Tilings and Bussola for Making Decisions

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Abstract We introduce the yes-no decision model, where individuals can make the decision yes or no. We characterize the coherent and uncoherent strategies that are Nash equilibria. Each decision tiling indicates the way coherent and uncoherent Nash equilibria co-exist and change with the relative decision preferences of the individuals for the yes or no decision. There are 289 combinatorial classes of decision tilings, described by the decision bussola, which demonstrates the high complexity of making decision.

1 Introduction

The main goal in Planned Behavior or Reasoned Action theories, as developed in the works of Ajzen (see [2]) and Baker (see [3]), is to understand and forecast the way individuals turn intentions into behaviors. Almeida-Cruz-Ferreira-Pinto (see

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[1]) created a game theoretical model for reasoned action, inspired by the works of J. Cownley and M. Wooders (see [6]). They studied the way saturation, boredom and frustration can lead to uncoherent (or split or impasse) strategies, and no saturation situations can lead to coherent (or heard or no-split) strategies. Here, we study the yes-no decision model that is a simplified version of the Almeida-Cruz-Ferreira-Pinto decision model. In this model, there are just two possible decisions d that individuals can make. For instance, they have to choose between yes or no, i.e. $d \in \{Yes, No\}$. Each set of economical, educational, political, psychological and social variables gives rise to a decision tiling that indicates all the the coherent and no-coherent pure Nash equlibria and also the mixed Nash equilibria in terms of the relative decision preference (taste type) of the individuals for the yes or no decision (see [4, 9]). The yes-no decision model incorporates, in the preference neighbours matrix (crowding type), the preference that an individual has for having other individuals making the same decision as his. The crowding type information gives rise to 289 different combinatorial classes of decision tilings, reflecting the complexity of the yes-no decision model (see [8, 12]). The decision bussola encodes all the information of each combinatorial class of decision tilings and indicates the way small changes in economical educational, political, psychological or social variables can transform one decision tiling, into another, thus creating and annihilating individuals and collective behavior. In this chapter, we survey, in part, the work presented in [11, 10].

2 Yes-No Decision Model

The *yes-no decision model* has two types $\mathbf{T} = \{t_1, t_2\}$ of individuals $i \in \mathbf{I}$ that have to make one decision $d \in \mathbf{D} = \{Y, N\}$. Let $n_p \geq 1$ be the number of individuals with type t_p[1]. Let $\mathscr{L}$ be the *preference decision matrix* whose *coordinates* ω_p^d indicate how much an individual, with type t_p, likes, or dislikes, to make decision d

$$\mathscr{L} = \begin{pmatrix} \omega_1^Y & \omega_1^N \\ \omega_2^Y & \omega_2^N \end{pmatrix}.$$

The preference decision matrix indicates, for each type, the decision that the individuals prefer, i.e. the individuals taste type (see [1, 5, 6, 11]).

Let $\mathscr{N}_d$ be the *preference neighbors matrix* whose *coordinates* α_{pq}^d indicate how much an individual, with type t_p, likes, or dislikes, that an individual, with type t_q, makes decision d

$$\mathscr{N}_d = \begin{pmatrix} \alpha_{11}^d & \alpha_{12}^d \\ \alpha_{21}^d & \alpha_{22}^d \end{pmatrix}.$$

[1] Similarly, we can consider that there is a single individual with type t_p that has to make n_p decisions, or we can, also, consider a mixed model using these two possibilities.

The preference neighbors matrix indicates, for each type of individuals, whom they prefer, or not, to be with in each decision, i.e. the individuals crowding type (see [1, 5, 6, 11]).

We describe the individuals' decision by a *strategy map* $S : \mathbf{I} \to \mathbf{D}$ that associates to each individual $i \in \mathbf{I}$ its decision $S(i) \in \mathbf{D}$. Let $\mathbf{S}$ be the space of all strategies S. Given a strategy S, let $\mathscr{O}_S$ be the *strategic occupation matrix*, whose coordinates $l_p^d = l_p^d(S)$ indicate the number of individuals, with type t_p, that make decision d

$$\mathscr{O}_S = \begin{pmatrix} l_1^Y & l_1^N \\ l_2^Y & l_2^N \end{pmatrix}.$$

The *strategic occupation vector* $\mathscr{V}_S$, associated to a strategy S, is the vector $(l_1, l_2) = (l_1^y(S), l_2^y(S))$. Hence, l_1 (resp. $n_1 - l_1$) is the number of individuals, with type t_1, that make the decision Y (resp. N). Similarly, l_2 (resp. $n_2 - l_2$) is the number of individuals, with type t_2, that make the decision Y (resp. N). The set $\mathbf{O}$ of all possible *occupation vectors* is

$$\mathbf{O} = \{(l_1, l_2) : 0 \le l_1 \le n_1 \quad \text{and} \quad 0 \le l_2 \le n_2\}.$$

Let $U_1 : \mathbf{D} \times \mathbf{O} \to \mathbb{R}$ the *utility function*, of an individual with type t_1, be given by

$$\begin{aligned} U_1(Y; l_1, l_2) &= \omega_1^Y + \alpha_{11}^Y (l_1 - 1) + \alpha_{12}^Y l_2 \\ U_1(N; l_1, l_2) &= \omega_1^N + \alpha_{11}^N (n_1 - l_1 - 1) + \alpha_{12}^N (n_2 - l_2). \end{aligned}$$

Let $U_2 : \mathbf{D} \times \mathbf{O} \to \mathbb{R}$ the *utility function*, of an individuals with type t_2, be given by

$$\begin{aligned} U_2(Y; l_1, l_2) &= \omega_2^Y + \alpha_{22}^Y (l_2 - 1) + \alpha_{21}^Y l_1 \\ U_2(N; l_1, l_2) &= \omega_2^N + \alpha_{22}^N (n_2 - l_2 - 1) + \alpha_{21}^N (n_1 - l_1). \end{aligned}$$

Given a strategy $S \in \mathbf{S}$, the *utility* $U_i(S)$, of an individual i with type $t_{p(i)}$, is given by $U_{p(i)}(S(i); l_1^y(S), l_2^y(S))$.

Definition 1. A strategy $S^* : \mathbf{I} \to \mathbf{D}$ is a *Nash equilibrium* if, for every individual $i \in \mathbf{I}$ and for every strategy S, with the property that $S^*(j) = S(j)$ for every individual $j \in I \setminus \{i\}$, we have

$$U_i(S^*) \ge U_i(S).$$

Let $x = \omega_1^Y - \omega_1^N$ be the *horizontal relative decision preference* of the individuals with type t_1 and let $y = \omega_2^Y - \omega_2^N$ be the *vertical relative decision preference* of the individuals with type t_2. The *Nash equilibrium domain* $E(S)$ of a strategy S is the set of all pairs (x, y) for which S is a Nash Equilibrium.

Definition 2. Let $A_{ij} = \alpha_{ij}^Y + \alpha_{ij}^N$, for $i, j \in \{1, 2\}$, be the coordinates of the *partial threshold order matrix*.

As we will show, the partial thresholds encode all the relevant information for the existence of Nash equilibria that are no-coherent strategies.

