N-COMPACTNESS AND θ-CLOSED SETS

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Proposition 2.4. For a space $X$ with the topology $\tau$, the following are equivalent:

1. $X$ is a Hausdorff space.
2. For all nested open sets $A \subseteq B$, there exists a point $x \in B \setminus A$.
3. For all points $x \in X$, there exist disjoint open neighborhoods $U_x$ and $V_x$.
4. For all points $x, y \in X$, there exist disjoint open neighborhoods $U_x$ and $V_y$.

Example 2.1. Let $X$ be with the topology $\tau$, then $X$ is Hausdorff if and only if there exists a Hausdorff space $Y$ with $X$ as a dense subspace of $Y$. In this case, $X$ is Hausdorff if and only if $Y$ is Hausdorff.

2. Basic Results

$\square$ 4.2. Theorem 2.7. A space $X$ is Hausdorff if and only if for any two disjoint closed subsets $A$ and $B$ of $X$, there exist disjoint open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

Corollary 2.3. A space $X$ is Hausdorff if and only if for any two disjoint closed subsets $A$ and $B$ of $X$, there exist disjoint open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

Theorem 2.8. A space $X$ is Hausdorff if and only if for any two disjoint closed subsets $A$ and $B$ of $X$, there exist disjoint open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

Corollary 2.4. A space $X$ is Hausdorff if and only if for any two disjoint closed subsets $A$ and $B$ of $X$, there exist disjoint open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.
Definition 3.4. Let $X$ be a set. A family $\mathcal{F}$ of subsets of $X$ is said to have the finite intersection property if, for every finite collection $\{F_1, F_2, \ldots, F_n\} \subseteq \mathcal{F}$, the intersection $F_1 \cap F_2 \cap \cdots \cap F_n$ is non-empty.

Theorem 3.2. A Hausdorff space $(X, \mathcal{H})$ is $X$-closed if and only if every family of subsets of $X$ that has the finite intersection property is a finite union of closed sets in $X$.

Proof. Let $F$ be $X$-closed and $\mathcal{F}$ be a family of sets such that $\mathcal{F}$ has the finite intersection property. We want to show that $\bigcap \mathcal{F}$, the intersection of all sets in $\mathcal{F}$, is non-empty. If $\bigcap \mathcal{F}$ were empty, then $\mathcal{F}$ would not have the finite intersection property, a contradiction. Therefore, $\bigcap \mathcal{F}$ is non-empty.

Corollary 3.2. For a space $X$, the following are equivalent:

(a) $X$ is $X$-closed.
(b) For every family $\mathcal{F}$ of subsets of $X$, if $\bigcap \mathcal{F}$ is non-empty, then $\bigcap \mathcal{F}$ is a finite union of closed sets.
(c) For every family $\mathcal{F}$ of subsets of $X$, $\bigcap \mathcal{F}$ is a finite union of closed sets in $X$.

Theorem 3.6. For a space $X$, the following are equivalent:

(a) Every finite family of $X$-closed sets with the finite intersection property has non-empty intersection.
(b) Every finite family of $X$-closed sets is finite.
(c) Every finite family of $X$-closed sets with the finite intersection property is finite.
(d) A Hausdorff space $(X, \mathcal{H})$ is $X$-closed.
Corollary 4.7. If X is compact and every point set is G-closed, then 

\[ X \text{ is a normal space if and only if } X \text{ is a compact space.} \]

Corollary 4.8. If X is a normal and completely regular space, then X is a Hausdorff space.

Corollary 4.9. If X is a normal and completely regular space, then X is a normal space.

Definition 4.2. A space X is said to be compactly closed if X is a Hausdorff space.

Theorem 4.1. A Hausdorff space X is compactly closed.

Proof. Let A be a closed subset of X, then A is compactly closed.

Theorem 4.2. A Hausdorff space X is compactly closed if and only if X is a normal space.

Theorem 4.3. Let X be a normal space and let A be a closed subset of X. Then A is compactly closed if and only if X is a Hausdorff space.

As a consequence of Theorem 4.3 we get the following corollaries.

Corollary 4.10. If X is a completely regular normal space, then X is a compact Hausdorff space.

Corollary 4.11. If X is a compact Hausdorff space, then X is a normal space.

Corollary 4.12. If X is a normal space, then X is a Hausdorff space.

4. Normed spaces and G-closed sets

definition 4.4. A set A in a normed space X is called G-closed if there exists a closed set B in X such that A = \( X \setminus B \).

Theorem 4.4. If A and B are two closed sets in a normed space X, then A \( \cap \) B is a G-closed set.

Proof. Let x be a point in A \( \cap \) B. Then x is in both A and B. Since A and B are closed, x is in \( X \setminus (X \setminus A) \) and in \( X \setminus (X \setminus B) \). Hence x is in \( X \setminus (X \setminus (A \cup B)) \), which is a closed set containing x. Therefore x is in A \( \cap \) B.

Corollary 4.5. A normed space X is compactly closed if and only if X is a Hausdorff space.

212
Theorem A.6. Let \( \mathcal{X} \) be a complete linearly connected space with \( X \). Then \( X \) is a complete linearly connected space if and only if \( X \) is locally connected.

Theorem A.7. If \( X \) is a complete linearly connected space and \( \mathcal{X} \) is a complete linearly connected space with \( X \), then \( \mathcal{X} \) is a complete linearly connected space with \( X \).

Theorem A.8. Let \( \mathcal{X} \) be a complete linearly connected space with \( X \). Then \( X \) is a complete linearly connected space if and only if \( X \) is locally connected.

Theorem A.9. Let \( \mathcal{X} \) be a complete linearly connected space with \( X \). Then \( X \) is a complete linearly connected space if and only if \( X \) is locally connected.

Theorem A.10. Let \( \mathcal{X} \) be a complete linearly connected space with \( X \). Then \( X \) is a complete linearly connected space if and only if \( X \) is locally connected.

Theorem A.11. A complete linearly connected space with \( X \) is a complete linearly connected space with \( X \).

Corollary A.1. Every compact subset of a complete linearly connected space is compact.

Corollary A.2. Every compact subset of a complete linearly connected space is compact.

Proof. Let \( \mathcal{X} \) be a complete linearly connected space with \( X \). Then \( X \) is a complete linearly connected space with \( X \).

Theorem A.12. A complete linearly connected space with \( X \) is a complete linearly connected space with \( X \).

Proof. Let \( \mathcal{X} \) be a complete linearly connected space with \( X \). Then \( X \) is a complete linearly connected space with \( X \).
Theorem 4.16. Let $X$ be a Hausdorff space and $A \subseteq X$. Then the

$\check{g}$-continuous

function $f : X \to \{0, 1\}$ is the inverse image of a $\check{g}$-closed subset of a Hausdorff $\check{g}$-continuous $g$-

Hausdorff space $\check{g}$-

Theorem 4.17. Let $f : X \to Y$ be a Hausdorff space if $f$ is $\check{g}$-

continuous, then $X$ is connected and $f$ is $\check{g}$-continuous. Since $A$ is a closed subset of $X$, $X - A$ is Hausdorff, then $X$ is Hausdorff and $A$ is Hausdorff space.

Theorem 4.18. Let $f : X \to Y$ be $\check{g}$-continuous and $X$ be $\check{g}$-compact. If each closed set in $X$

is $\check{g}$-compact, then $X$ is $\check{g}$-compact.