GEOMETRIC APPROACHES AND BIFURCATIONS IN THE DICHOTOMOUS DECISION MODEL

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Abstract. Resorting to the dichotomous decision model, where individuals can make alternative decisions, we study two geometric approaches to construct all possible decisions tilings. Each decision tiling indicates the way the Nash equilibria co-exist and change with the relative decision preferences of the individuals. We find the Nash domains for the pure and mixed strategies and characterize the space of all parameters where the pure Nash equilibria are either cohesive or disparate. We show how the coordinates of the influence matrix together with the total number of individuals affect significantly the occurrence of bifurcations with and without overlaps between the pure strategies.

1. Introduction

The dichotomous decision model is a recent game theoretical model introduced by Mousa et al. in 2014 (see [11]). In this model, there are just two possible decisions that individuals can make. For instance, they have to choose between yes or no, i.e. $d \in D = \{Yes, No\}$. The individuals will have to make decisions according to their preferences. The preferences have the interesting feature of taking into account not only how much the individuals like or dislike a certain decision but also the other individuals' decisions. This decision model has a wide applications in real life and can be used to understand better the social interaction (see [10, 15]), tourism industry (see [5, 6]) and economical and political revolutions (see [3, 9, 14]).

The dichotomous decision model is a modified version of the game theoretical model introduced by Pinto et al. (see [2]) who developed a psychological game model for reasoned action theories inspired by the works of J. Cownley and M. Wooders (see [7]). They studied the way saturation, boredom and frustration can lead to desperate strategies (if the individuals of same group will make different decisions), and no saturation situations can lead to cohesive strategies (if all the individuals belonging to a same group will make the same decision). Ajzen (see [1]) and Baker (see [4]) predict the way individuals turn intentions into behaviors and this prediction is the main goal in Planned Behavior or Reasoned Action theories.

Mousa et al. [15] show that groups are formed by individuals with the same utility, and a group is cohesive if every individual has a gain in his utility when other individuals of the same group make the same decision as his. Furthermore, they show that individuals in a same group can make

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different decisions at certain Nash equilibria. In a dynamical version of the decision model (see [11]), the authors exhibit solutions that are periodic attracting cycles and so the individuals can keep changing the probabilities that they use to make a decision or another around some thresholds. These thresholds show the appearance of hysteretic-like behavior in the decision models. As in dynamics [8, 16], small changes in the parameters might imply the appearance and disappearance of the pure Nash equilibria.

The dichotomous decision model has been extended to a general model in [8], and other future extension formulation for the decision model would be to include some kind of stochastic pattern in the model parameters. A recent research articles that handel a stochastic decision problems for individuals introduced by Mousa et al. (see [12, 13]).

In this paper, we study two geometric approaches to construct all possible Nash eqilibria for the decisions tilings. We characterize the space of all parameters for the dichotomous decision model where the pure and mixed strategies are Nash equilibria and we find the corresponding Nash domains. We will see how the coordinates of the influence matrix together with the total number of individuals encode all the relevant information for the existence of Nash equilibria strategies. The existence of these equilibria are related also to size effect of the relative decision preferences for the individuals. The two approaches rises in making 289 different combinatorial classes of decision tilings by capturing the information that rises from the crowding type of individuals, reflecting the complexity of the yes-no decision model (see [10]).

This paper is organized as follow. In Section 2 we review the dichotomous decision model and some main results introduced in [11]. In Section 3 we study two different strategic approaches to construct geometrically all possible tilings and determine the Nash domains for the pure and mixed strategies. We conclude in Section 5.

2. The Dichotomous Decision Model

In this section, we review the *dichotomous decision model* introduced in [11] with some main results. In section 2.1 we introduce the decision model. In section 2.2 we study the pure Nash equilibria and in section 2.3 we study the mixed Nash equilibria.

2.1. **Model set up.** The model has two types $\mathbf{T} = \{t_1, t_2\}$ of individuals. Let $I_1 = \{1, \ldots, n_1\}$ be the set of all individuals with type t_1, and let $I_2 = \{1, \ldots, n_2\}$ be the set of all individuals with type t_2. Let $I = I_1 \sqcup I_2$ be the disjoint union. The individual $i \in \mathbf{I}$ has to make one decision $d \in \mathbf{D} = \{Y, N\}$[1].

Let $\mathcal{L}$ be the *preference decision matrix* whose *coordinates* ω_p^d indicate how much an individual with type t_p likes or dislikes, to make decision $d \in \mathbf{D}$

$$\mathcal{L} = \begin{pmatrix} \omega_1^Y & \omega_1^N \\ \omega_2^Y & \omega_2^N \end{pmatrix}.$$

[1]Similarly, we can consider that there is a single individual with type t_p that has to make n_p decisions, or we can also consider a mixed model using these two possibilities.

The coordinates of the preference decision matrix indicates for each type of individuals the decision that the individuals prefer, i.e. the taste type of the individuals (see [10, 11, 15]).

Let $\mathcal{N}_d$ be the *preference neighbors matrix* whose *coordinates* α^d_{pq} indicate how much an individual with type t_p who decides d likes or dislikes that an individual with type t_q also makes decision d

$$\mathcal{N}_d = \begin{pmatrix} \alpha^d_{11} & \alpha^d_{12} \\ \alpha^d_{21} & \alpha^d_{22} \end{pmatrix}.$$

The coordinates of the preference neighbors matrix indicate, for each type of individuals whose decision is d, whom they prefer, or do not prefer, to be with in each decision, i.e. the crowding type of the individuals (see [5, 7, 11]).

Definition 2.1 ([11]). *The (pure) decision of the individuals is a (pure) strategy map $S : \mathbf{I} \to \mathbf{D}$ that associates to each individual $i \in \mathbf{I}$ its decision $S(i) \in \mathbf{D}$.*

Let $\mathbf{S}$ be the space of all strategies S. For a given a strategy $S \in \mathbf{S}$, let $\mathcal{O}_S$ be the *strategic decision matrix* whose coordinates $l^d_p = l^d_p(S)$ indicate the number of individuals with type t_p, who make decision d

$$\mathcal{O}_S = \begin{pmatrix} l^Y_1 & l^N_1 \\ l^Y_2 & l^N_2 \end{pmatrix}.$$

Definition 2.2 ([11]). *Let $S \in \mathbf{S}$. The strategic decision vector associated to a strategy S is the vector*

$$(l_1, l_2) = (l^y_1(S), l^y_2(S)),$$

where l_1 (resp. $n_1 - l_1$) is the number of individuals with type t_1 who make the decision Y (resp. N), and l_2 (resp. $n_2 - l_2$) is the number of individuals with type t_2 who make the decision Y (resp. N). Furthermore, the set $\mathbf{O}$ of all possible strategic decision vectors is defined by

$$\mathbf{O} = \{0, \dots, n_1\} \times \{0, \dots, n_2\} .$$

The *utility function* $U_1 : \mathbf{D} \times \mathbf{O} \to \mathbb{R}$ of an individual with type t_1 is defined by

$$\begin{aligned} U_1(Y; l_1, l_2) &= \omega^Y_1 + \alpha^Y_{11}(l_1 - 1) + \alpha^Y_{12} l_2; \\ U_1(N; l_1, l_2) &= \omega^N_1 + \alpha^N_{11}(n_1 - l_1 - 1) + \alpha^N_{12}(n_2 - l_2) \end{aligned}$$

and the *utility function* $U_2 : \mathbf{D} \times \mathbf{O} \to \mathbb{R}$ of an individual with type t_2 is defined by

$$\begin{aligned} U_2(Y; l_1, l_2) &= \omega^Y_2 + \alpha^Y_{22}(l_2 - 1) + \alpha^Y_{21} l_1; \\ U_2(N; l_1, l_2) &= \omega^N_2 + \alpha^N_{22}(n_2 - l_2 - 1) + \alpha^N_{21}(n_1 - l_1). \end{aligned}$$

Given a strategy $S \in \mathbf{S}$, the *utility* $U_i(S)$ of an individual i with type $t_{p(i)}$ is given by $U_{p(i)}(S(i); l^y_1(S), l^y_2(S))$.