3 Evolutionary Dynamics and Yes-No Decision Models

We implement the *evolutionary deterministic yes-no decision models* as follows (see [11]): Fix an infinite sequence (i_t, d_t), with $t \in \mathbb{N}$, of pairs $(i_t, d_t) \in \mathbf{I} \times \mathbf{D}$ with the property that every pair, contained in $\mathbf{I} \times \mathbf{D}$, occurs in the sequence infinitely often. Given a strategy $S_t : \mathbf{I} \to \mathbf{D}$, at moment t, the strategy $S_{t+1} : \mathbf{I} \to \mathbf{D}$ is defined as follows: (i) $S_{t+1} = S_t|\mathbf{I} \setminus \{i_{t+1}\}$; (ii) $S_{t+1}(i_{t+1}) = d_{t+1}$, if i_{t+1} increases its utility by making decision d_{t+1} instead of $S_t(i_{t+1})$ (knowing that $S_{t+1} = S_t|\mathbf{I} \setminus \{i_{t+1}\}$), and $S_{t+1}(i) = S_t(i)$, otherwise. Hence, the Nash equilibria are the fixed points, and vice-versa, of the evolutionary decision deterministic models.

We implement the *evolutionary stochastic yes-no decision models* as follows: Let P be a probability distribution that assigns a positive probability to each pair $(i,d) \in \mathbf{I} \times \mathbf{D}$. Given a strategy $S_t : \mathbf{I} \to \mathbf{D}$, at moment t, we choose randomly a pair (i,D) according to the probability distribution P. The strategy $S_{t+1} : \mathbf{I} \to \mathbf{D}$ is defined as follows: (i) $S_{t+1} = S_t|\mathbf{I} \setminus \{i\}$; (ii) $S_{t+1}(i) = d$, if i increases its utility by deciding d instead of $S_t(i)$ (knowing that $S_{t+1} = S_t|\mathbf{I} \setminus \{i\}$), and $S_{t+1}(i) = S_t(i)$, otherwise. Hence, the Nash equilibria are the absorbing states, and vice-versa, of the evolutionary decision stochastic model.

4 (Coherent, Coherent) Strategies

A *(coherent, coherent) strategy*[2] is a strategy in which all individuals, with the same type, prefer to make the same decision (see [11]). A *(coherent, coherent) strategy* is described by a map $C : \mathbf{T} \to \mathbf{D}$ that, for every individual i, with type $t_{p(i)}$, indicates its decision $C(p(i))$. Hence, a (coherent, coherent) strategy $C : \mathbf{T} \to \mathbf{D}$ determines an unique strategy $S : \mathbf{I} \to \mathbf{D}$ given by $S(i) = C(p(i))$.

We observe that there are four (coherent, coherent) strategies:

- (Y,Y) *strategy*: all individuals make the decision Y;
- (Y,N) *strategy*: all individuals, with type t_1, make the decision Y, and all individuals, with type t_2, make the decision N;
- (N,Y) *strategy*: all individuals, with type t_1, make the decision N and all individuals, with type t_2, make the decision Y;
- (N,N) *strategy*: all individuals make the decision N.

The *horizontal* $H(Y,Y)$ and *vertical* $V(Y,Y)$ *strategic thresholds* of the (Y,Y) strategy are given by

$$H(Y,Y) = -\alpha_{11}^{Y}(n_1 - 1) - \alpha_{12}^{Y} n_2 \quad \text{and} \quad V(Y,Y) = -\alpha_{22}^{Y}(n_2 - 1) - \alpha_{21}^{Y} n_1.$$

[2] or equivalently, *(no-split, no-split) strategy* or *(heard, heard) strategy*

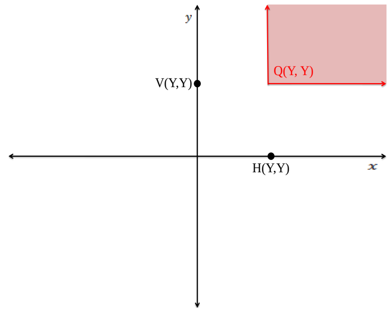

Fig. 1 (Y,Y) Nash equilibria domain $Q(Y,Y)$.

The (Y,Y) *Nash equilibria domain* $Q(Y,Y)$ is the right-upper quadrant (see Figure 1)

$$Q(Y,Y) = \{(x,y) : x \geq H(Y,Y) \quad \text{and} \quad y \geq V(Y,Y)\}.$$

The horizontal $H(Y,N)$ and vertical $V(Y,N)$ *strategic thresholds* of the (Y,N) strategy are given by

$$H(Y,N) = -\alpha^Y_{11}(n_1 - 1) + \alpha^N_{12} n_2 \quad \text{and} \quad V(Y,N) = \alpha^N_{22}(n_2 - 1) - \alpha^Y_{21} n_1.$$

The (Y,N) *Nash equilibria domain* $Q(Y,N)$ is the right-lower quadrant (see Figure 2)

$$Q(Y,N) = \{(x,y) : x \geq H(Y,N) \quad \text{and} \quad y \leq V(Y,N)\}.$$

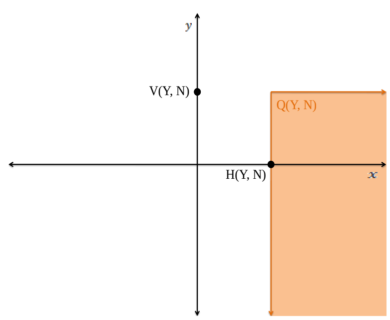

Fig. 2 (Y,N) Nash equilibria domain $Q(Y,N)$

The horizontal $H(N,Y)$ and vertical $V(N,Y)$ *strategic thresholds* of the (N,Y) strategy are given by

$$H(N,Y) = \alpha^N_{11}(n_1 - 1) - \alpha^Y_{12} n_2 \quad \text{and} \quad V(N,Y) = -\alpha^Y_{22}(n_2 - 1) + \alpha^N_{21} n_1.$$

The (N,Y) *Nash equilibria domain* $Q(N,Y)$ is the left-upper quadrant (see Figure 3)

$$Q(N,Y) = \{(x,y) : x \leq H(N,Y) \quad \text{and} \quad y \geq V(N,Y)\}.$$

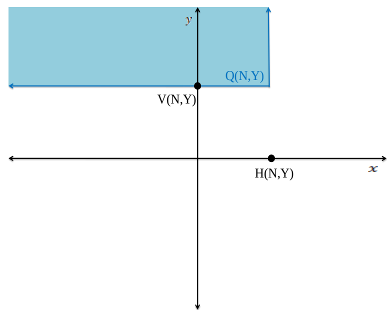

Fig. 3 (N,Y) Nash equilibria domain $Q(N,Y)$.

The horizontal $H(N,N)$ and vertical $V(N,N)$ strategic thresholds of the (N,N) strategy are given by

$$H(N,N) = \alpha_{11}^{N}(n_1 - 1) + \alpha_{12}^{N} n_2 \quad \text{and} \quad V(N,N) = \alpha_{22}^{N}(n_2 - 1) + \alpha_{21}^{N} n_1.$$

The (N,N) *Nash equilibria domain* $Q(N,N)$ is the left-lower quadrant (see figure 4)

$$Q(N,N) = \{(x,y) : x \leq H(N,N) \quad \text{and} \quad y \leq V(N,N)\}.$$

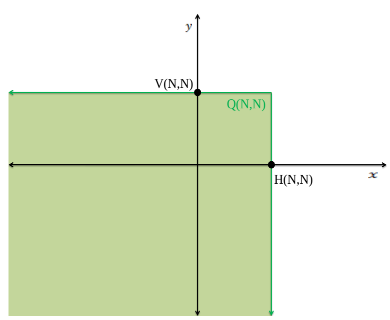

Fig. 4 (N,N) Nash equilibria domain $Q(N,N)$.

