a sequence where the real variables grow at the same rate, namely $k_{it+1} = g^* k_{it}$ and $c_{it+1} = g^* c_{it}$, where $g^* > 1$ and $g^* - 1 > 0$ are the (stationary) Balanced Growth Factor (BGF) and the corresponding rate.

In the money market, demand for balances depends on the cash-in-advance constraint. On the supply side, each agent of type i receives in period t an amount $\theta_i(M_{t+1} - M_t)$ of nominal balances from the monetary authority:

$$M_{t+1} - M_t = \theta_0(M_{t+1} - M_t) + \theta_1(M_{t+1} - M_t) = p_t\tau_{0t} + p_t\tau_{1t} \tag{17}$$

where

$$\tau_{it} = (m_{t+1}\pi_{t+1} - m_t)\theta_i \tag{18}$$

are the real balances per capita.

Then, money market clears. The equality between the aggregate supply and demand writes:

$$m_t = m_{1t} \tag{19}$$

As usual, in dynamic monetary models, money growth has both inflationary and real effects (growth of real balances):

$$\eta = \pi_{t+1} m_{t+1}/m_t \tag{20}$$

Replacing (19) in (20) gives:

$$\pi_{t+1} = \eta m_{1t}/m_{1t+1} \tag{21}$$

Along the BGP, the inflation factor $\pi = \eta/g^*$ is constant and the decomposition $\eta = g^*\pi$ holds. Aggregating side by side the budget constraints (3), now binding across individuals, we obtain

$$k_{0t+1} - \Delta k_{0t} + c_{0t} + m_{1t+1}\pi_{t+1} - m_{1t} + c_{1t} = r_t k_{0t} + \tau_{0t} + w_t + \tau_{1t}$$

Replacing (19) in the LHS and (17) in the RHS, we obtain

$$k_{0t+1} - \Delta k_{0t} + c_{0t} + c_{1t} = r_t k_{0t} + w_t$$

Substituting the equilibrium prices (6) and (14), and noticing that $k_t \equiv K_t/L_t = k_{0t}$, we find the market clearing for goods:

$$k_{t+1} - \Delta k_t + c_{0t} + c_{1t} = Ak_t \tag{22}$$

that is, the sum of aggregate investment and aggregate consumption is equal to the aggregate production.

Within standard exogenous growth models with infinite-lived agents, the heterogeneity of discounting ($\beta_1 < \beta_0$) implies the Ramsey conjecture (1928) later proved by Becker (1980): impatient agents hold no capital in the long run. While more patient individuals tend to accumulate savings, the others tend to consume more and thus spend their incomes and even their capital. As individuals attempt to adjust their consumption profile and impatient agents consume more and more of their capital stock, capital is transferred to the most patient individuals.

In our monetary economy where patient agents do not hold money, $\beta_1 < \beta_0$ is not enough to ensure this degenerate distribution of capital along a BGP. We need to consider some additional restriction on the parameter γ that characterizes the borrowing constraint: γ needs to be low enough to ensure that patient agents hold capital. In the case such a restriction is not satisfied and the discount rates are sufficiently close, impatient agents hold the whole capital stock and patient agents do not own any capital. Since, we are in the regime where patient agents neither hold money nor work, they have no mean to consume and the equilibrium fails to exist. Therefore, the only possibility is an equilibrium where the dichotomy takes place: the patient agents own the whole capital stock while the impatient ones hold money.

We introduce restrictions to ensure the non-negativity of the growth rate along the BGP as well as the positivity of interest rate.

Assumption 1. $1/(\Delta + sA) < \beta_0 < (\Delta + sA)^{(1-\sigma)/\sigma}$ *and* $\gamma < \bar{\gamma}$ *with*

$$\bar{\gamma} \equiv \frac{(\Delta + sA)[\beta_0(\Delta + sA)]^{\sigma}(\beta_0 - \beta_1)}{\eta\beta_0(\Delta + sA) - \beta_1[\beta_0(\Delta + sA)]^{\sigma}} \tag{23}$$

Indeed, under Assumption 1:

$$\frac{c_{0t+1}}{c_{0t}} = \frac{c_{1t+1}}{c_{1t}} = g^* = [\beta_0(\Delta + sA)]^{\sigma} > 1$$

and the nominal interest rate turns out to be positive: $\pi(\Delta + sA) > 1$ with $\pi = \eta/g^*$ (Fisher decomposition and zero lower bound). As a result, since Assumption 1 implies the following inequality

$$[\beta_0(\Delta + sA)]^{\sigma} > \left(\beta_1 \frac{\Delta + sA - \gamma}{1 - \gamma\pi}\right)^{\sigma}$$

there exists $T > 0$ such that $k_{1t} = 0$ for every $t > T$: only the patient agents hold capital in the long run.

Around the BGP, Eqs. (15) and (16) write

$$m_{1t+1}\pi_{t+1} - m_{1t} + c_{1t} = w_t + \tau_{1t} \tag{24}$$

$$c_{1t} = m_{1t} \tag{25}$$

and the transversality condition (10) becomes:

$$\lim_{t\to+\infty} \beta_1^t c_{1t}^{-1/\sigma} \pi_{t+1} m_{1t+1} = 0 \tag{26}$$

The following ratios are convenient to study the endogenous growth:

$$g_{t+1} \equiv \frac{k_{t+1}}{k_t}, \quad x_{0t} \equiv \frac{m_{0t}}{k_t}, \quad x_{1t} \equiv \frac{m_{1t}}{k_t} \tag{27}$$

As in the initial (Romer, 1986) model, the BGP is the only possible equilibrium and there is no room for transitional dynamics.

Proposition 1. *Under Assumption 1, the unique equilibrium is the BGP with* $g_t = g^* \equiv [\beta_0(\Delta + sA)]^{\sigma} > 1$ *and*

$$\begin{aligned}
(x_{0t}, x_{1t}) &= \left(0, \frac{(1-s)A}{1 + (\eta - 1)\theta_0}\right) \\
c_{00} &= k_0\left[\Delta + sA + \frac{(1-s)A(\eta-1)\theta_0}{1 + (\eta-1)\theta_0} - g^*\right] > 0 \\
c_{10} &= \frac{(1-s)Ak_0}{1 + (\eta-1)\theta_0} > 0
\end{aligned} \tag{28}$$

In this equilibrium, only the patient agents own capital while only the impatient ones hold money.

Proof. The proof is available in the technical appendix published online. □

Since the capitalists' borrowing constraints do not bind, they hold no money and the parameter γ ends up to have no impact on the growth rate g^* and the initial consumptions c_{i0}. The fact that the BGP emerges as the only possible equilibrium means that, under rational expectations, the forward-looking variable (consumption) adjusts and the economy jumps from the very beginning on the BGP. Thus, any transition is ruled out. Our conclusion recovers the main result in Ferraris and Watanabe (2011). They show indeed that when agents are below their borrowing limits, capital fluctuations never occur.

4.4. Monetary policy

Along the BGP, the monetary policy does not affect the growth factor g^* but has an impact on the consumption and the welfare of both types of agents. The effects on $c_{it} = g^{*t}c_{i0}$ and

$$U_i \equiv \sum_{t=0}^{\infty} \beta_i^t u(c_{it}) = \sum_{t=0}^{\infty} \beta_i^t \frac{\left(c_{i0}g^{*t}\right)^{1-1/\sigma}}{1 - 1/\sigma} = \frac{\sigma}{\sigma - 1}\frac{c_{i0}^{1-1/\sigma}}{1 - \beta_i g^{*1-1/\sigma}} \tag{29}$$

are qualitatively the same because, under the transversality condition $\beta_i g^{*1-1/\sigma} < 1$, $\partial U_i/\partial c_{i0} > 0$. However, the impacts on the initial consumptions of agents differ.

Proposition 2. *Under Assumption 1, along the BGP, we have*

$$\frac{\partial U_0}{\partial \eta} > 0 \text{ and } \frac{\partial U_1}{\partial \eta} < 0 \tag{30}$$

and, if $\eta > 1$,

$$\frac{\partial U_0}{\partial \theta_0} > 0 \text{ and } \frac{\partial U_1}{\partial \theta_0} < 0$$

Proof. The proof is available in the technical appendix published online. □

The impact of the money growth factor η deserves an economic interpretation. The patient agent receives more money today and bears no opportunity cost of holding money, because she spends these balances today (indeed, she does need money to finance consumption tomorrow). The impatient agents bear an opportunity cost in order to finance future consumption with money. This opportunity cost is given by the nominal interest rate and increases in the monetary growth rate because of the inflation. Her cost is greater than the benefit of additional monetary transfers to impatient agents from the monetary authority. Since patient agents are richer than impatient ones as they own the whole stock of capital, we can interpret the social inequalities as a difference between the utility levels along the BGP of both classes of agents. As shown here, these inequalities increase in η. The share θ_0 of money "helicoptered" to patient agents has a similar effect.

5. Patient agents pay cash

Consider now the second regime where the patient agent's liquidity constraint is also binding in the long run. The household is aware of her credit function, which is a sort of institutional constraint: it represents either the credit market endogenous organization or some legal constraints. Both types of agents have a binding cash-in-advance so that both hold money.

5.1. Patient agent

Patient agent's consumption smoothing over time is affected by the liquidity constraint. The Euler equation captures the intertemporal arbitrage and includes now the monetary effects through the inflation rate. Indeed, the capitalist also bears the opportunity cost of holding money. In order to understand the effects of inflation on the nominal interest rate and this opportunity cost, we compute the patient agent's Euler equation. From (6) and the first-order conditions (7)–(9) with $\mu_{0t} > 0$, we derive the Euler inequality:

$$\left(\frac{c_{0t+1}}{c_{0t}}\right)^{1/\sigma} \geq \beta_0 \frac{\pi_t}{\pi_{t+1}} \frac{\Delta + sA - \gamma}{1 - \gamma \pi_t} \tag{31}$$

which holds with equality if $k_{0t} > 0$, and the constraints:

$$m_{0t+1}\pi_{t+1} - m_{0t} + k_{0t+1} - \Delta k_{0t} + c_{0t} = r_t k_{0t} + \tau_{0t} \tag{32}$$

$$c_{0t} - \gamma k_{0t} = m_{0t} \tag{33}$$

Comparing (31) with (11), we observe that inflation matters now. Since patient agent's wealth accumulation coincides with the aggregate capital accumulation, inflation has a direct effect on economic growth.

5.2. Impatient agent

As in the first regime, Eqs. (14), (15) and (16) and the transversality condition (10) still apply.

5.3. Intertemporal equilibrium and balanced growth path

Eqs. (17) and (18) still hold. However, now the capitalists hold money. The money market clears:

$$m_t = m_{0t} + m_{1t} \tag{34}$$

Replacing (34) in (20) yields the inflation dynamics:

$$\pi_{t+1} = \eta \frac{m_t}{m_{t+1}} = \eta \frac{m_{0t} + m_{1t}}{m_{0t+1} + m_{1t+1}} \tag{35}$$

Eq. (22) (market clearing for goods) still holds.

