ON WEAK INJECTIVITY AND WEAK PROJECTIVITY

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Abstract

Given a right R-module M, a module $Q \in \sigma[M]$ is said to be weakly injective in $\sigma[M]$ if for every finitely generated submodule N of the M-injective hull $\widehat{Q}$, N is contained in a submodule Y of $\widehat{Q}$ such that $Y \simeq Q$. Weakly projective modules in $\sigma[M]$ are defined dually. Several characterizations of (weakly) semisimple modules are given in terms of tight and cotight modules in $\sigma[M]$.

1. INTRODUCTION

Throughout this paper all rings are associative with identity and all modules are unitary. Any terminology used but not defined in this paper will be standard. Sources for standard terminology include [1, 4, 13]. We denote the category of all right R-modules by Mod-R and for any $M \in$ Mod-R, $\sigma[M]$ stands for the full subcategory of Mod-R whose objects are submodules of M-generated modules (see [13]). Given a module X_R the injective hull of X in Mod-R (resp., in $\sigma[M]$) is denoted by $E(X)$ (resp., $\widehat{X}$). The purpose of this

paper is to further the study of the concepts of weak injectivity (projectivity) in $\sigma[M]$ studied in [3], and [14].

Given two modules Q and $N \in \sigma[M]$, we call Q weakly N - injective in $\sigma[M]$ if for every homomorphism $\varphi : N \to \widehat{Q}$, there exists a homomorphism $\widehat{\varphi} : N \to Q$ and a monomorphism $\sigma : Q \to \widehat{Q}$ such that $\varphi = \sigma\widehat{\varphi}$. Equivalently, there exists a submodule X of $\widehat{Q}$ such that $\varphi(N) \subset X \simeq Q$. A module $Q \in \sigma[M]$ is called *weakly injective* in $\sigma[M]$ if for every finitely generated submodule N of the M-injective hull $\widehat{Q}$, N is contained in a submodule Y of $\widehat{Q}$ such that $Y \simeq Q$. Equivalently, if Q is weakly N-injective for all finitely generated modules N in $\sigma[M]$.

A module X is *N-tight* in $\sigma[M]$ if every quotient of N which is embeddable in the M-injective hull of X is embeddable in X. A module X is *tight* in $\sigma[M]$ if it is tight in $\sigma[M]$ relative to all finitely generated submodules of its M-injective hull.

Given two modules Q and $N \in \sigma[M]$, we call Q *weakly N-projective* in $\sigma[M]$ if for every homomorphism $\varphi : P(Q) \to N$, where $P(Q)$ is the $\sigma[M]$-projective cover, there exists a homomorphism $\widehat{\varphi} : Q \to N$ and an epimorphism $\sigma : P(Q) \to Q$ such that $\varphi = \widehat{\varphi}\sigma$. Equivalently, if for every homomorphism $\varphi : P(Q) \to N$, there exists a submodule X of $ker(\varphi)$ such that $P(Q)/X \simeq Q$. A module $Q \in \sigma[M]$ is called *weakly projective* in $\sigma[M]$ if it is weakly N- projective for all finitely M-generated modules N in $\sigma[M]$. Given two modules Q and $N \in \sigma[M]$, we call Q *N-cotight* in $\sigma[M]$ if for every epimorphism $\varphi : P(Q) \to N$, where $P(Q)$ is the $\sigma[M]$-projective cover, there exists an epimorphism $\widehat{\varphi} : Q \to N$. A module M is *(weakly) semisimple* if every module X in $\sigma[M]$ is (weakly) injective in $\sigma[M]$.

2. Weak- Projectivity (Cotightness) in $\sigma[M]$.

In this section we study some of the basic results on weak projectivity (cotightness) in $\sigma[M]$.

Theorem 2.1. Let $N, Q \in \sigma[M]$. If Q has a projective cover P in $\sigma[M]$ via an epimorphism $\pi : P \to Q$. Then Q is N-projective in $\sigma[M]$ if and only if for every homomorphism $\varphi : P \to N$, there exists $\widehat{\varphi} : Q \to N$ such that $\widehat{\varphi}\pi = \phi$. Equivalently, $\phi(ker\pi) = 0$.