The representations of the domains $Q(Y,Y)$, $Q(Y,N)$, $Q(N,Y)$, and $Q(N,N)$ in the plan (x,y) determine the *decision tilings*. Let $U(Y,Y) \subset Q(Y,Y)$, $U(Y,N) \subset Q(Y,N)$, $U(N,Y) \subset Q(N,Y)$, and $U(N,N) \subset Q(N,N)$ be the regions with unique Nash equilibrium. In Figure 5, we represent three decision tilings, 1) with the coherent uniqueness Nash equlibria domains $U(Y,Y)$, $U(Y,N)$, $U(N,Y)$, and $U(N,N)$ colored red, orange, blue and green, respectively, 2) regions without coherent Nash equilibrium colored purple and 3) regions with two, three and four Nash equilibria colored yellow, brown and pink, respectively. In the left tiling, there is an unbounded region without coherent Nash equilibrium. In the central tiling, for every relative decision preferences, there is a unique coherent Nash equilibrium, except along the axis, where there are two coherent Nash equilibria, and at the origin, where there are four coherent Nash equilibria. In the right tiling, there are regions with one, two, three and four coherent Nash equilibria.

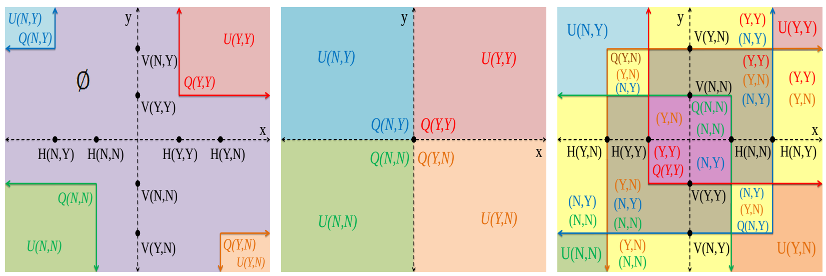

Fig. 5 Three examples of strategic thresholds and decision tilings; *left*: $A_{11} < 0$, $A_{12} > 0$, $B_{12} < 0$, $A_{22} < 0$, $A_{21} > 0$, $B_{21} < 0$; *center*: $A_{11} = A_{12} = A_{21} = A_{22} = 0$; *right*: $A_{11} > 0$, $A_{12} < 0$, $B_{12} > 0$, $A_{22} > 0$, $A_{21} < 0$, $B_{21} > 0$.

5 (Uncoherent, Coherent) Strategies

An *(uncoherent, coherent) strategy*[3] is a strategy in which all individuals, with type t_2, prefer to make the same decision, but individuals, with type t_1, split between the two decision Y and N (see [10]). Hence, the (uncoherent, coherent) strategies can be of two types:

- (l,Y) *strategy*: all the individuals, with type t_2, and l individuals, with type t_1, make decision Y, and $n_1 - l$ individuals, with type t_1, make decision N.
- (l,N) *strategy*: l individuals, with type t_1, choose decision Y, but all the individuals, with type t_2, and $n_1 - l$ individuals, with type t_1, choose decision N.

We define the *left horizontal threshold* $H_L(l,Y)$ and the *right horizontal threshold* $H_R(l,Y)$ of the (l,Y) strategy by

$$H_L(l,Y) = -\alpha_{11}^Y(l-1) - \alpha_{12}^Y n_2 + \alpha_{11}^N(n_1 - l)$$

$$H_R(l,Y) = -\alpha_{11}^Y l - \alpha_{12}^Y n_2 + \alpha_{11}^N(n_1 - l - 1).$$

[3] or equivalently, *(split, no-split)* or *(no-heard, heard) strategy*

We define the *vertical threshold* $V(l,Y)$ of the (l,Y) strategy by

$$V(l,Y) = -\alpha_{21}^{Y} l + \alpha_{21}^{N}(n_1 - l) - \alpha_{22}^{Y}(n_2 - 1)$$

The (l,Y) *Nash equilibria domain* $E(l,Y)$ strategy is a Nash Equilibrium if, and only if, $(x,y) \in E(l,Y)$, where

$$E(l,Y) = \{(x,y) : H_L(l,Y) \le x \le H_R(l,Y) \quad \text{and} \quad y \ge V(l,Y)\}.$$

Hence, $E(l,Y)$ is the *Nash Equilibrium domain* of the (l,Y) strategy (see Figure 6).

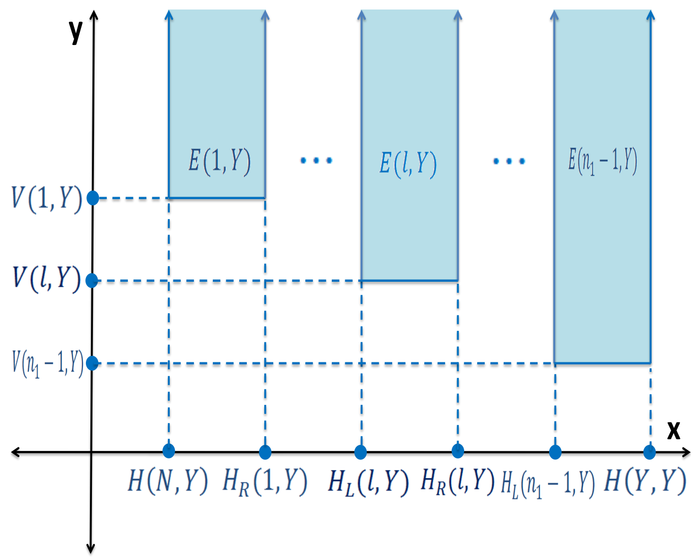

Fig. 6 (Uncoherent, coherent) Nash equilibria, $A_{11} < 0$ and $A_{21} > 0$.

We define the *left horizontal threshold* $H_L(l,N)$ and the *right horizontal threshold* $H_R(l,N)$ of the (l,N) strategy by

$$H_L(l,N) = -\alpha_{11}^{Y}(l-1) + \alpha_{12}^{N} n_2 + \alpha_{11}^{N}(n_1 - l)$$

$$H_R(l,N) = -\alpha_{11}^{Y} l + \alpha_{12}^{N} n_2 + \alpha_{11}^{N}(n_1 - l - 1).$$

We define the *vertical threshold* $V(l,N)$ of the (l,N) strategy by

$$V(l,N) = -\alpha_{21}^{Y} l_1 + \alpha_{21}^{N}(n_1 - l) + \alpha_{22}^{N}(n_2 - 1)\ .$$

The (l,N) strategy is a Nash Equilibrium if, and only if, $(x,y) \in E(l,N)$, where

$$E(l,N) = \{(x,y) : H_L(l,N) \le x \le H_R(l,N) \quad \text{and} \quad y \le V(l,N)\}.$$

Hence, $E(l,N)$ is the *Nash Equilibrium domain* of the (l,N) strategy (see Figure 7).

