

ALMOST CONTINUITY IMPLIES CLOSURE CONTINUITY[†]

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Abstract. The purpose of this note is to answer in the affirmative a long standing open question raised by Singal and Singal—whether every almost continuous function is closure continuous (θ -continuous).

Introduction. Among other generalizations of continuity, the concepts of weak and closure continuity have been studied by many mathematicians: D. R. Andrew, J. Chew, L. Herrington, N. Levine, P. E. Long, T. Noire, J. Porter, M. Saleh, J. Tong, E. K. Whittlesy and others. In 1961, Levine introduced the concept of weak continuity as a generalization of continuity: later in 1966, Andrew and Whittlesy [2] introduced the concept of closure continuity which is stronger than weak continuity. Indeed closure continuity was introduced many years earlier by S. Fomin [3] precisely in 1941 as θ -continuity, but it seems that Andrew and Whittlesy were not aware of the paper by Fomin. In 1968, Singal and Singal [9] introduced almost continuity as another generalization and raised the following question in Remark 3.3: is every almost continuous function θ -continuous? In this short note we answer this question positively.

Definitions and notation. Let A be a subset of a topological space X . The closure and the interior of A in X are denoted, respectively, by \bar{A} , A° . A function $f : X \rightarrow Y$ is *closure continuous* (θ -continuous) at $x \in X$ if, given any open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(\bar{U}) \subseteq \bar{V}$. If this condition is satisfied at each $x \in X$, then f is said to be *closure continuous* (θ -continuous). A function $f : X \rightarrow Y$ is said to be *almost continuous in the sense of Singal and Singal* if for each point $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subseteq \bar{V}^\circ$.

THEOREM 1. *Let $f : X \rightarrow Y$ be almost continuous. Then f is closure continuous.*

Proof. Let $x \in X$ and let V be an open set containing $f(x)$. By almost continuity of f , there exists an open set U containing x such that $f(U) \subseteq \bar{V}^\circ$. Let $y \in \bar{U}$. For any open set W containing $f(y)$ there exists, by almost continuity of f , an open set A containing y such that $f(A) \subseteq \bar{W}^\circ$. Since $y \in \bar{U}$, we have $A \cap U \neq \emptyset$. Therefore, $\emptyset \neq f(A \cap U) \subseteq \bar{V}^\circ \cap \bar{W}^\circ \subseteq \bar{W}$. Since $\bar{V}^\circ \cap \bar{W}^\circ$ is open we have $\bar{V}^\circ \cap \bar{W}^\circ \cap W \neq \emptyset$: that is, $\bar{V}^\circ \cap W \neq \emptyset$. Since this is true for every open set containing $f(y)$ we have $f(y) \in \bar{V}$. Also since this is true for every $y \in \bar{U}$ we obtain $f(\bar{U}) \subseteq \bar{V}$: that is, f is closure continuous.

Recall that a subset A of a space X is called *closure (almost) compact* if every open cover of A has a finite subcollection whose closures cover A . Closure compactness was introduced as $H(i)$ -spaces in [8] as a generalization of absolutely closed (H -closed) spaces in [1].

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Clearly every compact set is closure compact but not conversely as is shown in the next example.

EXAMPLE 1. Let X be any uncountable space with the cocountable topology. Then every subset of X is closure compact, but the only compact subsets of X are the finite ones.

The next theorem shows that closure compactness is preserved under closure continuous functions.

THEOREM 2. *Let $f: X \rightarrow Y$ be closure continuous and let K be a closure compact subset of X . Then $f(K)$ is a closure compact subset of Y .*

Proof. Let \mathcal{V} be an open cover of $f(K)$. For each $k \in K, f(k) \in V_k$ for some $V_k \in \mathcal{V}$. By closure continuity of f , there exists an open set U_k containing x such that $f(\overline{U}_k) \subseteq \overline{V}_k$. The collection $\{U_k : k \in K\}$ is an open cover of K and so, since K is closure compact, there is a finite subcollection $\{U_k : k \in K_0\}$, where K_0 is a finite subset of K , and $\{\overline{U}_k : k \in K_0\}$ covers K . Clearly $\{\overline{V}_k : k \in K_0\}$ covers $f(K)$ and thus $f(K)$ is a closure compact subset of Y .

COROLLARY 1. *Let $f: X \rightarrow Y$ be almost continuous and let K be a closure compact subset of X . Then $f(K)$ is a closure compact subset of Y .*

As a consequence of the corollary, we get Theorem 3.3 and Lemmas 3.2, 3.3 in [10] and Theorem 3.4 in [4].

REFERENCES

1. P. Alexandroff and P. Urysohn, Mèmoire sur les espaces topologiques compacts, *Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. I* **14** (1929), 1–96.
2. D. R. Andrew and E. K. Whittlesy, Closure continuity, *Amer. Math. Monthly* **73** (1966), 758–759.
3. S. V. Fomin, Extensions of topological spaces, *C. R. Dokl. Akad. Sci. URSS (M. S.)* **32** (1941), 114–116.
4. L. Herrington, Properties of nearly-compact spaces, *Proc. Amer. Math. Soc.* **45** (1974), 431–436.
5. P. E. Long and E. E. McGehee, Properties of almost continuous functions, *Proc. Amer. Math. Soc.* **24** (1970), 175–180.
6. P. E. Long and D. A. Carnahan, Comparing almost continuous functions, *Proc. Amer. Math. Soc.* **38** (1973), 413–418.
7. T. Noire, On weakly continuous mappings, *Proc. Amer. Math. Soc.* **46**(1) (1974), 120–124.
8. C. T. Scarborough and A. H. Stone, Products of nearly compact spaces, *Trans. Amer. Math. Soc.* **124** (1966), 131–147.
9. M. K. Singal and A. R. Singal, Almost continuous mappings, *Yokohama Math. J.* **16** (1968), 63–73.
10. M. K. Singal and A. Mathur, On nearly compact spaces, *Boll. Un. Mat. Ital.* **4**(2) (1969), 702–710.

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