If $\beta_0 > \beta_1$, considering simultaneously Eqs. (14) and (31), we get now a transitional dynamics: along the transition, Eq. (31) holds with equality while Eq. (14) with a strict inequality. Put differently, as in the first regime, the Ramsey conjecture holds: impatient agents own no capital and their consumption depends only on money holding (see Eqs. (24) and (25)).

We may then compute the dynamical system that drives the transitional dynamics. We use again the variables g_{t+1}, x_{0t} and x_{1t} defined above.

Proposition 3. *When the patient agent cash-in-advance constraint is binding, the intertemporal equilibrium satisfies the system:*

$$\eta x_{1t+1} + (1-\eta)\theta_1(x_{0t+1} + x_{1t+1}) = (1-s)A\frac{x_{0t+1} + x_{1t+1}}{x_{0t} + x_{1t}} \tag{36}$$

$$\frac{g_t}{g_{t+1}}\frac{\eta x_{1t} + (1-\eta)\theta_1(x_{0t} + x_{1t})}{\eta x_{1t+1} + (1-\eta)\theta_1(x_{0t+1} + x_{1t+1})}\left[1 - \gamma\frac{\eta}{g_t}\frac{(1-s)A}{\eta x_{1t} + (1-\eta)\theta_1(x_{0t} + x_{1t})}\right]$$
$$= \beta_0(sA + \Delta - \gamma)\left(\frac{x_{0t} + \gamma}{x_{0t+1} + \gamma}\frac{1}{g_{t+1}}\right)^{1/\sigma} \tag{37}$$

with $x_{0t} \equiv m_{0t}/k_t$, $x_{1t} \equiv m_{1t}/k_t$. $g_t = k_t/k_{t-1}$ *is given by*

$$g_t = g(x_{0t}, x_{1t}) \equiv \Delta + sA - \gamma - (1-s)A\frac{\eta x_{0t} + (1-\eta)\theta_0(x_{0t} + x_{1t})}{\eta x_{1t} + (1-\eta)\theta_1(x_{0t} + x_{1t})} \tag{38}$$

Proof. The proof is available in the technical appendix published online. □

Eqs. (36) and (37) form a two-dimensional dynamic system in (x_{0t}, x_{1t}). The steady state of system (36)–(37) represents the BGP. Thus, this system not only captures the long run (the BGP) but also the short run (equilibrium transition around the BGP). The economy may experience endogenous fluctuations along the transition. [10]

Along the BGP, the Euler Eqs. (14) and (31) imply

$$\frac{c_{0t+1}}{c_{0t}} = \frac{c_{1t+1}}{c_{1t}} = g^* = \left(\beta_0\frac{\Delta + sA - \gamma}{1 - \gamma\pi}\right)^{\sigma} > \left(\beta_1\frac{\Delta + sA - \gamma}{1 - \gamma\pi}\right)^{\sigma}$$

with $\pi = \eta/g^*$. To ensure the BGP existence and uniqueness, such that all the variables (balances, capital and consumption) grow at a constant rate, we introduce an assumption closely related to Assumption 1.

Assumption 2. $1/(\Delta + sA) < \beta_0 < (\Delta + sA)^{(1-\sigma)/\sigma}$

Proposition 4. *Under Assumption 2, there are* $\bar{\eta} > 1$ *and* $\bar{\gamma} > 0$ *such that, when* $\eta \in [1, \bar{\eta})$ *and* $\gamma \in (0, \bar{\gamma})$, *a unique solution* $g^* > 1$ *exists to equation*

$$g = \gamma\eta + \beta_0(\Delta + sA - \gamma)g^{\frac{\sigma-1}{\sigma}} \equiv \varphi(g) \tag{39}$$

This solution is a BGP with $k_t = k_0 g^{*t}$, $c_{0t} = c_{00}g^{*t}$, $c_{1t} = c_{10}g^{*t}$ *and*

$$x_0 = \frac{[1 + (\eta-1)\theta_0](\Delta + sA - \gamma - g^*) + (1-s)A(\eta-1)\theta_0}{\eta} > 0 \tag{40}$$

$$x_1 = \frac{(1-s)A[1 + (\eta-1)\theta_1] + \theta_1(\eta-1)(\Delta + sA - \gamma - g^*)}{\eta} > 0 \tag{41}$$

$$c_{00} = k_0(\gamma + x_0) > 0 \tag{42}$$

$$c_{10} = k_0 x_1 > 0 \tag{43}$$

Proof. The proof is available in the technical appendix published online. □

We observe that the constraint $c_{0t}/k_{0t} > \gamma$ always holds along the BGP. Indeed, Eqs. (40) and (42) in Proposition 4 entails that $c_{0t}/k_{0t} = c_{00}/k_0$ which is larger than γ.

[10] Boucekkine et al. (2005) show the occurrence of oscillatory dynamics in an AK model with vintage capital while (Boucekkine and Ruiz-Tamarit, 2008) observe transitional dynamics in the Uzawa–Lucas model. See also Caballe and Santos (1993) or Nishimura et al. (2008) for related results in two-sector models.

5.4. Monetary policy

One may wonder whether a monetary expansion or a financial markets regulation have a positive impact on economic growth and citizens' welfare.

5.4.1. Effects on growth

Focus first on the effects on the growth rate. The monetary growth factor is the product of the inflation and the growth factor (decomposition (35)). The monetary policy determines the magnitude of inflation and, by consequence, of growth. Let us address, for simplicity, the policy issue only along the BGP.

Proposition 5. *Under Assumption 2, the impact of both the monetary growth and the sensitivity to collateral are positive:*

$$\frac{\partial g^*}{\partial \eta} = \frac{\sigma\gamma}{1-\gamma\pi(1-\sigma)} > 0, \ \frac{\partial g^*}{\partial \gamma} = \frac{\sigma g^*}{1-\gamma\pi(1-\sigma)} \frac{\pi(\Delta + sA) - 1}{\Delta + sA - \gamma} > 0 \tag{44}$$

with $\eta = \pi g^*$.

Proof. The proof is available in the technical appendix published online. □

Consider the impact of η. Issuing money raises the inflation and the nominal interest rate. The opportunity cost of holding money increases and agents hold less money, consume less and accumulate more capital. In our model, money growth is growth-enhancing.

Consider the impact of γ. If γ is larger, a higher capital level lowers the money demand and the opportunity cost of holding money (which is positive under the assumption of positive nominal interest rate). Households accumulate more capital and economic growth is enhanced at the end.

5.4.2. Effects on welfare

The growth rate does not capture the social welfare. Indeed, agents are heterogenous and there is a trade-off between the initial consumption and the growth rate (consuming less at the beginning promotes higher savings and investment). To study both the effects of η and of γ, reconsider the expression (29) where c_{00} and c_{10} are given by (42) and (43).

Proposition 6. *Under Assumption 2, the impact of η (monetary growth) and γ (credit sensitivity to collateral) on the welfare U_i of a social class (patient capitalists or impatient workers) is ambiguous. The impact is disentangled in a positive effect on the growth rate and an ambiguous effect on the initial consumption c_{0i}.*

(i) *There exist $\bar{\gamma} > 0$ and $\bar{\theta}_0 \in (0,1)$ such that when $\gamma \in (0,\bar{\gamma})$ and $\theta_0 \in \left(0,\bar{\theta}_0\right)$, then $\partial U_0/\partial\eta < 0$ and $\partial U_1/\partial\eta > 0$.*

(ii) *There exist $\tilde{\gamma} > 0$ and $\tilde{\theta}_1 \in (0,1)$ such that when $\gamma \in (0,\tilde{\gamma})$ and $\theta_1 \in (0,\tilde{\theta}_1)$, then $\partial U_0/\partial\gamma > 0$ and $\partial U_1/\partial\gamma > 0$.*

Proof. The proof is available in the technical appendix published online. □

The positive effect on the growth rate is highlighted in Proposition 5 . The ambiguous effects on consumptions entail at the end an ambiguous impact on welfare. However, some additional condition lead to an unambiguous result. Differently from the first regime, money growth may now worsen the welfare of capitalists because they also bear the opportunity cost of holding balances, and improve the welfare of workers. As the patient agents own the total stock of capital and, thus, are richer than the impatient ones, we reinterpret this decrease in the utility differential along the BGP between the two groups as a reduction of social inequalities. Nevertheless, the welfare of both agents is positively affected by the credit sensitivity to collateral. Although, Proposition 6 is derived under some sufficient conditions based on the shares θ_i of money hold by each agents, we show in Section 5.6 the plausibility of these effects through a numerical simulation based on a yearly calibration jointly with $\theta_0 = \theta_1 = 1/2$. We also provide additional economic interpretations.

5.5. Sunspot fluctuations and stabilization policy

A benevolent monetary authority takes into account individual preferences to implement a welfare-oriented monetary policy. Loosely speaking, individuals are risk averse and dislike inefficient fluctuations. In particular, they dislike expectation-driven fluctuations possibly occurring when multiple equilibria (indeterminacy and sunspots) arise from market imperfections. In this respect, the stabilization issue is a second scope for monetary policy and regulation.

Local indeterminacy occurs when the number of eigenvalues inside the unit circle exceeds the number of predetermined variables. Since there are no pre-determined variables in our dynamical system (36)–(37), local indeterminacy requires at least one eigenvalue inside the unit circle. In the case of a two-dimensional system, we do not study directly the eigenvalues but, instead, the trace and determinant (T, D) of the Jacobian matrix evaluated at the steady state. In the spirit of Grandmont et al. (1998), we draw the lines $D = 1$, $D = T - 1$, $D = -T - 1$ in the plane (T, D) as shown in the following Fig. 1:

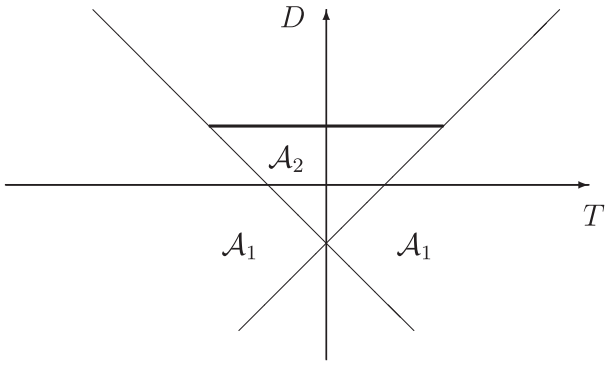

Fig. 1. Areas of local indeterminacy.

Local indeterminacy then arises when the pair (T, D) lies inside the areas $\mathcal{A}_1$ (with one eigenvalue within the unit circle) or $\mathcal{A}_2$ (with two eigenvalues within the unit circle) shown in Fig. 1.