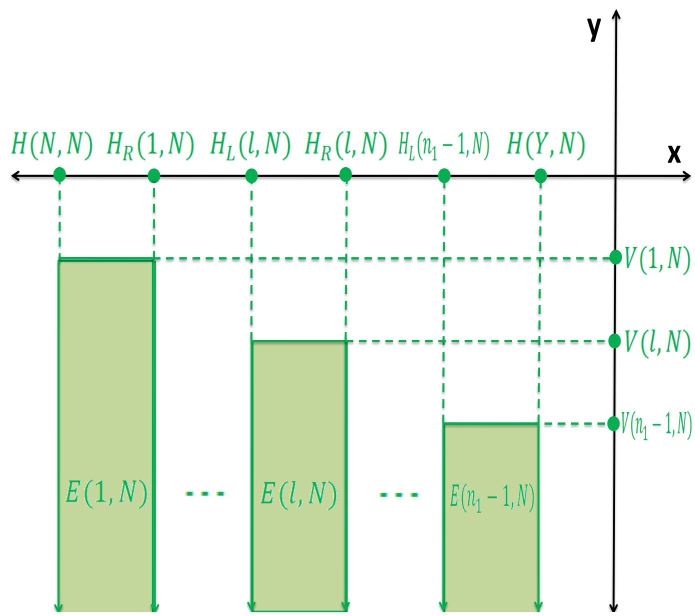

Fig. 7 (Uncoherent, coherent) Nash equilibria, $A_{11} < 0$ and $A_{21} > 0$.

Since $H_R(l,Y) = H_L(l,Y) - A_{11}$ and $H_R(l,N) = H_L(l,N) - A_{11}$, we have

- if $A_{11} > 0$, there are no (l,Y) and (l,N) Nash equilibria, for every $l \in \{1,\ldots,n_1 - 1\}$;
- if $A_{11} \leq 0$, there are (l,Y) and (l,N) Nash equilibria, for every $l \in \{1,\ldots,n_1 - 1\}$.

Hence, the following equalities determine the domains of the (l,Y) Nash equilibria (see Figure 6):

$$H_R(l,Y) = H_L(l+1,Y) \quad , \quad H_L(1,Y) = H(N,Y) \quad , \quad H_R(n_1 - 1,Y) = H(Y,Y);$$

$$V(l,N) = V(l+1,N) + A_{21} \ , \ V(1,Y) = V(N,Y) - A_{21} \ , \ V(n_1 - 1,Y) = V(Y,Y) + A_{21}.$$

Similarly, the following equalities determine the domains of the (l,N) strategies (see Figure 7):

$$H_R(l,N) = H_L(l+1,N) \quad , \quad H_L(1,N) = H(N,N) \quad , \quad H_R(n_1 - 1,N) = H(Y,N);$$

$$V(l,N) = V(l+1,N) + A_{21} \ , \ V(1,N) = V(N,N) - A_{21} \ , \ V(n_1 - 1,N) = V(Y,N) + A_{21}.$$

6 (Coherent, Uncoherent) Strategies

A *(coherent, uncoherent) strategy*[4] is a strategy in which all individuals, with type t_1, prefer to make the same decision, but individuals, with type t_2, split between the two decisions Y and N (see [10]). Hence, the (coherent, uncoherent) strategies can be of two types:

- (Y,l) *strategy*: all the individuals, with type t_1, and l individuals, with type t_2, make decision Y, and $n_2 - l$ individuals, with type t_2, make decision N.
- (N,l) *strategy*: l individuals, with type t_2, choose decision Y, but all the individuals, with type t_1, and $n_2 - l$ individuals, with type t_2, choose decision N.

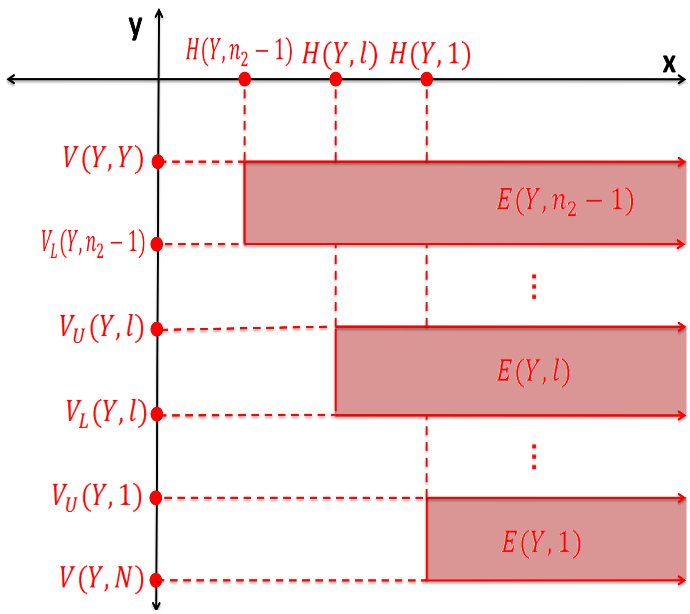

Fig. 8 (Coherent, uncoherent) Nash equilibria, $A_{22} < 0$ and $A_{12} > 0$.

We define the *lower vertical threshold* $V_L(Y,l)$ and the *upper vertical threshold* $V_U(Y,l)$ of the (Y,l) strategy by

$$V_L(Y,l) = -\alpha_{22}^{Y}(l-1) - \alpha_{21}^{Y} n_1 + \alpha_{22}^{N}(n_2 - l)$$

$$V_U(Y,l) = -\alpha_{22}^{Y} l - \alpha_{21}^{Y} n_1 + \alpha_{22}^{N}(n_2 - l - 1).$$

We define the *horizontal threshold* $H(Y,l)$ of the (Y,l) strategy by

$$H(Y,l) = -\alpha_{12}^{Y} l + \alpha_{12}^{N}(n_2 - l) - \alpha_{11}^{Y}(n_1 - 1).$$

[4] or equivalently, *(no-split, split)* or *(heard, no-heard) strategy*

The (Y,l) *Nash equilibria domain* $E(Y,l)$ strategy is a Nash Equilibrium if, and only if, $(x,y) \in E(Y,l)$, where

$$E(Y,l) = \{(x,y) : V_L(Y,l) \leq y \leq V_U(Y,l) \quad \text{and} \quad x \geq H(Y,l)\}.$$

Hence, $E(Y,l)$ is the *Nash Equilibrium domain* of the (Y,l) strategy (see Figure 8).

We define the *lower vertical threshold* $V_L(N,l)$ and the *upper vertical threshold* $V_U(N,l)$ of the (N,l) strategy by

$$V_L(N,l) = -\alpha_{22}^{Y}(l-1) + \alpha_{21}^{N} n_1 + \alpha_{22}^{N}(n_2 - l)$$

$$V_U(N,l) = -\alpha_{22}^{Y} l + \alpha_{21}^{N} n_1 + \alpha_{22}^{N}(n_2 - l - 1).$$

We define the *horizontal threshold* $H(N,l)$ of the (N,l) strategy by

$$H(N,l) = -\alpha_{12}^{Y} l + \alpha_{12}^{N}(n_2 - l) + \alpha_{11}^{N}(n_1 - 1) \ .$$

The (N,l) *Nash equilibria domain* $E(N,l)$ strategy is a Nash Equilibrium if, and only if, $(x,y) \in E(N,l)$, where

$$E(N,l) = \{(x,y) : V_L(N,l) \leq y \leq V_U(N,l) \quad \text{and} \quad x \leq H(N,l)\}.$$

Hence, $E(Y,l)$ is the *Nash Equilibrium domain* of the (Y,l) strategy (see Figure 9).