Definition 2.3 ([11]). *The horizontal relative decision preference of the individuals with type t_1 is define by*

$$x = \omega^Y_1 - \omega^N_1$$

and the vertical relative decision preference of the individuals with type t_2 is defined by

$$y = \omega_2^Y - \omega_2^N \ .$$

If $x > 0$, the individuals with type t_1 prefer to decide Y, without taking into account the influence of the others. If $x = 0$, the individuals with type t_1 are indifferent to decide Y or N, without taking into account the influence of the others. If $x < 0$, the individuals with type t_1 prefer to decide N, without taking into account the influence of the others.

Definition 2.4 ([11])**.** *For $i, j \in \{1, 2\}$, we define the coordinates of the influence matrix by*

$$A_{ij} = \alpha_{ij}^Y + \alpha_{ij}^N \ .$$

If $A_{ij} > 0$, the individuals with type t_j have a *positive influence* over the utility of the individuals with type t_i. If $A_{ij} = 0$, the individuals with type t_j are *indifferent* for the utility of the individuals with type t_i. If $A_{ij} < 0$, the individuals with type t_j have a *negative influence* over the utility of the individuals with type t_i.

Definition 2.5 ([11])**.** *A strategy $S^* : \mathbf{I} \to \mathbf{D}$ is a* Nash equilibrium *if, for every individual $i \in \mathbf{I}$ and for every strategy S, with the property that $S^*(j) = S(j)$ for every individual $j \in I \setminus \{i\}$, we have*

$$U_i(S^*) \geq U_i(S) \ .$$

Furthermore, the Nash equilibrium domain $E(S)$ of a strategy S is the set of all pairs (x, y) for which S is a Nash Equilibrium.

2.2. **Pure Nash equilibria.** The *pure* strategies are either *cohesive* strategies or *disparate* strategies.

Definition 2.6 ([11])**.** *A cohesive strategy is a pure strategy in which all individuals with the same type prefer to make the same decision. A disparate strategy is a pure strategy that is not cohesive, i.e. a pure strategy in which there are some individuals with the same type who prefer to make different decisions.*

Lemma 2.7 ([11])**.** *The* Nash domain $\boldsymbol{N}(Y, Y)$ *of the cohesive strategy (Y, Y) is given by*

$$\boldsymbol{N}(Y, Y) = \{(x, y) : x \geq H(Y, Y) \ \ \text{and} \ \ y \geq V(Y, Y)\} \ ,$$

where the horizontal $H(Y, Y)$ *and* vertical $V(Y, Y)$ strategic thresholds *of the (Y, Y) strategy are given by*

$$H(Y, Y) = -\alpha_{11}^Y (n_1 - 1) - \alpha_{12}^Y n_2 \ \ \text{and} \ \ V(Y, Y) = -\alpha_{22}^Y (n_2 - 1) - \alpha_{21}^Y n_1 \ .$$

Hence, the cohesive strategy (Y, Y) is a *Nash equilibrium* if, and only if, $(x, y) \in \mathbf{N}(Y, Y)$. Moreover, the *Nash domain* $\mathbf{N}(Y, Y)$ is a right-upper quadrant in the xy-plane (see Figure 1).

Lemma 2.8 ([11])**.** *The* Nash domain $\boldsymbol{N}(Y, N)$ *of the cohesive strategy (Y, N) is given by*

$$\boldsymbol{N}(Y, N) = \{(x, y) : x \geq H(Y, N) \ \ \text{and} \ \ y \leq V(Y, N)\} \ ,$$

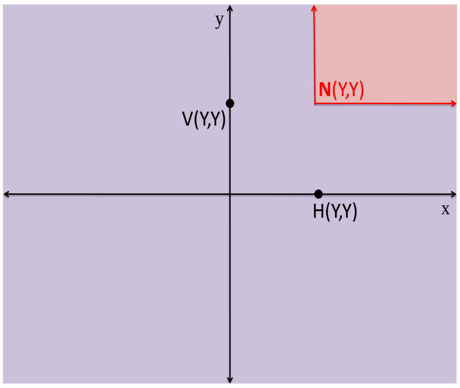

FIGURE 1. Cohesive Nash equilibria domain $\mathbf{N}(Y,Y)$.

where the horizontal $H(Y,N)$ *and* vertical $V(Y,N)$ strategic thresholds *of the* (Y,N) *strategy are given by*

$$H(Y,N) = -\alpha_{11}^{Y}(n_1-1) + \alpha_{12}^{N}n_2 \quad \textit{and} \quad V(Y,N) = \alpha_{22}^{N}(n_2-1) - \alpha_{21}^{Y}n_1 \ .$$

Hence, the cohesive strategy (Y,N) is a *Nash equilibrium* if, and only if, $(x,y) \in \mathbf{N}(Y,N)$. Moreover, the *Nash domain* $\mathbf{N}(Y,N)$ is a right-lower quadrant in the xy-plane (see Figure 2).

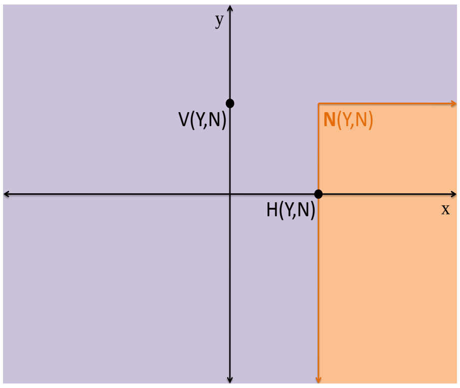

FIGURE 2. Cohesive Nash equilibria domain $\mathbf{N}(Y,N)$.

Lemma 2.9 ([11])**.** *The* Nash domain $\boldsymbol{N}(N,Y)$ *of the cohesive strategy* (N,Y) *is given by*

$$\boldsymbol{N}(N,Y) = \{(x,y) : x \leq H(N,Y) \quad \textit{and} \quad y \geq V(N,Y)\} \ ,$$

where the horizontal $H(N,Y)$ *and* vertical $V(N,Y)$ strategic thresholds *of the* (N,Y) *strategy are*

$$H(N,Y) = \alpha_{11}^{N}(n_1-1) - \alpha_{12}^{Y}n_2 \quad \textit{and} \quad V(N,Y) = -\alpha_{22}^{Y}(n_2-1) + \alpha_{21}^{N}n_1 \ .$$

Hence, the cohesive strategy (N,Y) is a *Nash equilibrium* if, and only if, $(x,y) \in \mathbf{N}(N,Y)$. Moreover, the *Nash domain* $\mathbf{N}(N,Y)$ is a left-upper quadrant in the xy-plane (see Figure 3).

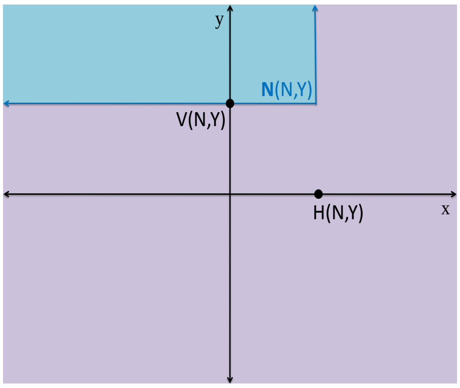

FIGURE 3. Cohesive Nash equilibria domain $\mathbf{N}(N, Y)$.

Lemma 2.10 ([11]). *The* Nash domain $\boldsymbol{N}(N, N)$ *of the cohesive strategy* (N, N) *is given by*

$$\boldsymbol{N}(N, N) = \{(x, y) : x \leq H(N, N) \ \ \textit{and} \ \ y \leq V(N, N)\} \ ,$$

where the horizontal $H(N, N)$ *and* vertical $V(N, N)$ *strategic thresholds of the* (N, N) *strategy are*

$$H(N, N) = \alpha_{11}^{N}(n_1 - 1) + \alpha_{12}^{N} n_2 \ \ \textit{and} \ \ V(N, N) = \alpha_{22}^{N}(n_2 - 1) + \alpha_{21}^{N} n_1 \ .$$

Hence, the cohesive strategy (N, N) is a *Nash equilibrium* if, and only if, $(x, y) \in \mathbf{N}(N, N)$. Moreover, the *Nash domain* $\mathbf{N}(N, N)$ is a left-lower quadrant in the xy-plane (see Figure 4).

