

A NOTE ON TIGHTNESS

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Abstract. The purpose of this note is to prove a results of Jain and López-Permouth under a weaker conditions replacing R -weak injectivity by R -tightness and even getting a simpler proof.

1. Introduction. Throughout this paper all rings are associative with identity and all modules are unitary right modules. We denote the category of all right R -modules by $\text{Mod-}R$. Given a module M_R the injective hull of M in $\text{Mod-}R$ is denoted by $E(M)$. The purpose of this paper is to further the study of the concept of tightness [1], [4]. Following Jain and López-Permouth, given two modules M and $N \in \text{Mod-}R$, a module M is N -tight if every quotient of N which is embeddable in $E(M)$ is embeddable in M . A module is *tight* if it is tight relative to all finitely generated submodules of $E(M)$.

A ring R is called right *CEP*-ring if every cyclic right R -module is essentially embeddable in a projective module. In this paper we assume all modules are unital right R -modules unless otherwise indicated.

We start first with some basic results that will be needed in this note.

LEMMA 1. *Let R be an artinian ring, and let N, M be finitely generated modules. If M is N -tight and N is M -tight and $\text{Soc}(M) \simeq \text{Soc}(N)$ then $M \simeq N$.*

Proof. Let $\sigma: N \rightarrow E(M)$ be the monomorphism induced by the isomorphism between $\text{Soc}(M)$ and $\text{Soc}(N)$. Since M is N -tight, N is embeddable in M . Similarly, M is embeddable in N . Since M and N are finitely generated over artinian ring, $M \simeq N$.

LEMMA 2. [2, 3, 4] *A right CEP-ring is right artinian. All projective indecomposable right modules over a right CEP-ring are uniform.*

LEMMA 3. *Let R be a right artinian ring such that all indecomposable projective right R -modules are uniform and R -tight. Then the following holds:*

- (i) every simple right R -module is isomorphic to the socle of an indecomposable projective module,
- (ii) every simple right R -module is embeddable in $\text{Soc}(R)$,
- (iii) if P and Q are projective right modules with $\text{Soc}(P) \simeq \text{Soc}(Q)$ then $P \simeq Q$.

Proof. The proof follows from Lemma 1 and [3, Lemma 5. 1].

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The next Theorem was proved by Jain and López-Permouth in [3] with the assumption that every indecomposable projective right module is weakly R -injective. We show that the theorem is true under a weaker condition. In fact all we need is to have every indecomposable projective right module is R -tight.

THEOREM. *A ring R is a right CEP-ring if and only if the following holds:*

- (i) R is right artinian,
- (ii) every indecomposable projective right module is uniform and R -tight.

Proof. Let R be a CEP-ring. By Lemma 2, R is right artinian. Let P be an indecomposable projective right module. Once more Lemma 2 implies that $Soc(P)$ is simple. Let $\sigma : R/I \rightarrow E(P)$ be a monomorphism. Then $Soc(R/I) \simeq Soc(E(P)) = Soc(P)$. Since R is a CEP-ring, R/I embeds essentially in some projective module, say, Q . Thus $Soc(P) \simeq Soc(Q)$. Hence by Lemma 3, $P \simeq Q$, and thus R/I embeds in P , proving that P is R -tight. Conversely, assume R satisfies the two conditions.

Write $R = \bigoplus_{i=1}^n e_i R$ as a direct sum of indecomposable right ideals. Let I be a right ideal of R . By Lemma 3, $Soc(R/I) \simeq \bigoplus_{i=1}^k Soc(e_i R)^{n_i}$. Let $P = \bigoplus_{i=1}^k (e_i R)^{n_i}$. Since $Soc(P) = Soc(E(P))$, the above isomorphism between $Soc(R/I)$ and $\bigoplus_{i=1}^k Soc(e_i R)^{n_i} = Soc(P)$ may be looked upon as an essential embedding $\varphi : Soc(R/I) \rightarrow E(P)$. This extends into an essential embedding $\hat{\varphi} : R/I \rightarrow E(P)$. By tightness, there exists an embedding $\sigma : R/I \rightarrow P$ which is essential in P , proving that R is a CEP-ring.

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