We introduce three reduced variables:

$$(y_0, z_0, a_0) \equiv \left(\frac{x_0}{x_0+\gamma}, \frac{x_0}{x_0+x_1}, \frac{(1-\eta)\theta_1 z_0}{\eta(1-z_0)+(1-\eta)\theta_1} \right) \tag{45}$$

and the elasticities of the growth rate $g(x_0, x_1)$ as in (38):

$$(\varepsilon_0, \varepsilon_1) \equiv \left(\frac{\partial g}{\partial x_0}\frac{x_0}{g}, \frac{\partial g}{\partial x_1}\frac{x_1}{g} \right) \tag{46}$$

For simplicity, we denote

$$h = 1/(1-\gamma\pi) > 1 \tag{47}$$

with $\eta = \pi g^*$. (72

Lemma 7. *We linearize system (36)–(37) around the steady state (40)–(41) to obtain the Jacobian matrix:*

$$J = \begin{bmatrix} \frac{\sigma(1-a_0-\varepsilon_0)+\varepsilon_0}{(z_0-a_0)(\sigma-y_0)} & \frac{\sigma}{\sigma-y_0} \\ \frac{y_0+\varepsilon_0-\sigma(a_0+\varepsilon_0)}{(z_0-a_0)(\sigma-y_0)} & \frac{\sigma}{\sigma-y_0} \end{bmatrix} \begin{bmatrix} z_0 & 1-z_0 \\ h(a_0+\varepsilon_0)-\frac{y_0}{\sigma} & h(1-a_0-\varepsilon_0) \end{bmatrix} \tag{48}$$

Proof. The proof is available in the technical appendix published online. □

The determinant and the trace of this Jacobian matrix are given by

$$D = \frac{y_0(1-z_0)+h\sigma(z_0-a_0-\varepsilon_0)}{(\sigma-y_0)(z_0-a_0)} \text{ and } T = 1+D+\frac{1+(h-1)\sigma}{(\sigma-y_0)(z_0-a_0)}\varepsilon_0$$

We choose the elasticity of intertemporal substitution σ as a convenient bifurcation parameter to analyze the stability properties of the steady state. In order to simplify this analysis, we adjust the scaling parameter A to normalize g^*. In this case, moving σ has no impact on g^*, the Normalized Steady State (NSS).

Proposition 8. *Under Assumption 2, there exist $\bar{\gamma} > 0$ and a unique value $A^*(\sigma, g^*)$ such that, when $\gamma \in (0, \bar{\gamma})$ and $A = A^*(\sigma, g^*)$, any given $g^* > 1$ is a NSS solution of Eq. (39) for any $\sigma > 0$.*

Proof. The proof is available in the technical appendix published online. □

Considering the NSS, we find two important properties.

Lemma 9. *Let Assumption 2 hold, $\gamma \in (0, \bar{\gamma})$, where $\bar{\gamma}$ is given by Proposition 8, and $A = A^*(\sigma, g^*)$. Thus, $T \neq 1+D$ and the eigenvalue are different from 1 whatever $\sigma > 0$. Moreover, there exists $\bar{\sigma} > 0$ such that: (1) $1-T+D < 0$ with $D < 0$ when $\sigma < \bar{\sigma}$; (2) $D = \pm\infty$ and $T = \pm\infty$ when $\sigma = \bar{\sigma}$; (3) $1-T+D > 0$ with $D > 1$ when $\sigma > \bar{\sigma}$.*

Proof. The proof is available in the technical appendix published online. □

This property rests on the uniqueness of the steady state and implies the impossibility of transcritical, saddle-node or pitchfork bifurcations, and the impossibility of local indeterminacy as well in the area $\mathcal{A}_2$ (with two characteristic roots less than 1 in absolute value). However, local indeterminacy in the area $\mathcal{A}_1$ may arise through a flip bifurcation generating cycles of period two.

Lemma 10. *Let Assumption 2 hold, $\gamma \in (0, \bar{\gamma})$, where $\bar{\gamma}$ is given by Proposition 8, and $A = A^*(\sigma, g^*)$. Then, there exist $\bar{\eta} > 1$, $\sigma^* \in (0, \bar{\sigma})$ and $\epsilon > 0$ such that, when $\eta \in [1, \bar{\eta})$, $1 + T + D|_{\sigma \in (0,\sigma^*)} > 0$, $1 + T + D|_{\sigma=\sigma^*} = 0$ and $1 + T + D|_{\sigma \in (\sigma^*,\sigma^*+\epsilon)} < 0$.*

Proof. The proof is available in the technical appendix published online. □

These Lemmas allows us to prove the main proposition.

Proposition 11. *Let Assumption 2 hold, $\gamma \in (0, \bar{\gamma})$ where $\bar{\gamma}$ is given by Proposition 8, and $A = A^*(\sigma, g^*)$. Then, there exist $\bar{\eta} > 1$, $\sigma^* > 0$ and $\epsilon > 0$ such that, when $\eta \in [1, \bar{\eta})$, the steady state g^* is a saddle point for $\sigma \in (0, \sigma^*)$ (equilibrium indeterminacy) and a source for $\sigma \in (\sigma^*, \sigma^* + \varepsilon)$ (equilibrium determinacy). Moreover, the system generically undergoes a flip bifurcation at $\sigma = \sigma^*$ and two-period cycles arise.*

Proof. Follows from Lemmas 9 and 10. □

The economic interpretation of Proposition 11 rests on the comparison between Ramsey equilibrium models with or without cash-in-advance. In our model, capital accumulation is driven by the patient agent's Euler equation. When she is constrained, the model behaves as a cash-in-advance Ramsey model with representative agent (see Stockman, 1981; Svensson, 1985 and Cooley and Hansen, 1989 among others). In this case, cycles and indeterminacy may arise.[11]

The intuition for two-period cycles and indeterminacy in the Ramsey models à la (Stockman, 1981) is straightforward. Focus for simplicity on the limit case $\gamma = n_1 = 0$ (the patient agent becomes the representative agent and the collateral no longer matters).

We explain first the occurrence of cycles. Let k_t increase from its steady-state value and the income Ak_t as well. Under a weak intertemporal substitution σ, the income effect prevails and raises the current consumption c_t. If σ is sufficiently weak, the response in terms of c_t exceeds the increase in $(\Delta + sA)k_t$ and, according to the budget constraint, $k_{t+1} = (\Delta + sA)k_t - c_t$ decreases. Thus, a decrease in k_{t+1} follows an increase in k_t and, eventually, cycles of period two arise.

The possibility of self-fulfilling prophecies rests on a different argument. Let households anticipate an increase in the future price p_{t+1}, that is, in the inflation factor π_{t+1}. Let us show that this expectation may be self-fulfilling. Reconsider the Euler equation (31) with equality and the cash-in-advance constraint (33), set for simplicity $\gamma = 0$ and omit the individual index 0:

$$\frac{c_{t+1}}{c_t} = \left[\beta(\Delta + sA)\frac{\pi_t}{\pi_{t+1}}\right]^{\sigma} \tag{49}$$

$$c_{t+1} = m_{t+1} \tag{50}$$

where $\Delta + sA$ is the gross real interest rate. From (49), we obtain the elasticity of the inflation factor w.r.t. future consumption c_{t+1}:

$$\varepsilon_{\pi c} \equiv \frac{d\pi_{t+1}}{dc_{t+1}}\frac{c_{t+1}}{\pi_{t+1}} = -\frac{1}{\sigma} \tag{51}$$

A higher expected price p_{t+1} lowers the real balances $m_{t+1} \equiv M_{t+1}/p_{t+1}$ and, according to Eq. (50), the future consumption. The drop in c_{t+1} determines an increase in π_{t+1} according to (50). However, this change in inflation realizes the initial prophecy of higher inflation only if it is large enough. This happens when the elasticity of intertemporal substitution σ in the left-hand side of (51) is sufficiently low, as shown, among others, by Cooley and Hansen (1989).

Our conclusions are similar to the results by Ferraris and Watanabe (2011) obtained in a monetary search model *à la* Lagos and Wright (2005) with credit constraint. They show that when agents attain their borrowing limits, fluctuations in capital occur through cycles of higher order, chaos and sunspot equilibria. The simultaneous existence of endogenous growth and endogenous fluctuations can be also related to Matsuyama (2001) where similar results are obtained in an economy with a representative infinitely-lived agent and growth driven by investment and innovation changes over time. It is finally worth noting that the same type of conclusions could be also obtained in a usual monetary economy such as Grandmont et al. (1998) where local indeterminacy and endogenous cycles are shown to easily arise. They consider indeed a (Woodford, 1986) model where the structure of society based on the co-existence of patient capitalists and impatience workers is similar to our model.

5.6. A numerical illustration

We consider a yearly calibration. Following the standard practice in the Real Business Cycle literature, we set $\beta_0 = 0.98$, implying a net annual return on capital of around 2%, and $\beta_1 = 0.95$ in order to consider a slight discounting difference between patient and impatient agents. We assume $\Delta = 9/10$, implying a 10% annual depreciation rate of physical capital. As explained previously, in order to calibrate the degree of increasing return to scale to 0.33, the point estimate obtained

[11] See also Fukuda (1993), and Michener and Ravikumar (1998).

by Basu and Fernald (1997) for the degree of increasing returns to scale in the durable manufacturing industry, we set, as in Mankiw et al. (1992) $s = 2/3$, implying that k covers physical and human capital. This value is usual in the endogenous growth literature. The money growth rate η will be adequately calibrated to imply an inflation rate $\pi - 1 = \eta/g^* - 1$ of 2% approximately, the standard inflation target of monetary authorities in OECD countries. As there is no obvious way to calibrate the value of γ, we consider a moderate credit sensitivity to collateral setting $\gamma = 0.01$. Following (Kocherlakota, 1996; Campbell and Woodford, 1999; Vissing-Jorgensen, 2002) and the most recent estimates provided by Havranek (2015), we assume that the elasticity of intertemporal substitution σ lies in the interval (0.1, 0.2). Finally, considering these calibrations, the value of A is chosen to ensure a balanced growth rate $g^* - 1$ approximately between 2% and 4%.

Setting $A = 0.5$, the corresponding BGP interval when $\sigma \in (0.1, 0.2)$ is $g^* \in (1.019, 1.039)$ and the associated growth rate of money supply ensuring $\pi \approx 1.02$ is $\eta \in (1.04, 1.06)$. Using this calibration, the welfare effects of the money growth factor η on the two classes of agents are reversed with respect to those obtained in the first regime (where patient agents hold no money):

$$\frac{\partial U_0}{\partial \eta} < 0 \text{ and } \frac{\partial U_1}{\partial \eta} > 0$$

(Proposition 6).[12] The negative impact on patient agents comes from the negative effect of η on their initial consumption c_{00}. This happens because now, in the second regime, patient agents bear an opportunity cost to finance future consumption with money. This cost appears to be larger than the benefit of additional money transfers from the monetary authority. Conversely, impatient agents have a positive impact on their initial consumption c_{10} as they have a relatively lower opportunity cost of holding money. We notice, however, that the effect on the social welfare is positive (i.e. $\partial U_0/\partial\eta + \partial U_1/\partial\eta > 0$). As seen above, because the most patient agents are richer (they own the whole stock of capital), a higher money growth rate improves the total welfare while decreasing the social inequality (lag between the utilities of both the types evaluated along the BGP).

According to the Proposition 6, the welfare impacts of credit sensitivity to collateral have the same sign

$$\frac{\partial U_0}{\partial \gamma} > 0 \text{ and } \frac{\partial U_1}{\partial \gamma} > 0$$

For both agents, a higher credit sensitivity to collateral lowers the money holdings and their opportunity cost. This results in higher consumption and welfare at the end.