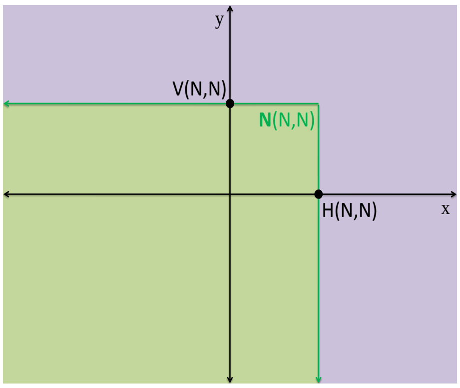

FIGURE 4. Cohesive Nash equilibria domain $\mathbf{N}(N, N)$.

2.3. **Mixed Nash equilibria.** Let $I = I_1 \sqcup I_2$ be the disjoint union. We describe the (mixed) decision of the individuals by a *(mixed) strategy map* $S : \mathbf{I} \to [\mathbf{0}, \mathbf{1}]$ that associates to each individual $i \in \mathbf{I_1}$ the probability $p_i = S(i)$ to decide $Y \in \mathbf{D}$ and to each individual $j \in \mathbf{I_2}$ the probability $q_j = S(j)$ to decide $Y \in \mathbf{D}$. Hence, each individual $i \in \mathbf{I_1}$ decides $N \in \mathbf{D}$ with probability $1 - p_i = 1 - S(i)$ and each individual $j \in \mathbf{I_2}$ decides $N \in \mathbf{D}$ with probability $1 - q_j = 1 - S(j)$. We assume that the decisions of the individuals are independent.

Define $P = \sum_{i=1}^{n_1} p_i$, $Q = \sum_{j=1}^{n_2} q_j$, $P_i = P - p_i$ and $Q_j = Q - q_j$. For every individual $i \in \mathbf{I_1}$, the *Y-fitness function* $f_{Y,1} : [0,1] \times [0,n_1] \times [0,n_2] \to \mathbb{R}^+$ is given by

$$f_{Y,1}(p_i; P, Q) = \omega_1^Y + \alpha_{11}^Y P_i + \alpha_{12}^Y Q \ ;$$

and the *N-fitness function* $f_{N,1} : [0,1] \times [0,n_1] \times [0,n_2] \to \mathbb{R}^+$ is given by

$$f_{N,1}(p_i; P, Q) = \omega_1^N + \alpha_{11}^N (n_1 - 1 - P_i) + \alpha_{12}^N (n_2 - Q) \ .$$

For every individual $j \in \mathbf{I_2}$, the *Y-fitness function* $f_{Y,2} : [0,1] \times [0,n_1] \times [0,n_2] \to \mathbb{R}^+$ is given by

$$f_{Y,2}(q_j; P, Q) = \omega_2^Y + \alpha_{22}^Y Q_j + \alpha_{21}^Y P \ ;$$

and the *N-fitness function* $f_{N,2} : [0,1] \times [0,n_1] \times [0,n_2] \to \mathbb{R}^+$ is given by

$$f_{N,2}(q_j; P, Q) = \omega_2^N + \alpha_{22}^N (n_2 - 1 - Q_j) + \alpha_{21}^N (n_1 - P) \ .$$

Lemma 2.11 ([11])**.** *Let $S : \mathbf{I} \to [\mathbf{0}, \mathbf{1}]$ be a mixed strategy. For every individual $i \in \mathbf{I_1}$, the utility function $U_1 : [0,1] \times [0,n_1] \times [0,n_2] \to \mathbb{R}^+$ is given by*

$$U_1(p_i; P, Q) = p_i \ f_{Y,1}(p_i; P, Q) + (1 - p_i) \ f_{N,1}(p_i; P, Q) \ .$$

For every individual $j \in \mathbf{I_2}$, the utility function $U_2 : [0,1] \times [0,n_1] \times [0,n_2] \to \mathbb{R}^+$ is given by

$$U_2(q_j, P, Q) = q_j \ f_{Y,2}(q_j; P, Q) + (1 - q_j) \ f_{N,2}(q_j; P, Q) \ .$$

Definition 2.12 ([11])**.** *A strategy $S^* : \mathbf{I} \to [\mathbf{0}, \mathbf{1}]$ is a* (mixed) Nash equilibrium*, if*

$$U_i(S^*) \geq U_i(S)$$

for every individual $i \in \mathbf{I}$ and for every strategy $S \in \boldsymbol{S}$ with the property that $S^*(j) = S(j)$, for every individual $j \in I \setminus \{i\}$.

Lemma 2.13 ([11])**.** *Let $S : \mathbf{I} \to [\mathbf{0}, \mathbf{1}]$ be a mixed Nash equilibrium.*

(i) *If $0 < p_i < 1$, then $x = -A_{11}(P - p_i) - A_{12}Q + H(N,N)$.*
(ii) *If $0 < q_j < 1$, then $y = -A_{21}P - A_{22}(Q - q_j) + V(N,N)$.*

Hence, if $A_{11} \neq 0$, then there is not a mixed Nash equilibrium with the property that $0 < p_{i_1} \neq p_{i_2} < 1$. Furthermore, if $A_{22} \neq 0$, then there is not a mixed Nash equilibrium with the property that $0 < q_{j_1} \neq q_{j_2} < 1$.

Definition 2.14 ([11])**.** *The* $(l_1, k_1, p; l_2, k_2, q)$ mixed strategic set *is the set of all strategies $S : \mathbf{I} \to [\mathbf{0}, \mathbf{1}]$ with the following properties:*

(i) *$l_1 = \#\{i \in I_1 : p_i = 1\}$ and $k_1 = \#\{i \in I_1 : p_i = p\}$;*
(ii) *$l_2 = \#\{j \in I_2 : q_j = 1\}$ and $k_2 = \#\{j \in I_2 : q_j = q\}$;*
(iii) *$n_1 - (l_1 + k_1) = \#\{i \in I_1 : p_i = 0\}$ and $n_2 - (l_2 + k_2) = \#\{j \in I_2 : q_j = 0\}$.*

For $p, q \in \{0, 1\}$, we observe that the $(l_1, k_1, p; l_2, k_2, q)$ mixed strategic set is equal to the $(l_1 + pk_1, l_2 + qk_2)$ pure strategic set.

Remark 2.15 ([11])**.** *By Lemma 2.13, supposing that $A_{11} \neq 0$ and $A_{22} \neq 0$, a mixed strategy S is a Nash equilibrium, if S is contained in some $(l_1, k_1, p; l_2, k_2, q)$ mixed strategic set.*

Since individuals with the same type are identical, if a mixed strategy contained in the $(l_1, k_1, p; l_2, k_2, q)$ mixed strategic set is a Nash equilibrium, then all the strategies in the $(l_1, k_1, p; l_2, k_2, q)$ mixed strategic set are Nash equilibria.

Definition 2.16 ([11])**.** *An* $(l_1, k_1, p; l_2, k_2, q)$ mixed Nash equilibrium (set) *is an* $(l_1, k_1, p; l_2, k_2, q)$ *strategic set whose strategies are Nash equilibria. The* (mixed) Nash domain $\boldsymbol{N}(l_1, k_1, p;\ l_2, k_2, q)$ *is the set of all pairs* (x, y) *for which the* $(l_1, k_1, p; l_2, k_2, q)$ *strategic set is a mixed Nash equilibrium set.*

An $(l_1, k_1, p; l_2, k_2, q)$ *strict mixed Nash equilibrium set* is a mixed Nash equilibrium set that does not contain pure strategies, i.e. $(p, q) \in [0, 1]^2 \setminus \{0, 1\}^2$. A *strict mixed Nash domain* $\mathbf{N}(l_1, k_1, p; l_2, k_2, q)$ is the mixed Nash domain of a strict mixed Nash equilibrium set.

3. Geometric approaches in constructing Tilings

In this section, we study two strategic approaches to construct Nash domains. The two approaches are the global approach and the local approach. In the global approach, we will construct all possible tilings using the coordinates of the *influence matrix*. In the local approach, we will characterize all possible orders for the domains of the *pure and mixed Nash equilibria* in tilings using the coordinates of the *influence matrix* too.

