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Robust D-Stability

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Abstract—Motivated by the paper by Johnson, the 13 sufficient conditions are extended to robust D-stability of $n \times n$ matrices including variant types of matrices.

Keywords—D-stability, Robust D-stability, Strong D-stability, M-matrix, Oscillatory matrix, Hadamard product.

1. INTRODUCTION

The concept of D-stability was studied in [1–4]. This concept has proved useful in large-scale systems [5] and in multiparameter singular perturbations [4,6]. Because of the lack of an effective method for characterising D-stability, Johnson [3] introduced four observations and 13 sufficient conditions involving a number of important matrices. Recall that a matrix $A \in R^{n\times n}$ is D-stability if for any positive diagonal matrix $\mathbf{D} \in \mathcal{D}_n$ the eigenvalues of $\mathbf{DA}$ have strictly negative real parts. That a D-stability matrix is also stable is easily shown if $\mathbf{D} = \mathbf{I}_{n\times n}$, the $n \times n$ identity matrix. Let $\mathbf{A}$ be a stable matrix representing a nominal design of an engineering system $\dot{\mathbf{x}} = \mathbf{A}x$. The determination of a stability margin is necessary for any complete design. It should be a measure of the smallest perturbation matrix $\mathbf{G} \in R^{n\times n}$ [7]. An important question is the following: let $\mathbf{A}$ be D-stability, does there exist an $\alpha > 0$ such that $(\mathbf{A} + \mathbf{G})$ is also D-stability whenever $\|\mathbf{G}\| < \alpha$? The notion of strong D-stability was introduced in [8].

DEFINITION 1. STRONG D-STABILITY OR ROBUST D-STABILITY. The matrix $\mathbf{A} \in R^{n\times n}$ is strongly D-stability if there is an $\alpha > 0$ such that $(\mathbf{A} + \mathbf{G})$ is D-stability for each $\mathbf{G} \in R^{n\times n}$ of sufficiently small norm α.

In this paper, the 13 sufficient conditions of D-stability introduced by Johnson [3] are extended to robust D-stability.

NOTATIONS. The following notations are used through out this paper: $\mathcal{C}^n$ denotes the n-dimensional complex space. $\mathbf{A} \in R^{n\times n}$ is an $n \times n$ real square matrix, $\mathbf{D} \in \mathcal{D}_n$ is an $n \times n$ diagonal matrix with positive entries, and $\mathbf{G} \in R^{n\times n}$ is the perturbation $n \times n$ real matrix, $\|\mathbf{G}\| < \alpha$, α is

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sufficiently small. $\mathcal{C}^n$ denotes the complex Euclidean space. $\sigma(\mathbf{A})$ denotes the spectrum of $\mathbf{A}$ and $\lambda_i \in \sigma(\mathbf{A})$ is an eigenvalue of $\mathbf{A}$. The four quadrants (including their boundaries) in the complex plane are denoted Q_i, $i = 1, \dots, 4$, numbering counterclockwise beginning with the upper right.

REMARK. A necessary condition for the zeros of the characteristic polynomial to lie in the left half of the complex plane is that all of its coefficients be positive with no missing terms, i.e., the coefficients are all greater than zero. Therefore, all principal minors of order j are of the sign $(-1)^j$.

2. SUFFICIENT CONDITIONS FOR ROBUST D-STABILITY

In this section, 13 sufficient conditions for strong D-stability are given. They are numbered as their counterpart in [3]. The proofs are given in Section 4.

A matrix $\mathbf{A}$ is strongly D-stability, i.e., $\mathbf{A} + \mathbf{G}$ is D-stability if it satisfies one of the following conditions.

CONDITION 1. $(\mathbf{DA} + \mathbf{A}^\top \mathbf{D})$ is negative definite.

The M-matrices [3,9,10] in the next condition have nonnegative off-diagonal entries and all its j^{th}-order principal minors are of the sign $(-1)^j$.

CONDITION 2. If $\mathbf{A}$ is an M-matrix.