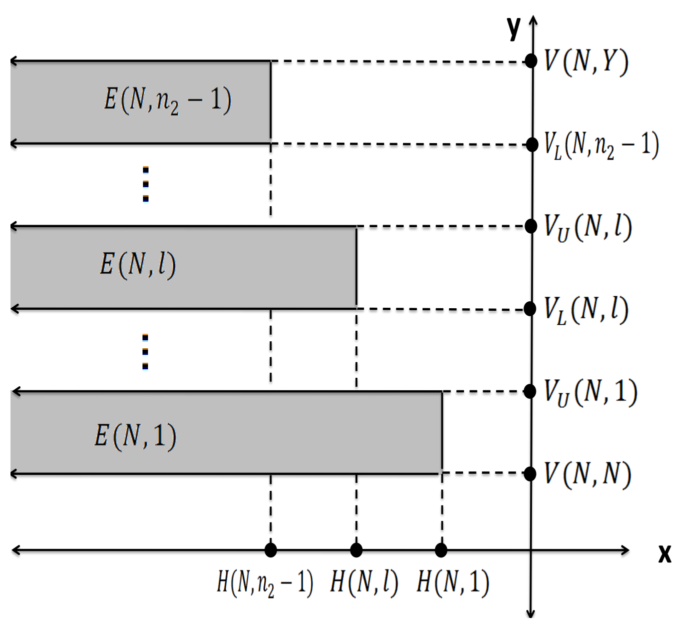

Fig. 9 (Coherent, uncoherent) Nash equilibria, $A_{22} < 0$ and $A_{12} > 0$.

Since $V_U(Y,1) = V_L(Y,1) - A_{22}$ and $V_R(N,1) = V_L(N,1) - A_{22}$, we have

- if $A_{22} > 0$, there are no (Y,l) and (N,l) Nash equilibria, for every $l \in \{1,\ldots,n_2 - 1\}$;
- if $A_{22} \le 0$, there are (Y,l) and (N,l) Nash equilibria, for every $l \in \{1,\ldots,n_2 - 1\}$.

Hence, the following equalities determine the domains of the (Y,l) Nash equilibria (see Figure 8):

$$V_U(Y,l) = V_L(Y,l+1) \quad , \quad V_L(Y,1) = V(Y,N) \quad , \quad V_U(Y,n_2-1) = V(Y,Y);$$

$$H(Y,l) = H(Y,l+1) + A_{12} \, , \, H(Y,1) = H(Y,N) - A_{12} \, , \, H(Y,n_2-1) = H(Y,Y) + A_{12}.$$

Similarly, the following equalities determine the domains of the (N,l) strategies (see Figure 9):

$$V_U(N,l) = V_L(N,l+1) \quad , \quad V_L(N,1) = V(N,N) \quad , \quad V_U(N,n_2-1) = V(N,Y);$$

$$H(N,l) = H(N,l+1) + A_{12} \, , \, H(N,1) = H(N,N) - A_{12} \, , \, H(N,n_2-1) = H(N,Y) + A_{12}.$$

7 (Uncoherent, Uncoherent) Strategies

An *(uncoherent, uncoherent) strategy*[5] is a strategy in which individuals, with type t_1 and type t_2, split between the two decisions Y and N (see [10]).
There are $(n_1 - 1)(n_2 - 1)$ (uncoherent, uncoherent) strategies:

- (l_1,l_2) *strategy*: l_1 individuals, with type t_1, and l_2 individuals, with type t_2, make decision Y, and $n_1 - l_1$ individuals, with type t_1, and $n_2 - l_2$ individuals, with type t_2, make decision N, for $l_1 \in \{1,\ldots,n_1-1\}$ and $l_2 \in \{1,\ldots,n_2-1\}$.

We define the *left horizontal threshold* $H_L(l_1,l_2)$ and the *right horizontal threshold* $H_R(l_1,l_2)$ of the (l_1,l_2) strategy by

$$H_L(l_1,l_2) = \alpha_{11}^N n_1 + \alpha_{12}^N n_2 + \alpha_{11}^Y - (\alpha_{12}^Y + \alpha_{12}^N) l_2 - (\alpha_{11}^Y + \alpha_{11}^N) l_1$$

$$H_R(l_1,l_2) = \alpha_{11}^N n_1 + \alpha_{12}^N n_2 - \alpha_{11}^N - (\alpha_{12}^Y + \alpha_{12}^N) l_2 - (\alpha_{11}^Y + \alpha_{11}^N) l_1 \ .$$

We define the *down vertical threshold* $V_D(l_1,l_2)$ and the the *up vertical threshold* $V_U(l_1,l_2)$ of the (l_1,l_2) strategy by

$$V_D(l_1,l_2) = \alpha_{22}^N n_2 + \alpha_{21}^N n_1 + \alpha_{22}^Y - (\alpha_{21}^Y + \alpha_{21}^N) l_1 - (\alpha_{22}^Y + \alpha_{22}^N) l_2$$

$$V_U(l_1,l_2) = \alpha_{22}^N n_2 + \alpha_{21}^N n_1 - \alpha_{22}^N - (\alpha_{21}^Y + \alpha_{21}^N) l_1 - (\alpha_{22}^Y + \alpha_{22}^N) l_2 \ .$$

The (l_1,l_2) strategy is a Nash Equilibrium if, and only if, $(x,y) \in E(l_1,l_2)$, where

$$E(l_1,l_2) = \{(x,y) : H_L(l_1,l_2) \le x \le H_R(l_1,l_2) \quad \text{and} \quad V_D(l_1,l_2) \le y \le V_U(l_1,l_2)\}.$$

[5] or equivalently, *(split, split)* or *(no-heard, no-heard) strategy*

Hence, $E(l_1,l_2)$ is the *Nash Equilibrium domain* of the (l_1,l_2) strategy (see Figure 10).

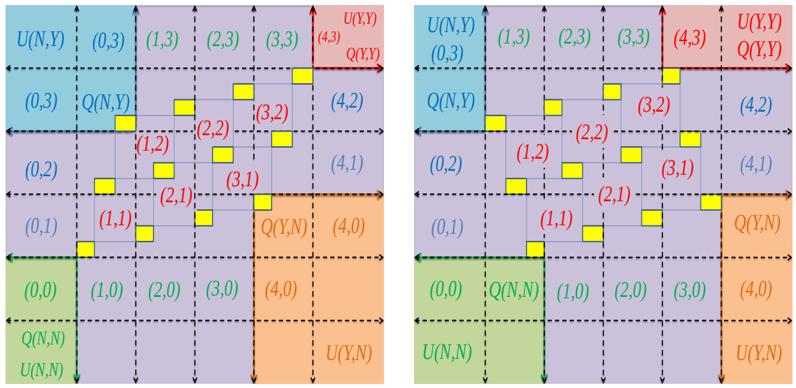

Fig. 10 (Uncoherent, uncoherent) Nash equilibria.