Focus now on the stability issue, that is Proposition 11. Consider the role of monetary policy and regulation on the occurrence of local indeterminacy and growth cycles. η and γ have an impact on σ^* and affect in turn the extent of the indeterminacy range $(0, \sigma^*)$. Thus, the monetary policy (regulation) stabilizes the economy if η (γ) has a negative impact on σ^*, that is if the room for indeterminacy and expectation-driven fluctuations shrinks. The signs of derivatives $\partial\sigma^*/\partial\eta$ and $\partial\sigma^*/\partial\gamma$ are *a priori* ambiguous. We need to address the stabilization issue simulating the shape of the function $\sigma^*(\eta, \gamma)$.

We consider a plausible growth rate of 2% ($g^* = 1.02$), compatible with any $\sigma > 0$ after the scaling parameter normalization ($A = A^*(\sigma, g^*)$), and $\eta = 1.0404$ to ensure $\pi = 1.02$.

This calibration gives $\sigma^* \approx 0.123$ as a (flip) bifurcation value for the elasticity of intertemporal substitution in consumption. This value clearly belongs to the empirical ranges provided by Kocherlakota (1996), Campbell and Woodford (1999), Vissing-Jorgensen (2002) and Havranek (2015).

Fig. 2 illustrates the role of η and γ for local indeterminacy: the bifurcation value σ^* is represented as a function of η and γ around $(\eta, \gamma) = (1.0404, 0.01)$.

The function σ^* is increasing in η and decreasing in γ: the indeterminacy range $(0, \sigma^*)$ widens with η and shrinks with γ. The higher the monetary growth (η), the higher the inflation rate and the opportunity cost of holding money (nominal interest rate), the heavier the burden on liquidity-constrained agents. Thus, the larger the credit market imperfection, the more likely the occurrence of endogenous fluctuations because of the local indeterminacy. Conversely the sensitivity of credit grants to collateral (γ) reduces the market imperfection, because, when γ increases, the collateral matters more and the patient agents' liquidity constraint weakens.

It is worth noting here that the qualitative properties of these two curves are preserved for a large set of values of the share of capital s but the bifurcation value of the elasticity of intertemporal substitution in consumption σ^* is affected. In particular, if we focus on the standard case $s = 1/3$ where the collateral constraint is only based on physical capital, σ^* is still decreasing with respect to γ and increasing with respect to η but it ranges within some interval centered around 0.033.

6. Conclusion

We have considered a Ramsey equilibrium model with heterogenous agents and liquidity constraints. Heterogenous discount rates imply a social segmentation in the long run: patient agents end up to own the entire capital stock. Money is needed to finance future consumption of agents with insufficient collateral. Impatient agents need money.

[12] It is worth noting that although we use conditions on the shares θ_i to establish Proposition 6, the same conclusions hold while $\theta_0 = \theta_1 = 1/2$.

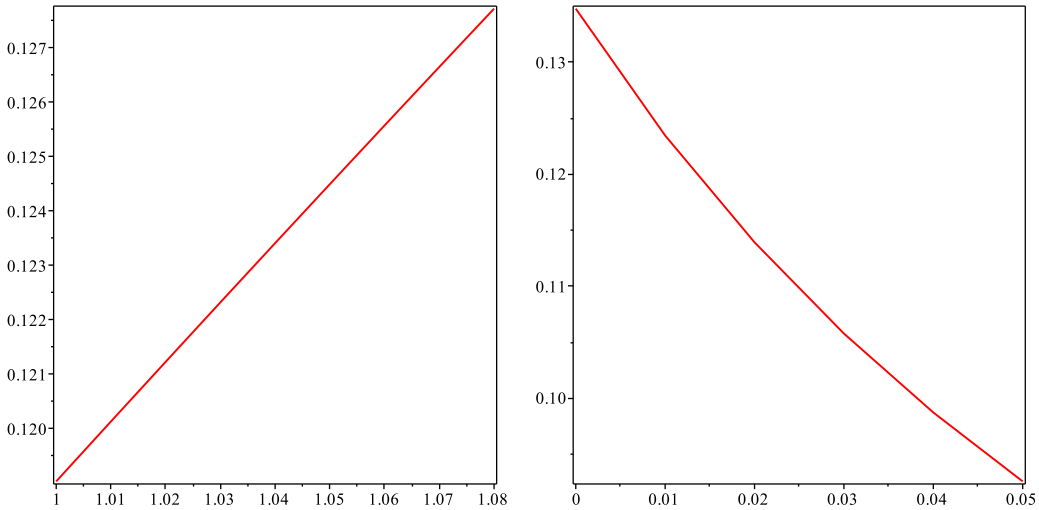

Fig. 2. Monetary policy (η) and regulation (γ).

In this economy, two regimes emerge according to the possibility that patient agents hold balances or not. We have studied the role of monetary policy (captured by η) and market regulation (synthesized by γ) in both these cases.

(1) Workers need money while patient agents buy on credit. Money growth improves the welfare of patient agents (they receive balances), but worsens that of impatient agents (they bear the opportunity cost of holding money: the higher the inflation, the higher the interest rate). Because the distribution of capital is degenerated along the BGP (patient agents own the capital stock), the inequality gap is better represented by the difference between the utility levels along the BGP. We find an increasing gap in η. The equilibrium jumps on the BGP from the beginning as in the seminal endogenous growth literature: the lack of transition rules out any cycle and stabilization policy.
(2) Patient agents also need money to consume. Under a higher inflation, the monetary growth (η) is growth-enhancing simply because patient agents renounce to balances, consume less and accumulate more capital. The higher sensitivity of credit grant to collateral (γ) also promotes economic growth because patient agents accumulate more capital to collateralize the consumption purchases and reduce their money demand. However, in general, money growth and credit sensitivity to collateral may have an ambiguous effect on the initial consumptions of patient and impatient agents and on their welfare as well. A numerical simulation allowed us to show that an accommodative monetary policy may increase the social welfare along the BGP while reducing the utility of patient agents who bear now the opportunity cost of holding money. Patient agents remain richer than impatient ones because of capital ownership. Therefore, our conclusion refers to a decrease in inequalities viewed as a narrower gap between utilities along the BGP. Conversely, a larger credit sensitivity to collateral improves the welfare of both the social categories.

In the second regime, the economy experiences an equilibrium transition. Therefore, the study of equilibrium multiplicity (local indeterminacy) and growth cycles makes sense. In this respect, the stabilization issue becomes a relevant concern in the political agenda. In our model, two policy instruments are available: money growth (η) and regulation (γ). The indeterminacy range widens with η and shrinks with γ. Indeed, a quantitative easing raises the inflation rate and the opportunity cost of holding money, increasing the burden of liquidity constraint and the credit market imperfection in turn. Conversely the sensitivity of credit grants to collateral γ reduces this imperfection: when γ goes up, the collateral matters more and the capitalists' liquidity constraint weakens.

Our results then allow us to provide some prudent policy recommendation. Since 2009, the monetary policy of central banks in OECD countries is very accommodative through quantitative easing programs. The main objective of such policy is, first, to provide liquidities in the economy in order to compensate for the sharp decrease in assets prices after the starting of the subprime crisis, and, second, to boost aggregate demand and inflation in order to avoid any deflationary spiral. We have shown that increasing the money supply can increase growth and inflation, and affect households' welfare in such a way that inequalities are reduced. However, expectation-driven fluctuations and aggregate volatility become also more likely as this has been observed over the last months. Stimulating investments in financial assets, this monetary policy indeed favors the occurrence of bubbles and instability. We prove in this paper that raising the sensitivity of credit grants to collaterals (in the sense of γ) for all type of agents could be a convenient way to shelter the economy from the destabilizing effects of quantitative easing. In this respect, the Basel III agreements reinforce the role of collateral of banks (up to 2019) and opportunely complements the quantitative easing policy implemented by the European Central Bank since September 2012. These devices appear then as two crucial facets of a coordinated policy that aims to enhance the economic growth without increasing the social inequalities and promoting undesirable expectations-driven fluctuations.

Appendix A. Technical Appendix of "Collateral and growth cycles with heterogeneous agents"

A.1. Proof of Proposition 1

Under Assumption 1, condition $\gamma \in (0, \bar{\gamma})$ implies

$$\frac{\beta_0}{\beta_1} > \frac{(\Delta + sA - \gamma)[\beta_0(\Delta + sA)]^\sigma}{(\Delta + sA)\left([\beta_0(\Delta + sA)]^\sigma - \gamma\eta\right)}$$

Along the BGP, the inflation factor π is constant and given by $\eta/g^* = \pi$ with $g^* = [\beta_0(\Delta + sA)]^\sigma$. The inequality is equivalent to

$$\beta_0(\Delta + sA) > \beta_1 \frac{\Delta + sA - \gamma}{1 - \gamma\pi}$$

Therefore, if $\gamma > \bar{\gamma}$ we get

$$\beta_0(\Delta + sA) < \beta_1 \frac{\Delta + sA - \gamma}{1 - \gamma\pi}$$

Along the BGP

$$\frac{c_{0t+1}}{c_{0t}} = \frac{c_{1t+1}}{c_{1t}} = g^*$$

Then, we obtain from inequalities (11) and (14):

$$\beta_0(\Delta + sA) < g^{*1/\sigma} = \beta_1 \frac{\Delta + sA - \gamma}{1 - \gamma\pi}$$

In this case, patient agents hold no capital. Since they hold no money (first regime) and do not work, they do not consume, a contradiction.

Conversely, if $\gamma \in (0, \bar{\gamma})$, the patient agents hold capital while the impatient ones do not:

$$\frac{c_{0t+1}}{c_{0t}} = \frac{c_{1t+1}}{c_{1t}} = g^* = [\beta_0(\Delta + sA)]^\sigma > \left(\beta_1 \frac{\Delta + sA - \gamma}{1 - \gamma\pi}\right)^\sigma$$

Thus, there exists $T > 0$ such that $k_{1t} = 0$ for every $t > T$ and only the patient agents hold capital in the long run.