In order to proceed, we need to introduce some auxiliary and generalized results.

Theorem 3.1. *The* (l_1, l_2) *strategy is a Nash Equilibrium if and only if* $(x, y) \in N(l_1, l_2)$*, where*

$$N(l_1, l_2) = \{(x, y) : H_L(l_1, l_2) \le x \le H_R(l_1, l_2) \ \text{ and } \ V_D(l_1, l_2) \le y \le V_U(l_1, l_2)\},$$

the left horizontal threshold $H_L(l_1, l_2)$ *and the right horizontal threshold* $H_R(l_1, l_2)$ *of the* (l_1, l_2) *strategy are given by*

$$H_L(l_1, l_2) = \alpha_{11}^N n_1 + \alpha_{12}^N n_2 + \alpha_{11}^Y - (\alpha_{12}^Y + \alpha_{12}^N) l_2 - (\alpha_{11}^Y + \alpha_{11}^N) l_1$$

$$H_R(l_1, l_2) = \alpha_{11}^N n_1 + \alpha_{12}^N n_2 - \alpha_{11}^N - (\alpha_{12}^Y + \alpha_{12}^N) l_2 - (\alpha_{11}^Y + \alpha_{11}^N) l_1 \ ,$$

the down vertical threshold $V_D(l_1, l_2)$ *and the the upper vertical threshold* $V_U(l_1, l_2)$ *of the* (l_1, l_2) *strategy are given by*

$$V_D(l_1, l_2) = \alpha_{22}^N n_2 + \alpha_{21}^N n_1 + \alpha_{22}^Y - (\alpha_{21}^Y + \alpha_{21}^N) l_1 - (\alpha_{22}^Y + \alpha_{22}^N) l_2$$

$$V_U(l_1, l_2) = \alpha_{22}^N n_2 + \alpha_{21}^N n_1 - \alpha_{22}^N - (\alpha_{21}^Y + \alpha_{21}^N) l_1 - (\alpha_{22}^Y + \alpha_{22}^N) l_2 \ .$$

Proof. The (l_1, l_2) strategy is a Nash equilibrium if, and only if, the following four inequalities hold

$$U_1(Y; l_1, l_2) \ge U_1(N; l_1 - 1, l_2) \ , \ U_1(N; l_1, l_2) \ge U_1(Y; l_1 + 1, l_2)$$

and

$$U_2(Y; l_1, l_2) \ge U_2(N; l_1, l_2 - 1) \ , \ U_2(N; l_1, l_2) \ge U_2(Y; l_1, l_2 + 1) \ .$$

Hence, the proof of Theorem 3.1 follows by rearranging the terms in the previous inequalities. □

Hence, $N(l_1, l_2)$ is the *Nash Equilibrium domain* of the (l_1, l_2) strategy (see Figure 5).

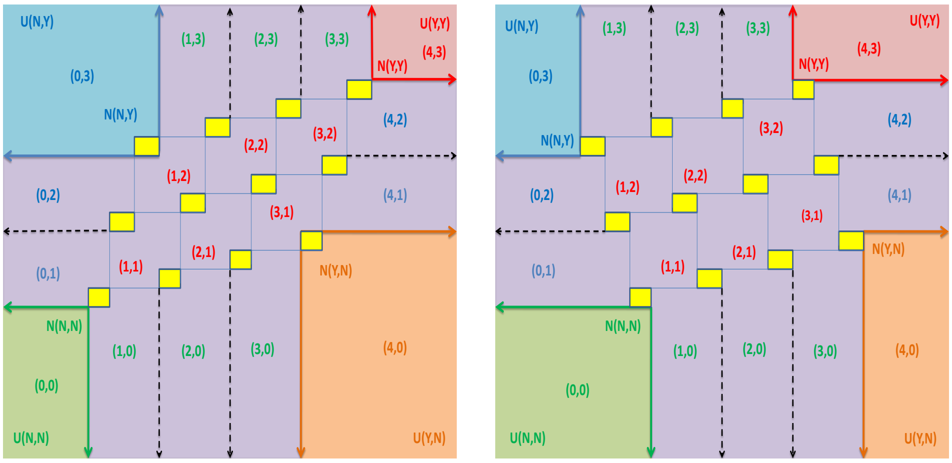

FIGURE 5. Disparate Nash equilibria when $n_1 = 4$ and $n_2 = 3$. *Left*: $A_{11} < 0$, $A_{12} < 0$, $A_{21} < 0$ and $A_{22} < 0$. The yellow rectangles include two pure Nash equilibria and a mixed Nash equilibrium. *Right*: $A_{11} < 0$, $A_{12} > 0$, $A_{21} < 0$ and $A_{22} < 0$. The yellow rectangles have no pure Nash equilibrium but include a mixed Nash equilibrium.

Each geometric graph in Figure 5 is called a tilings results by joining the four quadrants described in Figures 1, 2, 3 and 4 in one geometric graph. The horizontal preferences x for individuals of type t_1 is being the x-axis and the vertical preferences y for individuals of type t_2 is being the y-axis. Each tiling indicates the way the horizontal thresholds $H(Y,Y)$, $H(Y,N)$, $H(N,Y)$, $H(N,N)$ are ordered along the horizontal x-axis and the way the way the vertical thresholds $V(Y,Y)$, $V(Y,N)$, $V(N,Y)$, $V(N,N)$ are ordered along the vertical y-axis. The order of these horizontal thresholds and vertical thresholds give rise to the Nash equilibria location. Thus, determining the Nash domain for each strategy. More detailed about the construction of these tilings will be discussed in the coming section.

The following thresholds determine the domains of the (l_1, l_2) disparate Nash equilibria.

$$\begin{aligned}
H_R(l_1,l_2) &= H_L(l_1+1,l_2), & H_L(l_1,l_2) &= H_L(l_1,l_2+1)+A_{12},\\
V_U(l_1,l_2) &= V_D(l_1,l_2+1), & H_R(l_1,l_2) &= H_R(l_1,l_2+1)+A_{12},\\
V_U(l_1,l_2) &= V_D(l_1,l_2)-A_{22}, & V_D(l_1,l_2) &= V_D(l_1+1,l_2)+A_{21},\\
H_R(l_1,l_2) &= H_L(l_1,l_2)-A_{11}, & V_U(l_1,l_2) &= V_U(l_1+1,l_2)+A_{21}\ .
\end{aligned}$$

We observe that (see Figure 5) if $A_{11} > 0$ or $A_{22} > 0$, then there are no (l_1, l_2) Nash Equilibria, for every $l_1 \in \{1, \ldots, n_1 - 1\}$ and $l_2 \in \{1, \ldots, n_2 - 1\}$. However, if $A_{11} \leq 0$ and $A_{22} \leq 0$, then there are (l_1, l_2) Nash Equilibria, for every $l_1 \in \{1, \ldots, n_1 - 1\}$ and $l_2 \in \{1, \ldots, n_2 - 1\}$.

Lemma 3.2. *The Nash domains satisfy the following properties:* $N(n_1, n_2) = N(Y,Y)$, $N(n_1, 0) = N(Y,N)$, $N(0, n_2) = N(N,Y)$, $N(0,0) = N(N,N)$.