Since $H_R(l_1,l_2) = H_L(l_1,l_2) - A_{11}$ and $V_U(l_1,l_2) = V_D(l_1,l_2) - A_{22}$, we have that

- if $A_{11} > 0$ or $A_{22} > 0$, there are no (l_1,l_2) Nash Equilibria, for every $l_1 \in \{1,\ldots,n_1-1\}$ and $l_2 \in \{1,\ldots,n_2-1\}$;
- if $A_{11} \leq 0$ and $A_{22} \leq 0$, there are (l_1,l_2) Nash Equilibria, for every $l_1 \in \{1,\ldots,n_1-1\}$ and $l_2 \in \{1,\ldots,n_2-1\}$.

Hence, the following equalities determine the domains of the (l_1,l_2) Nash Equilibria (see Figure 10):

$$H_R(l_1,l_2) = H_L(l_1+1,l_2) \quad \text{and} \quad V_U(l_1,l_2) = V_D(l_1,l_2+1).$$

In the left tiling of Figure 10, we have

$$\mathcal{N}_Y = \begin{pmatrix} -1 & -\frac{1}{3} \\ \frac{1}{2} & -1 \end{pmatrix} \text{ and } \mathcal{N}_N = \begin{pmatrix} -1 & -\frac{1}{3} \\ 1 & -1 \end{pmatrix}.$$

The yellow rectangles are regions with two pure Nash equilibria and one mixed Nash equilibrium. In the right tiling of Figure 10, we have

$$\mathcal{N}_Y = \begin{pmatrix} -1 & \frac{1}{3} \\ -\frac{1}{2} & -1 \end{pmatrix} \text{ and } \mathcal{N}_N = \begin{pmatrix} -1 & \frac{1}{3} \\ -1 & -1 \end{pmatrix}.$$

The yellow rectangles are regions with no pure Nash equilibrium and one mixed Nash equilibrium.

8 Bifurcations and Combinatorial Equivalent Tilings

Let $A_{ij} = \alpha^Y_{ij} + \alpha^N_{ij}$, for $i, j \in \{1,2\}$, be the coordinates of the *partial threshold order matrix*. We observe that

$$H(N,Y) \leq H(Y,Y) \Leftrightarrow A_{11} \leq 0 \Leftrightarrow H(N,N) \leq H(Y,N);$$

$$H(Y,N) \leq H(Y,Y) \Leftrightarrow A_{12} \leq 0 \Leftrightarrow H(N,N) \leq H(N,Y);$$

$$V(N,Y) \leq V(Y,Y) \Leftrightarrow A_{21} \leq 0 \Leftrightarrow V(N,N) \leq V(Y,N);$$

$$V(Y,N) \leq V(Y,Y) \Leftrightarrow A_{22} \leq 0 \Leftrightarrow V(N,N) \leq V(N,Y).$$

Let $B_{11}(n_1,n_2) = A_{11}(n_1-1) - A_{12}n_2$, $B_{12}(n_1,n_2) = A_{11}(n_1-1) + A_{12}n_2$, $B_{21}(n_1,n_2) = A_{22}(n_2-1) + A_{21}n_1$ and $B_{22}(n_1,n_2) = A_{22}(n_2-1) - A_{21}n_1$ be the coordinates of the *balanced threshold weight matrix*. We observe that

$$H(N,Y) \leq H(Y,N) \Leftrightarrow B_{11}(n_1,n_2) \leq 0;$$

$$H(N,N) \leq H(Y,Y) \Leftrightarrow B_{12}(n_1,n_2) \leq 0;$$

$$V(N,N) \leq V(Y,Y) \Leftrightarrow B_{21}(n_1,n_2) \leq 0;$$

$$V(Y,N) \leq V(N,Y) \Leftrightarrow B_{22}(n_1,n_2) \leq 0.$$

We say that a decision tiling is *structurally stable*, if all the horizontal and vertical thresholds are pairwise distinct. We say that a decision tiling is *a bifurcation*, if there are, at least, two horizontal thresholds that coincide or there are, at least, two vertical thresholds that coincide (see Figures 11 and 12).
We say that a decision tiling is *structurally horizontal (resp. vertical) stable*, if all the horizontal (resp. vertical) thresholds are pairwise distinct. A bifurcation is *horizontally (resp. vertically) single* if, and only if, two horizontal (resp. vertical) thresholds coincide. A bifurcation is *horizontally (resp. vertically) double* if, and only if, two pairs of horizontal (resp. vertical) thresholds coincide. A bifurcation is *horizontally (resp. vertically) degenerated* if all horizontal (resp. vertical) thresholds coincide.

Two decision tilings are *combinatorial equivalent*, if the lexicographic orders of the horizontal and vertical thresholds along the axis are the same in both tilings. The *parameter space PS* is the set

$$PS = \{\underline{\alpha} = (\alpha^Y_{11}, \alpha^Y_{12}, \alpha^Y_{21}, \alpha^Y_{22}, \alpha^N_{11}, \alpha^N_{12}, \alpha^N_{21}, \alpha^N_{22}) \in \mathbb{R}^8\}.$$

The *bifurcation parameter space BPS*

$$BPS = \{\underline{\alpha} \in PS : A_{ij} = 0 \vee B_{ij} = 0, \quad \text{with} \quad i,j \in \{1,2\}\}$$

is the set of all parameters corresponding to bifurcation decision tilings. All parameters, in a same connected component of $PS \setminus BPS$, determine decision tilings that are combinatorial equivalent.

Fig. 11 A single horizontal and vertical bifurcations. $A_{11} > 0$ and $A_{22} > 0$; B_{11} and B_{22} changing signs.

Next we characterize the different orders for the horizontal and vertical thresholds.

Case $H(N,Y) < H(Y,N)$*:* If $A_{11} < 0$ and $A_{12} > 0$, then

$$H(N,Y) < H(Y,Y) < H(Y,N) \text{ and } H(N,Y) < H(N,N) < H(Y,N).$$

Hence, the horizontal threshold $H(N,Y)$ is the smallest one and the horizontal threshold $H(Y,N)$ is the largest. Therefore, the only indeterminacy to solve this case

Fig. 12 A single horizontal and vertical bifurcations. $A_{11} < 0$ and $A_{22} < 0$; B_{11} and B_{22} changing signs.

is the order between the thresholds $H(N,N)$ and $H(Y,Y)$. If $B_{12}(n_1,n_2) < 0$, then $H(N,N) < H(Y,Y)$. If $B_{12}(n_1,n_2) = 0$, then $H(N,N) = H(Y,Y)$. If $B_{12}(n_1,n_2) > 0$, then $H(Y,Y) < H(N,N)$. If $A_{11} = 0$ and $A_{12} > 0$, then $H(N,Y) = H(Y,Y)$ and $H(N,N) = H(Y,N)$ (see Figure 13).

