Uniqueness for a one-dimensional inverse parabolic problem

Alaeddin Elayyan
Department of Mathematics, Birzeit University PO Box 14, Birzeit, West Bank, Palestine

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Abstract. We prove the uniqueness of a discontinuous principal coefficient of a one-dimensional second-order parabolic equation of the form $a_0 + \chi(Q^*) b$ with known smooth $a_0$ and unknown $b = b(x)$ from all possible lateral boundary measurements of solutions of this equation. In the proofs we make use of singular solutions of parabolic equations.

0. Introduction

We consider the problem of recovery of the coefficient $a$ of the parabolic equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( a \frac{\partial u}{\partial x} \right) = 0 \quad \text{in} \quad Q = (0, 1) \times (0, T)$$

(0.1)

with the initial and boundary conditions

$$u = 0 \quad \text{on} \quad \Omega \times 0, \quad \text{where} \quad \Omega = (0, 1)$$

(0.2)

$$u = g \quad \text{on} \quad S = 0, 1 \times (0, T)$$

(0.3)

when $h = \partial u / \partial v$ is given for all (regular) $g$ on $S$ which satisfy the following compatibility conditions: $g(0) = g(1) = 0$. In this paper we prove the uniqueness of the recovery of discontinuous $a = a_0 + \chi(Q^*) b$, where $\chi(Q^*)$ is the indicator function of an open set $Q^* \subset Q$ with the Lipschitz lateral boundary $\partial_x Q^*$ changing with time, $a = a_0(x)$ and $b = b(x)$ are, respectively, given and unknown $C^2[0, 1]$-functions. In addition, we prove uniqueness in the case of a piece-wise constant $a$ when the Dirichlet-to-Neumann map is only given on one part of the boundary. Uniqueness for multidimensional parabolic equations ($n \geq 2$) has been proven in [3] where the connectedness of $Q \setminus Q^*$ is assumed for every fixed $t$, which we cannot assume for the one-dimensional problem. For elliptic equations nonuniqueness for the one-dimensional case has been shown in the paper of Isakov [6], but for higher dimensions ($n \geq 2$), uniqueness has been proven in the papers of Kohn and Vogelius [9] (piece-wise analytic $a$) and by Isakov [6] (Lipschitz $Q^*$ and smooth $b$). Also, for elliptic equations, when one is making use of only one set of $u, \partial u / \partial v$ on $\partial \Omega$, some partial global uniqueness results for $Q^*$ have been obtained by Friedman and Isakov [5]. In connection with parabolic equations we can also refer to the paper of Bellout [2] studying local stability in the inverse problem. This inverse parabolic problem is fundamental, in particular, in the search for ground water [11], and is important for many engineering applications.

We introduce some notation. For standard notation we refer to Friedman [4] and Ladyzenskaja et al [10]. For an open set $Q$ in the layer $\mathbb{R} \times (0, T)$, the lateral boundary $\partial_x Q$, is the $x$-boundary, that is the closure of the set $\partial Q \setminus \{ \tau = 0 \text{ or } T \}$. We say that $Q$ is $x$-Lipschitz if its $x$-boundary is locally the graph of a function $x = \gamma(t)$ that is Lipschitz,
and $a$ is piece-wise Lipschitz constant on $Q$ if there exists a cover $\{Q_j\}_{j=1}^n$ of $Q$ such that $a = \sum \chi(Q_j)a_j$, $Q_j$ are open $x$-Lipschitz sets, $Q_j \subset \Omega \times (-T, 2T)$, $a_j \neq a_{j+1}$ on $\partial Q_j$, $a_j$ are constants, $j = 1, \ldots, n$.

1. Statement of results

We are interested in finding an open set $Q_j$ and a function $b_j$ entering the parabolic initial-boundary value problem

\[
\frac{\partial u_j}{\partial t} - \frac{\partial}{\partial x} \left( a_j \frac{\partial}{\partial x} u_j \right) = 0 \quad \text{in} \quad Q
\]

\[
u_j = g \quad \text{on} \quad S
\]

\[
u_j = 0 \quad \text{on} \quad \Omega \times 0
\]  

where

\[
a_j = a_0 + \chi(Q_j)b_j, \quad b_j \neq 0 \quad \text{on} \quad \partial Q_j.
\]  

We are also interested in finding an open set $Q_j$ and a function $a_j$ entering the parabolic initial-boundary value problem

\[
\frac{\partial u_j}{\partial t} - \frac{\partial}{\partial x} \left( a_j \frac{\partial}{\partial x} u_j \right) = 0 \quad \text{in} \quad Q
\]

\[
u_j = g \quad \text{on} \quad S' = 1 \times (0, T)
\]

\[
u_j = 0 \quad \text{on} \quad S'' = 0 \times (0, T)
\]

\[
u_j = 0 \quad \text{on} \quad (0, 1) \times 0
\]

where $a$ is piece-wise constant.

It is well known that there is a unique (generalized) solution, $u_j$, of these problems when $g \in C^{2+\ell}(S)$, and $u_j \in C^{\ell}(Q)$ for some $\ell \in (0, 1)$, $\frac{\partial}{\partial x} u_j \in L_2(Q)$, and $\in C(Q \setminus Q_j)$. For this and for other results about the direct parabolic problem, (1.1)–(1.4), we refer the reader to the books of Friedman [4] and Ladyzenskaja et al [10, pp 153, 204, 227].

**Theorem 1.** Suppose that $a_j = a_0 + \chi(Q_j)b_j$, $j = 1, 2$, $b_j \neq 0$ on $\partial Q_j$, $a_0 = a_0(x)$ and $b = b(x)$ are, respectively, given and unknown $C^2[0, 1]$-functions, and $Q_1$, $Q_2$ are open $x$-Lipschitz sets, $Q_j \subset \Omega \times (-T, 2T)$.

If solutions $u_j$ to the initial-boundary value problems (1.1)–(1.3) satisfy the equality

\[
\frac{\partial u_1}{\partial v} = \frac{\partial u_2}{\partial v} \quad \text{on} \quad S \text{ (v is a normal)}
\]

for all $g \in C^2(\tilde{S})$, then

\[
a_1 = a_2 \quad \text{on} \quad Q.
\]

This result guarantees the uniqueness of the reconstruction of $Q_j$ from all possible lateral measurements for an arbitrary $T > 0$.