CONDITION 3. $\mathbf{A}$ is D-stability and $\mathbf{DA} = \mathbf{B} = (b_{ij})$ satisfies

$$b_{ii} < - \sum_{\substack{1 \le j \le \\ j \ne i}} |b_{ij}|, \qquad 1 \le i \le n. \tag{1}$$

CONDITION 4. $\mathbf{A} = (a_{ij})$ is triangular and $a_{ii} < 0$, $i = 1, \dots, n$,

CONDITION 5. A sign stable matrix $\mathbf{A}$ without any zero element is strongly D-stable.

EXAMPLE.
$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \text{ is sign stable since the matrix } \begin{bmatrix} 0 & a \\ -b & -c \end{bmatrix}$$
is stable for any positive a, b, and c, but not strongly D-stable.

CONDITION 6. $\mathbf{A}$ is a Jacobi matrix and all its j^{th}-order principal minors are of sign $(-1)^j$.

The next condition considers oscillatory matrices which are a version of sign symmetric matrices [3,10,11].

CONDITION 7. $\mathbf{A}$ is an oscillatory matrix.

CONDITION 8. For each $x \in \mathcal{C}^n$, $x \ne 0$ there is $\mathbf{D}$ such that $\operatorname{Re}(x^* \mathbf{DA} x) < 0$.

The Hadamard product of the real positive definite symmetric matrix $\mathbf{H}$ and the stable matrix $\mathbf{A} \in R^{n \times n}$, denoted $\mathbf{H} \circ \mathbf{A}$, has its field of values $F(\mathbf{H} \circ \mathbf{A})$ (numerical range) [11,12] in the left half of the complex plane, $Q_2 \cup Q_3$, i.e., $\mathbf{A}$ is Schur stable. Schur stable matrices are also D-stability matrices [3,12]. The convex hull of the points of the spectrum $\sigma(\mathbf{H} \circ \mathbf{A})$ is a subset of $F(\mathbf{H} \circ \mathbf{A})$, and consequently, $\mathbf{DA}$ is stable. The field of values of a stable matrix $\mathbf{A}$ are denoted $F(\mathbf{A}) = \{c_A \mid c_A \in (Q_2 \cup Q_3)\}$.

CONDITION 9. If the Hadamard product of $\mathbf{H}$ and $\mathbf{A} + \mathbf{G}$, $(\mathbf{H} \circ (\mathbf{A} + \mathbf{G}))$ is Schur stable for each positive definite symmetric matrix $\mathbf{H}$ and a perturbation $\mathbf{G}$, $\|\mathbf{G}\| < \alpha$, then $\mathbf{A} + \mathbf{G}$ is in the closure of D-stability.

CONDITION 10. All the j^{th}-order principal minors of $\mathbf{A}$ are of sign$(-1)^j$ and each pair of symmetrically placed minors has positive product.

CONDITION 11. $\mathbf{A} \in R^{2 \times 2}$, is strongly D-stability if and only if all its j^{th}-order principal minors are of sign $(-1)^j$.

CONDITION 12. $\mathbf{A} = (a_{ij}) \in R^{3\times 3}$, all the j^{th}-order principal minors of $\mathbf{A}$ are of sign $(-1)^j$ and

$$a_{11}a_{22}a_{33} < \frac{a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}}{2}.$$

CONDITION 13. If $n \leq 4$ and $\mathbf{A} \in R^{n\times n}$ satisfies the GKK condition.

The GKK condition is a weaker version of sign symmetric property (see [3] and the references therein).

