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ORIGINAL RESEARCH

Dynamics of nonlinear difference equation $x_{n+1} = \frac{\beta x_n+\gamma x_{n-k}}{A+Bx_n+Cx_{n-k}}$

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Abstract The main goal of this paper is to investigate the boundedness, invariant intervals, semi-cycles and global attractivity of all nonnegative solutions of the equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}, \quad n \in \mathbb{N}_0,$$

where the parameters β, γ, A, B and C and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0$ are non-negative real numbers, $k = \{1, 2, \ldots\}$. We give a detailed description of the semi-cycles of solutions, and determine conditions that satisfy the global asymptotic stability of the equilibrium points.

1 Introduction

In this paper we consider studying and investigating the difference equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{1}$$

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where the parameters β, γ, A, B and C are non-negative real numbers with at least one parameter is non zero and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0$ are non-negative real numbers for which the solution is defined and $k \in \{1, 2, \ldots\}$.

Our concentration is on boundedness, invariant intervals, periodic character, the character of semi-cycles and global asymptotic stability of the zero and the positive equilibrium points of Eq. (1).

Changing the variables

Before investigating the characteristics of Eq. (1), it is more convenient to reduce the number of parameters by a change of variables.

Assume that $x_n = \frac{\gamma}{C} y_n$, and substitute it in Eq. (1) we get:

$$\frac{\gamma}{C} y_{n+1} = \frac{\beta \frac{\gamma}{C} y_n + \frac{\gamma^2}{C} y_{n-k}}{A + B\frac{\gamma}{C} y_n + C\frac{\gamma}{C} y_{n-k}}.$$

Tacking a common factor $\frac{\gamma^2}{C}$ in the numerator and γ in the denominator we get,

$$\frac{\gamma}{C} y_{n+1} = \frac{\frac{\gamma^2}{C}\left[\frac{\beta}{\gamma} y_n + y_{n-k}\right]}{\gamma\left[\frac{A}{\gamma} + \frac{B}{C} y_n + y_{n-k}\right]}.$$

Which implies;

$$y_{n+1} = \frac{\frac{\beta}{\gamma} y_n + y_{n-k}}{\frac{A}{\gamma} + \frac{B}{C} y_n + y_{n-k}}.$$

By assuming $p = \frac{\beta}{\gamma}$, $q = \frac{B}{C}$ and $r = \frac{A}{\gamma}$, we get the following equation:

$$y_{n+1} = \frac{p y_n + y_{n-k}}{r + q y_n + y_{n-k}}, \qquad n \in \mathbb{N}_0. \tag{2}$$

Equilibrium points

Definition 1.1 The equilibrium point $\bar{y}$ of the equation

$$y_{n+1} = f(y_n, y_{n-1}, \ldots, y_{n-k}), \qquad n \in \mathbb{N}_0 \tag{3}$$

is the point that satisfies the condition

$$\bar{y} = f(\bar{y}, \bar{y}, \ldots, \bar{y}).$$

To find the equilibrium points of Eq. (2) using the definition, let $f(\bar{y}, \bar{y}) = \bar{y}$, then we solve the following equation

$$\bar{y} = \frac{p\bar{y} + \bar{y}}{r + q\bar{y} + \bar{y}}.$$

Cross multiplication and rearranging the terms, we get

$$\bar{y}\big[(r - p - 1) + (q + 1)\bar{y}\big] = 0.$$

Hence, the equilibrium points of Eq. (2) are

$$\bar{y} = 0, \quad \text{and} \quad \bar{y} = \frac{p + 1 - r}{q + 1} \text{ where } p + 1 > r.$$

2 Linearization of the difference equation

To find the linearization of our Eq. (2) about the equilibrium points, consider

$$f(u, v) = \frac{pu + v}{r + qu + v}.$$

Thus,

$$f_u(u, v) = \frac{p(r + qu + v) - q(pu + v)}{(r + qu + v)^2} = \frac{pr + (p - q)v}{(r + qu + v)^2}.$$

Which implies;

$$f_u(\bar{y}, \bar{y}) = \frac{pr + (p - q)\bar{y}}{(r + q\bar{y} + \bar{y})^2} = \frac{pr + (p - q)\bar{y}}{(r + (q + 1)\bar{y})^2}.$$

In the same way.

$$\begin{aligned} f_v(u, v) &= \frac{(r + qu + v) - (pu + v)}{(r + qu + v)^2} \\ &= \frac{r + (q - p)u}{(r + qu + v)^2}. \end{aligned}$$

Substituting $\bar{y}$ we get,

$$f_v(\bar{y}, \bar{y}) = \frac{r + (q - p)\bar{y}}{(r + q\bar{y} + \bar{y})^2} = \frac{r + (q - p)\bar{y}}{(r + (q + 1)\bar{y})^2}.$$

The linearized equation is

$$z_{n+1} = f_u(\bar{y}, \bar{y})\, z_n + f_v(\bar{y}, \bar{y})\, z_{n-k}.$$

So,

$$z_{n+1} = \frac{pr + (p-q)\bar{y}}{(r + (q+1)\bar{y})^2}\, z_n + \frac{r + (q-p)\bar{y}}{(r + (q+1)\bar{y})^2}\, z_{n-k}.$$

That is;

$$z_{n+1} - \frac{pr + (p-q)\bar{y}}{(r + (q+1)\bar{y})^2}\, z_n - \frac{r + (q-p)\bar{y}}{(r + (q+1)\bar{y})^2}\, z_{n-k} = 0. \tag{4}$$

And the characteristic equation is

$$\lambda^{n+1} - \frac{pr + (p-q)\bar{y}}{(r + (q+1)\bar{y})^2}\, \lambda^n - \frac{r + (q-p)\bar{y}}{(r + (q+1)\bar{y})^2}\, \lambda^{n-k} = 0.$$

Which implies;

$$\lambda^{k+1} - \frac{pr + (p-q)\bar{y}}{(r + (q+1)\bar{y})^2}\, \lambda^k - \frac{r + (q-p)\bar{y}}{(r + (q+1)\bar{y})^2} = 0. \tag{5}$$

3 Local stability of the equilibrium points

Definition 3.1 [11] Let $\bar{y}$ be an equilibrium point of Eq. (3).

(i) The equilibrium point $\bar{y}$ of Eq. (3) is called *locally stable* (or *stable*) if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $y_{-k}, \ldots, y_{-1}, y_0 \in I$ with

$$\sum_{i=-k}^{0} |y_i - \bar{y}| < \delta,$$

we have $|y_n - \bar{y}| < \epsilon$ for all $n \geqslant -k$.

(ii) The equilibrium point $\bar{y}$ of Eq. (3) is called locally *asymptotically stable* (*asymptotic stable*) if it is *locally stable*, and if there exists $\gamma > 0$ such that for all $y_{-k}, \ldots, y_{-1}, y_0 \in I$ with

$$\sum_{i=-k}^{0} |y_i - \bar{y}| < \gamma,$$

we have $\lim\limits_{n\to\infty} y_n = \bar{y}$.

(iii) The equilibrium point $\bar{y}$ of Eq. (3) is called a *global attractor* if for every $y_{-k}, \ldots, y_{-1}, y_0 \in I$, we have $\lim\limits_{n\to\infty} y_n = \bar{y}$.

(iv) The equilibrium point $\bar{y}$ of Eq. (3) is called *globally asymptotically stable* if it is *locally stable* and a *global attractor*.

(v) The equilibrium point $\bar{y}$ of Eq. (3) is called *unstable* if it is *not stable*.
(vi) The equilibrium point $\bar{y}$ of Eq. (3) is called a *source*, or a *repeller*, if there exists $r > 0$ such that for all $y_{-k}, \ldots, y_{-1}, y_0 \in I$ with $\sum_{n=-k}^{0} |y_i - \bar{y}| < \gamma$, there exists $N \geqslant 1$ such that $|y_N - \bar{y}| \geqslant r$.

Theorem 3.1 [10] *Let I be some interval of real numbers and let*

$$f : I^{k+1} \to I$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, \ldots, x_1, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-k}), \quad n \in \mathbb{N}_0 \tag{6}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Theorem 3.2 [9] *Assume that $a, b \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$, then*

$$|a| < 1 - b < 2$$

is a necessary and a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+1} - ay_n - by_{n-k} = 0, \quad n \in \mathbb{N}_0. \tag{7}$$

Theorem 3.3 [10] *Assume that all the roots of the characteristic equation of the above Eq. (7) lie inside the unit circle, then the equilibrium point is* locally asymptotically stable.

3.1 Local stability of the zero equilibrium point

Let us find the linearized equation about the zero equilibrium point associated with Eq. (2). By substituting $\bar{y} = 0$ in the linearized Eq. (4), we find the linearized equation associated with Eq. (2) about $\bar{y} = 0$,

$$z_{n+1} - \frac{p}{r} z_n - \frac{1}{r} z_{n-k} = 0. \tag{8}$$

Now, let us apply Theorem 3.2 to the above Eq. (8), so we have

$$\overbrace{\left|\frac{p}{r}\right| < 1 - \frac{1}{r}}^{\text{part (I)}} \hspace{-4em} \underbrace{\phantom{1 - \frac{1}{r}} < 2}_{\text{part (II)}}. \tag{9}$$

It is clear that $\frac{p}{r}$ is positive, since p and r are nonnegative, so

$$\frac{p}{r} < 1 - \frac{1}{r} \text{ implies } p + 1 < r.$$

Now, the right hand inequality in (9),

$$1-\frac{1}{r}<2 \text{ implies } -1<r,$$

which is true for all nonnegative values of r.