Proof. Only if direction. Let $\varphi : P \to N$ be a homomorphism. We shall first show that $\varphi(Ker\pi) = 0$. Let $T = \varphi(Ker\pi)$ and let $\pi_T : N \to N/T$ be the natural projection. Then φ induces $\hat{\varphi}\ (q) = \pi_T\varphi(p)$, where $q = \pi(p)$. It follows that $\hat{\varphi}\ \pi = \pi_T\varphi$. Since Q is $N-$projective in $\sigma[M]$, there exists a map $\beta : Q \to N$ such that $\hat{\varphi} = \pi_T\varphi$. Clearly, $(\varphi - \beta\pi)P \subseteq T$. We claim that $\phi(ker\pi) = 0$.

Let $X = \{p \in P|\varphi(p) = \beta\pi(p)\}$. We shall show that $X = P$. Let $x \in P$. Since $(\varphi - \beta\pi)(x) \in T = \varphi(Ker\pi)$, there exists $k \in Ker\pi$ such that $(\varphi - \beta\pi)(x) = \varphi(k)$. Thus, $\varphi(x - k) = \beta\pi(x - k)$, since $\pi(k) = 0$. Therefore $x - k \in X$. Thus, $P(M) = X + ker\varphi$. This interns imply that $P(M) = X$, since $ker\varphi \ll P(M)$. Therefore, $(\varphi - \beta\pi)P(M) = 0$. In particular, $(\varphi - \beta\pi)Ker\pi = 0$, yielding $\varphi(\ker\pi) = 0$. Equivalently, there exists $\varphi' : Q \to N$ such that $\varphi'\pi = \varphi$.

Conversely, let $\Phi : Q \to N/K$ be a homomorphism. The projectivity of P implies the existence of $\Phi' : P \to N$ such that $\Phi\pi = \pi_K\Phi'$. By our hypothesis there exists $\Phi'' : Q \to N$ such that $\Phi''\pi = \Phi'$. It follows easily that $\pi_K\Phi'' = \Phi$, proving that Q is $N-$projective in $\sigma[M]$.

The next theorem is a very useful characterization of weak projectivity.

Theorem 2.2. Let $N, Q \in \sigma[M]$. If Q has a projective cover P in $\sigma[M]$ via an epimorphism $\pi : P \to Q$. Then Q is weakly N-projective in $\sigma[M]$ if and only if for every homomorphism $\varphi : P \to N$ there exists $X \subset ker\varphi$ such that $P/X \simeq Q$.

Proof. Let $\varphi : P \to N$ be a homomorphism. Assume that Q is weakly $N-$ projective in $\sigma[M]$ and let $\widehat{\varphi} : Q \to N$ be the homomorphism and $\sigma : P \to N$ the epimorphism as in the definition of weak $N-$projectivity. Since $\varphi = \widehat{\varphi}\sigma$, $\ker\sigma \subseteq \ker\varphi$. Thus the implication is proven by taking $X = \ker\sigma$. Conversely, if $X \subseteq P$ satisfies the conditions in the statement of the theorem, then the isomorphism $P/X \cong Q$, composed with the natural projection $\pi_X : P \to P/X$ is an epimorphism σ satisfies that $\ker\sigma = X \subseteq \ker\varphi$. It follows that the mapping $\widehat{\varphi} : Q \to N$ given by $\widehat{\varphi}(q) = \varphi(p)$, whenever $\sigma(p) = q$ is well-defined and satisfies $\varphi = \widehat{\varphi}\sigma$, proving that Q is weakly $N-$projective.

For cotightness, following similar proof as in Theorem 2.2 one gets the following characterization.

Theorem 2.3. Let $N, Q \in \sigma[M]$. If Q has a projective cover P in $\sigma[M]$ via an epimorphism $\pi : P \to Q$. Then Q is N-cotight in $\sigma[M]$ if and only if for every homomorphism $\varphi : P \to N$ there exists $X \subset ker\varphi$ and $K \subset Q$ such that $P/X \simeq Q/K$.

The class of weak projectivity in $\sigma[M]$ is closed under submodules and quotient modules.