**Theorem 2.** Suppose that $a_1$, $a_2$ are two piece-wise Lipschitz constants, and suppose that solutions $u_j$ to the initial-boundary value problems (1.1)–(1.3) satisfy the equality

\[
\frac{\partial u_1}{\partial v} = \frac{\partial u_2}{\partial v} \quad \text{on} \quad S'
\]

for all $g \in C^2(\tilde{S}')$, then

\[
a_1 = a_2 \quad \text{on} \quad Q.
\]
The paper is organized as follows: in section 2 we show that the validity of equality (1.6) for all Dirichlet boundary data \( g \) implies certain integral relations which can be interpreted as orthogonality relations. To prove uniqueness, in sections 4 and 5 we modify an approach from [6] (use of singular solutions with the pole in those orthogonality relations to obtain a contradiction when the pole converges to the boundary of one of the domains \( Q_j \)). To show that some integrals in these relations are bounded while one of them is not we use the estimates of integrals of singular solutions given in section 3. In the last section we prove theorem 2.

2. Orthogonality relations

In this section we assume that the conditions of theorem 1 are satisfied and obtain some auxiliary relations which will be used in its proof.

Denote by \( \tilde{Q}_{3t} \) the component of the open set \( \Omega \setminus (\tilde{Q}_{1t} \cup \tilde{Q}_{2t}) \) whose boundary meets \( S \). Here \( Q_{j\theta} = Q_j \cap \{t = \theta\} \), \( j = 1, 2 \). Let \( Q_3 = \cup Q_{3t} \) over \( 0 < t < T \) and let \( Q_4 = \Omega \setminus \tilde{Q}_3 \).

**Lemma 2.1.**

\[
\int_{Q_3} b_1 \frac{\partial v_1}{\partial x} \frac{\partial u_j^*}{\partial x} \, dx \, dt = \int_{Q_4} b_2 \frac{\partial v_1}{\partial x} \frac{\partial u_j^*}{\partial x} \, dx \, dt
\]  

(2.1)

for all solutions \( v_1 \) to equation (1.1), with \( j = 1 \) near \( \tilde{Q}_4 \), that are zero when \( t < 0 \), and solutions \( u_j^* \) to the adjoint equation

\[
\frac{\partial u_j^*}{\partial t} + \frac{\partial}{\partial x} \left( a_2 \frac{\partial u_j^*}{\partial x} \right) = 0 \quad \text{near } \tilde{Q}_4, \text{ that are zero when } t > T.
\]

**Proof.** From well known results about the regularity of solutions to the parabolic initial-boundary value problem (1.1)–(1.3) it follows that \( u_j \) is in \( C^{2,1}(Q_3) \) and in \( H^{2,1}(Q_5) \), where \( Q_5 = V \times (0, T) \) and \( V \) is a vicinity of \( \partial \Omega \) in \( \Omega \). Due to conditions (1.2), (1.5), both \( u_1 \) and \( u_2 \) have the same Cauchy data on \( S \) and satisfy the same parabolic equation in \( Q_3 \). Thus, from the uniqueness of continuation for second-order parabolic equations (see e.g. [7], corollary 1.2.4) we conclude that \( u_1 = u_2 \) on \( Q_3 \). Letting \( u = u_2 - u_1 \) and subtracting equations (1.1) with \( j = 1 \) from those with \( j = 2 \) we get

\[
\frac{\partial}{\partial x} \left( (a_0 + b_2 \chi(Q_2)) \frac{\partial u}{\partial x} \right) - \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( (b_1 \chi(Q_1) - b_2 \chi(Q_2)) \frac{\partial u_1}{\partial x} \right) \quad \text{in } Q. \quad (2.2)
\]

Now, using the definition of a weak solution to the parabolic equation under consideration we obtain

\[
\int_Q \left( (a_0 + b_2 \chi(Q_2)) \frac{\partial u}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial u}{\partial t} \psi \right) = \int_Q \left( (b_1 \chi(Q_1) - b_2 \chi(Q_2)) \frac{\partial u_1}{\partial x} \frac{\partial \psi}{\partial x} \right)
\]  

(2.3)

for any function \( \psi \) from \( H_0^{1,1}(Q) \). Since \( u \) and \( \chi(Q_j) \) are zero outside \( \tilde{Q}_4 \cap \{t < T\} \) this relation remains valid for any function \( \psi \) from \( H^{1,1}(Q_5) \) (where \( Q_6 \) is an arbitrary vicinity of \( Q_4 \) that is zero when \( t > T \)).

If \( \psi = u_j^* \) is an \( H^{1,1}(Q_6) \) solution to the adjoint equation from lemma 2.1, then integrating the left side of (2.3) by parts with respect to \( t \) and using the definition of a weak solution to this adjoint equation with the test function \( u \) which is zero outside \( \tilde{Q}_4 \cap \{t < T\} \), we conclude that the left-hand side in (2.3) is zero. So we have the relation (2.1) with \( u_1 \) instead of \( v_1 \).

Now, by using the Runge property we extend equality (2.1) onto all \( v_1 \) solving equation (1.1) with \( j = 1 \) near \( \tilde{Q}_4 \), and satisfying the initial condition (1.3). Denote the space of such \( v_1 \) by \( X \). It is sufficient to prove that solutions \( u_1 \) to the initial-boundary value problem (1.1)–(1.3) with \( j = 1 \) (for various \( g \) supported in \( S \)) approximate in \( L_2(Q_7) \) any solution from \( X \). We
denote the space of solutions to (1.1)–(1.3) by \( X_1 \). Indeed, let \( v_1 \in X \). Then we can similarly approximate it by solutions from \( X \) in \( L_2(\Omega) \), where \( \Omega \) is a Lipschitz domain containing \( Q_4 \) with \( \text{dist}(\partial_x Q_4, Q_4) > 0 \). From well known interior Schauder-type estimates for parabolic equations it follows that these solutions from \( X_1 \) will approximate \( v_1 \) in \( H^{1,0}(\Omega) \).