The above discussion yields the following theorem.

Theorem 3.4 *The zero equilibrium point of Eq.* (2) *is* locally asymptotically stable *under the condition* $p+1<r$. *Otherwise, it is* unstable.

3.2 Local stability of the positive equilibrium point

Theorem 3.5 *The positive equilibrium point* $\bar{y}=\frac{p+1-r}{q+1}$ *of Eq.* (2) *is* locally asymptotically stable *for all values of the parameters* p, q *and* r *provided that all roots of Eq.* (5) *lie inside the unit circle.*

Theorem 3.6 *Assume that* $p+1>r$, *then the positive equilibrium point* $\bar{y}=\frac{p+1-r}{q+1}$ *of Eq.* (2) *is* locally asymptotically stable *when*

$$q+r<3p+1+qr+pq.$$

Otherwise, it is unstable.

Proof First, substitute $\bar{y}=\frac{p+1-r}{q+1}$ in the linearized Eq. (4) to get the linearized equation about $\bar{y}=\frac{p+1-r}{q+1}$,

$$z_{n+1}-\frac{pr+(p-q)\frac{p+1-r}{q+1}}{\left(r+(q+1)\frac{p+1-r}{q+1}\right)^2}z_n-\frac{r+(q-p)\frac{p+1-r}{q+1}}{\left(r+(q+1)\frac{p+1-r}{q+1}\right)^2}z_{n-k}=0. \tag{10}$$

Simplifying the above equation, we get

$$z_{n+1}-\frac{p-q+qr}{qp+q+p+1}z_n-\frac{-p+q+r}{qp+q+p+1}z_{n-k}=0. \tag{11}$$

By applying Theorem 3.2 on the linearized Eq. (11), we have

$$a=\frac{p-q+qr}{qp+q+p+1} \quad \text{and} \quad b=\frac{-p+q+r}{qp+q+p+1}.$$

Now, we need to verify the inequality,

$$\overbrace{\left|\frac{p-q+qr}{qp+q+p+1}\right|<\underbrace{1-\frac{-p+q+r}{qp+q+p+1}<2}_{\text{second side}}}^{\text{first side}}. \tag{12}$$

First, assume that the left of the first side is positive, then

$$\frac{p-q+qr}{qp+q+p+1} < 1 - \frac{-p+q+r}{qp+q+p+1}$$
$$\Rightarrow \frac{p-q+qr}{qp+q+p+1} < \frac{qp+2p+1-r}{qp+q+p+1},$$

and so, $p - q + qr < qp + 2p + 1 - r$, which implies, $r < p + 1$, which is the assumption.

Now, assume that the left of the first side of inequality (12) is negative, then

$$-\frac{p-q+qr}{qp+q+p+1} < \frac{qp+2p+1-r}{qp+q+p+1}.$$

And so, $-p+q-qr < qp+2p+1-r \Rightarrow q+r < 3p+1+qr+pq$, as needed.

The second side of the inequality,

$$1 - \frac{-p+q+r}{qp+q+p+1} < 2,$$

then

$$\frac{qp+2p+1-r}{qp+q+p+1} < 2.$$

That is,

$$qp+2p+1-r < 2qp+2q+2p+2$$

which implies, $0 < qp + 2q + 1 + r$, which is true for all positive values of p, q and r. □

4 Boundedness

Definition 4.1 We say that a solution x_n of a difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n \in \mathbb{N}_0. \tag{13}$$

is *bounded and persists* if there exist positive constants P and Q such that

$$P \leqslant x_n \leqslant Q \quad \text{for} \quad n = -k, -k+1, \ldots$$

Theorem 4.1 *Assume that B and C are greater than zero, then every nonnegative solution of Eq.* (1) *is bounded from above by a positive constant.*

Proof First case, assume that $A = 0$, then by using Eq. (1), we can write the following:

$$\begin{aligned} x_{n+1} &= \frac{\beta x_n + \gamma x_{n-k}}{B x_n + C x_{n-k}} \\ &\leqslant \frac{\max(\beta, \gamma)(x_n + x_{n-k})}{\min(B, C)(x_n + x_{n-k})} = \frac{\max(\beta, \gamma)}{\min(B, C)}. \end{aligned}$$

The second case, assume that $A > 0$, then

$$\begin{aligned} x_{n+1} &= \frac{\beta x_n + \gamma x_{n-k}}{A + B x_n + C x_{n-k}} \\ &\leqslant \frac{\max(\beta, \gamma) + \beta x_n + \gamma x_{n-k}}{A + B x_n + C x_{n-k}} \\ &\leqslant \frac{\max(\beta, \gamma)(1 + x_n + x_{n-k})}{\min(A, B, C)(1 + x_n + x_{n-k})} \\ &= \frac{\max(\beta, \gamma)}{\min(A, B, C)}. \end{aligned}$$

The proof is complete. □

5 Invariant intervals

Definition 5.1 (*Invariant Interval*) An *Invariant Interval* of the difference Eq. (13) is an interval with the property that if $k + 1$ consecutive terms of the solution fall in I, then all subsequent terms of the solution also belong to I. In other words, I is an *invariant interval* for (13) if $x_{N-k+1}, \ldots, x_{N-1}, x_N \in I$ for some $N \geqslant 0$, then $x_n \in I$ for every $n > N$.

Assume that $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of Eq. (2), then the following identities are easily established:

$$y_{n+1} - 1 = (p - q)\frac{y_n - \frac{r}{p-q}}{r + q y_n + y_{n-k}}, \quad n \in \mathbb{N}_0, \tag{14}$$

$$y_{n+1} - \frac{p}{q} = \frac{q - p}{q}\frac{y_{n-k} - \frac{pr}{q-p}}{r + q y_n + y_{n-k}}, \quad n \in \mathbb{N}_0. \tag{15}$$

If $p = q$, then the numerator in (14) can be written in the form

$$(p - q)\, y_n - r = -r.$$

Thus, if $p + 1 > r$ and $p = q$, then the unique equilibrium is $\bar{y} = \frac{p+1-r}{q+1} = \frac{p+1-r}{p+1}$, and the following identities hold:

$$y_{n+1} - 1 = \frac{-r}{r + p\, y_n + y_{n-k}}, \quad n \in \mathbb{N}_0, \tag{16}$$

$$y_{n+1} - \bar{y} = \frac{r}{p+1} \frac{p\,(y_n - \bar{y}) + (y_{n-k} - \bar{y})}{r + p\,y_n + y_{n-k}}, \quad n \in \mathbb{N}_0. \tag{17}$$

When $p = q + r$, the unique equilibrium is $\bar{y} = 1$ and identity (14) becomes

$$y_{n+1} - 1 = \frac{r\,(y_n - 1)}{r + q\,y_n + y_{n-k}}, \quad n \in \mathbb{N}_0, \tag{18}$$

and when $q = p + qr$, the unique equilibrium is $\bar{y} = \frac{p}{q}$ and identity (15) becomes

$$y_{n+1} - \frac{p}{q} = \frac{r\left(y_{n-k} - \frac{p}{q}\right)}{r + q\,y_n + y_{n-k}}, \quad n \in \mathbb{N}_0. \tag{19}$$

Theorem 5.1 [7] *Assume that $f(x, y)$ is defined as*

$$f(x, y) = \frac{p\,x + y}{r + q\,x + y}. \tag{20}$$

Then the following statements hold true:

(i) $f(x, x)$ *is* strictly increasing *in x in* $[0, \infty)$.
(ii) *Assume $p = q$, then $f(x, y)$ is* strictly increasing *in each of its arguments.*
(iii) *Assume $p > q$, then $f(x, y)$ is* strictly increasing *in each of its arguments for $x < \frac{r}{p-q}$ and it is* strictly increasing *in x and* decreasing *in y for $x \geqslant \frac{r}{p-q}$.*
(iv) *Assume $p < q$, then $f(x, y)$ is* strictly increasing *in each of its arguments for $y < \frac{pr}{q-p}$, and it is* strictly increasing *in y, and* decreasing *in x for $y \geqslant \frac{pr}{q-p}$.*

Proof (i) Note that $f(x, x) = \frac{(p+1)x}{r+(q+1)x}$ is strictly increasing function in the interval $[0, \infty)$, since the derivative of f, $f'(x, x) = \frac{r(p+1)}{(r+(q+1)x)^2}$ is always positive for $x \geqslant 0$.