Proof. We prove $N(n_1, n_2) = N(Y, Y)$ and the proof for the other Nash domains follows similarly. Substituting l_1 by n_1 and l_2 by n_2 in the horizontal and vertical thresholds stated in Theorem 3.1, we have that

$$\begin{aligned} H_L(n_1, n_2) &= \alpha_{11}^N n_1 + \alpha_{12}^N n_2 + \alpha_{11}^Y - (\alpha_{12}^Y + \alpha_{12}^N) n_2 - (\alpha_{11}^Y + \alpha_{11}^N) n_1 \\ &= -\alpha_{11}^Y (n_1 - 1) - \alpha_{12}^Y n_2 \\ &= H(Y, Y) \end{aligned}$$

and

$$\begin{aligned} V_D(n_1, n_2) &= \alpha_{22}^N n_2 + \alpha_{21}^N n_1 + \alpha_{22}^Y - (\alpha_{21}^Y + \alpha_{21}^N) n_1 - (\alpha_{22}^Y + \alpha_{22}^N) n_2 \\ &= -\alpha_{22}^Y (n_2 - 1) - \alpha_{21}^Y n_1 \\ &= V(Y, Y) \ . \end{aligned}$$

Hence, $N(n_1, n_2) = N(Y, Y)$ and we conclude the proof. □

3.1. **Global approach.** We will see that the coordinates of the *influence matrix* together with the total number of individuals play a significant role to determine the Nash domains for a given strategy. We will also denote to the Nash domains $N(l_1, l_2)$ by $Q(l_1, l_2)$ as being referred to the quadrants. We notify that a pair of thresholds $(H(Y, Y), V(Y, Y))$ (respectively, $(H(Y, N), V(Y, N))$, $(H(N, Y), V(N, Y))$, $(H(N, N), V(N, N)))$ form a corner for the quadrant $Q(Y, Y)$ (respectively, $Q(Y, N)$, $Q(N, Y)$, $Q(N, N)$).

We summarize the global approach by the following remark which provides a strategy for constructing all possible tilings.

Remark 3.3 (Golden Tiling)**.** *Let* $S_1 = (A_{12}, A_{22})$ *and* $S_2 = (A_{11}, A_{21})$*. Every tiling is determined by a corner of quadrant and a vector of stairs* (S_1, S_2) *together with the total number of individuals.*

We now emphasise Remark 3.3 by referring to the Figures 6, 7, 8 and 9 and by ordering the following steps:

- connect the losangles between the corner of the quadrants $Q(Y, Y)$, $Q(Y, N)$, $Q(N, Y)$ and $Q(N, N)$;
- use the coordinates of the influence matrix (A_{22}, A_{12}) to construct the left and right green ladders boundaries of the losangles (see Figures 7 and 8);
- use the coordinates of the influence matrix (A_{11}, A_{21}) to construct the upper and down blue ladders boundaries of the losangles (see Figures 6 and 9);
- we repeat the second and third items in a similar fashion, but with different locations;
- the ladders intersect the losangles in the points upper-down
$$\frac{j_2 A_{12}}{n_2}$$
for $j_2 = 0, 1, \ldots, n_2$;
- the ladders intersect the losangles in the points left-right
$$\frac{j_1 A_{21}}{n_1}$$
for $j_1 = 0, 1, \ldots, n_1$.

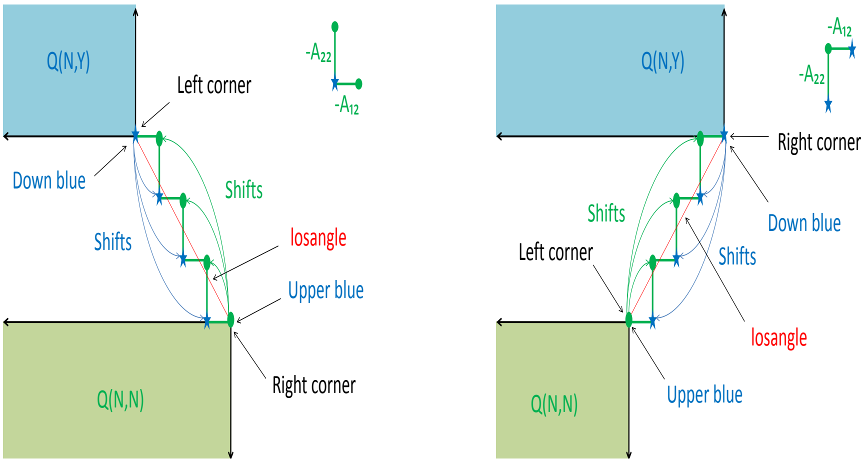

FIGURE 6. ***Left: Left green boundaries shift*** 1: The rule: Go in the boundaries in the horizontal dimension of the right corner and come in from outside of the horizontal boundaries in the horizontal dimension of the left corner. ***Right: Left green boundaries shift*** 2: The rule: Go out from the boundaries in the horizontal dimension of the left corner and come in from inside the horizontal boundaries in the horizontal dimension of the right corner.

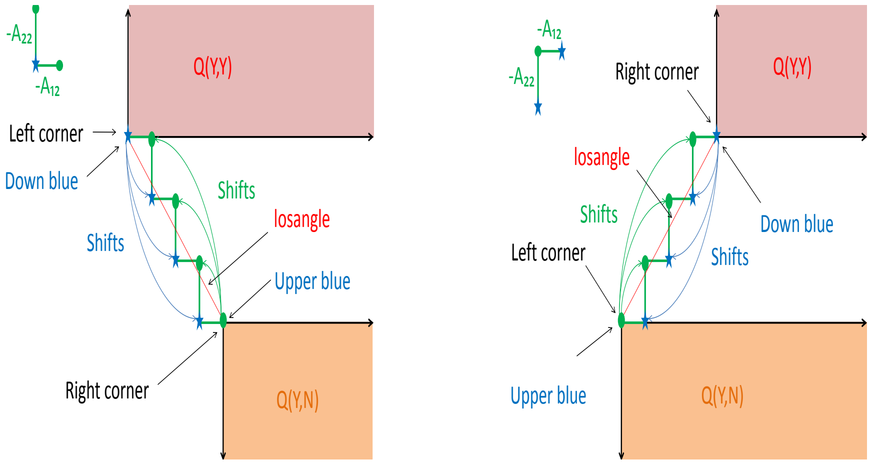

FIGURE 7. ***Left: Right green boundaries shift*** 1: The rule: Go in the boundaries in the horizontal dimension of the left corner and come in from outside of the horizontal boundaries in the horizontal dimension of the right corner. ***Right: Right green boundaries shift*** 2: The rule: Go out from the boundaries in the horizontal dimension of the right corner and come in from inside the horizontal boundaries in the horizontal dimension of the left corner.

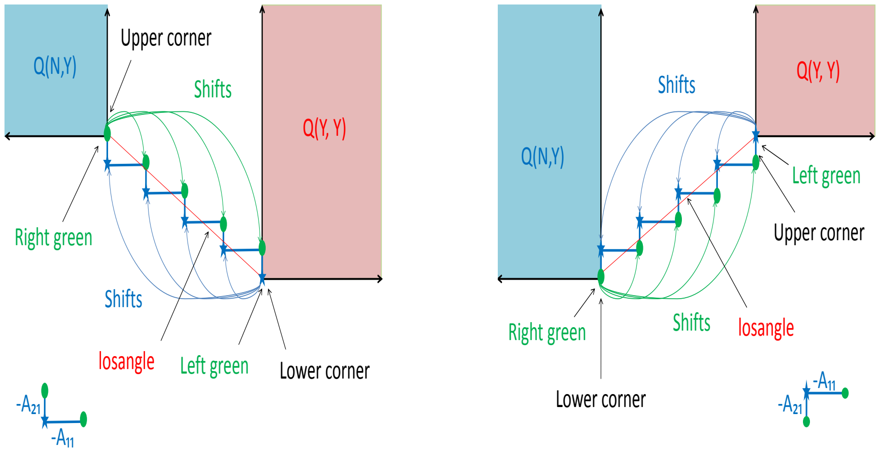

FIGURE 8. ***Left: Down blue boundaries shift*** 1: The rule: Go out of the boundaries in the vertical dimension of the upper corner and come in from inside the vertical boundaries in the vertical dimension of the lower corner. ***Right: Down blue boundaries shift*** 2: The rule: Go in the boundaries in the vertical dimension of the lower corner and come in from outside the vertical boundaries in the vertical dimension of the upper corner.