Focus on the equilibrium budget constraints (12) and (24). Substituting $k_{0t} = k_t$ and (18) in (12), and (25) and (18) in (24) we obtain, respectively:

$$k_{t+1} - (\Delta + r)k_t + c_{0t} = \theta_0(m_{1t+1}\pi_{t+1} - m_{1t}) \tag{52}$$

$$\theta_1 m_{1t} + (1 - \theta_1)m_{1t+1}\pi_{t+1} = w_t \tag{53}$$

Replacing (21) in (52) and (53), we get

$$k_{t+1} - (\Delta + r)k_t + c_{0t} = \theta_0(\eta - 1)m_{1t} \tag{54}$$

$$[\theta_1 + (1 - \theta_1)\eta]m_{1t} = (1 - s)Ak_t \tag{55}$$

as $w_t = (1 - s)Ak_t$. Dividing (55) by k_t, we obtain the constant ratios:

$$(x_{0t}, x_{1t}) = \left(0, \frac{(1 - s)A}{1 + (\eta - 1)\theta_0}\right) \tag{56}$$

and, dividing (54) by k_t:

$$\frac{c_{0t}}{k_t} = \Delta + sA + \frac{(1 - s)A(\eta - 1)\theta_0}{1 + (\eta - 1)\theta_0} - g_{t+1} \tag{57}$$

The patient agents' Euler Eu. (11) becomes $c_{0t+1}/c_{0t} = [\beta_0(\Delta + sA)]^\sigma$. The only possible equilibrium (compatible with non-negative quantities and the transversality condition) is the BGP: $k_t = k_0 g^{*t}$ and $c_{0t} = c_{00}g^{*t}$ with $g^* = [\beta_0(\Delta + sA)]^\sigma$. Under Assumption 1, we have $g^* \geq 1$ and the transversality condition (13):

$$\lim_{t \to +\infty} \left(\beta_0 g^{*\frac{\sigma-1}{\sigma}}\right)^t = 0$$

There is no transition and the initial consumption of the patient agent jumps on the BGP: (57) gives

$$c_{00} = k_0\left[\Delta + sA + \frac{(1-s)A(\eta-1)\theta_0}{1+(\eta-1)\theta_0} - g^*\right] \tag{58}$$

Assumption 1 implies $c_{00} > 0$.

The consumption of impatient agents along the BGP becomes $c_{1t} = c_{10}g^{*t}$. From (22), evaluated at time $t = 0$, we find

$$g^* - \Delta + \frac{c_{00}}{k_0} + \frac{c_{10}}{k_0} = A$$

and, using (58):

$$c_{10} = \frac{(1-s)Ak_0}{1+(\eta-1)\theta_0} \tag{59}$$

□

A.2. Proof of Proposition 2

We compute the derivatives of the initial consumption levels given in Proposition 1 for both the agents:

$$\frac{\partial c_{00}}{\partial \eta} = \theta_0\frac{(1-s)Ak_0}{[1+(\eta-1)\theta_0]^2} > 0 \text{ and } \frac{\partial c_{10}}{\partial \eta} = -\theta_0\frac{(1-s)Ak_0}{[1+(\eta-1)\theta_0]^2} < 0$$

Using the expressions (29), we derive

$$\frac{\partial U_i}{\partial \eta} = \frac{c_{i0}^{-1/\sigma}}{1-\beta_i g^{*1-1/\sigma}}\frac{\partial c_{i0}}{\partial \eta}$$

We finally observe that, if $\eta > 1$, then $\partial c_{00}/\partial\theta_0 > 0$ and $\partial c_{10}/\partial\theta_0 < 0$. From (29), we also find

$$\frac{\partial U_i}{\partial \theta_0} = \frac{c_{i0}^{-1/\sigma}}{1-\beta_i g^{*1-1/\sigma}}\frac{\partial c_{i0}}{\partial \theta_0}$$

□

A.3. Proof of Proposition 3

We reconsider the equilibrium budget constraints. Replacing $k_{0t} = k_t$, (18) and (34) in (32), and (25), (18) and (34) in (24) we obtain, respectively:

$$[(1-\theta_0)m_{0t+1} - \theta_0 m_{1t+1}]\pi_{t+1} + \theta_0(m_{0t} + m_{1t}) = (\Delta + r)k_t - k_{t+1} + m_{0t} - c_{0t} \tag{60}$$

$$[(1-\theta_1)m_{1t+1} - \theta_1 m_{0t+1}]\pi_{t+1} + \theta_1(m_{0t} + m_{1t}) = w_t \tag{61}$$

In the second regime (capitalists hold money), the CIA constraint is binding:

$$c_{0t} = m_{0t} + \gamma k_t \tag{62}$$

and (60) reduces to

$$[(1-\theta_0)m_{0t+1} - \theta_0 m_{1t+1}]\pi_{t+1} + \theta_0(m_{0t} + m_{1t}) = (\Delta + r - \gamma)k_t - k_{t+1} \tag{63}$$

(21) becomes

$$m_{0t} + m_{1t} = (m_{0t+1} + m_{1t+1})\pi_{t+1}/\eta \tag{64}$$

Replacing (64) in (63) and (61), we find, respectively,

$$\begin{aligned}\left[m_{0t+1} + \frac{1-\eta}{\eta}\theta_0(m_{0t+1} + m_{1t+1})\right]\pi_{t+1} &= (\Delta + sA - \gamma)k_t - k_{t+1}\\ \left[m_{1t+1} + \frac{1-\eta}{\eta}\theta_1(m_{0t+1} + m_{1t+1})\right]\pi_{t+1} &= (1-s)Ak_t\end{aligned} \tag{65}$$

Patient agents' Euler Eq. (31) jointly with constraint (62) becomes

$$\frac{m_{0t+1} + \gamma k_{t+1}}{m_{0t} + \gamma k_t} = \left(\beta_0\frac{\pi_t}{\pi_{t+1}}\frac{\Delta + r - \gamma}{1-\gamma\pi_t}\right)^\sigma \tag{66}$$

Dividing (63), (65) and (66) by k_{t+1} and replacing (27), we obtain at time t:

$$\left[x_{0t} + \frac{1-\eta}{\eta}\theta_0(x_{0t} + x_{1t})\right]\pi_t = \frac{\Delta + sA - \gamma}{g_t} - 1 \tag{67}$$

$$\left[x_{1t} + \frac{1-\eta}{\eta}\theta_1(x_{0t} + x_{1t})\right]\pi_t = \frac{(1-s)A}{g_t} \tag{68}$$

$$\frac{x_{0t} + \gamma}{x_{0t+1} + \gamma}\left(\beta_0 \frac{\pi_t}{\pi_{t+1}} \frac{\Delta + sA - \gamma}{1 - \gamma\pi_t}\right)^{\sigma} = g_{t+1} \tag{69}$$

Dividing (67) and (68) side by side, we get (38). From (64) and (65), we have (36). Deriving π_t from (68) and replacing π_t and π_{t+1} in (69), we eventually obtain (37). □

A.4. Proof of Proposition 4

Eq. (36) becomes at the steady state:

$$\eta x_1 + (1-\eta)\theta_1(x_0 + x_1) = (1-s)A \tag{70}$$

Replacing (70) in the Euler Eq. (37) evaluated at the steady state, we find

$$g = \gamma\eta + \beta_0(\Delta + sA - \gamma)g^{\frac{\sigma-1}{\sigma}} \equiv \varphi(g) \tag{71}$$

that is an implicit equation which determines the balanced growth rate g^*. Substituting (70) in (38) evaluated at the steady state, we get

$$\eta x_0 + (1-\eta)\theta_0(x_0 + x_1) = \Delta + sA - \gamma - g^* \tag{72}$$

Eqs. (70) and (72) form a linear system:

$$\begin{bmatrix} \eta + (1-\eta)\theta_0 & (1-\eta)\theta_0 \\ (1-\eta)\theta_1 & \eta + (1-\eta)\theta_1 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} \Delta + sA - \gamma - g^* \\ (1-s)A \end{bmatrix}$$

to solve with respect to (x_0, x_1). Noticing that $\theta_0 + \theta_1 = 1$, we obtain

$$x_0 = \frac{(\Delta + sA - \gamma - g^*)[1 + (\eta-1)\theta_0] + (1-s)A(\eta-1)\theta_0}{\eta} \tag{73}$$

$$x_1 = \frac{(1-s)A[1 + (\eta-1)\theta_1] + \theta_1(\eta-1)(\Delta + sA - \gamma - g^*)}{\eta} \tag{74}$$

where g^* is given by (71).

The asset ratio is given by $m/k = x_0 + x_1 = A + \Delta - \gamma - g^*$. The consumption of patient agents along the BGP is given by $c_{0t} = c_{00}g^{*t}$ and the binding cash-in-advance along the BGP implies: $c_{00} = (x_0 + \gamma)k_0$. Similarly, the consumption of impatient agents along the BGP is given by $c_{1t} = c_{10}g^{*t}$ and the binding cash-in-advance along the BGP implies: $c_{10} = m_0 = x_1 k_0$.

The existence of a monetary equilibrium requires the positivity of variables (and nominal interest rate) and the transversality condition.

(1) Positivity of variables x_0 and x_1 needs, respectively:

$$g^* < \Delta + sA - \gamma + \frac{(1-s)A(\eta-1)\theta_0}{1 + (\eta-1)\theta_0} \equiv g_1 \tag{75}$$

$$g^* < \Delta + sA - \gamma + \frac{(1-s)A[1 + (\eta-1)\theta_1]}{(\eta-1)\theta_1} \equiv g_2 \tag{76}$$

Inequality (75) implies (76) as $g_2 > g_1$. Moreover, if $g^* < g_1$, we get $c_{00} > 0$ and $c_{10} > 0$.

(2) We need to have a positive nominal interest rate to get a monetary equilibrium. Indeed, money is required to be a dominated asset. We define the nominal interest factor as $i_t \equiv \pi_t(\Delta + sA)$. The zero lower bound entails $i_t - 1 > 0$ (a productive economy exists if money is a dominated asset). Since patient agents hold the stock of capital in the long run, we need condition (31) to be satisfied with equality at the steady state, that is:

$$g^* = \left(\beta_0 \frac{\Delta + sA - \gamma}{1 - \gamma\pi}\right)^{\sigma} \tag{77}$$

Noting that $i > 1$ implies $\Delta + sA > 1/\pi$. We obtain from (77): $g^* \geq (\beta_0/\pi)^{\sigma}$. Since, along the BGP, (20) becomes $\eta = \pi g^*$, the inequality ensuring a positive nominal interest rate writes

$$\eta > \beta_0 g^{*(\sigma-1)/\sigma} \tag{78}$$

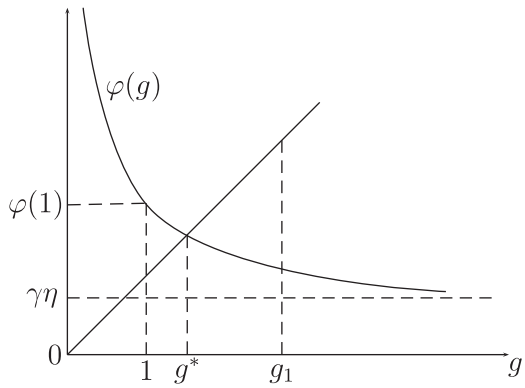

Fig. A.1. Existence and uniqueness of $g^* \in (1, g_1)$ when $\sigma < 1$.

(3) The Balanced Growth Transversality Condition (BGTC) requires

$$\beta_0 g^{*(\sigma-1)/\sigma} < 1 \tag{79}$$

Noting that $g^* = \gamma\eta + (\Delta + sA - \gamma)\beta_0 g^{*(\sigma-1)/\sigma}$, we get $g^* < \Delta + sA + \gamma(\eta - 1) \equiv g_3$. We observe that, when $\eta = 1$, $g_3 > g_1$. Moreover, since $\eta \geq 1$, (79) implies (78).

We conclude that there exists $\bar{\eta} > 1$ such that when $\eta \in [1, \bar{\eta})$, any BGP $g^* < g_1$ satisfies the positivity constraints of all the variables, the positivity of the nominal interest rate and the BGTC.