Case $H(Y,Y) < H(N,N)$*:* If $A_{11} > 0$ and $A_{12} > 0$, then

$$H(Y,Y) < H(N,Y) < H(N,N) \text{ and } H(Y,Y) < H(Y,N) < H(N,N).$$

Hence, the horizontal threshold $H(Y,Y)$ is the smallest one and the horizontal threshold $H(N,N)$ is the largest. Therefore, the only indeterminacy to solve this case

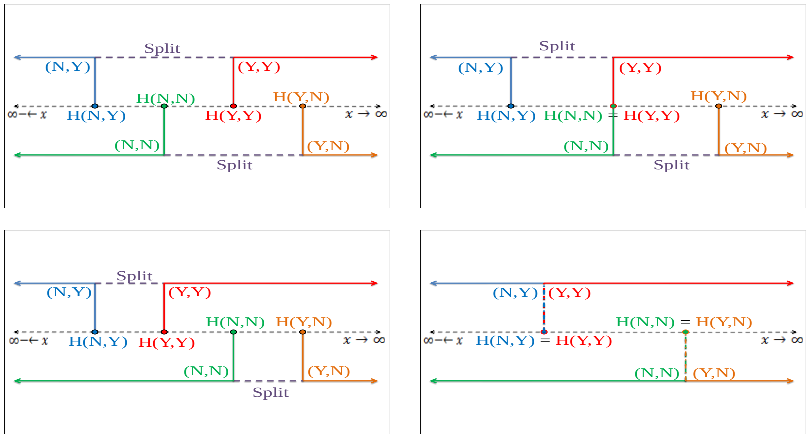

Fig. 13 $A_{11} \leq 0$ and $A_{12} > 0$

is the order between the thresholds $H(N,Y)$ and $H(Y,N)$. If $B_{11}(n_1,n_2) < 0$, then $H(N,Y) < H(Y,N)$. If $B_{11}(n_1,n_2) = 0$, then $H(N,Y) = H(Y,N)$. If $B_{11}(n_1,n_2) > 0$, then $H(Y,N) < H(N,Y)$. If $A_{12} = 0$ and $A_{11} > 0$, then $H(Y,Y) = H(Y,N)$ and $H(N,Y) = H(N,N)$ (see Figure 14).

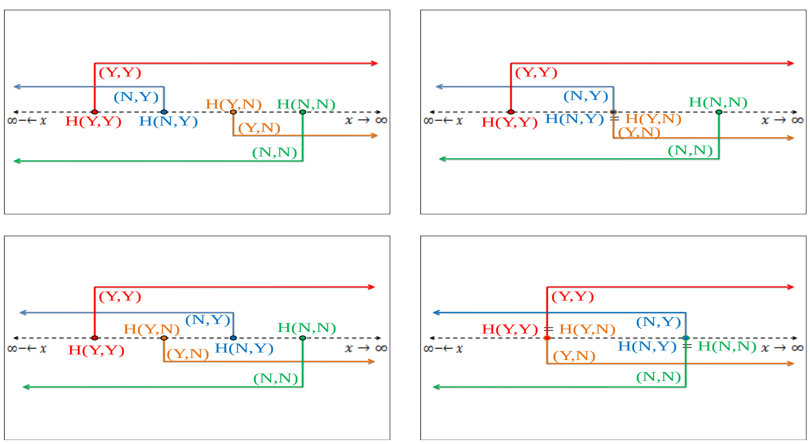

Fig. 14 $A_{11} > 0$ and $A_{12} \geq 0$

Case $H(Y,N) < H(N,Y)$: If $A_{11} > 0$ and $A_{12} < 0$, then

$$H(Y,N) < H(Y,Y) < H(N,Y) \ \textit{and} \ H(Y,N) < H(N,N) < H(N,Y).$$

Hence, the horizontal threshold $H(Y,N)$ is the smallest one and the horizontal threshold $H(N,Y)$ is the largest. Therefore, the only indeterminacy to solve this case is the order between the thresholds $H(Y,Y)$ and $H(N,N)$. If $B_{12}(n_1,n_2) > 0$, then $H(Y,Y) < H(N,N)$. If $B_{12}(n_1,n_2) = 0$, then $H(Y,Y) = H(N,N)$. If $B_{12}(n_1,n_2) < 0$, then $H(N,N) < H(Y,Y)$. If $A_{11} = 0$ and $A_{12} < 0$, then $H(Y,N) = H(N,N)$ and $H(N,Y) = H(Y,Y)$ (see Figure 15).

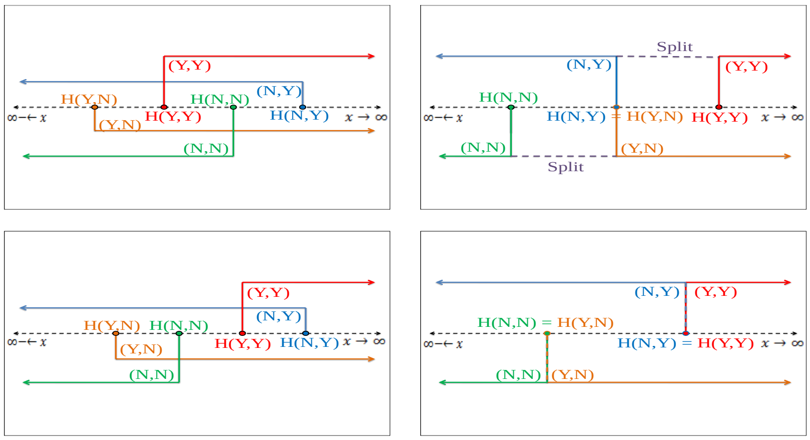

Fig. 15 $A_{11} \geq 0$ and $A_{12} < 0$

Case $H(N,N) < H(Y,Y)$*:* If $A_{11} < 0$ and $A_{12} < 0$, then

$$H(N,N) < H(Y,N) < H(Y,Y) \text{ and } H(N,N) < H(N,Y) < H(Y,Y).$$

Hence, the horizontal threshold $H(N,N)$ is the smallest one and the horizontal threshold $H(Y,Y)$ is the largest. Therefore, the only indeterminacy to solve this case is the order between the thresholds $H(Y,N)$ and $H(N,Y)$. If $B_{11}(n_1,n_2) > 0$, then $H(Y,N) < H(N,Y)$. If $B_{11}(n_1,n_2) = 0$, then $H(Y,N) = H(N,Y)$. If $B_{11}(n_1,n_2) < 0$, then $H(N,Y) < H(Y,N)$. If $A_{12} = 0$ and $A_{11} < 0$, then $H(N,N) = H(N,Y)$ and $H(Y,Y) = H(Y,N)$ (see Figure 16).

Case $H_{[(N,N)=(N,Y)=(Y,N)=(Y,Y)]}$ If $A_{12} = 0$ and $A_{11} = 0$, we obtain $H(N,N) = H(N,Y) = H(Y,N) = H(Y,Y)$. Hence, in this case, we have determined all the no-split strategies that are Nash equilibria in terms of the horizontal relative preferences decision x (see figure 17).