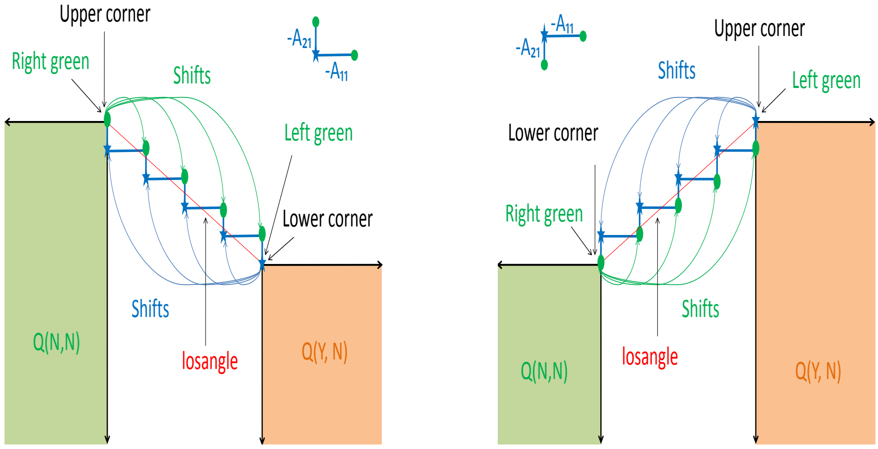

FIGURE 9. ***Left: Upper blue boundaries shift*** 1: The rule: Go in of the boundaries in the vertical dimension of the upper corner and come in from outside the vertical boundaries in the vertical dimension of the lower corner. ***Right: Upper blue boundaries shift*** 2: The rule: Go out the boundaries in the vertical dimension of the lower corner and come in from inside the vertical boundaries in the vertical dimension of the upper corner.

We remark that shifts in the left green ladders boundaries of the losangles are different from right green ladders boundaries of the losangles; shifts in the upper blue ladders boundaries of the losangles are different from down blue ladders boundaries of the losangles; and down blue stars start in blue stars and they end in the green circles, but upper blue stars start in the green circles and end in the blue stars.

We see that there are eight different boundaries kind of shifts: left green boundaries shift 1, left green boundaries shift 2, right green boundaries shift 1, right green boundaries shift 2, down blue boundaries shift 1, down blue boundaries shift 2, upper blue boundaries shift 1 and upper blue boundaries shift 2.

Recall that $\mathbf{O}$ is the set of all possible strategic occupation vectors. Let the horizontal and vertical set of strategies be given, respectively, by

$$O_H = \{(0, l_2)\} \cup \{(n_1, l_2)\} \text{ and } O_V = \{(l_1, 0)\} \cup \{(l_1, n_2)\}$$

for every $l_1 \in \{0, 1, \ldots, n_1\}$ and $l_2 \in \{0, 1, \ldots, n_2\}$.

The following theorem determines the conditions that guarantees the existence of a strictly mixed Nash equilibria for a given tiling.

Theorem 3.4. *Given an influence matrix A and a point of stairs $S = (S_1, S_2)$. The corresponding tiling $T(A, S)$ has the following properties:*

(i) *if $A_{12}A_{21} > 0$, then there is a strictly mixed strategies only in the Nash equilibria domain $N(l_1, l_2)$ for every pure strategy $(l_1, l_2) \in \mathbf{O} \setminus \{O_H \cup O_V\}$;*

(ii) *if $A_{12}A_{21} < 0$, then there is a strictly mixed strategies only outside the Nash equilibria domain $N(l_1, l_2)$ for every pure strategy $(l_1, l_2) \in \mathbf{O} \setminus \{O_H \cup O_V\}$;*

(iii) *if $A_{12}A_{21} = 0$, then there are no strictly mixed strategies for every pure strategy $(l_1, l_2) \in \mathbf{O}$.*

Proof. By Contradiction. We proof case (i) and the proof of cases $(ii)-(iii)$ follow similarly. Assume that there is a strictly mixed Nash equilibrium strategy $S : \mathbf{I} \to [\mathbf{0}, \mathbf{1}]$ in the Nash equilibria domain $N(l_1, l_2)$ for some occupation vector $(l_1, l_2) \in \{O_H \cup O_V\}$. Note that $A_{12}A_{21} > 0$ implies that either $A_{12} > 0$ and $A_{21} > 0$ (individuals of a certain type affect positively the other type of individuals to chair a particular decision) or $A_{12} < 0$ and $A_{21} < 0$ (individuals of a certain type affect negatively the other type of individuals to chair a particular decision). If $N(l_1, l_2) = N(0, 0)$, then $p_i = q_j = 0$ for all $i = 0, 1, \ldots, n_1$ and $j = 0, 1, \ldots, n_2$ which contradicts the fact that $S : \mathbf{I} \to [\mathbf{0}, \mathbf{1}]$ is a strictly mixed Nash equilibrium strategy. Similarly, if $N(l_1, l_2) = N(n_1, n_2)$, then $p_i = q_j = 1$ for all $i = 0, 1, \ldots, n_1$ and $j = 0, 1, \ldots, n_2$ which contradicts the fact that $S : \mathbf{I} \to [\mathbf{0}, \mathbf{1}]$ is a strictly mixed Nash equilibrium strategy. If $N(l_1, l_2) = N(0, l_2)$ (resp. $N(l_1, l_2) = N(l_1, 0)$), then $p_i = 0$ for all $i = 0, 1, \ldots, n_1$ (resp. $q_j = 0$ for all $j = 0, 1, \ldots, n_2$) which is a contradiction. □

In Figure 10, we show an example of two rotated tilings in which the horizontal thresholds $H(Y, Y)$, $H(Y, N)$, $H(N, Y)$, $H(N, N)$ are ordered along the horizontal x-axis and the vertical thresholds $V(Y, Y)$, $V(Y, N)$, $V(N, Y)$, $V(N, N)$ are ordered along the vertical y-axis. The influence matrix for the

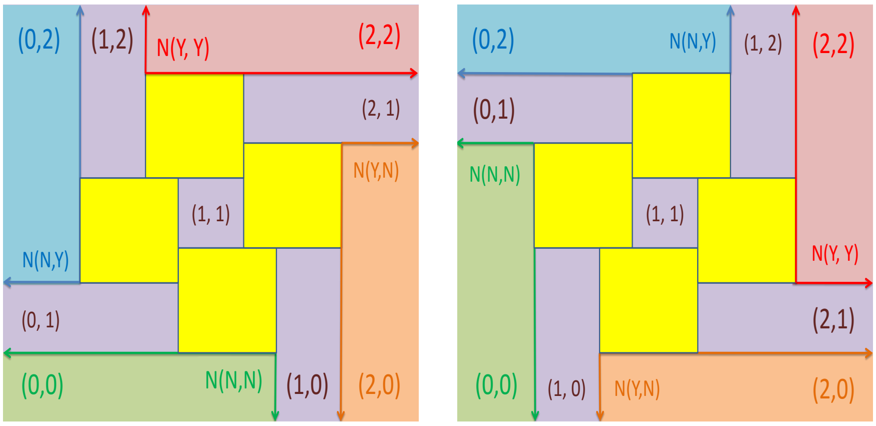

FIGURE 10. Pure and mixed Nash equilibria.

left tiling and the influence matrix for the right tiling are, respectively, given by

$$A = \begin{pmatrix} -2 & 3 \\ -3 & -2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}.$$

Hence, small changes in the coordinates of the *influence matrix* can create a different tiling. In [10], it was shown that there are 289 combinatorial classes of decision tilings, described by the decision bussola, which demonstrates the high complexity of making decision.

3.2. **Local Approach.** The local approach uses the signs of the coordinates of the influence matrix to determine the domains of the pure and mixed strategies in all tilings (see Figure 11). We observe that changing the signs of the pairs (A_{11}, A_{21}) and (A_{12}, A_{22}) imply different orders for the pure strategies (l_1, l_2). Let

$$E_{ij} = -A_{ij}$$

for all $i, j = \{1, 2\}$. Let us define the horizontal axis by E_{12} and the vertical axis by E_{21}. The sign of the pair (E_{12}, E_{21}) determines a certain order of pure strategies (l_1, l_2). Note that there are four possible orders for the pure strategies that are not located along any axis which are given by a small white rectangles in Figure 11.