(ii)-(iv) By calculating the partial derivatives of the function $f(x, y)$, we have:

$$f_x(x, y) = \frac{pr - (q - p)y}{(r + qx + y)^2},$$
$$f_y(x, y) = \frac{r - (p - q)x}{(r + qx + y)^2},$$

from which these statements easily follow. □

Theorem 5.2 *Assume that $p > q$, $p + 1 > r$, and that $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of Eq.* (2). *Then the following statements are true:*

(i) $y_n \leqslant \frac{p}{q}$ *for all $n \in \mathbb{N}$.*
(ii) *If $p \geqslant q + r$ and for some $N \geqslant 0$, $y_N > \frac{r}{p-q}$, then $y_n > 1$ for all $n > N$.*
(iii) *If $p = q + r$ and for some $N \geqslant 0$, $y_N = 1$, then $y_n = 1$ for all $n > N$.*
(iv) *If $p \leqslant q + r$ and for some $N \geqslant 0$, $y_N < \frac{r}{p-q}$, then $y_n < 1$ for all $n > N$.*

(v) *If* $p \leqslant q + \frac{qr}{p}$, *then Eq.* (2) *possesses an* invariant interval $[0, \frac{p}{q}]$ *and* $\bar{y} \in [0, \frac{p}{q}]$, *moreover, the interval* $[0, 1]$ *is also* an invariant interval *for Eq.* (2) *and* $\bar{y} \in (0, 1)$.

(vi) *If* $q + \frac{qr}{p} < p < q + r$, *then Eq.* (2) *possesses an* invariant interval $[0, \frac{r}{p-q}]$ *and* $\bar{y} \in [0, \frac{r}{p-q}]$, *moreover, the interval* $[0, 1]$ *is also an* invariant interval *for Eq.* (2) *and* $\bar{y} \in (0, 1)$.

(vii) *If* $p > q + r$, *then Eq.* (2) *possesses an* invariant interval $[\frac{r}{p-q}, \frac{p}{q}]$ *and* $\bar{y} \in [\frac{r}{p-q}, \frac{p}{q}]$, *moreover, the interval* $[0, \frac{p}{q}]$ *is also* an invariant interval *for Eq.* (2) *and* $\bar{y} \in (0, \frac{p}{q})$.

Proof (i) By writing the identity (15) in the form

$$y_{n+1} - \frac{p}{q} = \frac{q-p}{q} \frac{y_{n-k} + \frac{pr}{p-q}}{r + qy_n + y_{n-k}}, \quad n \in \mathbb{N}_0,$$

and since $p > q$, it implies that $y_{n+1} - \frac{p}{q} \leqslant 0$, which implies that $y_{n+1} \leqslant \frac{p}{q}$ for all $n \in \mathbb{N}$.

(ii) Since $p \geqslant q + r$, implies that $\frac{r}{p-q} \leqslant 1$. By using the identity (14)

$$y_{N+1} - 1 = (p-q)\frac{y_N - \frac{r}{p-q}}{r + qy_N + y_{N-k}},$$

and the assumption $p > q$, then the right side of this identity is greater than zero, implies $y_{N+1} > 1$.
For the next term y_{N+2},

$$y_{N+2} - 1 = (p-q)\frac{y_{N+1} - \frac{r}{p-q}}{r + qy_{N+1} + y_{N-k+1}},$$

then the right side of the identity greater than zero since $y_{N+1} > 1 \geqslant \frac{r}{p-q}$, implies $y_{N+2} > 1$. By induction $y_n > 1$ for all $n > N$.

(iii) Since $p = q + r$, this implies that $\frac{r}{p-q} = 1$. By using the identity (14)

$$y_{N+1} - 1 = (p-q)\frac{y_N - \frac{r}{p-q}}{r + qy_N + y_{N-k}},$$

and the assumption $y_N = \frac{r}{p-q}$, then the right side of the identity equal zero, implies $y_{N+1} = 1$.
For the next term y_{N+2},

$$y_{N+2} - 1 = (p-q)\frac{y_{N+1} - \frac{r}{p-q}}{r + qy_{N+1} + y_{N-k+1}},$$

then the right side of the identity equal zero since $y_{N+1} = \frac{r}{p-q} = 1$, which implies $y_{N+2} = 1$. By induction, $y_n = 1$ for all $n > N$.

(iv) Since $p \leqslant q + r$, implies that $\frac{r}{p-q} \geqslant 1$. By using the identity (14)

$$y_{N+1} - 1 = (p-q)\frac{y_N - \frac{r}{p-q}}{r + qy_N + y_{N-k}},$$

and the assumption $p > q$, then the right side of the identity less than zero, implies $y_{N+1} < 1$.
For the next term y_{N+2},

$$y_{N+2} - 1 = (p-q)\frac{y_{N+1} - \frac{r}{p-q}}{r + qy_{N+1} + y_{N-k+1}},$$

then the right side of the identity less than zero since $y_{N+1} < 1 \leqslant \frac{r}{p-q}$, implies $y_{N+2} < 1$. By induction $y_n < 1$ for all $n > N$,

(v) Since $q < p \leqslant q + \frac{qr}{p}$, from the left side $1 < \frac{p}{q}$, and from the right side $p - q \leqslant \frac{qr}{p}$ which implies $\frac{p-q}{r} \leqslant \frac{q}{p}$. Then we have that $1 < \frac{p}{q} \leqslant \frac{r}{p-q}$.
Since $f(x, y) = \frac{px+y}{r+qx+y}$ is nondecreasing in x and y for each $x, y \in (0, \frac{p}{q}]$, then

$$y_1 = f(y_0, y_{-k}) \leqslant f\left(\frac{r}{p-q}, \frac{r}{p-q}\right) = 1,$$

which implies that $y_1 \in [0, 1] \subset [0, \frac{p}{q}]$, and $y_2 = f(y_1, y_{-k+1}) \leqslant f\left(1, \frac{r}{p-q}\right) \leqslant f\left(\frac{r}{p-q}, \frac{r}{p-q}\right) = 1$. By the induction, we have that $y_n \in [0, 1] \subset \left[0, \frac{p}{q}\right]$ for every $n \in \mathbb{N}$.
On the other hand, the condition $1 < \frac{r}{p-q}$ is equivalent to $\frac{p+1-r}{q+1} < 1$, that is $\bar{y} < 1$ from which it follows that $\bar{y} \in (0, 1)$.

(vi) Similar to the above, since $q + \frac{qr}{p} < p < q + r$, it implies that $\frac{qr}{p} < p - q < r$, and so $\frac{q}{p} < \frac{p-q}{r} < 1$. Then we have $1 < \frac{r}{p-q} < \frac{p}{q}$.
And the function $f(x, y)$ in nondecreasing in x and y for each $x, y \in (0, \frac{r}{p-q}]$, which implies that

$$y_1 = f(y_0, y_{-k}) \leqslant f\left(\frac{r}{p-q}, \frac{r}{p-q}\right) = 1,$$

which implies that $y_1 \in [0, 1] \subset [0, \frac{r}{p-q}]$, and

$$y_2 = f(y_1, y_{-k+1}) \leqslant f\left(1, \frac{r}{p-q}\right) \leqslant f\left(\frac{r}{p-q}, \frac{r}{p-q}\right) = 1,$$

which implies that $y_2 \in [0, 1] \subset [0, \frac{r}{p-q}]$. By the induction, we have that $y_n \in [0, 1] \subset [0, \frac{r}{p-q}]$ for every $n \in \mathbb{N}$.
Now, note that the condition $p < q + r$ implies that

$$\frac{p+1-r}{q+1} < \frac{q+r+1-r}{q+1} = 1.$$

Which means that $\bar{y} \in (0, 1)$, as desired.

(vii) It is easy to see that the function $f(x, y)$ is strictly increasing in x for each fixed $y \in (0, \infty)$, and nonincreasing in y for each fixed $x \geqslant \frac{r}{p-q}$. From this and (i), we have

$$1 = f\left(\frac{r}{p-q}, \frac{p}{q}\right) \leqslant y_1 = f(y_0, y_{-k}) \leqslant \frac{p}{q}.$$

Since $p > q + r$, it implies that $\frac{r}{p-q} < 1$, and since $p > q \Rightarrow 1 < \frac{p}{q}$. So we have $\frac{r}{p-q} < 1 < \frac{p}{q}$. From this, we have $y_1 \in [1, \frac{p}{q}] \subset [\frac{r}{p-q}, \frac{p}{q}]$. By the induction, it follows that $y_n \in [1, \frac{p}{q}] \subset [\frac{r}{p-q}, \frac{p}{q}]$, for every $n \in \mathbb{N}$. Further, we have that $p > q + r$ which implies that

$$\bar{y} = \frac{p+1-r}{q+1} > \frac{q+r+1-r}{q+1} = 1,$$

as well as $\frac{p+1-r}{q+1} < \frac{p}{q}$, which means that $\bar{y} \in (1, \frac{p}{q})$, as desired. □

Theorem 5.3 *Assume that $p = q + r$, $p + 1 > r$, and that $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of Eq.* (2). *Then the following statements are true:*

(i) $y_n \leqslant \frac{p}{q}$ *for all* $n \in \mathbb{N}$.
(ii) If for some $N \geqslant 0$, $y_N > 1$, *then* $y_n > 1$ *for all* $n > N$.
(iii) If for some $N \geqslant 0$, $y_N < 1$, *then* $y_n < 1$ *for all* $n > N$.