Proposition 2.4. For modules $N, L \in \sigma[M]$, the following conditions are equivalent:

(a) L is weakly N-projective in $\sigma[M]$;

(b) L is weakly K-projective in $\sigma[M]$ for every submodule K of N;

(c) L is weakly N/K-projective in $\sigma[M]$ for every submodule K of N;

(d) for every submodule K of N, and for every epimorphism $\varphi : P(L) \to K$, where $P(L)$ is the $\sigma[M]$-projective cover, there exists an epimorphism $\widehat{\varphi} : K \to L$ and an epimorphism $\sigma : P(L) \to L$ such that $\varphi = \widehat{\varphi}\sigma$.

Proof. $(a) \Rightarrow (b)$. Let K be a submodule of N and let $\varphi : P(L) \to K$ be a homomorphism. Since L is weak N-projective, $f = i_K\varphi$ factors through L by an epimorphism $\sigma : P(L) \to L$ and a homomorphism $\widehat{f} : L \to N$. Since σ is onto, the range of $\widehat{f}$ equals the range of f and so is contained in K. Define $\widehat{\varphi} : Q \to K$ by $\widehat{\varphi}(q) = \widehat{f}(q)$. Then it follows that $\varphi = \widehat{\varphi}\sigma$.

$(b) \Rightarrow (c)$. Let K be a submodule of N and let $\varphi : P(L) \to N/K$ be a homomorphism. By the projectivity of $P(L)$,there exists a homomorphism $\widehat{\varphi} : P(L) \to N$ such that $\varphi = \pi_K\widehat{\varphi}$. Since L is weakly $N-$projective, there exists an epimorphism $\sigma : P(L) \to L$ and a homomorphism $\widehat{\sigma} : L \to N$ such that $\widehat{\varphi} = \widehat{\sigma}\sigma$. It follows that $\varphi = \pi_K\widehat{\sigma}\sigma$, proving that L is weakly $N/K - projective$.

$(c) \Rightarrow (d)$ and $(d) \Rightarrow (a)$ are straightforward.

Finite direct sums of weakly projectives (cotights) in $\sigma[M]$ and superfluous covers of weakly projective modules are also weakly projective in $\sigma[M]$.

Proposition 2.5. For modules N, L and $K \in \sigma[M]$, we have the following:

(a) if L and K are weakly N-projective (cotight) in $\sigma[M]$, then $L \oplus K$ is weakly N-projective (cotight) in $\sigma[M]$,

(b) if L is weakly N-projective in $\sigma[M]$ and K is a superfluous cover of L then K is weakly N-projective in $\sigma[M]$,

(c) if a module X in $\sigma[M]$ is weakly projective relative to its projective cover in $\sigma[M]$, then X is projective in $\sigma[M]$. Consequently, a finitely generated weakly projective module in $\sigma[M]$ is indeed projective in $\sigma[M]$.

Proof. Straightforward from the definition.

Proposition 2.6. Let $\{X_i\}_I$ be a class of weakly N-projectives (cotight) in $\sigma[M]$ and $\bigoplus_I X_i$ has a projective cover in $\sigma[M]$. Then $\bigoplus_I X_i$ is weakly N-projective (cotight) in $\sigma[M]$.

Proof. The proof follows directly from the fact that in this case $P(\bigoplus_I X_i) = \bigoplus_I P(X_i)$.

The next theorem shows the difference between weak-projectivity and cotightness in $\sigma[M]$.

Theorem 2.7. Given modules $N, Q \in \sigma[M]$, and assume Q is supplemented and has a projective cover $\pi : P \to Q$ in $\sigma[M]$. Then Q is weakly N-projective in $\sigma[M]$ if and only if for every submodule K of N and for every epimorphism $\varphi : P \to K$, there exists an epimorphism $\widehat{\varphi} : Q \to K$ such that for every supplement L' of $ker\widehat{\varphi}$ in Q, there exists a submodule L of P such that $P/L \simeq Q/L'$ and $L + ker\varphi = P$.