Remark 3.5 (Rotating Pure Nash Domains)**.** *Given the location of the pure strategies in the small whit rectangles. We remark that*

(i) *if the signs of the coordinates (E_{12}, E_{21}) is $(+, +)$, then the pure strategies are rotated to make new ordering given by the small red rectangles that appear in Figure 12;*
(ii) *if the signs of the coordinates (E_{12}, E_{21}) is $(+, -)$, then the pure strategies are rotated to make new ordering given by the small orange rectangles appear in Figure 13;*
(iii) *if the signs of the coordinates (E_{12}, E_{21}) is $(-, +)$, then the pure strategies are rotated to make new ordering given by the small green rectangles appear in Figure 14;*

(iv) *if the signs of the coordinates* (E_{12}, E_{21}) *is* $(-,-)$, *then the rotated to make new ordering given by pure strategies are the small blue rectangles appear in Figure 15.*

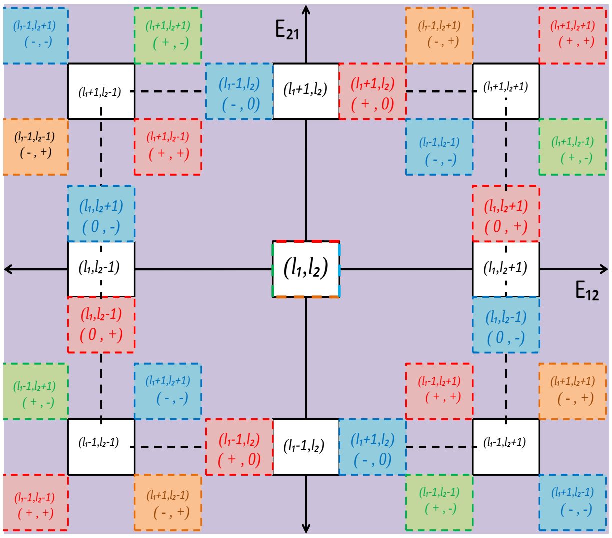

FIGURE 11. Rotating pure Nash domains using the local approach. E_{12} is located along the horizontal axis E_{21} is located along the vertical axis.

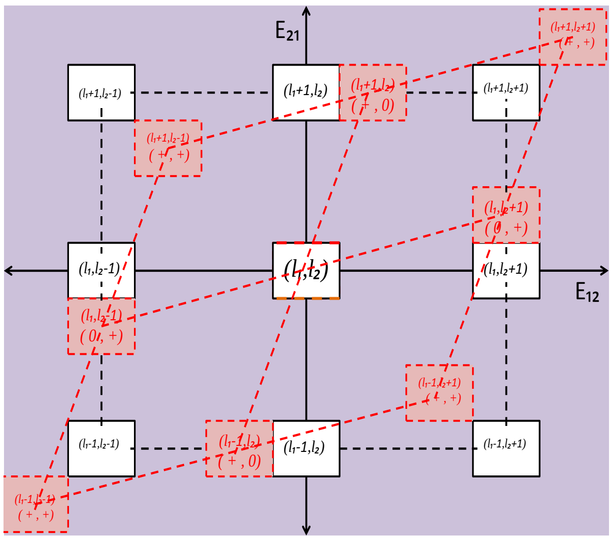

FIGURE 12. Rotating the pure strategies when the signs of (E_{12}, E_{21}) is $(+,+)$. The new order of the pure strategies moves to the small red rectangles.

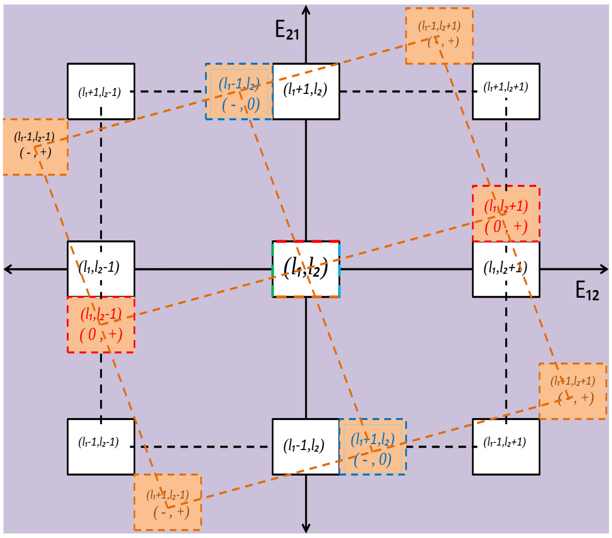

FIGURE 13. Rotating the pure strategies when the signs of (E_{12}, E_{21}) is $(+, -)$. The new order of the pure strategies moves to the small orange rectangles.

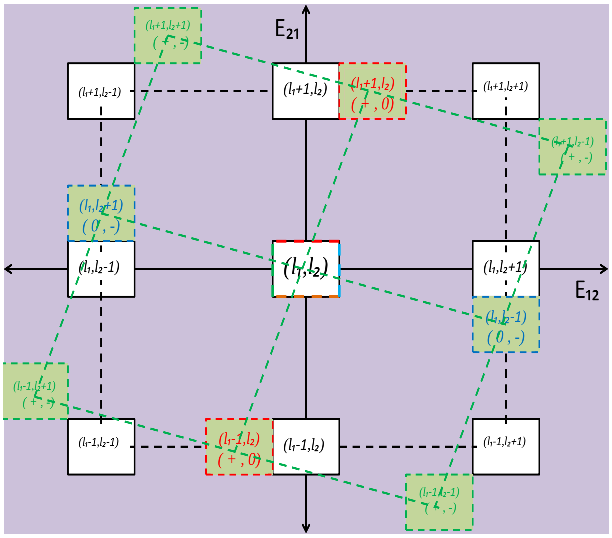

FIGURE 14. Rotating the pure strategies when the signs of (E_{12}, E_{21}) is $(-, +)$. The new order of the pure strategies moves to the small green rectangles.

4. Mixed strategies in local approach

We study geometrically two cases where mixed strategies co-exist. We present the first case in section 4.1 where no intersection between the pure strategies occurs. Second case will be introduced in section 4.2 where an intersection between the pure strategies occurs.

4.1. **No intersections between pure the strategies.** Without loss of generality, we will consider the case where the signs of (E_{12}, E_{12}) is $(+, +)$

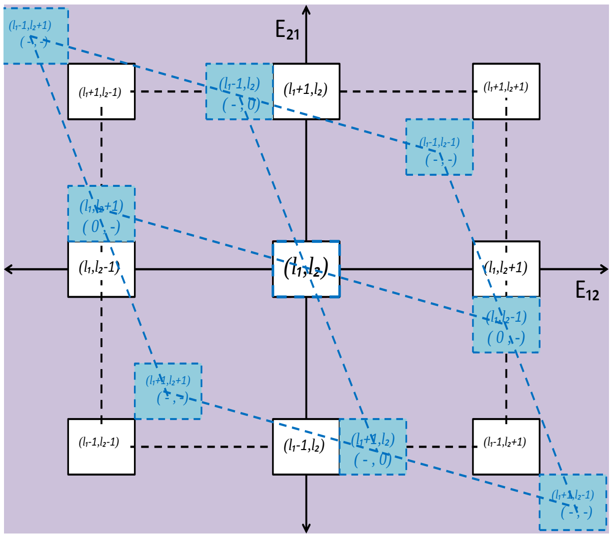

FIGURE 15. Rotating the pure strategies when the signs of (E_{12}, E_{21}) is $(-,-)$. The new order of the pure strategies moves to the small blue rectangles.

and focus on the mixed strategies that occurs in the corresponding Figure 12. The other three cases follow in a similar way.

Recall that $p \in [0,1]$ is the probability of an individual of type t_1 makes decision Y and $q \in [0,1]$ is the probability of an individual of type t_2 makes decision Y.

Theorem 4.1. *Consider the case where (E_{12}, E_{21}) is $(+,+)$. Then there is a mixed strategy $(l_1 + p, l_2 + q)$ with*

$$p = \frac{q_1}{\sqrt{|A_{21}|^2 + |A_{11}|^2}}$$

and

$$q = \frac{q_2}{\sqrt{|A_{12}|^2 + |A_{22}|^2}}$$

for every $1 < l_1 < n_1 - 1$ and $1 < l_2 < n_2 - 1$, where q_1 and q_2 are non-negative real values.