To prove \( L_2 \)-approximation, in view of the Hahn–Banach theorem, it is sufficient to show that if \( f \) from the dual space \( L_2(Q_4) \) is orthogonal to \( X_1 \), then \( f \) is orthogonal to \( X \).

Let \( \Omega = (0, 1) \), and \( \Omega_0 \) be a bounded interval such that \( \Omega \subset \Omega_0 \), \( \Omega \neq \Omega_0 \). Let \( K(x, t; y, s) \) be the Green function to the first initial value problem for the operator \( \frac{\partial}{\partial t} + \frac{\partial}{\partial x}(a \frac{\partial}{\partial x}) \) in \( Q_0 = \Omega_0 \times (0, T) \). Let \( f \) be orthogonal to \( X_1 \). The Green potential

\[
U(x, t; f) = \int_{Q_4} f K(x, t; )
\]

is equal to zero on \( Q_0 \setminus \bar{Q}_4 \) because the function \( u_1 = K(x, t; ) \) belongs to \( X_1 \) if \((x, t) \in Q_0 \setminus \bar{Q}_4 \). Since \( \text{supp} \, f \subset \bar{Q}_4 \), this potential is a solution to the equation

\[
-\frac{\partial}{\partial t} (a_0 \nabla u) = \frac{\partial u}{\partial t} \quad \text{on} \quad Q_0 \setminus \bar{Q}_4.
\]

The coefficient \( a_0 \) belongs to \( C^1(\bar{Q}_0) \), so this equation has the property of the unique continuation. Therefore, \( U(:, f) = 0 \) on \( Q_0 \setminus \bar{Q}_4 \). Now, let \( v \in X_1 \), then \( v \) is a solution to the homogeneous equation near \( Q_3 \cup \partial_x Q_5 \), where \( Q_3 \) is an open set with a \( C^\infty \) lateral boundary and \( \text{dist}(\partial_x Q_5, \partial_x Q_4) > 0 \). Using the representation of \( v \) by a single-layer potential, we obtain

\[
v(y, s) = \int_{\partial_x Q_5} g K(; y, s) \, d\Gamma
\]

for some \( g \in C(\partial_x Q_3) \). By using this representation (2.4) and the Fubini theorem we yield

\[
\int_{Q_4} f v = \int_{\partial_x Q_5} g U(:, f) = 0
\]

because \( U(:, f) = 0 \) on \( \partial_x Q_5 \). Accordingly, relation (2.1) is valid for any \( v_1 \) satisfying the conditions of lemma 2.1. Hence the proof is complete. \( \square \)

Assuming that

\[
Q_1 \neq Q_2
\]

then we may assume that \( Q_1 \) is not contained in \( Q_2 \); hence, we conclude that there is a point \((x_0, t_0) \in \partial Q \setminus \bar{Q}_2 \) such that \((x_0, t_0) \in \partial_x Q_3 \). By considering \( g = 0 \) for \( t < t_0 \) and using the translations \( t \to t - t_0, x \to x - x_0 \), we can reduce the general case to \( t = 0, x = 0 \). We can choose a ball \( B = (0, \varepsilon) \) and a cylinder \( Z = B(0, \varepsilon) \) such that \( \bar{B} \subset \Omega, Z \) does not intersect \( \bar{Q}_2 \), and \((\partial_x Q_1) \cap \bar{Z} \) is a Lipschitz curve. Due to the well known variants of the Whitney extension theorem there is a \( C^2(\bar{Q}_1 \cup \bar{Z}) \)-function \( a_3 \) that coincides with \( a_1 \) on \( Q_1 \). Extend \( a_3 \) onto \( Q \setminus (\bar{Q}_1 \cup Z) \) as \( a_2 \).

Lemma 2.2. Under the conditions of lemma 2.1,

\[
\int_{Q_1} b_1 \frac{\partial u_3}{\partial x} \frac{\partial u_3}{\partial x} = \int_{Q_2} b_2 \frac{\partial u_3}{\partial x} \frac{\partial u_3}{\partial x}
\]

for any solution \( u_3 \) to the equation \( \frac{\partial}{\partial t}(a_3 \frac{\partial u_3}{\partial x}) - \frac{\partial}{\partial x}(u_3) = 0 \) near \( \bar{Q}_4 \) which is zero when \( t < 0 \) and for any solution \( u_3^* \) from lemma 2.1.
Proof. Consider \( u_3 \) and let \( Q_8 \) be an open set with a \( C^\infty \)-boundary \( \partial_x Q_8 \) and containing \( Q_4 \) with \( \text{dist}(\partial_x Q_8, Q_4) > 0 \) such that \( u_3 \) is a solution to the equation

\[
\frac{\partial}{\partial x} \left( a_3 \frac{\partial u_3}{\partial x} \right) - \frac{\partial}{\partial t} (u_3) = 0 \quad \text{near } Q_8.
\]

Introduce a sequence of open sets \( Q_{4k} \) such that (i) \( Q_{4k} \setminus Z = Q_4 \setminus Z \), (ii) the (Hausdorff) distance from \( \partial Q_{4k} \cap Z \) is less than \( 1/k \) and \( \partial_x Q_{4k} \cap Z \) does not intersect \( Q_4 \). Define a coefficient \( a_{3k} \) as \( a_{3k} \) on \( Q_8 \setminus (Q_{4k} \setminus Q_4) \) and as \( a_0 \) on \( Q_{4k} \setminus Q_4 \). Since \( \partial Q_4 \cap Z \) is a Lipschitz surface we have

\[
\text{meas}_n \{ a_{3k} \neq a_3 \} \to 0 \quad \text{as} \quad k \to +\infty. \tag{2.6}
\]

Let \( u_{3k} \) be solutions to the initial-boundary value problems:

\[
\frac{\partial}{\partial x} \left( a_{3k} \frac{\partial u_{3k}}{\partial x} \right) - \frac{\partial}{\partial t} (u_k) = 0 \quad \text{in } Q_8, \quad u_{3k} = u_3 \quad \text{on } \partial_x Q_8.
\]

\[
 u_{3k} = 0 \quad \text{on } Q_8 \cap \{ t = 0 \}.
\]