Proof By using Theorem 5.2, the proof is a direct consequence of the assumptions and identity (18). □

Theorem 5.4 *Assume that $q > p$, $p + 1 > r$, and that $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of Eq.* (2). *Then the following statements are true:*

(i) $y_n \leqslant 1$ *for all* $n \in \mathbb{N}$.
(ii) If for some $N \geqslant 0$, $y_N > \frac{pr}{q-p}$, *then* $y_{N+k+1} > \frac{p}{q}$.
(iii) If for some $N \geqslant 0$, $y_N = \frac{pr}{q-p}$, *then* $y_{N+k+1} = \frac{p}{q}$.
(iv) If for some $N \geqslant 0$, $y_N < \frac{pr}{q-p}$, *then* $y_{N+k+1} < \frac{p}{q}$.
(v) If $q \leqslant p + pr$, *then Eq.* (2) *possesses an invariant interval* $[0, 1]$ *and* $\bar{y} \in [0, 1]$, *moreover, the interval* $[0, \frac{p}{q}]$ *is also an invariant interval for Eq.* (2) *and* $\bar{y} \in \left(0, \frac{p}{q}\right)$.

(vi) *If $p + pr < q < p + qr$, then Eq. (2) possesses an invariant interval $[0, \frac{pr}{q-p}]$ and $\bar{y} \in [0, \frac{pr}{q-p}]$, moreover, the interval $[0, \frac{p}{q}]$ is also an invariant interval for Eq. (2) and $\bar{y} \in \left(0, \frac{p}{q}\right)$.*

(vii) *If $q > p + qr$, then Eq. (2) possesses an invariant interval $[\frac{pr}{q-p}, 1]$ and $\bar{y} \in [\frac{pr}{q-p}, 1]$, moreover, the interval $[\frac{p}{q}, 1]$ is also an invariant interval for Eq. (2) and $\bar{y} \in \left(\frac{p}{q}, 1\right)$.*

Proof (i) If we write identity (14) in the form

$$y_{n+1} - 1 = (p - q)\frac{y_n + \frac{r}{q-p}}{r + qy_n + y_{n-k}}, \quad n \in \mathbb{N}_0,$$

and use the assumption $q > p$, then the right side of the above identity is less than zero, which means $y_{n+1} - 1 \leqslant 0 \Rightarrow y_{n+1} \leqslant 1$.

(ii) By using the identity (15)

$$y_{n+k+1} - \frac{p}{q} = \frac{q - p}{q}\frac{y_n - \frac{pr}{q-p}}{r + qy_{n+k} + y_n}, \quad n \in \mathbb{N}_0$$

and the assumptions $y_N > \frac{pr}{q-p}$ and $q > p$, implies that the right side of the identity is greater than zero, which implies that $y_{N+k+1} > \frac{p}{q}$.

(iii) As in (ii), by using the identity (15) and the assumptions $y_N = \frac{pr}{q-p}$ and $q > p$, implies that the right side of the identity is zero, which implies that $y_{N+k+1} = \frac{p}{q}$.

(iv) Similarly, by identity (15) and the assumptions $y_N < \frac{pr}{q-p}$ and $q > p$, implies that the right side of the identity is less that zero, which implies that $y_{N+k+1} < \frac{p}{q}$.

(v) Since $p < q \leqslant p + pr$, from left side $\frac{p}{q} < 1$ and from right side $q - p \leqslant pr \Rightarrow 1 \leqslant \frac{pr}{q-p}$, so we have $\frac{p}{q} < 1 \leqslant \frac{pr}{q-p}$. By Theorem 5.1 (iv) the function $f(x, y)$ is strictly increasing in y for each fixed $x \in (0, \infty)$, and nondecreasing in x for each fixed $y \in (0, 1]$, we have

$$y_1 = f(y_0, y_{-k}) \leqslant f\left(\frac{pr}{q-p}, \frac{pr}{q-p}\right) = \frac{p}{q} < 1,$$

which implies that $y_1 \in [0, \frac{p}{q}] \subset [0, 1]$.
By the induction, $y_n \in [0, \frac{p}{q}] \subset [0, 1]$ for every $n \in \mathbb{N}$.
On the other hand, we have

$$\bar{y} = \frac{p + 1 - r}{q + 1} < \frac{p}{q} < 1,$$

as desired.

(vi) Since $p + pr < q < p + qr$, implies $pr < q - p < qr$ then $1 < \frac{q-p}{pr} < \frac{qr}{pr}$, so we have $\frac{p}{q} < \frac{pr}{q-p} < 1$. By Theorem 5.1 (iv), as above, we have

$$y_1 = f(y_0, y_{-k}) \leqslant f\left(\frac{pr}{q-p}, \frac{pr}{q-p}\right) = \frac{p}{q} < 1,$$

which implies that $y_1 \in [0, \frac{p}{q}] \subset [0, \frac{pr}{q-p}]$.
By the induction, $y_n \in [0, \frac{p}{q}] \subset [0, \frac{pr}{q-p}]$ for every $n \in \mathbb{N}$.
On the other hand, we have

$$\bar{y} = \frac{p+1-r}{q+1} < \frac{p}{q} < \frac{pr}{q-p},$$

as desired.

(vii) Since $q > p + qr > p$, we have $\frac{pr}{q-p} < \frac{p}{q} < 1$. On the other hand, by Theorem 5.1 (iv), we have for this case, the function $f(x, y)$ is strictly increasing in y for each fixed $x \in (0, \infty)$, and nonincreasing in x for each fixed $y > \frac{pr}{q-p}$. From this and (i) it follows that

$$\frac{p}{q} = f\left(1, \frac{pr}{q-p}\right) \leqslant y_1 = f(y_0, y_{-k}) \leqslant 1,$$

which implies that $y_1 \in [\frac{p}{q}, 1] \subset [\frac{pr}{q-p}, 1]$.
By the induction, $y_n \in [\frac{p}{q}, 1] \subset [\frac{pr}{q-p}, 1]$ for every $n \in \mathbb{N}$.
On the other hand, $q > p + qr$ implies that

$$\frac{p}{q} < \bar{y} = \frac{p+1-r}{q+1} < \frac{p+1-r}{p+1} < 1,$$

as claimed. □

6 Existence of two cycles

Definition 6.1 We say that a solution $\{x_n\}_{n=-k}^{\infty}$ of the difference Eq. (13) is *periodic* if there exists a positive integer p such that $x_{n+p} = x_n$, for every $n \geq -k$. The smallest such positive integer p is called the *prime period* of the solution of the difference equation.

We study here the periodic solution of our equation,

$$x_{n+1} = \frac{px_n + x_{n-k}}{r + qx_n + x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{21}$$

Let us assume that the two periodic nonnegative solutions of our equation will be in the form

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

- If k is odd then $x_{n+1} = x_{n-k}$, so we get

$$\psi = \frac{p\,\phi + \psi}{r + q\,\phi + \psi} \quad \text{and} \quad \phi = \frac{p\,\psi + \phi}{r + q\,\psi + \phi}.$$

This yields

$$\psi(r + q\,\phi + \psi) = p\,\phi + \psi \tag{22}$$
$$\phi(r + q\,\psi + \phi) = p\,\psi + \phi. \tag{23}$$

By subtracting Eq. (23) from Eq. (22), we get the following

$$r(\psi - \phi) + (\psi^2 - \phi^2) = p(\phi - \psi) + (\psi - \phi)$$
$$\Rightarrow\ (\psi - \phi)(r + p - 1) + (\psi^2 - \phi^2) = 0$$

Which implies

$$(\psi - \phi)(r + p - 1 + (\psi + \phi)) = 0.$$

Then either $\psi = \phi$ or $\psi + \phi = 1 - (r + p)$. Then in this case there is no two periodic nonnegative solution for Eq. (21) unless $\psi + \phi = 1 - (r + p)$.
Let us now take k to be even and see what we will get.

- If k is even then $x_n = x_{n-k}$, so we get

$$\psi = \frac{p\,\phi + \phi}{r + q\,\phi + \phi} \quad \text{and} \quad \phi = \frac{p\,\psi + \psi}{r + q\,\psi + \psi}.$$

This yields

$$\psi(r + q\,\phi + \phi) = \phi(p + 1) \tag{24}$$
$$\phi(r + q\,\psi + \psi) = \psi(p + 1) \tag{25}$$

By subtracting Eq. (25) from Eq. (24), we get the following

$$r(\psi - \phi) + (\psi - \phi)(p + 1) = 0$$
$$(\psi - \phi)(r + p + 1) = 0.$$

Then either $\psi = \phi$, or $r + p + 1 = 0$ which is impossible, since r and p are nonnegative variables. Then in this case, there exists no two periodic nonnegative solution for our Eq. (21).

From the above discussion, we have the following theorem.

Theorem 6.1 *There exists no two periodic nonnegative solution for the difference equation*

$$x_{n+1} = \frac{px_n + x_{n-k}}{r + qx_n + x_{n-k}}, \quad n \in \mathbb{N}_0,$$

unless if k is odd and $r + p < 1$.