3. PROOFS OF CONDITIONS 1, ..., 13

PROOF OF CONDITION 1. $\mathbf{A}$ is D-stability [2,3]. The proof of strong D-stability is similar to that of proposition in [8]. ∎

PROOF OF CONDITION 2. Nonnegative off-diagonal M-matrices satisfy Condition 3. ∎

PROOF OF CONDITION 3. Inequality 1 implies that the logarithmic measure, $\mu_\infty(\mathbf{DA}) < 0$ which implies $\mu_\infty(\mathbf{A}) < 0$. For strong D-stability of $\mathbf{A}$ the logarithmic measure satisfies $\mu_\infty(\mathbf{DA}) + \mu_\infty(\mathbf{DG}) < 0$ which implies $\mathrm{Re}\{\lambda_i(\mathbf{D}(\mathbf{A}+\mathbf{G})) \mid i = 1,\dots,n\} \leq \mu_\infty(\mathbf{D}(\mathbf{A}+\mathbf{G})) \leq \mu_\infty(\mathbf{DA}) + \mu_\infty(\mathbf{DG}) < 0$. This is equivalent to $d_{ii}(a_{ii} + g_{ii}) < -(\sum_{j\neq i} |d_{ii}a_{ij}| + \sum_{j\neq i} |d_{ii}g_{ij}|) \leq -\sum_{j\neq i} |d_{ii}(a_{ij} + g_{ij})| < 0$, i.e., $\mathbf{A}+\mathbf{G}$ is D-stable. ∎

PROOF OF CONDITION 4. For a suitable $\mathbf{D}$ this condition satisfies Condition 3. ∎

PROOF OF CONDITION 5. If for a sign stable matrix $\mathbf{A}$, $(\mathbf{DA})$ has no zero element, the perturbed matrix $\mathbf{D}(\mathbf{A}+\mathbf{G})$ still has the same sign pattern for any perturbation $\mathbf{G}$ with $\|G\| < \alpha$, α is sufficiently small. ∎

PROOF OF CONDITION 6. Let $\mathbf{DA} = \mathbf{B} = (b_{ij})$. $\mathbf{B}$ is still Jacobi and its j^{th} principal minors are of sign $(-1)^j$, i.e., $\mathbf{B}$ and also the $\mathbf{B}+\mathbf{B}^\top$ are stable. Consequently, $\mathbf{DA}+\mathbf{A}^\top\mathbf{D} < 0$, and thus, Condition 1 is satisfied. Hence, there exists a matrix $\mathbf{G}$, $\|\mathbf{G}\| < \alpha$ such that $\mathbf{D}(\mathbf{A}+\mathbf{G}) + \mathbf{A}+\mathbf{G})^\top\mathbf{D} < 0$, i.e., $\mathbf{A}$ is strongly D-stability.

Notice that some choices of $\mathbf{D}$ may yield a number of off-diagonal entries, of the symmetric matrix $\mathbf{C} = \mathbf{B}+\mathbf{B}^\top$, that are zeros. In particular, let

$$\mathbf{D} = \mathrm{diag}(d_1,\dots,d_n),$$

$$d_1 = 1,\dots,d_k = \left|\frac{\prod_{i=2}^{k} a_{(i-1),i}}{\prod_{i=2}^{k} a_{i,(i-1)}}\right|,$$

$$d_{k+1} = d_k \left|\frac{a_{k,(k+1)}}{a_{(k+1),k}}\right|, \qquad k = 2,\dots,n.$$

The off-diagonal entries of $\mathbf{C}$ are either $c_{ij} = 2b_{ij}$, if $\mathrm{sign}\,(b_{ij}) = \mathrm{sign}\,(b_{ji})$ or $c_{ij} = 0$ if $\mathrm{sign}\,(b_{ij}) \neq \mathrm{sign}\,(b_{ji})$. Let m be the number of c_{ij}s that are zeros, then $\mathbf{C}$ can be represented as a composite system of $m+1$ isolated subsystems $(\mathbf{C}_i \mid i = 1,\dots,m+1)$. The submatrices $\mathbf{C}_i$ are also Jacobi matrices with principal minors of sign $(-1)^j$. Hence, $\mathbf{C}_i$ is strongly D-stability. The strong D-stability of $\mathbf{C}$ follows from [13, Theorems 1,2]. Consequently, $\mathbf{A}$ is strongly D-stability . ∎

PROOF OF CONDITION 7. The principal minors of the matrix $\mathbf{A}$ are continuous functions. Hence, D-stability is preserved under small perturbations and consequently $\mathbf{A}$ is strongly D-stability. ∎

To prove Condition 8, a concept from topology, *partition of unity*, is used [14, p. 222].