Proof. Assume Q is weakly N-projective in $\sigma[M]$ and let $\varphi : P \to K$ be an epimorphism onto a submodule $K \subset N$. Then there exists an epimorphism $\sigma : P \to Q$ and a homomorphism $\widehat{\varphi} : Q \to K$ such that $\varphi = \widehat{\varphi}\sigma$. Let L' be a supplement of $\ker\widehat{\varphi}$ in Q and let $L = \sigma^{-1}(l')$. For an arbitrary $p \in P$, $\sigma(p)$ may be written as $\sigma(p) = l' + k'$, with $l' \in L'$ and $k' \in ker\widehat{\varphi}$. It follows that $\varphi(p) = \widehat{\varphi}\sigma(p) = \widehat{\varphi}(l') + \widehat{\varphi}(k') = \widehat{\varphi}(l')$. Choose $p_1 \in \sigma^{-1}(l') \subset L$. Then $\sigma(p_1) = l'$. On the other hand, $\varphi(p_1) = \widehat{\varphi}\sigma(p_1) = \widehat{\varphi}(l') = \varphi(p)$. So $p - p_1 \in \ker\varphi$ and so $L + ker\varphi = P$. The fact that $P/L \cong Q/L'$ follows since L is the kernel of the epimorphism $\pi_{L'}\sigma : P \to Q/L'$. Conversely, let us assume that for every submodule $K \subset N$ and for every epimorphism $\varphi : P \to K$ there exists an epimorphism $\widehat{\varphi} : Q \to K$ such that $\varphi = \widehat{\varphi}\sigma$ and for every supplement L' of $ker\widehat{\varphi}$ in Q, there exists a submodule $L \subset P$ such that $P/L \simeq Q/L'$ and $L + ker\varphi = P$. Let $\varphi : P \to K$ be an epimorphism and $\widehat{\varphi} : Q \to K$ the corresponding epimorphism. All we need is to produce another epimorphism $\sigma : P \to Q$ such that $\varphi = \widehat{\varphi}\sigma$. Let L' be a supplement for $ker\widehat{\varphi}$ and let L be the corresponding submodule of P. Let $\psi : P/L \to M/L'$ be an isomorphism. The Chinese remainder theorem yields that the map $m + ker\widehat{\varphi} \cap L' \to (M + ker\widehat{\varphi}, m + L')$ is an isomorphism between $M/(Ker\widehat{\varphi} \cap L')$ and $M/ker\widehat{\varphi} \times M/L'$. Also, $M/ker\widehat{\varphi} \cong K$ via $m + ker\widehat{\varphi} \to \widehat{\varphi}(m)$. So, one gets an isomorphism $\Psi : M/Ker\widehat{\varphi} \cap L' \to K \times M/L'$ such that $\Psi(m + Ker\widehat{\varphi} \cap L') = (\widehat{\varphi}(m), \pi_{L'}(m))$. The isomorphism Ψ induces an onto map $\Phi = \Psi\pi_L : P \to M/L'$. Since $L + ker\varphi = P$, the map $\alpha : P \to K \times M/L'$ given by $\alpha(p) = (\varphi(p), \Phi(p))$ is onto. The induced epimorphism $\alpha' = \Psi^{-1}\alpha : P \to M/(Ker\widehat{\varphi} \cap L')$ may then be lifted to a map $\sigma : P \to Q$. Since $Ker\widehat{\varphi} \cap L' \ll Q$, σ is indeed an epimorphism. It only remains to show that $\varphi = \widehat{\varphi}\sigma$. Let us refer for the rest of the proof to $\pi_{Ker\widehat{\varphi}\cap L'}$ simply as π. We do know that $\pi\sigma = \sigma' = \Psi^{-1}\alpha$, hence $\Psi\pi\sigma = \alpha$. Let $p \in P$ be arbitrary. Then $\Psi(\sigma(p) + Ker\widehat{\varphi} \cap L') = \alpha(p) = (\varphi(p), \Phi(p))$. On the other hand, $\Psi(\sigma(p) + Ker\widehat{\varphi} \cap L') = (\widehat{\varphi}\sigma(p), \sigma(p) + L')$. Comparing the first component in both expressions yields the desired equality, proving that Q is weakly N-projective in $\sigma[M]$.

Corollary 2.8. Given modules $N, Q \in \sigma[M]$. If Q is hollow then Q is N-cotight in $\sigma[M]$ iff Q is weakly N-projective in $\sigma[M]$.

Proposition 2.9. Given modules $N, Q \in \sigma[M]$. If Q is self-projective and N-cotight in $\sigma[M]$, then Q is indeed N-projective in $\sigma[M]$.