Since \( a_{3k} = a_0 + \chi(Q_1) b_1 \) near \( Q_1 \) relation (2.1) is valid for any \( u_1 = u_{3k} \). The difference \( u_k = u_{3k} - u_3 \) satisfies the equation

\[
\frac{\partial}{\partial x} \left( a_{3k} \frac{\partial u_{3k}}{\partial x} \right) - \frac{\partial}{\partial t} (u_k) = \frac{\partial}{\partial x} \left( (a_3 - a_{3k}) \frac{\partial u_{3k}}{\partial x} \right) \quad \text{in } Q_8
\]

and \( u_k = 0 \) on \( \partial Q_8 \cap \{ t < T \} \) because \( u_{3k} \) and \( u_3 \) coincide on the lateral boundary of \( Q_8 \) and when \( t = 0 \). From the definition of a weak solution to this initial-boundary value problem with the test function \( u_k \) we have

\[
\int_{Q_8} a_{3k} \frac{\partial u_k}{\partial x} \frac{\partial u_k}{\partial x} + \int_{Q_8 \cap \{ t = T \}} u_k^2/2 = \int_{Q_8} (a_3 - a_{3k}) \frac{\partial u_{3k}}{\partial x} \frac{\partial u_k}{\partial x}.
\]

According to the assumptions, \( \varepsilon < a_{3k} \) for certain positive \( \varepsilon \). Using this inequality, dropping the second integral in the left-hand side, and majorating the right-hand side by the inequality \( x \cdot y \leq \varepsilon^{-1}/2 \vert x \vert^2 + \varepsilon/2 \vert y \vert^2 \) we obtain

\[
\int_{Q_8} \varepsilon \frac{\partial u_k}{\partial x} \leq C(\varepsilon) \int_{Q_8} \vert a_3 - a_{3k} \vert^2 \frac{\partial u_{3k}}{\partial x}^2 + \varepsilon/2 \int_{Q_8} \varepsilon \frac{\partial u_k}{\partial x}^2.
\]

Since \( \frac{\partial}{\partial x} u_k \) belongs to \( L_2(Q_8) \) we conclude from (2.6) that the first integral in the right-hand side tends to zero. Therefore, \( \frac{\partial}{\partial x} u_k \) converges to zero in \( L_2(Q_8) \). Putting \( u_1 = u_{3k} = u_3 + u_k \) into the relation (2.1) and letting \( k \to \infty \), we complete the proof of lemma 2.2. \( \Box \)

3. Estimates of integrals of singular solutions

We make use of the solutions \( u_3, u_2 \) with singularities outside \( Q_4 \). Solutions of elliptic equations of second order with arbitrary power singularities were constructed by Alessandrini [1]. To simplify obtaining bounds on the integrals of such solutions we introduce new variables. According to our assumptions, \( \partial_x Q_1 \) near the origin is the graph of a Lipschitz function \( x = q_1(t) \) which can be assumed to be defined and Lipschitz on \( \mathbb{R} \). The substitution

\[
x = x^* + q_1(t), \quad t = t^*
\]

transforms equations (1.1) into similar equations with additional first-order differentiation with respect to \( x^* \) multiplied by a Lipschitz function of \( t \). The domains \( Q_j \) are transformed onto domains with similar properties and with the additional property that the points \( (0, t), 0 < t < T \) belong to \( \partial_x Q_1 \). Further on we drop the superscript ".
Let $K^+$ be the fundamental solution of the Cauchy problem for the forward parabolic equation $\frac{\partial}{\partial t} (a_1 \frac{\partial u}{\partial x}) - \frac{\partial}{\partial t} (u) = 0$ in $\ast$-coordinates. Let $K^-$ be the fundamental solution of the backward Cauchy problem for the backward parabolic equation $\frac{\partial}{\partial t} (a_2 \frac{\partial u}{\partial x}) + \frac{\partial}{\partial t} (u) = 0$ in these coordinates. It is known that

$$K^+ = K_1^+ + K_0^+ \quad K^- = K_1^- + K_0^- \quad (3.1)$$

where $K_1^+, K_1^-$ are the principal parts of $K^+$, $K^-$ (parametics) and $K_0^+, K_0^-$ are the remainders. The principal parts are

$$K_1^+(x, t; y, \tau) = \frac{C}{(t - \tau)^{1/2}} \exp \left( -\frac{|x - y|^2}{a_3(x)(t - \tau)} \right)$$

$$K_1^-(x, t; y, \tau) = \frac{C}{(t - \tau)^{1/2}} \exp \left( -\frac{|x - y|^2}{a_0(x)(t - \tau)} \right). \quad (3.2)$$

From the known bounds of fundamental solutions of parabolic equations ([10], p.377) we have

$$\left| \frac{\partial}{\partial x} K_0^+(x, t; y, \tau) \right| \leq \frac{C}{(t - \tau)^{1/2}} \exp \left( -\frac{|x - y|^2}{C(t - \tau)} \right)$$

$$\left| \frac{\partial}{\partial x} K_0^-(x, t; y, \tau) \right| \leq \frac{C}{(t - \tau)^{1/2}} \exp \left( -\frac{|x - y|^2}{C(t - \tau)} \right). \quad (3.3)$$

When $(y, 0), (y, \tau)$ are outside $Q$, the functions $K^+(y, 0), K^-(y, \tau)$ are $(x, t)$-solutions to the homogeneous parabolic equations with bounded measurable coefficients satisfying zero initial and final conditions. Using lemma 2.2 with $u_3 = K^+(y, 0), u_3^* = K^-(y, \tau)$ we get

$$\int_{Q_t \cap \Omega} b_1 \frac{\partial}{\partial x} K^+(y, 0) \frac{\partial}{\partial x} K^-(y, \tau) = - \int_{Q_t \setminus \Omega} b_1 \frac{\partial}{\partial x} K^+(y, 0) \frac{\partial}{\partial x} K^-(y, \tau)$$

$$+ \int_{Q_t^*} b_2 \frac{\partial}{\partial x} K^+(y, 0) \frac{\partial}{\partial x} K^-(y, \tau). \quad (3.4)$$

From the estimates (3.3) and similar estimates for $\frac{\partial}{\partial x} K_1^+, \frac{\partial}{\partial x} K_1^-$ we conclude that the integrands are majorated by an integrable function uniformly with respect to $y$ outside $Q_1$, by the Lebesgue dominated convergence theorem we may let $y \to 0$, and replace $y$ in (3.4) by zero.