PROOF OF CONDITION 8. Since $\mathbf{A}$ is D-stability, $\mathrm{Re}(x^*\mathbf{DA}x) < 0$, $\forall x \neq 0$, $x \in \mathcal{C}^n$. Hence, $(x^*(\mathbf{DA}+\mathbf{A}^\top\mathbf{D})x)/2 < 0$. It suffices to prove that $\mathrm{Re}(x^*\mathbf{DA}x) + \mathrm{Re}(x^*\mathbf{DG}x) < 0$, for each $x \neq 0$ and $\forall \|\mathbf{G}\| < \alpha$, where α depends on x and the matrix $\mathbf{D}$. First let $x \in S$, $S = \{x \in$

$C^n : xx^* \leq 1\}$. Since S is compact, there is a finite cover $\{O_{ii}\}$, $i = 1, \ldots, n$, and there exists a partition of unity $\{\phi_{ii}\}$ dominated by $\{O_{ii}\}$, where $\{\phi_{ii}\}$ is an indexed family of continuous functions [14, p. 222]. Let $\mathbf{D(x)} = \sum_{i=1}^{n} \phi_{ii}(x)\mathbf{D}$, then $\mathbf{D(x)}$ is a positive diagonal matrix and continuous function on S, $x \in S$, $\mathrm{Re}(x^*\mathbf{D(x)A}x) = \sum_{ii=1}^{n} \phi_{ii}(x)\mathrm{Re}(x^*\mathbf{DA}x) < 0$. Since S is compact, there exist positive numbers γ and β where, $-\gamma := \max_{x\in S} \mathrm{Re}(x^*\mathbf{D(x)A}x) < 0$ $\beta := \max_{x\in S} |D(x)| = \|D(x)\|$. Let $\alpha < \gamma/\beta$. Then for each $\mathbf{G}$ with $\|\mathbf{G}\| < \alpha$, and for each $x \neq 0$, $x \in S$ the estimate $\mathrm{Re}(x^*\mathbf{D(x)(A+G)}x) = \mathrm{Re}(x^*\mathbf{D(x)A}x) + \mathrm{Re}(x^*\mathbf{D(x)G}x) \leq -\gamma + \|x\|^2\|\mathbf{D}(x)\|\|\mathbf{G}\| < -\gamma + \beta\alpha^* < 0$ and $\mathrm{Re}(x^*\mathbf{D(x)(A+G)}x) < 0$.

Let $x \neq 0$, $x \in C^n$ with $\|x\| = \omega > 1$, substitute $x := x/\omega$ in $\mathrm{Re}(x^*\mathbf{D(x)A}x) + \mathrm{Re}(x^*\mathbf{D(x)G}x) = \omega^2(\mathrm{Re}(x^*/\omega)\mathbf{DA}(x/\omega)) + \mathrm{Re}((x^*/\omega)\mathbf{DG}(x/\omega)) < 0$. Thus, $\mathbf{A}$ is strongly D-stability in all the space C^n. ∎

PROOF OF CONDITION 9. Given a matrix $\mathbf{A}$ and a positive definite symmetric matrix $\mathbf{H} \in R^{n\times n}$, $F(\mathbf{H}) = \{c_H \mid c_H \in (Q_1 \cup Q_4)\}$. Clearly, $\sigma(\mathbf{H} \circ (\mathbf{A}+\mathbf{G})) \subseteq F(\mathbf{H} \circ (\mathbf{A}+\mathbf{G})) \subseteq F(\mathbf{H})F(\mathbf{A}+\mathbf{G})$. Since $\mathrm{Re}\{c_H\} > 0$, then $(\mathbf{A}+\mathbf{G})$ is Schur stable if $\mathrm{Re}(\{c_{(A+G)}\} < 0$. There exists a matrix $\mathbf{G}$, $\|G\| < \alpha$ such that $\sup(\mathrm{Re}\{c_G\}) < -\sup(\mathrm{Re}\{c_A\})$. Then $(\mathbf{A}+\mathbf{G})$ is Schur stable and, consequently, is in the closure of D-stability matrices [12], i.e., $\mathbf{A}$ strongly D-stable. ∎