7 Semi-cycle analysis

Here we give the definitions for the positive and negative semi-cycle of the solution of a difference equation, relative to an equilibrium point $\bar{y}$.

Definition 7.1 [9] Let $\{y_n\}_{n=-k}^{\infty}$ be a nonnegative solution of Eq. (13). A *positive semi-cycle* of a solution $\{y_n\}_{n=-k}^{\infty}$ of Eq. (13) consists of a "string" of terms $\{y_l, y_{l+1}, \ldots, y_m\}$, all greater than or equal to the equilibrium $\bar{y}$, with $l \geqslant -k$ and $m \leqslant \infty$, and such that

$$\text{either } l = -k, \text{ or } l > -k \quad \text{and} \quad y_{l-1} < \bar{y},$$

and

$$\text{either } m = \infty, \text{ or } m < \infty \quad \text{and} \quad y_{m+1} < \bar{y}.$$

Definition 7.2 [9] Let $\{y_n\}_{n=-k}^{\infty}$ be a nonnegative solution of Eq. (13). A *negative semi-cycle* of a solution $\{y_n\}_{n=-k}^{\infty}$ of Eq. (13) consists of a "string" of terms $\{y_l, y_{l+1}, \ldots, y_m\}$, all less than the equilibrium $\bar{y}$, with $l \geqslant -k$ and $m \leqslant \infty$, and such that

$$\text{either } l = -k, \text{ or } l > -k \quad \text{and} \quad y_{l-1} \geqslant \bar{y},$$

and

$$\text{either } m = \infty, \text{ or } m < \infty \quad \text{and} \quad y_{m+1} \geqslant \bar{y}.$$

Definition 7.3 A solution $\{y_n\}$ of Eq. (13) is called *non-oscillatory* if there exists $N \geqslant -k$ such that $y_n > \bar{y}$ for all $n \geqslant N$ or $y_n < \bar{y}$ for all $n \geqslant N$.
And a solution $\{y_n\}$ is called *oscillatory* if it is *not non-oscillatory*.

Theorem 7.1 [10] *Assume that* $f \in C[(0, \infty) \times (0, \infty), (0, \infty)]$ *is such that* $f(x, y)$ *is* increasing *in both arguments. Let* $\bar{x}$ *be a positive equilibrium of Eq.* (13). *Then, every oscillatory solution of Eq.* (13) *has semi-cycles of length k.*

Proof When $k = 1$, the proof is presented as theorem (1.7.3) in [10]. We just give the proof of the theorem for $k = 2$, the other cases for $k \geqslant 3$ are similar and we omit them.
Assume that $\{x_n\}$ is an oscillatory solution with three consecutive terms

$$x_{N-1} \geqslant \bar{x} \text{ , } x_N \geqslant \bar{x} \quad \text{and} \quad x_{N+1} \geqslant \bar{x}$$

with at least one of the inequalities being strict. The proof in the case of negative semi-cycle is similar and is omitted. Then by using the increasing character of $f(x, y)$ we obtain:

$$x_{N+2} = f(x_{N+1}, x_{N-1}) > f(\bar{x}, \bar{x}) = \bar{x},$$

which shows that the next term x_{N+2} also belongs to the positive semi-cycle. It follows by induction that all future terms of this solution belong to this positive semi-cycle, which is a contradiction. □

Theorem 7.2 [10] *Assume that* $f \in C[(0, \infty) \times (0, \infty), (0, \infty)]$ *is such that* $f(x, y)$ *is* increasing *in x for each fixed y, and is* decreasing *in y for each fixed x. Let* $\bar{x}$ *be a positive equilibrium of Eq.* (13)*. Then, except possibly for the first semi-cycle, every solution of Eq.* (13) *has semi-cycles of length at least* $k + 1$.

Proof When $k = 1$, the proof is presented as theorem (1.7.4) in Ref. [10]. We just give the proof of the theorem for $k = 2$, the other cases for $k \geqslant 3$ are similar and can be omitted.
Assume that $\{x_n\}$ is an oscillatory solution with three consecutive terms x_{N-1}, x_N, x_{N+1}, such that $x_{N-1} < \bar{x} < x_{N+1}$ or $x_{N-1} > \bar{x} > x_{N+1}$. We will assume that $x_{N-1} < \bar{x} < x_{N+1}$, the other case is similar and will be omitted. Then by using decreasing character of f we obtain

$$x_{N+2} = f(x_{N+1}, x_{N-1}) > f(\bar{x}, \bar{x}) = \bar{x}$$

Now, if $x_N \geqslant \bar{x}$ then the result follows. Otherwise $x_N < \bar{x}$. Hence

$$x_{N+3} = f(x_{N+2}, x_N) > f(\bar{x}, \bar{x}) = \bar{x}$$

which shows that it has at least *three* terms in the positive semi-cycle. □

Theorem 7.3 [10] *Assume that* $f \in C[(0, \infty) \times (0, \infty), (0, \infty)]$ *is such that* $f(x, y)$ *is* decreasing *in x for each fixed y, and is* increasing *in y for each fixed x. Let* $\bar{x}$ *be a positive equilibrium of Eq.* (13)*. Then, except possibly for the first semi-cycle, every solution of Eq.* (13) *has semi-cycles of length k.*

Proof When $k = 1$, the proof is presented as theorem (1.7.1) in Ref.[10]. We just give the proof of the theorem for $k = 2$, the other cases for $k \geqslant 3$ are similar and we omitted them. Let $\{x_n\}$ be a solution of Eq. (13) with at least three semi-cycles, then there exists $N \geqslant 0$ such that either $x_{N-1} < \bar{x} \leqslant x_{N+1}$ or $x_{N-1} \geqslant \bar{x} > x_{N+1}$. We will assume that $x_{N-1} < \bar{x} \leqslant x_{N+1}$, the other case is similar and will be omitted. Then by using the monotonic character of $f(x, y)$ we have

$$x_{N+2} = f(x_{N+1}, x_{N-1}) < f(\bar{x}, \bar{x}) = \bar{x},$$

and

$$x_{N+3} = f(x_{N+2}, x_N) > f(\bar{x}, \bar{x}) = \bar{x}.$$

Thus,

$$x_{N+2} < \bar{x} < x_{N+3}.$$

□

By using the Theorems 5.1–5.4 together with Theorems 7.1–7.3, it is easy to obtain the following results concerning semi-cycle analysis.

Theorem 7.4 *Assume that* $p = q$, $p+1 > r$, *and* $\{y_n\}_{n=-k}^{\infty}$ *is a nonnegative solution of Eq.* (2). *Then the following statements are true:*

(i) Except possibly for the first semi-cycle, every oscillatory solution of Eq. (2) has semi-cycles of length k.
(ii) If $(y_{-k} - \bar{y})(y_{-k+1} - \bar{y}) \cdots (y_0 - \bar{y}) > 0$, *then* $\{y_n\}_{n=-k}^{\infty}$ *is not an oscillatory solution.*

Theorem 7.5 *Assume that* $p > q$, $p+1 > r$, *and* $\{y_n\}_{n=-k}^{\infty}$ *is a nonnegative solution of Eq. (2). Then the following statements are true:*

(i) If $p \leqslant q + \frac{qr}{p}$, *then except possibly for the first semi-cycle, every oscillatory solution of Eq.* (2) *has semi-cycles of length k.*
(ii) If $q + \frac{qr}{p} < p < q + r$, *then except possibly for the first semi-cycle, every oscillatory solution of Eq.* (2) *which lies in the invariant interval* $[0, \frac{r}{p-q}]$ *has semi-cycles of length k.*
(iii) If $p > q + r$, *then every oscillatory solution of Eq.* (2) *which lies in the invariant interval* $[\frac{r}{p-q}, \frac{p}{q}]$ *has semi-cycles of length at least k.*
(iv) If $p = q + r$, *then Eq.* (2) *does not have oscillatory solutions.*

Theorem 7.6 *Assume that* $q > p$, $p+1 > r$, *and* $\{y_n\}_{n=-k}^{\infty}$ *is a nonnegative solution of Eq.* (2). *Then the following statements are true:*

(i) If $q \leqslant p + pr$, *then except possibly for the first semi-cycle, every oscillatory solution of Eq.* (2) *has semi-cycles of length k.*
(ii) If $p + pr < q < p + qr$, *then every oscillatory solution of Eq.* (2) *which lies in the invariant interval* $[0, \frac{pr}{q-p}]$ *has semi-cycles of length k.*
(iii) If $q > p + qr$, *then every oscillatory solution of Eq.* (2) *which lies in the invariant interval* $[\frac{pr}{q-p}, 1]$ *has semi-cycles of length k.*
(iv) If $q = p + qr$, *then every oscillatory solution of Eq.* (2) *has semi-cycles of length k. In particular, Eq.* (2) *does not have oscillatory solutions with* $(y_{-k} - \frac{p}{q})(y_{-k+1} - \frac{p}{q}) \cdots (y_0 - \frac{p}{q}) > 0$.