In Figure 18, the thresholds $H(Y,Y)$ (resp. $V(Y,Y)$) are marked by the red dots, the thresholds $H(Y,N)$ (resp. $V(N,Y)$) are marked by the orange dots, the thresholds $H(N,Y)$ (resp. $V(Y,N)$) are marked by the blue dots, and the thresholds $H(N,N)$ (resp. $V(N,N)$) are marked by the green dots. We have four horizontal (resp. vertical) thresholds whose order is determined in each direction of the *bussola*. The way the colored thresholds spiral in the bussola correspond to the way they change with

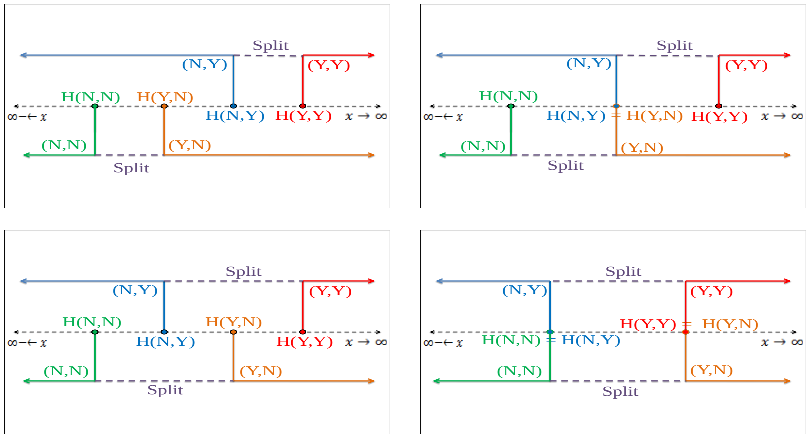

Fig. 16 $A_{11} < 0$ and $A_{12} \leq 0$

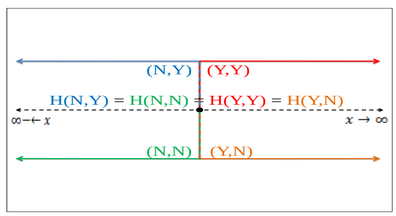

Fig. 17 Triple bifurcation

the coordinates of the partial threshold order matrix and with the coordinates of the threshold balanced weight matrix. Hence, a pair (d_1, d_2) of directions in the bussola determine unique decision tiling, up to combinatorial equivalence, and vice-versa. The bussula has the following properties:

- d_1 and d_2 are both in the north side of the bussola if, and only if, there are only (uncoherent, uncoherent) Nash equilibria in the corresponding tiling;
- d_1 is in the north side and d_2 is in the south side of the bussola if, and only if, there are (uncoherent, coherent) Nash equilibria in the corresponding tiling;
- d_1 is in the south side and d_2 is in the north side of the bussola if, and only if, there are (coherent, uncoherent) Nash equilibria in the corresponding tiling;
- d_1 and d_2 are both in the south side of the bussola if, and only if, there are (coherent, coherent) Nash equilibria in the corresponding tiling;

There are 64 combinatorial classes of structurally stable decision tilings and 225 combinatorial classes of bifurcation decision tilings.

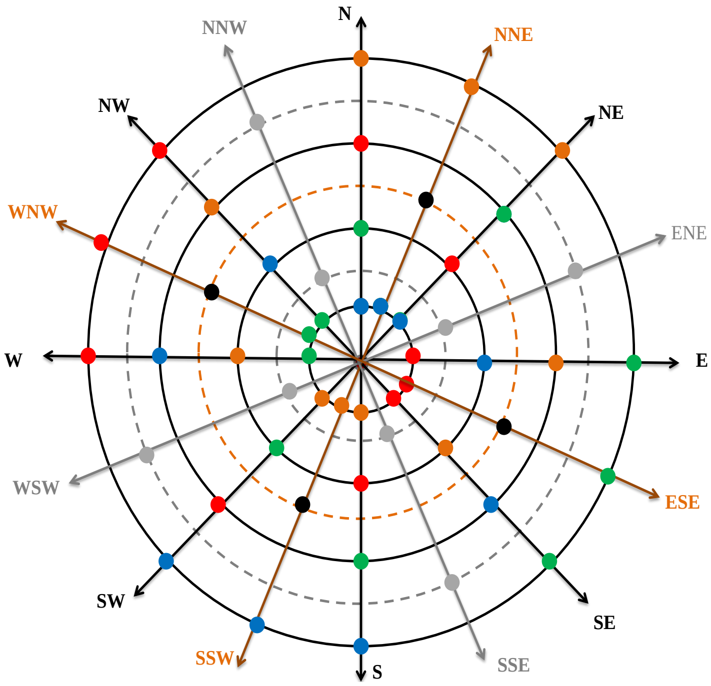

Fig. 18 Horizontal (or vertical) decision bussola

9 Conclusions

Small changes in the the coordinates of the partial threshold order matrix and of the threshold balanced weight matrix, when these coordinates are close to zero, can change their sign and, therefor, alter the order of the horizontal and vertical thresholds. These changes can create and annihilate coherent and uncoherent Nash equilibria giving rise to abrupt changes in individuals and collective behavior.

This work, after presented in ICM 2010, was highlighted by G.S. Mudur (see [7]) along with other works of Alberto Adrego Pinto, Stanley Osher, from University of California, and Philip Kumar Maini, from University of Oxford.

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References

1. Almeida, L., Cruz, J., Ferreira, H. and Pinto, A. A., Bayesian-Nash Equilibria in Theory of Planned Behavior. *Journal of Difference Equations and Applications*, 1-9 (2009).
2. Ajzen, I., Perceived Behavioral Control, Self-Efficacy, Locus of Control, and the Theory of Planned Behavior. *Journal of Applied Social Psychology*, 32 665 - 683 (2002).
3. Baker, S., Beadnell, B., Gillmore, M., Morrison, D., Huang, B. and S. Stielstra, The Theory of Reasoned Action and the Role of External Factors on Heterosexual Mens Monogamy and Condom Use. *Journal of Applied Social Psychology*, 38 1 97-134 (2008).
4. Brida, J., Defesa, M., Faias, M. and Pinto, A., A Tourists Choice Model. *Dynamics, Games and Science I* (eds.: M. Peixoto, A. A. Pinto and D. Rand). Proceedings in Mathematics series, Springer-Verlag, Chapter 10, 159-167 (2011).
5. Brida, J., Defesa, M., Faias, M. and Pinto, A. A., Strategic Choice in Tourism with Differentiated Crowding Types. *Economics Bulletin*, 30 (2) 15091515 (2010).
6. Conley, John P. and Wooders, Myrna H., Tiebout Economies with Differential Genetic Types and Endogenously Chosen Crowding Characteristics. *Journal of Economic Theory*, 98 261-294 (2001).
7. Mudur, G. S.: Maths for movies, medicine and markets. The Telegraph Calcutta, India, 20/09/2010 (2010).
8. Pinto, A. A.: *Game Theory and Duopoly Models*. Interdisciplinary Applied Mathematics Series. Springer, New York (2011).
9. Pinto, A. A., Faias, M. and Mousa, A. S., Resort Pricing and Bankruptcy. *Dynamics, Games and Science II* (eds.: M. Peixoto, A. A. Pinto and D. Rand). Proceedings in Mathematics series, Springer-Verlag, Chapter 39, 549-555 (2011).
10. Pinto, A. A. and Mousa, A. S., Uncoherent human decisions (2011).
11. Pinto, A.A., Mousa, A.S., Mousa, M.S., Samarah, R.M.: Tilings, Chromosomes and Bussola for Human Evolutionary Dynamical Yes-No Decision Models (2011).
12. Pinto, A. A., Rand D. A. and Ferreira F., *Fine Structures of Hyperbolic Diffeomorphisms*. Springer-Verlag Monograph (2010).