PROOF OF CONDITION 10. $\mathbf{A}$ is D-stability. The principal minors are continuous functions. Hence, for $\mathbf{G}$, with $\|\mathbf{G}\| < \alpha$, D-stability of $\mathbf{A}+\mathbf{G}$ is preserved. ∎

PROOF OF CONDITION 11. The characteristic polynomial of $\mathbf{DA}$ is $\lambda^2 - (d_1a_{11} + d_2a_{22})\lambda + d_1d_2(\det(\mathbf{A}))$. d_1a_{11} and d_2a_{22} are negative (1^{st}-order principal minors) and $d_1d_2\det(\mathbf{A})$ is positive (2^{nd}-order). Thus, $\mathbf{DA}$ is stable for any $\mathbf{D} \in \mathcal{D}^n$. This condition holds for sufficiently small perturbations, so A is strongly D-stability. ∎

PROOF OF CONDITION 12. Similar to the proof of Condition 11. ∎

PROOF OF CONDITION 13. Matrices with the GKK property [3] are a version of sign symmetric matrices. They are strongly D-stability. Proof is similar to Condition 7. ∎

REFERENCES

1. A.C. Enthoven and K.J. Arrow, A theorem on expectations and the stability of equilibrium, *Econometrica* **24**, 288–293, (1956).
2. K.J. Arrow and M. McManus, A note on dynamic stability, *Econometrica* **26**, 448–454, (1958).
3. C.R. Johnson, Sufficient conditions for D-stability, *Journal of Economic Theory* **9**, 53–62, (1974); *Systems and Control Letters* **5**, 279–283, (1985).
4. H.K. Khalil and P.V. Kokotovic, D-stability and multiparameter singular perturbation, *SIAM J. Control Optim.* **17**, 56–65, (1979).
5. D.D. Šiljak, *Large Scale Dynamic Systems: Stability and Structure*, North Holland, Amsterdam, (1978).
6. E.A. Abed and A.L. Tits, On the stability of multiple time-scal system, *INT. J. Control* **44** (1), 211–218, (1986).
7. J.M. Martin and G.A. Hewer, Smallest destabilizing perturbations for linear systems, *Int. J. Control* **45**, 1495–1504, (1987).
8. E.H. Abed, Strong D-stability, *Systems and Control Letters* **7** (3), 207–212, (June 1986).
9. M. Fiedler and V. Ptak, On matrices with non-positive off diagonal elements and positive principal minors, *Czech. Math. J.* **12**, 382–400, (1962).
10. A. Berman and R.J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Siam, Philadelphia, PA, (1994).
11. P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, Academic Press, New York, (1985).
12. C.R. Johnson, Hadamard products of matrices, *Linear and Multilinear Algebra* **1**, 295–307, (1974).
13. M. Araki and B. Kondo, Stability and transient behavior of composite nonlinear systems, *IEEE Trans. Automatic Control* **AC-24**, 537–541, (1972).
14. J.R. Munkres, *Topology, a first course*, Prentice-Hall, Englewood Cliffs, NJ, (1975).
15. C.R. Johnson, D-stability and real and complex quadratic forms, *Linear Algebra and its Applications* **9**, 89–94, (1974).
16. M. Vidyasagar, *Nonlinear Systems Analysis*, Prentice-Hall, NJ, (1987).
17. C.F. Kuen, Analysis and applications of strong D-stability, In *Thesis for the Degree of Master of Science*, Faculty of the Graduate School of the University of Maryland, (1987).