Using representation (3.1) we obtain from (3.4) that

$$|I_1| \leq |I_2| + |I_3| \quad (3.5)$$

where

$$I_1 = \int_{Q_t \cap \Omega} b_1 \frac{\partial}{\partial x} K_1^+(0, 0) \frac{\partial}{\partial x} K_1^- (0, \tau)$$

is formed from the principal parts of $K$ and the remainders are collected in

$$I_2 = \int_{Q_t \cap \Omega} b_1 \frac{\partial}{\partial x} K_0^+ (0, 0) \frac{\partial}{\partial x} K_0^- (0, \tau) + \frac{\partial}{\partial x} K_0^+ (0, 0) \frac{\partial}{\partial x} K_0^- (0, \tau)$$

and

$$I_3 = \int_{Q_t^*} b_2 \frac{\partial}{\partial x} K^+(0, 0) \frac{\partial}{\partial x} K^- (0, \tau) + \int_{Q_t^*} b_2 \frac{\partial}{\partial x} K^+(0, 0) \frac{\partial}{\partial x} K^- (0, \tau).$$

In the following three lemmas we bound $I_1$ from below and $I_2, I_3$ from above.

**Lemma 3.1.**

$$|I_1| \geq \frac{1}{C t^{1/2}}$$

where $C$ is independent of $t$. 

**Proof.** Using that $b_1(0) \neq 0$ and choosing $\varepsilon$ sufficiently small, we obtain

\[
|I_1| \leq C^{-1} \int_0^\varepsilon \int_0^\varepsilon \frac{\partial}{\partial x} K_1^+(x,t;0,0) \frac{\partial}{\partial x} K_1^-(x,t;0,0) \, dx \, dt
\]

\[
= C^{-1} \int_0^\varepsilon \int_0^\varepsilon \frac{x}{2t^{3/2}} \exp\left(-\frac{x^2}{a_3(x)t}\right) \frac{x}{(\tau - t)^{3/2}} \exp\left(-\frac{x^2}{a_0(x)(\tau - t)^{3/2}}\right) \, dx \, dt
\]

\[
\leq C^{-1} \int_0^\varepsilon \int_0^\varepsilon \frac{x^2}{t(\tau - t)^{3/2}} \exp\left(-\frac{x^2}{mt(\tau - t)}\right) \, dx \, dt,
\]

where $m = \inf(a_3, a_0)$ over $Q$.

Using the inequality

\[
\frac{1}{t\tau} \leq \frac{1}{t(\tau-t)} \leq \frac{2}{t\tau} \quad \text{when} \quad 0 < t < \tau/2
\]

from below we bound the above integral by

\[
C^{-1} \int_0^\varepsilon \int_0^\varepsilon \frac{x^2}{t(\tau - t)^{3/2}} \exp\left(-\frac{2x^2}{mt}\right) \, dx \, dt = \frac{1}{C\tau^{3/2}} \int_0^\varepsilon \int_0^\varepsilon \frac{x^2}{t^{3/2}} \exp\left(-\frac{2x^2}{mt}\right) \, dx \, dt
\]

\[
= \frac{1}{C\tau^{3/2}} \int_0^\varepsilon \int_0^\frac{\varepsilon}{\tau} \int_{w/2}^\infty \frac{x}{w^{1/2}} \exp\left(-\frac{2x^2}{mt}\right) \, dx \, dw \, dt
\]

\[
\leq \frac{1}{C\tau^{3/2}} \int_0^\varepsilon \int_0^\frac{\varepsilon}{\tau} \left(\int_0^\infty \frac{x}{w^{1/2}} \exp\left(-\frac{2x^2}{mt}\right) \, dx \right) w^{-1/2} \exp\left(-w/2\right) \, dw \, dt
\]

\[
\leq \frac{1}{C\tau^{3/2}} \int_0^\varepsilon \int_0^\frac{\varepsilon}{\tau} \frac{1}{w^{1/2}} \exp\left(-w/2\right) \, dw \, dt
\]

The proof is complete. \qed

**Lemma 3.2.**

\[|I_2| \leq C\]

where $C$ is independent of $t$.

**Proof.** We bound the integral of the first of the three functions forming $I_2$ as defined after (3.5):

\[|I_2| \leq C^{-1} \int_0^\varepsilon \int_0^\varepsilon \frac{\partial}{\partial x} K_1^+(x,t;0,0) \cdot \frac{\partial}{\partial x} K_1^-(x,t;0,0) \, dx \, dt
\]

\[= C^{-1} \int_0^\varepsilon \int_0^\varepsilon \frac{x}{2t^{3/2}} \exp\left(-\frac{x^2}{a_3(x)t}\right) \frac{1}{(\tau - t)^{3/2}} \exp\left(-\frac{x^2}{a_0(x)(\tau - t)^{3/2}}\right) \, dx \, dt.
\]

Using the inequality

\[
\frac{|x|}{t^{3/2}} \exp(-|x|^2/4t) \leq \frac{C}{t} \exp(-|x|^2/8t)
\]

we get

\[
\leq C^{-1} \int_0^\varepsilon \int_0^\varepsilon \frac{1}{t} \exp\left(-\frac{x^2}{a_3(x)t}\right) \frac{1}{(\tau - t)^{3/2}} \exp\left(-\frac{x^2}{a_0(x)(\tau - t)^{3/2}}\right) \, dx \, dt
\]

\[
\leq C^{-1} \int_0^\varepsilon \int_0^\varepsilon \frac{1}{t^{1/2}(\tau - t)^{1/2}} \exp\left(-\frac{x^2\tau}{Mt(\tau - t)}\right) \, dt \, dx,
\]

where $M = \sup(a_3, a_0)$ over $Q$. 