8 Analysis of global stability

Theorem 8.1 [10] *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \quad n \in \mathbb{N}_0, \tag{26}$$

where $k \in \{1, 2, \ldots\}$. Let $I = [a, b]$ be some interval of real numbers and assume that

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(u, v)$ is nondecreasing in each of its arguments u and v.
(b) The equation

$$f(y, y) = y$$

has a unique positive solution.

Then Eq. (26) has a unique equilibrium point $\bar{y} \in [a, b]$ and every solution of Eq. (26) converges to $\bar{y}$.

Proof Set

$$m_0 = a \quad \text{and} \quad M_0 = b,$$

and for $i = 1, 2, \ldots$ set

$$M_i = f(M_{i-1}, M_{i-1}) \quad \text{and} \quad m_i = f(m_{i-1}, m_{i-1}).$$

Now observe that for each $i \geqslant 0$,

$$m_0 \leqslant m_1 \leqslant \cdots \leqslant m_i \leqslant \cdots \leqslant M_i \leqslant \cdots \leqslant M_1 \leqslant M_0,$$

and

$$m_i \leqslant y_l \leqslant M_i \quad \text{for} \quad l \geqslant 2i + 1.$$

Set

$$m = \lim_{i \to \infty} m_i \quad \text{and} \quad M = \lim_{i \to \infty} M_i.$$

Then clearly

$$m \leqslant \liminf_{i \to \infty} y_i \leqslant \limsup_{i \to \infty} y_i \leqslant M,$$

and by the continuity of f,

$$m = f(m, m) \quad \text{and} \quad M = f(M, M).$$

In view of (b),

$$m = M = \bar{y},$$

from which the result follows. □

Theorem 8.2 [10] *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \quad n \in \mathbb{N}_0, \tag{27}$$

where $k \in \{1, 2, \ldots\}$. Let $I = [a, b]$ be some interval of real numbers and assume that

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) *$f(u, v)$ is* nondecreasing *in $u \in [a, b]$ for each $v \in [a, b]$ and* nonincreasing *in $v \in [a, b]$ for each $u \in [a, b]$.*
(b) *If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system*

$$m = f(m, M) \quad \text{and} \quad M = f(M, m),$$

then $m = M$.

Then Eq. (27) *has a unique equilibrium $\bar{y}$ and every solution of Eq. (27) converges to $\bar{y}$.*

Theorem 8.3 [10] *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \quad n \in \mathbb{N}_0, \tag{28}$$

where $k \in \{1, 2, \ldots\}$. Let $I = [a, b]$ be some interval of real numbers and assume that

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(u, v)$ *is* nonincreasing *in* $u \in [a, b]$ *for each* $v \in [a, b]$ *and* nondecreasing *in* $v \in [a, b]$ *for each* $u \in [a, b]$.
(b) If $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$m = f(M, m) \quad \text{and} \quad M = f(m, M),$$

then $m = M$.

Then Eq. (28) *has a unique equilibrium* $\bar{y} \in [a, b]$ *and every solution of Eq.* (28) *converges to* $\bar{y}$.

Proof Set

$$m_0 = a \quad \text{and} \quad M_0 = b$$

and for $i = 1, 2, \ldots$ set

$$m_i = f(M_{i-1}, m_{i-1}) \quad \text{and} \quad M_i = f(m_{i-1}, M_{i-1}).$$

Now observe that

$$m_0 \leqslant m_1 \leqslant \cdots \leqslant m_i \leqslant \cdots \leqslant M_i \leqslant \cdots \leqslant M_1 \leqslant M_0,$$

and

$$m_i \leqslant y_l \leqslant M_i \quad \text{for } l \geqslant 2i + 1.$$

Set

$$m = \lim_{i\to\infty} m_i \quad \text{and} \quad M = \lim_{i\to\infty} M_i.$$

Then clearly

$$m \leqslant \liminf_{i\to\infty} y_i \leqslant \limsup_{i\to\infty} y_i \leqslant M,$$

and by the continuity of f,

$$m = f(M, m) \quad \text{and} \quad M = f(m, M).$$

In view of (b) $m = M = \bar{y}$. □

Theorem 8.4 [10] *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \quad n \in \mathbb{N}_0, \tag{29}$$

where $k \in \{1, 2, \ldots\}$. *Let* $I = [a, b]$ *be some interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(u, v)$ *is nonincreasing in each of its arguments* u *and* v.
(b) If $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$m = f(M, M) \quad and \quad M = f(m, m),$$

then $m = M$.

Then Eq. (29) *has a unique equilibrium* $\bar{y} \in [a, b]$ *and every solution of Eq.* (29) *converges to* $\bar{y}$.

Proof Set

$$m_0 = a \quad \text{and} \quad M_0 = b$$

and for $i = 1, 2, \ldots$ set

$$m_i = f(M_{i-1}, M_{i-1}) \quad \text{and} \quad M_i = f(m_{i-1}, m_{i-1}).$$

Now observe that

$$m_0 \leqslant m_1 \leqslant \cdots \leqslant m_i \leqslant \cdots \leqslant M_i \leqslant \cdots \leqslant M_1 \leqslant M_0,$$

and

$$m_i \leqslant y_l \leqslant M_i \quad \text{for } l \geqslant 2i + 1.$$

Set

$$m = \lim_{i \to \infty} m_i \quad \text{and} \quad M = \lim_{i \to \infty} M_i.$$

Then clearly

$$m \leqslant \liminf_{i \to \infty} y_i \leqslant \limsup_{i \to \infty} y_i \leqslant M,$$

and by the continuity of f,

$$m = f(M, M) \quad \text{and} \quad M = f(m, m),$$

and so $m = M = \bar{y}$. □

8.1 Global stability of the zero equilibrium point

Theorem 8.5 *Assume that $p + 1 < r$, then the zero equilibrium of Eq.* (2) *is* globally asymptotically stable.

Proof Consider the function

$$f(x, y) = \frac{px + y}{r + qx + y}.$$

By using Theorem 5.1 (ii, iii, iv), note that $f(x, y)$ is nondecreasing in each of its arguments in the intervals

$$I = \begin{cases} \left[0, \frac{r}{p-q}\right] & \text{where } p > q \\ \left[0, \frac{pr}{q-p}\right] & \text{where } p < q \\ [0, \infty] & \text{where } p = q \end{cases}$$

Now, let $(m, M) \in I \times I$ is a solution of the system,

$$m = f(M, M) \quad \text{and} \quad M = f(m, m),$$

then

$$m = \frac{pM + M}{r + qM + M} \quad \text{and} \quad M = \frac{pm + m}{r + qm + m},$$
$$rm + qmM + mM = pM + M$$
$$rM + qmM + mM = pm + m.$$

This yields $(M - m)(r + p + 1) = 0$, then the only solution is $m = M$.
Then both conditions of Theorem 8.1 hold, therefore, every solution of Eq. (2) converges to $\bar{y}$ in the interval I.

As $\bar{y}$ is locally asymptotically stable under the condition $p + 1 < r$ as shown in Sect. 3.1, then by Definition 3.1, it is *globally asymptotically stable* on the interval I. □

8.2 Global stability of the positive equilibrium point

We studied the local stability of the *positive equilibrium point* under the following conditions:

$$p + 1 > r, \tag{30}$$
$$q + r < 3p + 1 + qr + pq. \tag{31}$$

Theorem 8.6 *Assume that $p > q$, (30) and (31) hold, then:*

(i) If $p \leqslant q + \frac{qr}{p}$, then the unique positive equilibrium of Eq. (2) on the interval $[0, \frac{p}{q}]$ *is* globally asymptotically stable.

(ii) If $q + \frac{qr}{p} < p < q + r$, then the unique positive equilibrium of Eq. (2) on the interval $[0, \frac{r}{p-q}]$ *is* globally asymptotically stable.

(iii) If $q + r < p \leqslant 1 + r$, then the unique positive equilibrium of Eq. (2) on the interval $[\frac{r}{p-q}, \frac{p}{q}]$ *is* globally asymptotically stable.

Proof (i) Consider the function $f(x, y) = \frac{px+y}{r+qx+y}$, by using Theorem 5.1 (iii), note that $f(x, y)$ is nondecreasing in each of its arguments in the interval $[0, \frac{r}{p-q}]$. By the assumption $q < p \leqslant q + \frac{qr}{p}$, implies that $\frac{p}{q} \leqslant \frac{r}{p-q}$, so $f(x, y)$ is nondecreasing in each of its arguments in the interval $[0, \frac{p}{q}] \subset [0, \frac{r}{p-q}]$.

Now, let $(m, M) \in [0, \frac{p}{q}] \times [0, \frac{p}{q}]$ be a solution of the system

$$m = f(M, M) \quad \text{and} \quad M = f(m, m),$$

then

$$m = \frac{pM + M}{r + qM + M} \quad \text{and} \quad M = \frac{pm + m}{r + qm + m},$$
$$rm + qmM + mM = pM + M$$
$$rM + qmM + mM = pm + m.$$

This yields $(M - m)(r + p + 1) = 0$, then the only solution is $m = M$. Therefore, both conditions of Theorem 8.1 hold, then every solution of Eq. (2) converges to $\bar{y}$ in the interval $[0, \frac{p}{q}]$.
As $\bar{y}$ is locally asymptotically stable under the conditions (30) and (31), then it is *globally asymptotically stable* on the interval $[0, \frac{p}{q}]$ by the Definition 3.1.