Proof. Let $\varphi : P \to N$. Since Q is $N-$cotight in $\sigma[M]$ there exists an epimorphism $\widehat{\varphi} : Q \to Im(\varphi)$ and by the projectivity of P, there exists a homomorphism $f : P \to Q$ such that $\varphi = \widehat{\varphi}f$. By self-projectivity of Q and Theorem 3.1, there exists a homomorphism $\widehat{f} : Q \to Q$ such that $f = \widehat{f}\pi$. It follows that $\varphi = \widehat{\varphi}\widehat{f}\pi$, proving that Q is $N-$projective.

A finitely generated direct summand S of the projective cover of a cotight module X in $\sigma[M]$ yields a direct summand (isomorphic to S) of X.

Proposition 2.10. Let Q be a cotight module in $\sigma[M]$ whose projective cover in $\sigma[M]$ has a finitely generated direct summand S. Then Q has a direct summand isomorphic to S.

Proof. Since S is finitely generated, Q is $S-$cotight. Thus the projection map $\pi_S : P(Q) \to S$ yields an epimorphism $\pi'_S : Q \to S$. Since S is projective we get $Q \cong S \oplus \ker \pi'_S$, proving our claim.

Proposition 2.11. Let M_R be locally noetherian, and let Q, N be finitely generated in $\sigma[M]$. If Q is N-cotight in $\sigma[M]$ and N is Q-cotight in $\sigma[M]$ and $Q/J(Q) \simeq N/J(N)$ then $Q \simeq N$.

Proof. Let $\sigma : P(Q) \to N$ be the epimorphism induced by the isomorphism between $Q/J(Q)$ and $N/J(N)$. Since Q is N-cotight in $\sigma[M]$, N is a homomorphic image of Q. Similarly, Q is a homomorphic image of N. Since Q and N are finitely generated over a locally noetherian module, $Q \simeq N$.

3. Weak- Injectivity (tightness) in $\sigma[M]$.

In this section we dualize most of the basic results on weak projectivity in $\sigma[M]$ given in the previous section and the proof is dualizable in most of these cases.

Proposition 3.1. Let $Q, N \in \sigma[M]$. Then Q is weakly N-injective in $\sigma[M]$ if and only if for every homomorphism $\varphi : N \to \widehat{Q}$, there exists a submodule X of $\widehat{Q}$ such that $\varphi(N) \subset X \simeq Q$.

The class of weak injectivity in $\sigma[M]$ is closed under submodules and quotient modules as it is shown in the next proposition.

Proposition 3.2. For modules N, $L \in \sigma[M]$, the following conditions are equivalent:

(a) L is weakly N-injective in $\sigma[M]$;
(b) L is weakly K-injective in $\sigma[M]$ for every submodule K of N;
(c) L is weakly N/K-injective in $\sigma[M]$ for every submodule K of N;
(d) for every submodule K of N, and for every monomorphism $\varphi : N/K \to \widehat{L}$, there exists a monomorphism $\widehat{\varphi} : N/K \to L$ and a monomorphism $\sigma : N/K \to \widehat{L}$ such that $\varphi = \sigma\widehat{\varphi}$.

Proposition 3.3. For modules N, L and $K \in \sigma[M]$, we have the following:
(a) if L and K are weakly N-injective (tight) in $\sigma[M]$ then $L \oplus K$ is weakly N-injective (tight) in $\sigma[M]$,
(b) if L is weakly N-injective in $\sigma[M]$ and L is an essential submodule of K then K is weakly N-injective in $\sigma[M]$.

Proposition 3.4. Given modules $N, Q \in \sigma[M]$, Q is weakly N-injective in $\sigma[M]$ if and only if for every submodule K of N and for every monomorphism φ: $N/K \to \widehat{Q}$, there exists a monomorphism $\widehat{\varphi} : N/K \to Q$, and for every complement L of $\widehat{\varphi}(N/K)$ in Q, there exists $L' \subset \widehat{Q}$ such that $L' \cap \varphi(N/K) = 0$ and $L' \simeq L$.

Corollary 3.5. Given modules $N, Q \in \sigma[M]$. If Q is uniform then Q is N-tight in $\sigma[M]$ iff Q is weakly N-injective in $\sigma[M]$.

Proposition 3.6. Given modules $N, Q \in \sigma[M]$. If Q is self-injective and N-tight in $\sigma[M]$, then Q is indeed N-injective in $\sigma[M]$.