where we used the symmetry of $t$ and that the maximum of $\exp(-\frac{x^2}{M(t-t')})$ was attained at $t = \tau/2$. Using inequality (3.6), we bound the integral from above by

$$
C^{-1} \int_0^\tau \int_0^{\tau/2} \frac{1}{t^{1/2}(t')^{1/2}} \exp\left(-\frac{x^2}{M} \right) \, dt \, dx = \frac{1}{C^{1/2}} \int_0^\tau \int_0^{\tau/2} \frac{1}{t} \exp\left(-\frac{x^2}{M} \right) \, dt \, dx
$$

$$
= \frac{1}{C^{1/2}} \int_0^\tau \int_0^{\infty} w^{-1} e^{-w} \, dw \, dz, \quad \text{where we substituted } w = 2x^2/mt
$$

$$
= \frac{1}{C^{1/2}} \int_0^\tau \int_0^{\infty} \left( \int_0^{\min(e,(Mt\tau)^{1/2})} 1 \, dx \right) w^{-1} e^{-w} \, dw
$$

$$
\leq \frac{1}{C^{1/2}} \int_0^\tau \int_0^{\infty} \left( \int_0^{(Mt\tau)^{1/2}} 1 \, dx \right) w^{-1} e^{-w} \, dw
$$

$$
= C \int_0^\infty w^{-1/2} e^{-w} \, dw = C.
$$

The proof is complete. \qed

Lemma 3.3.

$$|I_3| \leq C.$$

**Proof.** $I_3$ consists of two integrals. The first one is bounded by

$$C \int_{\varepsilon < x < R} \left| \frac{\partial}{\partial x} K^+ (\tau; 0, 0) \frac{\partial}{\partial x} K^- (\tau; 0, 0) \right|
$$

$$\leq C \int_{\varepsilon < x < R} \int_0^{\tau/2} \frac{1}{t(t-t')^{1/2}} \exp\left(-\frac{x^2 \tau}{M(t-t')} \right) \, dt \, dx.
$$

The bound on $\frac{\partial}{\partial x} K^+ \frac{\partial}{\partial x} K^- \mid$ follows from the direct differentiation of (3.2), the inequality

$$|x|(t-\tau)^{-3/2} \exp(-|x|^2/4(t-\tau)) \leq C(t-\tau)^{-1} \exp(-|x|^2/8(t-\tau))$$

and the bounds (3.3).

Applying, as above, inequality (3.6) we bound the last integral by

$$C \int_\varepsilon^R \int_0^{\tau/2} \frac{1}{t} \exp\left(-\frac{x^2}{M} \right) \, dt \, dx = \frac{C}{\tau} \int_\varepsilon^R \int_0^{\infty} w^{-1} e^{-w} \, dw \, dz
$$

when we use the substitution $w = x^2/(Mt)$. The function $w^{-1/2}$ is decreasing. Replacing it by its value at $2x^2/(Mt)$ we increase the integral, and it becomes

$$\leq \frac{C}{\tau} \int_\varepsilon^R \int_0^{\infty} \left( \frac{2x^2}{Mt} \right)^{-1} e^{-w} \, dw \, dz
$$

$$\leq C \int_\varepsilon^R x^{-2} e^{-\frac{3x^2}{Mt}} \, dw \, dz = C.
$$

\qed
4. Proof of theorem 1

Now we will finish the proof of the theorem. Using lemmas 1–3 it follows that:

\[
\frac{1}{\tau^{1/2}} \leq C.
\]

Choosing \( \tau \) small enough we get a contradiction. This contradiction shows that \( Q_1 = Q_2 \).

The next step of the proof is to show that

\[
b_1 = b_2 \quad \text{on} \quad \partial_x Q_1.
\] (4.1)

As in the proof for \( Q_j \) we assume the opposite. Then we can assume that the origin \( 0 \in \partial_x Q_1 \) and \( b_1(0) > b_2(0) \). By continuity, \( b_1(0) - b_2(0) > C^{-1} \) for some \( C \) in a certain ball \( B \) centred at the origin, let \( Z = B(0, T) \). Extend \( a_2 \) from \( Q_2 \) onto \( \mathbb{R}^n \) as a \( C^2 \)-function \( a_4 > 0 \). By repeating the proof of lemma 3.2 we obtain the following orthogonality relation:

\[
\int_{\partial_x Q_1} (b_1 - b_2) \frac{\partial}{\partial x} u_3 \frac{\partial}{\partial x} u_4^* = 0
\] (4.2)

for all solutions \( u_3 \) to the equation \( \frac{\partial}{\partial x} (a_3 \frac{\partial}{\partial x} u_3) - \frac{\partial}{\partial x} u_3 = 0 \) near \( Q_4 \) which are zero when \( t < 0 \) and for all solutions \( u_4^* \) to the adjoint equation near \( Q_4 \) which are zero when \( t > T \). Let \( K^+ \) be a fundamental solution to the forward Cauchy problem for the first equation and \( K^- \) be the fundamental solution to the backward Cauchy problem for the adjoint equation with the coefficient \( a_4 \). Using the representation (3.1) of these fundamental solutions, splitting \( Q_1 \) into \( Q_1 \cap Z \) and its complement, as in section 3, we obtain from (4.2) the inequality

\[
|I_4| \leq |I_5| + |I_6|
\] (4.3)

where

\[
I_4 = \int_{Q_1 \cap Z} (b_1 - b_2) \frac{\partial}{\partial x} K^+ \frac{\partial}{\partial x} K^-_1
\]

is related to the supposedly singular part and

\[
I_5 = \int_{Q_1 \setminus Z} (b_1 - b_2) \frac{\partial}{\partial x} K^+ \frac{\partial}{\partial x} K^-_1
\]

\[
I_6 = \int_{Q_1} (b_1 - b_2) \frac{\partial}{\partial x} K^+_0 \frac{\partial}{\partial x} K^-_0.
\]

It is easy to see that lemmas 3.1–3.3 are valid for \( I_4, I_5 \) and \( I_6 \), respectively. So we arrive at a contradiction, exactly as in the above proof, i.e. that \( Q_1 = Q_2 \).

This shows that the assumption about \( b_1, b_2 \) is wrong and that \( b_1 = b_2 \) on \( \partial_x Q_1 \).