(ii) As we have seen above, by Theorem 5.1 (iii) the function $f(x, y)$ is nondecreasing in each of its arguments in the interval $[0, \frac{r}{p-q}]$.
Let $(m, M) \in [0, \frac{r}{p-q}] \times [0, \frac{r}{p-q}]$ is a solution of the system

$$m = f(M, M) \quad \text{and} \quad M = f(m, m),$$

then

$$m = \frac{pM + M}{r + qM + M} \quad \text{and} \quad M = \frac{pm + m}{r + qm + m},$$
$$rm + qmM + mM = pM + M$$
$$rM + qmM + mM = pm + m.$$

This yields $(M - m)(r + p + 1) = 0$, then the only solution is $m = M$. Therefore, both conditions of Theorem 8.1 hold, then every solution of Eq. (2) converges to $\bar{y}$ in the interval $[0, \frac{r}{p-q}]$.

Since $\bar{y}$ is locally asymptotically stable under the conditions (30) and (31), then it is *globally asymptotically stable* on the interval $[0, \frac{r}{p-q}]$ by the Definition 3.1.

(iii) By Theorem 5.1 (iii) the function $f(x, y)$ is nondecreasing in x and nonincreasing in y in the interval $[\frac{r}{p-q}, \infty)$. By the assumption $q + \frac{qr}{p} < p < q + r$, implies $\frac{qr}{p} < p - q < r$ then $1 < \frac{r}{p-q} < \frac{p}{q}$, i.e. the interval $[\frac{r}{p-q}, \frac{p}{q}] \subset [\frac{r}{p-q}, \infty)$. Let $(m, M) \in [\frac{r}{p-q}, \frac{p}{q}] \times [\frac{r}{p-q}, \frac{p}{q}]$ be a solution of the system

$$m = f(m, M) \quad \text{and} \quad M = f(M, m),$$

then

$$m = \frac{pm + M}{r + qm + M} \quad \text{and} \quad M = \frac{pM + m}{r + qM + m},$$
$$rm + qm^2 + mM = pm + M$$
$$rM + qM^2 + mM = pM + m.$$

This yields

$$(M - m)[r + q(M + m) - p + 1] = 0$$

Then either $m = M$ or $M + m = \frac{p-(1+r)}{q}$ which contradicts the assumption $p < 1 + r$. So the only solution is $m = M$.

Then both conditions of Theorem 8.2 hold, therefore every solution of Eq. (2) converges to $\bar{y}$ in the interval $[\frac{r}{p-q}, \frac{p}{q}]$.

By Definition 3.1, $\bar{y}$ is locally asymptotically stable under the conditions (30) and (31), then it is *globally asymptotically stable* on the interval $[\frac{r}{p-q}, \frac{p}{q}]$. □

Theorem 8.7 *Assume that $p < q$, (30) and (31) hold, then:*

(i) If $q \leqslant p + pr$, then the unique positive equilibrium of Eq. (2) on the interval $[0, \frac{p}{q}]$ *is* globally asymptotically stable.

(ii) If $p + pr < q < p + qr$, then the unique positive equilibrium of Eq. (2) on the interval $[0, \frac{pr}{q-p}]$ *is* globally asymptotically stable.

(iii) If $q > p + qr$, then the unique positive equilibrium of Eq. (2) on the interval $[\frac{pr}{q-p}, 1]$ *is* globally asymptotically stable.

Proof (i) Consider the function

$$f(x, y) = \frac{px + y}{r + qx + y}.$$

By using Theorem 5.1 (iv), note that $f(x, y)$ is nondecreasing in each of its arguments in the interval $[0, \frac{pr}{q-p}]$.

By the assumption $p < q \leqslant p + pr$, implies that $\frac{pr}{q-p} \geqslant 1 > \frac{p}{q}$, so $f(x, y)$ is

nondecreasing in each of its arguments in the interval $[0, \frac{p}{q}] \subset [0, \frac{pr}{q-p}]$.
Now, let $(m, M) \in [0, \frac{p}{q}] \times [0, \frac{p}{q}]$ be a solution of the system

$$m = f(M, M) \quad \text{and} \quad M = f(m, m),$$

then

$$m = \frac{pM + M}{r + qM + M} \quad \text{and} \quad M = \frac{pm + m}{r + qm + m},$$
$$rm + qmM + mM = pM + M$$
$$rM + qmM + mM = pm + m.$$

This yields $(M - m)(r + p + 1) = 0$, then the only solution is $m = M$.
Then both conditions of Theorem 8.1 hold, therefore, every solution of Eq. (2) converges to $\bar{y}$ in the interval $[0, \frac{p}{q}]$.
As $\bar{y}$ is locally asymptotically stable under the conditions (30) and (31), then it is *globally asymptotically stable* on the interval $[0, \frac{p}{q}]$ by the Definition 3.1.

(ii) As we see above, by Theorem 5.1 (iv) the function $f(x, y)$ is nondecreasing in each of its arguments in the interval $[0, \frac{pr}{q-p}]$.
Let $(m, M) \in [0, \frac{pr}{q-p}] \times [0, \frac{pr}{q-p}]$ be a solution of the system

$$m = f(M, M) \quad \text{and} \quad M = f(m, m),$$

then

$$m = \frac{pM + M}{r + qM + M} \quad \text{and} \quad M = \frac{pm + m}{r + qm + m},$$
$$rm + qmM + mM = pM + M$$
$$rM + qmM + mM = pm + m.$$

This yields $(M - m)(r + p + 1) = 0$, then the only solution is $m = M$.
Therefore, both conditions of Theorem 8.1 hold, then every solution of Eq. (2) converges to $\bar{y}$ in the interval $[0, \frac{pr}{q-p}]$.
Since $\bar{y}$ is locally asymptotically stable under the conditions (30) and (31), then it is *globally asymptotically stable* on the interval $[0, \frac{pr}{q-p}]$ by the Definition 3.1.

(iii) By Theorem 5.1 (iv) the function $f(x, y)$ is nonincreasing in x and nondecreasing in y in the interval $[\frac{pr}{q-p}, 1] \subset [\frac{pr}{q-p}, \infty)$.
Let $(m, M) \in [\frac{pr}{q-p}, 1] \times [\frac{pr}{q-p}, 1]$ be a solution of the system

$$m = f(M, M) \quad \text{and} \quad M = f(m, m),$$

then

$$m = \frac{pM + M}{r + qM + M} \quad \text{and} \quad M = \frac{pm + m}{r + qm + m},$$
$$rm + qmM + mM = pM + M$$
$$rM + qmM + mM = pm + m.$$

This yields $(M - m)(r + p + 1) = 0$, then the only solution is $m = M$.
Therefore, both conditions of Theorem 8.3 hold, then every solution of Eq. (2) converges to $\bar{y}$ in the interval $[\frac{pr}{q-p}, 1]$.
By Definition 3.1, $\bar{y}$ is locally asymptotically stable under the conditions (30) and (31), then it is *globally asymptotically stable* on the interval $[\frac{pr}{q-p}, 1]$. □

Theorem 8.8 *Assume that* $p = q$, (30) *and* (31) *hold, then the unique positive equilibrium of Eq.* (2) *is* globally asymptotically stable.

Proof Consider the function

$$f(x, y) = \frac{px + y}{r + px + y}.$$

By using the derivative of $f(x, y)$ with respect to x and y, then $f(x, y)$ is nonincreasing in each of its arguments.

Now, let $(m, M) \in [0, \infty] \times [0, \infty]$ be a solution of the system

$$m = f(M, M) \quad \text{and} \quad M = f(m, m),$$

then

$$m = \frac{pM + M}{r + pM + M} \quad \text{and} \quad M = \frac{pm + m}{r + pm + m},$$
$$rm + qmM + mM = pM + M$$
$$rM + qmM + mM = pm + m.$$

This yields $(M - m)(r + p + 1) = 0$, then the only solution is $m = M$.

Therefore, both conditions of Theorem 8.4 hold, then every solution of Eq. (2) converges to $\bar{y}$.

As $\bar{y}$ is locally asymptotically stable under the conditions (30) and (31), then it is *globally asymptotically stable* by the Definition 3.1. □

9 Numerical approach

In order to support our theoretical discussions and to illustrate the main results, we consider several interesting numerical examples, those represent different types of qualitative behavior of solutions to nonlinear difference Eq. (2).