Let us prove the BGP existence and uniqueness. We determine g^* as solution of Eq. (71), that is $g^* = \varphi(g^*)$. Let us focus on a solution $g^* \in (1, g_1)$. To ensure that $g_1 > 1$, we require

$$\Delta + sA - 1 + \frac{(1-s)A(\eta-1)\theta_0}{1+(\eta-1)\theta_0} \equiv \bar{\gamma}_1 > \gamma$$

The assumption $\Delta + sA \geq 1/\beta_0$ implies $\bar{\gamma}_1 > 0$.

Consider first the case $\sigma = 1$ yielding $g^* = \gamma\eta + (\Delta + sA - \gamma)\beta_0$. Under $\Delta + sA \geq 1/\beta_0$ we get $g^* \geq 1$. Inequality $g^* < g_1$ is equivalent to

$$\gamma < \bar{\gamma}_2 \equiv \frac{(\Delta + sA)(1-\beta_0) + \frac{(1-s)A(\eta-1)\theta_0}{1+(\eta-1)\theta_0}}{\eta + 1 - \beta_0}$$

Let us now consider the case $\sigma < 1$. We get, $\varphi'(g) < 0$ for every $g > 0$. Moreover, $\lim_{g\to 0^+}\varphi(g) = +\infty$ and $\lim_{g\to+\infty}\varphi(g) = \gamma\eta$. The case is illustrated in the following Fig. A.1:

The existence and the uniqueness of a solution g^* of Eq. (71) are immediate. Moreover, $\Delta + sA \geq 1/\beta_0$ implies $\varphi(1) > 1$ and thus $g^* \geq 1$. We obtain also

$$\varphi(g_1) < \gamma\eta + \beta_0(\Delta + sA - \gamma)^{\frac{2\sigma-1}{\sigma}}$$

Moreover,

$$\gamma\eta + \beta_0(\Delta + sA - \gamma)^{\frac{2\sigma-1}{\sigma}} < \Delta + sA - \gamma$$

holds for $\gamma = 0$, and $\Delta + sA - \gamma < g_1$. Thus, there exists $\bar{\gamma}_3 > 0$ such that, when $\gamma \in (0, \bar{\gamma}_3)$, $\varphi(g_1) < g_1$ and, hence, $g^* < g_1$.

Let us finally consider the case $\sigma > 1$. We find $\varphi'(g) > 0$, $\varphi''(g) < 0$ for every $g > 0$. In addition, we have $\lim_{g\to 0^+}\varphi(g) = \gamma\eta > 0$, $\lim_{g\to+\infty}\varphi(g) = +\infty$ and $\lim_{g\to+\infty}\varphi'(g) = 0 < 1$. The following Fig. A.2 illustrates this case.

Thus, a unique solution g^* of Eq. (71) exists. As in the previous case, $\Delta + sA \geq 1/\beta_0$ implies $g^* > 1$. Finally, $\varphi(g_1) < g_1$ is equivalent to

$$\gamma\eta < \left[\Delta + sA - \gamma + \frac{(1-s)A(\eta-1)\theta_0}{1+(\eta-1)\theta_0}\right]\left\{1 - \frac{\beta_0(\Delta + sA - \gamma)}{\left[\Delta + sA - \gamma + \frac{(1-s)A(\eta-1)\theta_0}{1+(\eta-1)\theta_0}\right]^{1/\sigma}}\right\}$$

When $\gamma = 0$ this inequality simplifies

$$0 < \left[\Delta + sA + \frac{(1-s)A(\eta-1)\theta_0}{1+(\eta-1)\theta_0}\right]\left\{1 - \frac{\beta_0(\Delta + sA)}{\left[\Delta + sA + \frac{(1-s)A(\eta-1)\theta_0}{1+(\eta-1)\theta_0}\right]^{1/\sigma}}\right\}$$

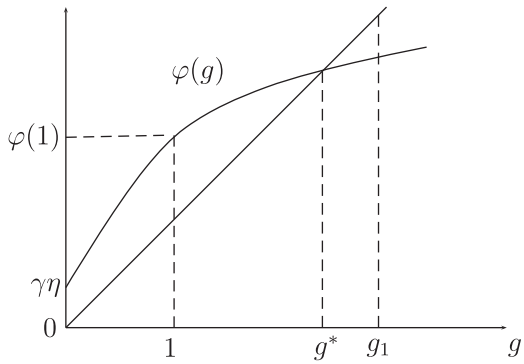

Fig. A.2. Existence and uniqueness of $g^* \in (1, g_1)$ when $\sigma > 1$.

The expression in the braces on the right-hand-side is positive if and only if

$$(\Delta + sA)\left[1 - \beta_0^\sigma (\Delta + sA)^{\sigma-1}\right] + \frac{(1-s)A(\eta-1)\theta_0}{1+(\eta-1)\theta_0} > 0$$

This inequality is satisfied under Assumption 2. Then, there exists $\bar{\gamma}_4 > 0$ such that when $\gamma \in (0, \bar{\gamma}_4)$, $g^* < g_1$.

Taking $\bar{\gamma} \equiv \min\{\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3, \bar{\gamma}_4\}$, we obtain the result. □

A.5. Proof of Proposition 5

Totally differentiating (39) with respect to γ, η, g and observing that

$$\beta_0 g^{*1-1/\sigma} = \frac{g^* - \gamma\eta}{\Delta + sA - \gamma} \tag{80}$$

we find

$$g^* \frac{\pi(\Delta + sA) - 1}{\Delta + sA - \gamma} d\gamma + \gamma d\eta + \frac{\gamma\pi(1-\sigma) - 1}{\sigma} dg^* = 0 \tag{81}$$

Setting $d\gamma = 0$ in (81), we get the partial derivative with respect to η in (44): the impact of monetary policy on the growth rate is always positive.

Setting $d\eta = 0$ in (81), we obtain the partial derivative with respect to γ in (44). The impact of regulation is positive ($\partial g^*/\partial\gamma > 0$) because the nominal interest rate is positive: $\pi(\Delta + sA) > 1$. □

A.6. Proof of Proposition 6

Consider the expression (29). The effect of η on U_0 is given by

$$\frac{\partial U_0}{\partial \eta} = \frac{c_{00}^{1-1/\sigma}}{1 - \beta_0 g^{*1-1/\sigma}} \left[\frac{1}{g^*}\frac{\partial g^*}{\partial \eta}\frac{\beta_0 g^{*1-1/\sigma}}{1 - \beta_0 g^{*1-1/\sigma}} + \frac{1}{c_{00}}\frac{\partial c_{00}}{\partial \eta}\right] \tag{82}$$

The trade-off between the initial consumption and the balanced growth rate is usual in the endogenous growth literature. The utility of patient agents increases because of the positive impact on the growth rate:

$$\frac{1}{g^*}\frac{\partial g^*}{\partial \eta}\frac{\beta_0 g^{*1-1/\sigma}}{1 - \beta_0 g^{*1-1/\sigma}} > 0$$

(see the inequality (44) and the transversality condition: $1 - \beta_0 g^{*1-1/\sigma} > 0$) and may decrease because of the ambiguous effect on the initial consumption. Indeed, using (42), we get

$$\frac{\partial c_{00}}{\partial \eta} = k_0 \left[\frac{(1-s)A\theta_0 - \theta_1(\Delta + sA - g^* - \gamma)}{\eta^2} - \frac{1 + (\eta-1)\theta_0}{\eta}\frac{\partial g^*}{\partial \eta}\right] \tag{83}$$

which can be positive or negative.

Consider the expression in the brackets in Eq. (82). Proposition 5 implies that, if γ is close enough to zero, the derivative $\partial g^*/\partial\eta$ is also close to zero and $\partial U_0/\partial\eta$ has the same sign as $\partial c_{00}/\partial\eta$. Consider now the term in the brackets in Eq. (83). If θ_0 is also sufficiently close to zero, then $\partial c_{00}/\partial\eta < 0$ and $\partial U_0/\partial\eta < 0$. Therefore, there exist $\hat{\gamma} > 0$ and $\hat{\theta}_0 \in (0, 1)$ such that, when $\gamma \in \left(0, \hat{\gamma}\right)$ and $\theta_0 \in (0, \hat{\theta}_0)$, then $\partial U_0/\partial\eta < 0$.

The impact of γ on U_0 is given by

$$\frac{\partial U_0}{\partial \gamma} = \frac{c_{00}^{1-1/\sigma}}{1-\beta_0 g^{*1-1/\sigma}}\left[\frac{1}{g^*}\frac{\partial g^*}{\partial \gamma}\frac{\beta_0 g^{*1-1/\sigma}}{1-\beta_0 g^{*1-1/\sigma}} + \frac{1}{c_{00}}\frac{\partial c_{00}}{\partial \gamma}\right] \tag{84}$$

There exists a trade-off between the initial consumption and the growth rate. The utility of patient agents increases because of the positive impact on the growth rate:

$$\frac{1}{g^*}\frac{\partial g^*}{\partial \gamma}\frac{\beta_0 g^{*1-1/\sigma}}{1-\beta_0 g^{*1-1/\sigma}} > 0$$

and can decrease because of the ambiguous effect on the initial consumption. Indeed, using (42), we get

$$\frac{\partial c_{00}}{\partial \gamma} = k_0\left[\frac{\eta-1}{\eta}\theta_1 - \frac{\partial g^*}{\partial \gamma}\frac{1+(\eta-1)\theta_0}{\eta}\right] \tag{85}$$

which may be positive or negative.