Proposition 3.7. Let M be a locally artinian module, and let N, Q be finitely generated modules in $\sigma[M]$. If Q is N-tight in $\sigma[M]$ and N is Q-tight in $\sigma[M]$ and $Soc(Q) \simeq Soc(N)$ then $Q \simeq N$.

Proof. Let $\sigma : N \to E(Q)$ be the monomorphism induced by the isomorphism between $Soc(Q)$ and $Soc(N)$. Since Q is N-tight in $\sigma[M]$, N is embeddable in Q. Similarly, Q is embeddable in N. Since Q and N are finitely generated over a locally artinian module, $Q \simeq N$.

4. A Characterization of Semisimple Modules.

Given a module M, it is easy to show that every module $K \in \sigma[M]$ is a direct summand of a tight module $Q = K \oplus (\widehat{K})^{(\alpha)}$ in $\sigma[M]$, where α is an infinite cardinal number. Similarly, if M is projective and perfect in $\sigma[M]$, then for every module $K \in \sigma[M]$, $K \oplus P(K)^{(\omega)}$, where ω is an infinite cardinal number is cotight in $\sigma[M]$. The proof of the next two theorems follows easily from the above disscution. First recall that a module M is (weakly) semisimple if every $K \in \sigma[M]$ is (weakly) injective in $\sigma[M]$.

Theorem 4.1. For a module M_R. The following are equivalent:
(a) M is semisimple;
(b) M is projective and perfect and every cotight module in $\sigma[M]$ is (quasi-) discrete;
(c) M is projective and perfect and every discrete module is cotight in $\sigma[M]$;
(d) M is projective and perfect and every cotight module in $\sigma[M]$ is (quasi-) continuous;
(e) every tight module in $\sigma[M]$ is (quasi-) discrete;
(f) every tight module in $\sigma[M]$ is (quasi-) continuous;
(g) every continuous module is cotight in $\sigma[M]$;
(h) every (direct summand of a) tight module in $\sigma[M]$ is (injective) projective in $\sigma[M]$;
(i) M is projective and perfect and every cotight module in $\sigma[M]$ is injective (projective) in $\sigma[M]$;
(j) M is projective and perfect in $\sigma[M]$ and every direct summand of a cotight module in $\sigma[M]$ is cotight in $\sigma[M]$;
(k) M is projective and perfect in $\sigma[M]$ and every (direct summand of a) cotight module in $\sigma[M]$ is quasi-projective in $\sigma[M]$;
(l) every direct summand of a tight module in $\sigma[M]$ is quasi-injective in $\sigma[M]$;
(m) M is projective and perfect in $\sigma[M]$ and every direct summand of a cotight module in $\sigma[M]$ is injective in $\sigma[M]$.

Theorem 4.2. For a module M_R. The following are equivalent:
(a) M is weakly semisimple;

(b) M is projective and perfect and every direct summand of a cotight module in $\sigma[M]$ is weakly injective in $\sigma[M]$;
(c) every direct summand of weakly injective module in $\sigma[M]$ is weakly injective in $\sigma[M]$.

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N-COMPACTNESS AND θ-CLOSED SETS

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Abstract

In this paper we introduce a new generalizations of $\delta-$closed and $\delta-$open sets. Using these sets, we obtain a new characterization of $H-$closed spaces. Among other results, it is shown that an N-compact space over which every one point set is $\theta-$closed is a completely regular normal space.

1. Introduction. The concepts of *$\delta-$closure and $\theta-$closure* operators were first introduced by Veli$\check{c}$ko [16]. Although $\theta-$interior and $\theta-$closure operators are not idempotents, the collection of all $\delta-$open sets in a topological space (X, Γ) forms a topology Γ_s on X, called the semiregularization topology of Γ weaker than Γ and the class of all regular open sets in Γ forms an open basis for Γ_s, and the collection of all $\theta-$open sets in a topological space (X, Γ) forms a topology Γ_θ on X weaker than Γ_s. So far, numerous applications of such operators have been found in studying different types of continuous like maps, axioms of separation, and above all, to many important types of compact like properties. For a set A in a space X, let us denote by $Int(A)$ or A^o and $cls(A)$ or $\overline{A}$ for the interior and the closure of A in X, respectively.

Following Veli$\check{c}$ko, a point x of a space X is called a *$\delta-$adherent point* of a subset A of X iff $Int(clsU) \cap A \neq \emptyset$, for every open set U containing x.