Let \( \Omega_0 \) be the intersection of all \( Q_{1\theta} \) over \( 0 < \theta < T \). Since \( b_1, b_2 \) do not depend on \( t \) and are equal on \( \partial_x Q_1 \), they coincide on \( Q_{1\theta} \setminus \Omega_0 \). Letting \( Q_0 = \Omega_0 \times (0, T) \) from (4.2) we obtain the relation

\[
\int_{Q_0} (b_1 - b_2) \frac{\partial}{\partial x} u_3 \frac{\partial}{\partial x} u_4^* = 0
\]

for all \( u_3, u_4^* \) in (4.4). As in the proof of lemma 3.2 it implies that

\[
\int_{Q_0} (b_1 - b_2) \frac{\partial}{\partial x} u_5 \frac{\partial}{\partial x} u_6^* = 0
\] (4.4)

for all solutions \( u_5 \) to the equation \( \frac{\partial}{\partial x} ((a_5 + b_1 \chi(Q_0)) \frac{\partial}{\partial x} u_5) - \frac{\partial}{\partial t} u_5 = 0 \) near \( Q_0 \) which are zero when \( t < 0 \) and for all solutions to the adjoint equation \( \frac{\partial}{\partial x} ((a_5 + b_2 \chi(Q_0)) \frac{\partial}{\partial x} u_6^*) + \frac{\partial}{\partial t} u_6^* = 0 \) near \( Q_0 \) which are zero when \( t > T \).
Now we will show that the equations for \( u_5, u_6 \) have the same lateral Dirichlet-to-Neumann maps. Let \( u_5, u_6 \) be the solutions to these equations with zero initial conditions and the same lateral Dirichlet data. By subtracting these equations and letting \( u = u_6 - u_5 \) we obtain
\[
\frac{\partial}{\partial x} \left( (a_0 + b_2 \chi(Q_0)) \frac{\partial}{\partial x} u \right) = \frac{\partial}{\partial x} \left( (b_1 - b_2) \chi(Q_0) \frac{\partial}{\partial x} u_5 \right) \quad \text{in} \quad Q.
\]
From the definition of a weak solution to this equation, we have
\[
\int_0^T a_0(u_5 \psi(1, t) - u_v \psi(0, t)) - \int_Q (a_0 + b_2 \chi(Q_0)) \frac{\partial}{\partial x} u \cdot \frac{\partial}{\partial x} \psi - \int \frac{\partial u}{\partial t} \cdot \psi
\]
for any function \( \psi \in H^{1,1}(Q) \). Using \( \psi = u_6^* \), integrating by parts in the third integral of the left-hand side and using again the definition of a weak solution to the equation
\[
\frac{\partial}{\partial t} \left( (a_0 + b_2 \chi(Q_0)) \frac{\partial}{\partial x} u_5^* \right) + \frac{\partial}{\partial x} u_5^* = 0
\]
with the test function \( u \) which is zero on \( \partial Q \cap \{ t < T \} \) we conclude that the sum of the second and third integrals in the left-hand side is zero. The right-hand side is zero due to (4.4). So the first integral in the left-hand side is zero. Since the lateral Dirichlet data \( \psi = u_6^* \) can be any function in \( C^\infty(\partial \Omega \times (0, T)) \) we get \( u_v = 0 \) on \( \partial \Omega \times (0, T) \). So, \( u_{3v} = u_{60} \) on the lateral boundary, which means that we have the same lateral Dirichlet-to-Neumann maps.

Take as the Dirichlet data \( g \), a function which does not depend on \( t \) when \( t > \tau \). Since the coefficients of the equations \( \frac{\partial}{\partial t} \left( (a_0 + b_1 \chi(Q_0)) \frac{\partial}{\partial x} u_j \right) - \frac{\partial}{\partial t} u_j = 0 \) are time independent, the solutions \( u_j(x, t) \) of the initial boundary value problems on \( \Omega \times (0, \infty) \) will be analytic with respect to \( t > \tau \). They have the same Cauchy data on \( \partial \Omega \times (0, \infty) \), therefore, as above by the uniqueness in the lateral Cauchy problem, \( u_5 = u_6 \) on \( \Omega \setminus Q_0 \times (0, T) \). By uniqueness of the analytic continuation, they are also equal on \( (\Omega \setminus Q_0) \times (0, \infty) \). Then we use theorem 9.2.1 in [8] to get \( b_1 = b_2 \) on \( \Omega_0 \), and so on \( Q \). The proof is complete. \( \square \)

5. Proof of theorem 2

To prove theorem 2 we need the following lemma.

Lemma 5.1.
\[
\int_Q (a_1 - a_2) \frac{\partial}{\partial x} u_1 \frac{\partial}{\partial x} u_2^* = 0 \quad \text{(5.1)}
\]
for all solutions \( u_1 \) to the equation \( \frac{\partial}{\partial x} (a_1 \frac{\partial}{\partial x} u_1) - \frac{\partial}{\partial t} u_1 = 0 \) in \( Q \) which are zero when \( t < 0 \) and satisfy (1.2') and for all solutions \( u_2^* \) to the adjoint equation \( \frac{\partial}{\partial x} (a_2 \frac{\partial}{\partial x} u_2^*) + \frac{\partial}{\partial x} u_2^* = 0 \) near \( Q \) which are zero when \( t > T \) and \( u_2^* = 0 \) on \( S'' \) and \( u_2^* = \psi \) on \( S' \) where \( \psi \in C_0^{2, \lambda}(S') \).

Proof of the lemma. Denote by \( \Lambda \) the Dirichlet-to-Neumann operator which maps the Dirchlet data \( u|_{\partial \Omega \times (0, T)} \) into the Neumann data \( (\frac{\partial}{\partial x} u)|_{\partial \Omega \times (0, T)} \).

Using [10], a generalized solution of equation (0.1) is a function \( u \in H^{1,1}(Q) \) which satisfies the identity
\[
\int_Q a \frac{\partial}{\partial x} u \frac{\partial}{\partial x} \phi - \int_Q u \phi_t = \int_S (\Lambda u) \phi \quad \text{(5.2)}
\]
for every \( \phi \in H^{1,1}(Q) \) that is zero for \( t = T \), and also zero in \( S'' \).
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Letting $\phi = u^*$ in (5.2), where $u^*$ is a solution to the adjoint equation

$$\frac{\partial}{\partial t}(u^*) + \frac{\partial}{\partial x} \left( a_j \frac{\partial}{\partial x} u^* \right) = 0 \quad \text{in} \quad Q$$

which is zero on $S''$, we have

$$\int_Q a_j \frac{\partial}{\partial x} u^* \frac{\partial}{\partial x} \phi + \int_Q u^* \frac{\partial}{\partial t} \phi = \int_S \Lambda u \cdot u^*.$$ 

Integrating the first integral by parts, we get

$$\int_S \Lambda u^* \cdot u = \int_S \Lambda u \cdot u^*$$

where we used that $u^*$ satisfies the adjoint equation (5.3).