Consider the term between brackets in Eq. (84). From Eqs. (40) and (42) in Proposition 4 we derive

$$c_{00} = k_0\frac{(\eta-1)[\gamma\theta_1+(1-s)A\theta_0]+(\Delta+sA-g^*)[1+(\eta-1)\theta_0]}{\eta} \tag{86}$$

Substituting this expression into (85) gives

$$\frac{1}{c_{00}}\frac{\partial c_{00}}{\partial \gamma} = \frac{(\eta-1)\theta_1 - \frac{\partial g^*}{\partial \gamma}[1+(\eta-1)\theta_0]}{(\eta-1)[\gamma\theta_1+(1-s)A\theta_0]+(\Delta+sA-g^*)[1+(\eta-1)\theta_0]} \tag{87}$$

Replacing (87) into (84) yields

$$\begin{aligned}\frac{\partial U_0}{\partial \gamma} = \frac{c_{00}^{1-1/\sigma}}{1-\beta_0 g^{*1-1/\sigma}}\Bigg[&\frac{(\eta-1)\theta_1}{(\eta-1)[\gamma\theta_1+(1-s)A\theta_0]+[1+(\eta-1)\theta_0][\Delta+sA-g^*]}\\ &+\frac{\partial g^*}{\partial \gamma}\frac{\beta_0 g^{*1-1/\sigma}[(\eta-1)[\gamma\theta_1+(1-s)A\theta_0]+[1+(\eta-1)\theta_0](\Delta+sA)]-g^*[1+(\eta-1)\theta_0]}{g^*[1-\beta_0 g^{*1-1/\sigma}]\{(\eta-1)[\gamma\theta_1+(1-s)A\theta_0]+[1+(\eta-1)\theta_0][\Delta+sA-g^*]\}}\Bigg]\end{aligned} \tag{88}$$

Eq. (39) writes

$$\beta_0 g^{*1-1/\sigma}(\Delta+sA) = g^* - \gamma\left[\eta-\beta_0 g^{*1-1/\sigma}\right] \tag{89}$$

Substituting this expression into (88), we obtain

$$\begin{aligned}\frac{\partial U_0}{\partial \gamma} = \frac{c_{00}^{1-1/\sigma}}{1-\beta_0 g^{*1-1/\sigma}}\Bigg[&\frac{(\eta-1)\theta_1}{(\eta-1)[\gamma\theta_1+(1-s)A\theta_0]+[1+(\eta-1)\theta_0][\Delta+sA-g^*]}\\ &+\frac{\partial g^*}{\partial \gamma}\frac{\beta_0 g^{*1-1/\sigma}(\eta-1)[\gamma\theta_1+(1-s)A\theta_0]-\gamma[1+(\eta-1)\theta_0]\left[\eta-\beta_0 g^{*1-1/\sigma}\right]}{g^*[1-\beta_0 g^{*1-1/\sigma}]\{(\eta-1)[\gamma\theta_1+(1-s)A\theta_0]+[1+(\eta-1)\theta_0][\Delta+sA-g^*]\}}\Bigg]\end{aligned} \tag{90}$$

Therefore, there exists $\tilde{\gamma} > 0$ such that $\partial U_0/\partial\gamma > 0$ when $\gamma \in (0, \tilde{\gamma})$.

Focus now on the welfare of impatient agents. The impact of η on U_1 is given by

$$\frac{\partial U_1}{\partial \eta} = \frac{c_{10}^{1-1/\sigma}}{1-\beta_1 g^{*1-1/\sigma}}\left[\frac{1}{g^*}\frac{\partial g^*}{\partial \eta}\frac{\beta_1 g^{*1-1/\sigma}}{1-\beta_1 g^{*1-1/\sigma}} + \frac{1}{c_{10}}\frac{\partial c_{10}}{\partial \eta}\right] \tag{91}$$

As for agents of type 0, there exists a trade-off between the initial consumption and the growth rate. The utility of impatient agents increases because of the positive impact on the growth rate:

$$\frac{1}{g^*}\frac{\partial g^*}{\partial \eta}\frac{\beta_1 g^{*1-1/\sigma}}{1-\beta_1 g^{*1-1/\sigma}} > 0$$

(inequality (44) and transversality condition: $1-\beta_1 g^{*1-1/\sigma} > 0$) and may decrease because of the ambiguous effect on the initial consumption. Indeed, using (43) and (83), we get

$$\frac{\partial c_{10}}{\partial \eta} = -k_0\frac{\partial g^*}{\partial \eta} - \frac{\partial c_{00}}{\partial \eta} \tag{92}$$

which can be positive or negative.

Consider the term between brackets in Eq. (91). Recall that if γ is sufficiently close to zero, the derivative $\partial g^*/\partial\eta$ is also close to zero and $\partial U_1/\partial\eta$ has the same sign as $\partial c_{10}/\partial\eta$. Consider now Eq. (92). If $\partial g^*/\partial\eta$ is close to zero, $\partial c_{00}/\partial\eta$ and $\partial c_{10}/\partial\eta$ work in opposite directions. Therefore, there exist $\bar{\gamma} \in (0, \hat{\gamma}]$ and $\bar{\theta}_0 \in (0, \hat{\theta}_0]$ such that, if $\gamma \in (0, \bar{\gamma})$ and $\theta_0 \in \left(0, \bar{\theta}_0\right)$, then $\partial U_1/\partial\eta > 0$.

Focus finally on $\partial U_1/\partial\gamma$. We have

$$\frac{\partial U_1}{\partial\gamma} = \frac{c_{10}^{1-1/\sigma}}{1-\beta_1 g^{*1-1/\sigma}}\left[\frac{1}{g^*}\frac{\partial g^*}{\partial\gamma}\frac{\beta_1 g^{*1-1/\sigma}}{1-\beta_1 g^{*1-1/\sigma}} + \frac{1}{c_{10}}\frac{\partial c_{10}}{\partial\gamma}\right] \tag{93}$$

There exists a trade-off between the initial consumption and the growth rate. The utility of impatient agents increases because of the positive impact on the growth rate:

$$\frac{1}{g^*}\frac{\partial g^*}{\partial\gamma}\frac{\beta_1 g^{*1-1/\sigma}}{1-\beta_1 g^{*1-1/\sigma}} > 0$$

and may decrease because of the ambiguous effect on the initial consumption. Indeed, using (43), we find

$$\frac{\partial c_{10}}{\partial\gamma} = -\theta_1\frac{\eta-1}{\eta}\left(1+\frac{\partial g^*}{\partial\gamma}\right)k_0 < 0 \tag{94}$$

Consider the expression in the brackets in Eq. (93). From Eqs. (41) and (43) in Proposition 4 we derive

$$c_{10} = k_0\frac{(1-s)A[1+(\eta-1)\theta_1]+\theta_1(\eta-1)(\Delta+sA-\gamma-g^*)}{\eta} \tag{95}$$

Replacing this expression into (94), we find

$$\frac{1}{c_{10}}\frac{\partial c_{10}}{\partial\gamma} = -\frac{\theta_1(\eta-1)\left(1+\frac{\partial g^*}{\partial\gamma}\right)}{(1-s)A[1+(\eta-1)\theta_1]+\theta_1(\eta-1)(\Delta+sA-\gamma-g^*)} \tag{96}$$

Substituting (96) into (93) yields

$$\begin{aligned}\frac{\partial U_1}{\partial\gamma} = \frac{c_{10}^{1-1/\sigma}}{1-\beta_1 g^{*1-1/\sigma}}\Bigg[&\frac{-(\eta-1)\theta_1}{(1-s)A[1+(\eta-1)\theta_1]+(\eta-1)\theta_1[\Delta+sA-\gamma-g^*]}\\ &+\frac{\partial g^*}{\partial\gamma}\frac{\beta_1 g^{*1-1/\sigma}\{(1-s)A[1+(\eta-1)\theta_1]+(\eta-1)\theta_1[\Delta+sA-\gamma]\}-g^*(\eta-1)\theta_1}{g^*\left(1-\beta_1 g^{*1-1/\sigma}\right)\{(1-s)A[1+(\eta-1)\theta_1]+(\eta-1)\theta_1[\Delta+sA-\gamma-g^*]\}}\Bigg]\end{aligned} \tag{97}$$

Therefore, there exists $\tilde{\theta}_1 \in (0,1)$ such that, when $\theta_1 \in (0,\tilde{\theta}_1)$, then $\partial U_1/\partial\gamma > 0$. □

A.7. Proof of Lemma 7

Eq. (70) simplifies:

$$\eta(1-z_0)+(1-\eta)\theta_1 = \frac{(1-s)A}{x_0+x_1} \tag{98}$$

Linearizing Eq. (36) and using (98), we obtain

$$(z_0-a_0)\frac{dx_{0t+1}}{x_0} - (z_0-a_0)\frac{dx_{1t+1}}{x_1} = z_0\frac{dx_{0t}}{x_0} + (1-z_0)\frac{dx_{1t}}{x_1} \tag{99}$$

In order to linearize the second equation, we apply the elasticities (46). Under Eqs. (70) and (37), evaluated at the steady state, writes:

$$1-\gamma\eta/g^* = \beta_0(sA+\Delta-\gamma)g^{*-1/\sigma} \tag{100}$$

Using Eqs. (98) and (100), we obtain

$$\left(a_0+\varepsilon_0-\frac{y_0+\varepsilon_0}{\sigma}\right)\frac{dx_{0t+1}}{x_0} + \left(1-a_0+\varepsilon_1-\frac{\varepsilon_1}{\sigma}\right)\frac{dx_{1t+1}}{x_1} = \left[h(a_0+\varepsilon_0)-\frac{y_0}{\sigma}\right]\frac{dx_{0t}}{x_0} + h(1-a_0+\varepsilon_1)\frac{dx_{1t}}{x_1} \tag{101}$$

where h is given by (47).

System (99)–(101) is represented by the following Jacobian matrix:

$$J = \begin{bmatrix} z_0-a_0 & a_0-z_0 \\ a_0+\frac{\sigma-1}{\sigma}\varepsilon_0-\frac{y_0}{\sigma} & 1-a_0+\frac{\sigma-1}{\sigma}\varepsilon_1 \end{bmatrix}^{-1}\begin{bmatrix} z_0 & 1-z_0 \\ h(a_0+\varepsilon_0)-\frac{y_0}{\sigma} & h(1-a_0+\varepsilon_1) \end{bmatrix} \tag{102}$$

Straightforward computations yield the expression of J in (48). From (38), noticing that

$$g^* = \Delta+sA-\gamma-(1-s)A\frac{\eta x_0+(1-\eta)\theta_0(x_0+x_1)}{\eta x_1+(1-\eta)\theta_1(x_0+x_1)}$$

and using $\theta_0+\theta_1 = 1$ and (70), we compute the elasticities ε_0 and ε_1:

$$\varepsilon_1 = -\varepsilon_0 = \frac{x_1}{g^*}\frac{\eta z_0}{\eta(1-z_0)+(1-\eta)\theta_1} \tag{103}$$

Replacing (103) in (102), we get (48) and we derive the determinant and the trace:

$$D = \frac{y_0(1-z_0) + h\sigma(z_0 - a_0 - \varepsilon_0)}{(\sigma - y_0)(z_0 - a_0)} \text{ and } T = 1 + D + \frac{1 + (h-1)\sigma}{(\sigma - y_0)(z_0 - a_0)}\varepsilon_0$$

Finally, from (40)–(41), we find

$$x_0 + x_1 = (\Delta + A - \gamma - g^*)$$

and, from (98),

$$\eta(1 - z_0) + (1 - \eta)\theta_1 = \frac{(1-s)A}{\Delta + A - \gamma - g^*} > 0 \tag{104}$$

Thus, $\varepsilon_1 = -\varepsilon_0 > 0$ for any $\eta > 1$. □

A.8. Proof of Proposition 8

We consider Eq. (39):

$$g = \gamma\eta + \beta_0(\Delta + sA - \gamma)g^{\frac{\sigma-1}{\sigma}} \equiv \varphi(g) \tag{105}$$

Then, any given $g^* > 1$ is solution if and only if $A = A^*(\sigma, g^*)$ with

$$A^*(\sigma, g^*) \equiv \frac{(g^* - \gamma\eta)g^{*\frac{1-\sigma}{\sigma}} - \beta_0(\Delta - \gamma)}{\beta_0 s}$$

Moreover, there exists $\bar{\gamma} > 0$ such that, for any $\gamma \in (0, \bar{\gamma})$, $A^*(\sigma, g^*) > 0$. □

A.9. Proof of Lemma 9

We know that $\varepsilon_0 < 0$ for any $\eta > 1$ and $h > 1$. We conclude that, for any $\sigma > 0$,

$$\frac{1 + (h-1)\sigma}{(\sigma - y_0)(z_0 - a_0)}\varepsilon_0 \neq 0$$

so that $T \neq 1 + D$: any bifurcation with an eigenvalue equal to 1 is ruled out. We also derive from (104) that $a_0 < 0$ for any $\eta > 1$ and $\sigma > 0$ so that $z_0 - a_0 > 0$.