Example 1 Assume that Eq. (2) holds, take $k = 4$, $p = 0.5$, $q = 3$ and $r = 1$. So the equation will be reduced to the following:

$$y_{n+1} = \frac{0.5\, y_n + y_{n-4}}{1 + 3\, y_n + y_{n-4}}.$$

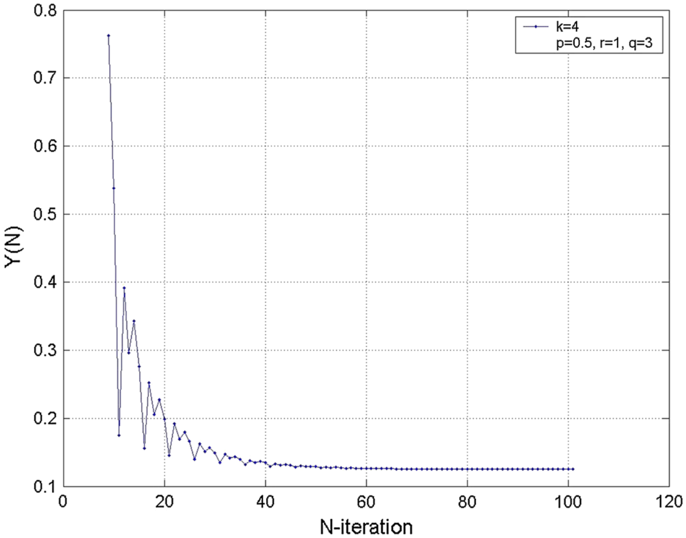

Fig. 1 The behavior of the equilibrium point of equation $y_{n+1} = \frac{0.5\, y_n + y_{n-4}}{1+3\, y_n + y_{n-4}}$

In this case, the values of p, q, and r satisfy the conditions of local stability (30) and (31), so by theory the value of $\bar{y} = \frac{p+1-r}{q+1} = \frac{0.5+1-1}{3+1} = 0.125$.
We assume the initial points $y_0 = 1$, $y_1 = 8$, $y_2 = 5$, $y_3 = 8$ and $y_4 = 3$, and the output of the numerical illustrated in Fig. 1, as expected as the theoretical results.

Example 2 In this example, let the parameters $p = 2$, $q = 5$ and $r = 4$, and $k = 3$. So the Eq. (2) will be reduced to the following:

$$y_{n+1} = \frac{2\, y_n + y_{n-3}}{4 + 5\, y_n + y_{n-3}}.$$

We assume the initial points $y_0 = 3$, $y_1 = 1$, $y_2 = 2$ and $y_3 = 0.5$.

In this case, the values of p, q, and r satisfy the condition of global stability of the zero equilibrium point ($p + 1 \leqslant r$), and this is clear numerically as illustrated in Fig. 2.

Example 3 Assume that Eq. (2) holds, take $k = 4$, $p = 0.25$, $q = 20$ and $r = 0.1$. So the equation will be reduced to the following:

$$y_{n+1} = \frac{0.25\, y_n + y_{n-4}}{0.1 + 20\, y_n + y_{n-4}}.$$

We assume the initial points $y_0 = 2$, $y_1 = 3$, $y_2 = 1$, $y_3 = 4$ and $y_4 = 3$.

In this case, the values of p, q, and r contradict the condition of local stability (31) such that

$$q + r > 3p + 1 + qr + pq$$

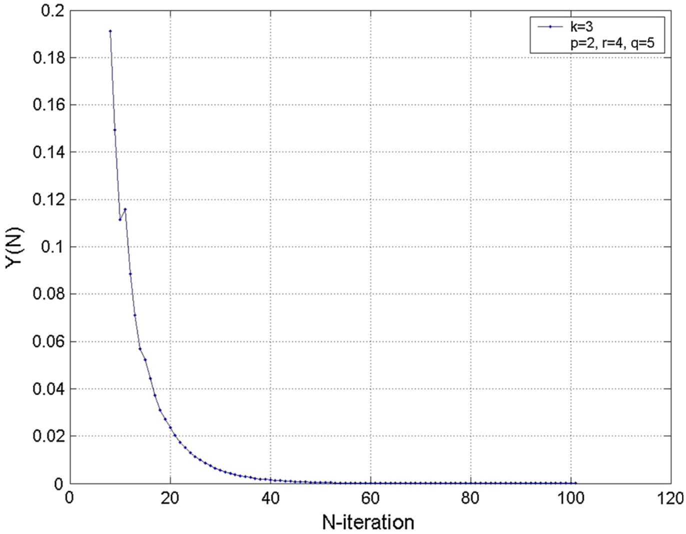

Fig. 2 The behavior of the zero equilibrium point of equation $y_{n+1} = \frac{2\,y_n + y_{n-3}}{4+5\,y_n + y_{n-3}}$

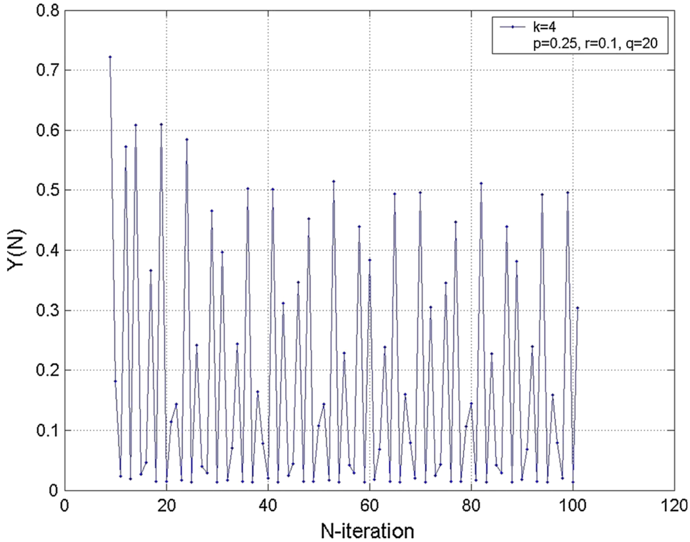

Fig. 3 The behavior of unstable solution of equation $y_{n+1} = \frac{0.25\,y_n + y_{n-4}}{0.1+20\,y_n + y_{n-4}}$

$$20 + 0.1 > 3 \times 0.25 + 1 + 20 \times 0.1 + 0.25 \times 20.$$

So we see from the Fig. 3 that there is no any stable solution for this case.

So, all what we have to say now is that our theoretical discussion was satisfied with the data we get from our numerical discussion. So we have correctly illustrated our study for the Eq. (2).

References

1. Das, S.E., Bayram, M.: Dynamics of a higher-order nonlinear rational difference equation. Int. J. Phys. Sci. **6**(12), 2950–2957 (2011)
2. Douraki, M., et al.: Dynamics of the difference equation $x_{n+1} = \frac{x_n + p x_{n-k}}{x_n + q}$. Comput. Math. Appl. **56**(1), 186–198 (2008)
3. El-Moneam, M.A., Zayed, E.M.E.: On the dynamics of the nonlinear rational difference equation $x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + \frac{bx_{n-k}}{dx_{n-k} - ex_{n-l}}$. J. Egypt. Math. Soc. **23**(3), 494–499 (2015)
4. Elaydi, Saber N.: An Introduction to Difference Equations, 3rd edn. Springer, New York (2005)
5. Grove, E.A., Ladas, G.: Periodicities in Nonlinear Difference Equations, volume Four of Advances in Discrete Mathematics and Applications, 1st edn. Chapman & Hall/CRC, Boca Raton (2005)
6. Hu, L.-X., et al.: Global asymptotical stability of a second order rational difference equation. Comput. Math. Appl. **54**(9–10), 1260–1266 (2007)
7. Hu, L.-X., et al.: Global asymptotic stability of a second order rational difference equation. J. Differ. Equ. Appl. **14**(8), 779–797 (2008)
8. Khaliq, Abdul, Elsayed, E.M.: Qualitative properties of difference equation of order six. $\sum$ Mathematics **4**(2), 24 (2016)
9. Kocic, V.L., Ladas, G.: Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, volume 256 of Mathematics and Its Applications. Kluwer Academic Publishers, The Netherlands (1993)
10. Kulenović, M.R.S., Ladas, G.: Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures. Chapman and Hall/CRC, Florida (2002)
11. Mazrooei-Sebdani, R., Dehghan, M.: Dynamics of a non-linear difference equation. Appl. Math. Comput. **178**(2), 250–261 (2006)
12. Saleh, M., Aloqeili, M.: On the rational difference equation $y_{n+1} = A + \frac{y_{n-k}}{y_n}$. Appl. Math. Comput. **171**(2), 862–869 (2005)
13. Saleh, M., et al.: On the dynamics of a rational difference equation $x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}$. Chaos Solitons Fractals **96**, 76–84 (2017)
14. Saleh, M., Farhat, A.: Global asymptotic stability of the higher order equation $x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}}$. Appl. Math. Comput. 1–14 (2016)
15. Wang, C., et al.: On the solution for a system of two rational difference equations. Comput. Anal. Appl. **20**(1), 175–186 (2016)
16. Zayed, E.M.E.: Dynamics of the nonlinear rational difference equation $x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}}$. Eur. J. Pure Appl. Math. **3**(2), 254–268 (2010)
17. Zayed, E.M.E., El-Moneam, M.A.: On the rational recursive sequence $x_{n+1} = \gamma x_{n-k} + \frac{Ax_n + Bx_{n-k}}{Cx_n - Dx_{n-k}}$. Bull. Iran. Math. Soc. **36**(1), 103–115 (2010)