Recalling that $(,)(S)$ denotes the scalar product in the Hilbert space of real-valued $L_2$ functions and using (5.5) we get that

$$(\Lambda u^*, u)(S) = (u^*, \Lambda u)(S)$$

for any solution $u$ to the first equation and any solution $u^*$ to the adjoint equation, which implies that

$$\Lambda^*_u = \Lambda$$

where $\Lambda^*_u$ denotes that adjoint of the Dirchlet-to-Neumann operator of the adjoint equation.

Now, using the definition of a weak solution to the adjoint equation (5.3) with $u^* = 0$ on $T$, and $u^* = \psi$ on $S'$ and zero on $S''$, we have that $u^*$ satisfies

$$\int_Q a_j \frac{\partial}{\partial x} u^* \frac{\partial}{\partial x} \phi + \int_Q u^* \frac{\partial}{\partial t} \phi = \int_S (\Lambda u^*) \phi$$

for every $\phi \in H^{1,1}(Q)$ that is zero for $t = 0$, and also zero in $S''$.

Now using (5.4) with $a = a_1$, $u = u_1$, $u^* = u_2^*$, we get

$$\int_Q a_1 \frac{\partial}{\partial x} u_1 \frac{\partial}{\partial x} u_2^* - \int_Q u_1 \frac{\partial}{\partial t} u_2^* = \int_S (\Lambda u_1) u_2^*$$

and using (5.8) with $a = a_2$, $u^* = u_2^*$, $\phi = u_1$, we get

$$\int_Q a_2 \frac{\partial}{\partial x} u_2^* \frac{\partial}{\partial x} u_1 + \int_Q u_2^* \frac{\partial}{\partial t} u_1 = \int_S (\Lambda u_2^*) u_1.$$ 

Now subtracting (5.10) from (5.9) and using (5.7) we get the result of lemma 5.1.

\[\square\]

**Proof of theorem 2.** Let $\{Q_j\}_{j=1}^n$ and $\{Q_k\}_{k=1}^m$ be two covers to which $a_1, a_2$ are piece-wise constant, respectively.

Let $Q'_j = Q_j \cap Q_k^2$.

Note that the collection of all $Q'_j$ constitute a cover of $Q$ relative to which $a_1, a_2$ are piece-wise constant. We collect $Q'_j$ into layers $M_k$ in the following way.

Let $Q_0 = Q$.

$M_1 = \{Q_j \subset Q : \partial Q'_j \cap S' \text{ has nonempty interior in } S'\}$.

On $M_1$, we have subdomains $Q'_j$ which have a boundary part in common with $S'$ where both $u_1, u_2$ have the same Cauchy data and since both functions satisfy the same parabolic equation in $M_1$, thus from the uniqueness of continuation of the second-order parabolic equation, we get $u_1 = u_2$ in $M_1$. Using lemma (5.1), if $Q_1 = Q \setminus M_1$ we have

$$\int_{Q_1} (a_1 - a_2) \frac{\partial}{\partial x} u_1 \frac{\partial}{\partial x} u_2^* = 0$$
for all solutions \( u_1 \) to the equation \( \frac{\partial}{\partial x} \left( a_1 \frac{\partial}{\partial x} u_1 \right) - \frac{\partial}{\partial t} u_1 = 0 \) in \( Q \) which are zero when \( t < 0 \) and satisfy (1.2') and for all solutions \( u_2^* \) to the adjoint equation \( \frac{\partial}{\partial x} \left( a_2 \frac{\partial}{\partial x} u_2^* \right) + \frac{\partial}{\partial t} u_2^* = 0 \) in \( Q \) which are zero when \( t > T \) and \( u_2^* = 0 \) on \( S' \) and \( u_2^* = \psi \) on \( S' \) where \( \psi \in C_0^\infty(S') \).

Using the Runge property, (5.11) still holds for all solutions \( u_2 \) to the equation

\[
\frac{\partial}{\partial x} \left( a_1 \frac{\partial}{\partial x} u_1 \right) - \frac{\partial}{\partial t} u_1 = 0 \quad \text{near } Q_1
\]

which are zero when \( t < 0 \) and satisfies (1.2') and for all solutions \( u_2^* \) to the adjoint equation

\[
\frac{\partial}{\partial x} \left( a_2 \frac{\partial}{\partial x} u_2^* \right) + \frac{\partial}{\partial t} u_2^* = 0 \quad \text{near } Q_1
\]

which are zero when \( t > T \) and \( u_2^* = 0 \) on \( S' \).

Set

\[
M_k = \cup \{ Q'_k \subset Q_{k-1} : \partial Q'_k \cap \partial M_{k-1} \text{ has nonempty interior in } \partial M_{k-1} \}, \quad k = 2, 3, \ldots
\]

\[
Q_k = Q_{k-1} \setminus M_k, \quad k = 2, 3, \ldots
\]

We show that \( a_1 = a_2 \) on \( M_k \).

Suppose the contrary. Then there exists a point \((x, t)\), a ball \( B \), and \( \varepsilon > 0 \) such that \( a_1(x, t) - a_2(x, t) > \varepsilon \) on \( B \), using (5.11), we have

\[
0 = \int_{Q_k} (a_1 - a_2) \frac{\partial}{\partial x} u_1 \frac{\partial}{\partial x} u_2^* + \int_{Q_k \setminus B} (a_1 - a_2) \frac{\partial}{\partial x} u_1 \frac{\partial}{\partial x} u_2^* + \int_{Q_k \setminus B} (a_1 - a_2) \frac{\partial}{\partial x} u_1 \frac{\partial}{\partial x} u_2^*
\]

using the same arguments as in the proof of theorem 1 and using Green's function as in [3], we get the contradiction. \( \square \)

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References