Moreover, we have $\partial A^*(\sigma, g^*)/\partial\sigma < 0$ with $A^*(0, g^*) = +\infty$ and

$$A^*(1, g^*) = \frac{g^* - \gamma\eta - \beta_0(\Delta - \gamma)}{\beta_0 s} \equiv A_1^*$$

with $A_1^* > 0$ when $\gamma \in (0, \bar{\gamma})$. x_0 and y_0 are increasing functions of A and x_0 respectively with

$$\lim_{\sigma\to 0} x_0 = +\infty, \quad \lim_{\sigma\to 1} x_0 = \frac{[1 + (\eta - 1)\theta_0](\Delta - \gamma - g^*) + A_1^*[s + (\eta - 1)\theta_0]}{\eta} \equiv x_0^1$$

$$\lim_{\sigma\to 0} y_0 = 1, \quad \lim_{\sigma\to 1} y_0 = \frac{x_0^1}{x_0^1 + \gamma}$$

It follows that the difference $\sigma - y_0$ is an increasing function of σ with $\lim_{\sigma\to 0}(\sigma - y_0) < 0$ and $\lim_{\sigma\to 1}(\sigma - y_0) > 0$. Thus, there exists $\bar{\sigma} \in (0, 1)$ such that $\sigma - y_0 = 0$ when $\sigma = \bar{\sigma}$. Moreover, from the expressions of D and T, we get that, when $\sigma = \bar{\sigma}$, $D = \pm\infty$ and $T = \pm\infty$.

Let first $\sigma > \bar{\sigma}$. It follows that

$$\frac{1 + (h-1)\sigma}{(\sigma - y_0)(z_0 - a_0)}\varepsilon_0 < 0$$

and, thus, $1 - T + D > 0$ with $D > 0$. Moreover, we have $D > 1$ if and only if

$$y_0(1 - a_0) + \sigma[(z_0 - a_0)(h - 1) - h\varepsilon_0] > 0$$

This inequality is always satisfied. Therefore, if $\sigma > \bar{\sigma}$, then $1 - T + D > 0$ with $D > 1$. Let eventually $\sigma < \bar{\sigma}$. It follows that

$$\frac{1 + (h-1)\sigma}{(\sigma - y_0)(z_0 - a_0)}\varepsilon_0 > 0$$

and $1 - T + D < 0$ with $D < 0$. □

A.10. Proof of Lemma 10

Let $\eta = 1$. We find from (45), (103), (104) and Proposition 4:

$$x_0 = \Delta + sA^* - \gamma - g^*, x_1 = (1-s)A^*, \quad z_0 = \frac{\Delta + sA^* - \gamma - g^*}{\Delta + A^* - \gamma - g^*}$$
$$\varepsilon_0 = -\frac{\Delta + sA^* - \gamma - g^*}{g^*}, \quad y_0 = \frac{\Delta + sA^* - \gamma - g^*}{\Delta + sA^* - g^*}, \quad a_0 = 0 \tag{106}$$

In this case,

$$1 + T + D = 2 + \frac{2y_0(1-z_0) + 2h\sigma z_0 + (1-\sigma - h\sigma)\varepsilon_0}{(\sigma - y_0)z_0}$$

By definition, when $\sigma = \bar{\sigma}$ we have $\sigma = y_0$. It follows that

$$2y_0(1-z_0) + 2h\sigma z_0 + (1-\sigma-h\sigma)\varepsilon_0 = y_0[2(1-z_0) + 2h(z_0 - \varepsilon_0) + (h-1)\varepsilon_0] + \varepsilon_0$$

Assume now that $\gamma = 0$ and, thus, $h = 1$ and $y_0 = 1$. (106) implies:

$$y_0[2(1-z_0) + 2h(z_0 - \varepsilon_0) + (h-1)\varepsilon_0] + \varepsilon_0 = 2 - \varepsilon_0 > 0$$

From the proof of Lemma 9, there exists $\bar{\eta}_1 > 1$ and $\bar{\gamma} > 0$ such that, when $\eta \in (1, \bar{\eta}_1)$ and $\gamma \in (0, \bar{\gamma})$, we have

$$\lim_{\sigma \to \bar{\sigma}_-} 1 + T + D = -\infty$$

Moreover, if $\eta = 1$, we obtain from (106) $A^*(0, g^*) = +\infty$ and the expressions of T and D

$$\lim_{\sigma \to 0} 1 + T + D \quad = \quad 2 - 2\lim_{\sigma \to 0} \frac{1-z_0}{z_0} - \lim_{\sigma \to 0} \frac{\varepsilon_0}{y_0 z_0}$$

with

$$\lim_{\sigma \to 0} \frac{1-z_0}{z_0} = \frac{1-s}{s} < 1$$

since $s \in (1/2, 1)$. Recalling that $\varepsilon_0 < 0$, we find

$$\lim_{\sigma \to 0} 1 + T + D > 0$$

Then, there exists $\bar{\eta}_2 > 1$ such that, when $\eta \in (1, \bar{\eta}_2)$, $\lim_{\sigma \to 0} 1 + T + D > 0$. We conclude that there exist $\sigma^* \in (0, \bar{\sigma})$ and $\epsilon > 0$ such that, if $\eta \in (1, \bar{\eta})$, then $1 + T + D > 0$ with $\sigma \in (0, \sigma^*)$, $1 + T + D = 0$ with $\sigma = \sigma^*$ and $1 + T + D < 0$ with $\sigma \in (\sigma^*, \sigma^* + \epsilon)$. The final result follows with $\bar{\eta} = \min\{\bar{\eta}_1, \bar{\eta}_2\}$. □

References

Barbar, R., Bosi, S., 2010. Collaterals and macroeconomic volatility. Res. Econ. 64, 146–161.
Basu, S., Fernald, J., 1997. Returns to scale in US production: estimates and implications. J. Pol. Econ. 105, 249–283.
Becker, R.A., 1980. On the long-run steady state in a simple dynamic model of equilibrium with heterogeneous households. Q. J. Econ. 95, 375–382.
Becker, R.A., Foias, C., 1987. A characterization of ramsey equilibrium. J. Econ. Theory 41, 173–184.
Becker, R.A., Foias, C., 1994. The local bifurcation of ramsey equilibrium. Econ. Theory 4, 719–744.
Bosi, S., Seegmuller, T., 2010. On the ramsey equilibrium with heterogeneous consumers and endogenous labor supply. J. Math. Econ. 46, 475–492.
Boucekkine, R., Licandro, O., Puch, L., del Rio, F., 2005. Vintage capital and the dynamics of the AK model. J. Econ. Theory 120, 39–72.
Boucekkine, R., Ruiz-Tamarit, R., 2008. Special functions for the study of economic dynamics : the case of the Lucas–Uzawa model. J. Math. Econ. 44, 33–54.
Caballe, J., Santos, M., 1993. On endogenous growth with physical and human capital. J. Pol. Econ. 101, 1042–1067.
Campbell, J., 1999. Asset prices, consumption and the business cycle. In: Taylor, J.B., Woodford, M. (Eds.), Handbook of Macroeconomics. North-Holland, Amsterdam, pp. 1231–1303.
Carlstrom, C.T., Fuerst, T.S., 2003. Money growth rules and price level determinacy. Rev. Econ. Dyn. 6.263–75
Cooley, T.F., Hansen, G.D., 1989. The inflation tax in a real business cycle model. Am. Econ. Rev. 79, 492–511.
Ferraris, L., Watanabe, M., 2008. Collateral secured loans in a monetary economy. J. Econ. Theory 143, 405–424.
Ferraris, L., Watanabe, M., 2011. Collateral fluctuations in a monetary economy. J. Econ. Theory 146, 1915–1940.
Fostel, A., Geanakoplos, J., 2008. Leverage cycles and the anxious economy. Am. Econ. Rev. 98, 1211–1244.
Fukuda, S., 1993. The emergence of equilibrium cycles in a monetary economy with a separable utility function. J. Monet. Econ. 32, 321–334.
Geanakoplos, J., Zame, W., 2007. Collateralized asset markets. Yale University Discussion Paper.
Grandmont, J.M., Pintus, P., de Vilder, R., 1998. Capital-labour substitution and competitive nonlinear endogenous business cycles. J. Econ. Theory 80, 14–59.
Grandmont, J.M., Younès, Y., 1972. On the role of money and the existence of a monetary equilibrium. Rev. Econ. Stud. 39, 355–372.
Havranek, T., 2015. Measuring intertemporal substitution: the importance of method choices and selective reporting. J. Eur. Econ. Assoc. 13, 1180–1204.
Kiyotaki, N., Moore, J., 1997. Credit cycles. J. Pol. Econ. 105, 211–248.
Kocherlakota, N., 1996. The equity premium: it's still a puzzle. J. Econ. Lit. 36, 42–71.
Lagos, R., Wright, R., 2005. A unified framework for monetary theory and policy analysis. J. Pol. Econ. 113, 463–484.
Le Van, C., Vailakis, Y., 2003. Existence of a competitive equilibrium in a one sector growth model with heterogeneous agents and irreversible investment. Econ. Theory 22, 743–771.
Lucas, R., Stokey, N., 1987. Money and interest in a cash-in-advance economy. Econometrica 55, 491–513.
Mankiw, N., Romer, D., Weil, D., 1992. A contribution to the empirics of economic growth. Q. J. Econ. 107, 407–437.

Matsuyama, K., 1999. Growing through cycles. Econometrica 67, 335–347.
Matsuyama, K., 2001. Growing through cycles in an infinitely lived agent economy. J. Econ. Theory 100, 220–234.
Matsuyama, K., 2007. Credit traps and credit cycles. Am. Econ. Rev. 97, 503–516.
Michener, R., Ravikumar, B., 1998. Chaotic dynamics in a cash-in-advance economy. J. Econ. Dyn. Control 22, 1117–1137.
Nishimura, K., Mino, K., Shimomura, K., Wang, P., 2008. Equilibrium dynamics in discrete-time endogenous growth models with social constant returns. Econ. Theory 34, 1–23.
Romer, P., 1986. Increasing returns and long-run growth. J. Pol. Econ. 94, 1002–1037.
Stockman, A., 1981. Anticipated inflation and the capital stock in a cash-in-advance economy. J. Monet. Econ. 8, 387–393.
Svensson, L., 1985. Money and asset prices in a cash-in-advance economy. J. Pol. Econ. 93, 919–944.
Vissing-Jorgensen, A., 2002. Limited asset market participation and the elasticity of intertemporal substitution. J. Pol. Econ. 110, 825–853.
Woodford, M., 1986. Stationary sunspot equilibria in a finance constrained economy. J. Econ. Theory 40, 128–137.