Proof. Note that if the mixed strategies $(l_1 \pm p, l_2 \pm q)$ are located along the horizontal and vertical axes (see the black rectangles in Figure 16), then they become pure and given by

$$\left(l_1 \pm \frac{A_{21}}{|A_{21}|},\ l_2 \pm \frac{A_{12}}{|A_{12}|} \right).$$

Considering the case where (E_{12}, E_{21}) is $(+,+)$. Thus, p and q may have now real values instated of being natural and their values are derived by applying the Pythagorean theorem among the three sides of a right triangles given in Figure 16, which ends the proof. $\square$

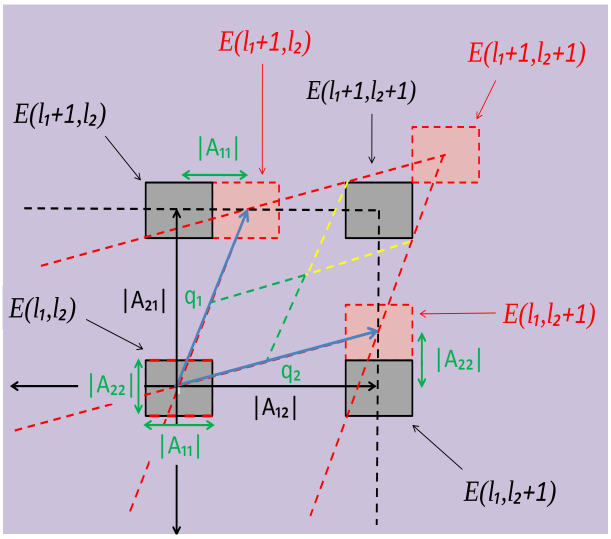

FIGURE 16. $(l_1 + p, l_2 + q)$ is the mixed strategy when (E_{12}, E_{21}) is $(+, +)$.

4.2. **Bifurcations between pure strategies.** In this section we study geometrically the bifurcations between the pure strategies and see the signs effect of the coordinates of the influence matrix. In Figures 17, 19, 20 and 18 we show all possible bifurcations between the pure strategies that may occur in the corresponding Figures 12, 13, 14 and 15, respectively.

In Figure 17, we show the bifurcations between the pure strategies when $(E_{12}, E_{21}) = (+, +)$. The blue, green and yellow rectangles represent the black rectangles (pure strategies) on the horizontal, vertical and diagonal axis in Figure 12, respectively. The red rectangles represent the red rectangles in Figure 12 and they describe the shifts in the black ones. We observe that there are three red overlaps between, where the mixed strategies may occur.

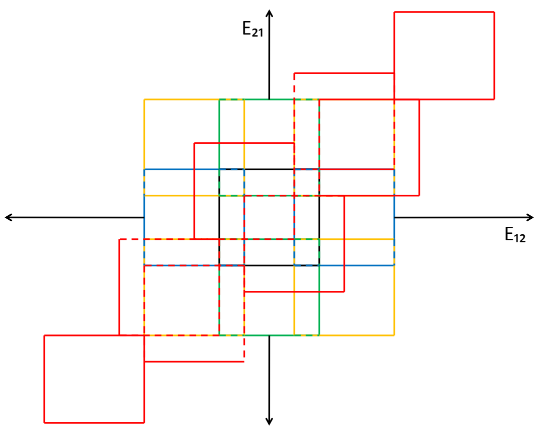

FIGURE 17. The bifurcations between the pure strategies when $(E_{12}, E_{21}) = (+, +)$.

In Figure 18, we show the bifurcations between the pure strategies when $(E_{12}, E_{21}) = (-,-)$. The blue, green and yellow rectangles represent the black rectangles (pure strategies) on the horizontal, vertical and diagonal axis in Figure 15, respectively. The red rectangles represent the blue rectangles in Figure 15 and they describe the shifts in the black ones. We observe that there are three red overlaps between, where the mixed strategies may occur.

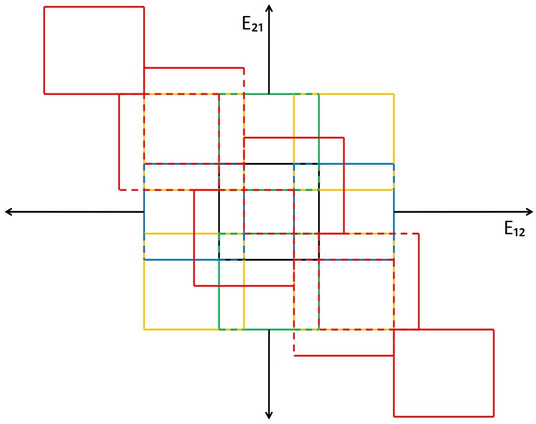

FIGURE 18. the bifurcations between the pure strategies when $(E_{12}, E_{21}) = (-,-)$.

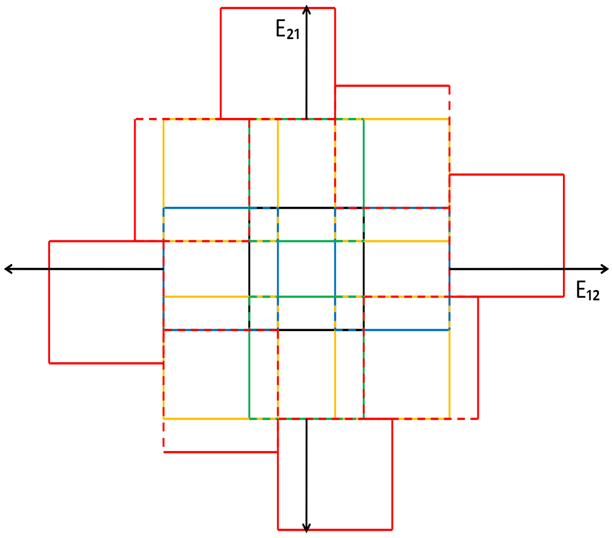

FIGURE 19. The bifurcations between the pure strategies when $(E_{12}, E_{21}) = (+,-)$.

In Figure 19, we show the bifurcations between the pure strategies when $(E_{12}, E_{21}) = (+,-)$. The blue, green and yellow rectangles represent the black rectangles (pure strategies) on the horizontal, vertical and diagonal axis in Figure 15, respectively. The red rectangles represent the orange rectangles in Figure 13 and they describe the shifts in the black ones. We observe that there are no overlaps between.

In Figure 20, we show the bifurcations between the pure strategies when $(E_{12}, E_{21}) = (-, +)$. The blue, green and yellow rectangles represent the black rectangles (pure strategies) on the horizontal, vertical and diagonal axis in Figure 14, respectively. The red rectangles represent the green rectangles in Figure 14 and they describe the shifts in the black ones. We observe that there are no overlaps between.

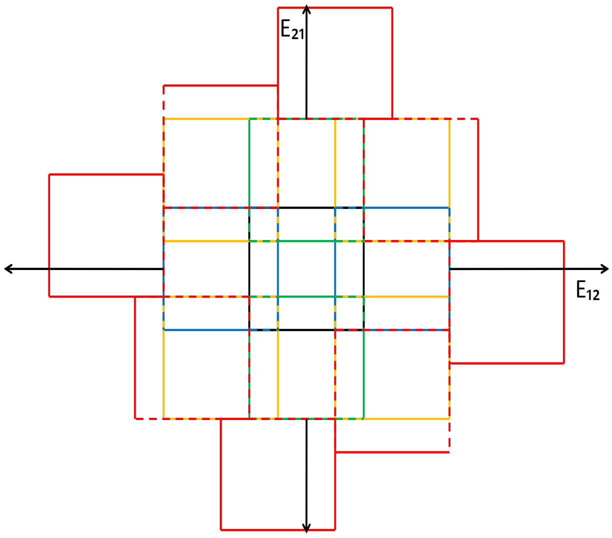

FIGURE 20. The bifurcations between the pure strategies when $(E_{12}, E_{21}) = (-, +)$.

5. Conclusions

Resorting to the dichotomous decision model presented in [11], two geometric approaches have been studied to construct all possible decisions tilings in which pure and mixed Nash equilibria co-exist and change with the relative decision preferences of the individuals. We have characterized all possible Nash domains for pure and mixed strategies and discussed the dependence of Nash equilibria on the parameters of the model. We have seen how the coordinates of the influence matrix and the total number of individuals can alter the order of the horizontal and vertical thresholds which allow the occurrence of bifurcations with and without overlaps between the pure strategies.

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