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# On the difference equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$ with $A < 0$

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**Abstract**

We find conditions for the global asymptotic stability of the unique negative equilibrium $\bar{y} = 1 + A$ of the equation

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}, \tag{0.1}$$

where $y_{-k}, y_{-k+1}, \ldots, y_0 \in (0, \infty)$, $A < 0$ and $k \in \{1, 2, 3, 4, \ldots\}$.


*Keywords:* Recursive sequence; Global asymptotic stability

## 1. Introduction

In [1] the periodicity of the difference equation

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $y_{-k}, \ldots, y_{-1}, y_0$, $A \in (0, \infty)$ and $k \in \{2, 3, 4, \ldots\}$ was studied. Our aim in this paper is to establish global asymptotic stability results for this difference equation with $A < 0$.

It was shown in [3] that for the case $k = 1$ the positive equilibrium $\bar{y} = 1 + A$ of Eq. (1.1) is globally asymptotically stable for $A > 1$. In [1], the periodicity of Eq. (1.1) was investigated. In this note, other related results of asymptotic, periodicity, and semi-cycles are investigated. We list below some definitions and basic results that will be needed in this paper (see [5,7,10]).

**Definition 1.1.** We say that a solution $\{y_n\}_{n=-k}^{\infty}$ of a difference equation $y_{n+1} = f(y_n, y_{n-1}, \ldots, y_{n-k})$ is *periodic* if there exists a positive integer $p$ such that $y_{n+p} = y_n$. The smallest such positive integer $p$ is called the prime period of the solution of the difference equation.

**Definition 1.2.** The equilibrium point $\bar{y}$ of the equation:

$$y_{n+1} = f(y_n, y_{n-1}, \ldots, y_{n-k}), \quad n = 0, 1, \ldots$$

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is the point that satisfies the condition

$$\bar{y} = f(\bar{y}, \bar{y}, \ldots, \bar{y}).$$

**Definition 1.3.** Let $\bar{y}$ be an equilibrium point of Eq. (1.1). Then the equilibrium point $\bar{y}$ is called

(1) locally stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $y_{-k}, y_{-k+1}, \ldots, y_0 \in I$ with $|y_{-k} - \bar{y}| + |y_{-k+1} - \bar{y}| + \cdots + |y_0 - \bar{y}| < \delta$, we have $|y_n - \bar{y}| < \epsilon$ for all $n \geqslant -1$,
(2) locally asymptotically stable if it is locally stable and if there exists $\gamma > 0$ such that for all $y_{-k}, y_{-k+1}, \ldots, y_0 \in I$ with $|y_{-k} - \bar{y}| + |y_{-k+1} - \bar{y}| + \cdots + |y_0 - \bar{y}| < \gamma$, we have $\lim_{n\to\infty} y_n = \bar{y}$,
(3) a global attractor if for all $y_{-k}, y_{-k+1}, \ldots, y_0 \in I$, we have $\lim_{n\to\infty} y_n = \bar{y}$,
(4) globally asymptotically stable if $\bar{y}$ is locally stable and $\bar{y}$ is a global attractor.

The linearized equation of Eq. (1.1) about the negative equilibrium $\bar{y} = 1 + A$ is

$$z_{n+1} - \frac{1}{1+A} z_n + \frac{1}{1+A} z_{n-k} = 0, \quad n = 0, 1, \ldots \tag{1.2}$$

The following result is a consequence of the conditions given in [6, page 12], see also [8,9].

**Lemma 1.4.** *Assume that $a, b \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$. Then*

$$|a| + |b| < 1 \tag{1.3}$$

*is a sufficient condition for the asymptotic stability of the difference equation*

$$y_{n+1} + a y_n + b y_{n-k} = 0, \quad n = 0, 1, \ldots \tag{1.4}$$

*Suppose in addition that one of the following two cases holds.*

*(a) $k$ odd and $b < 0$.*
*(b) $k$ even and $ab < 0$.*

*Then (1.3) is also a necessary condition for the asymptotic stability of Eq. (1.4).*

**Lemma 1.5.** *Assume that $a, b \in \mathbb{R}$. Then*

$$|a| < b + 1 < 2$$

*is a necessary and sufficient condition for the asymptotic stability of the difference equation*

$$y_{n+1} + a y_n + b y_{n-k} = 0, \quad n = 0, 1, \ldots \tag{1.5}$$

**Lemma 1.6.** *The difference equation*

$$y_{n+1} - b y_n + b y_{n-k} = 0, \quad n = 0, 1, \ldots \tag{1.6}$$

*is asymptotically stable iff $0 < |b| < 1/2\cos\left(\frac{k\pi}{k+2}\right)$.*

**Lemma 1.7.** *Consider Eq. (1.1). If $A < -2\cos\left(\frac{\pi}{k+2}\right) - 1$ then the unique negative equilibrium $\bar{y} = 1 + A$ of Eq. (1.1) is locally asymptotically stable, while if $A > -2\cos\left(\frac{\pi}{k+2}\right) - 1$ then the positive equilibrium is unstable.*

**Proof.** The proof is a direct consequence of the conditions in Lemma 1.6. □

The above lemmas lead to parts (a), (b) of the next theorem and the proof of part (c) is straightforward.

**Theorem 1.8.** *The following statements are true*:

(*a*) *The equilibrium point* $A+1$ *of Eq.* (1.1) *is locally asymptotically stable iff* $A<-3$.
(*b*) *The equilibrium point* $A+1$ *of Eq.* (1.1) *is unstable if* $0\geqslant A\geqslant -3$.
(*c*) *If a solution of Eq.* (1.1) *is eventually constant then* $y_n=A+1$, $n=-k,-k+1,\ldots$

## 2. Analysis of the global stability, boundedness and the semi-cycles of solutions of Eq. (1.1)

In this section, we show that every negative solution of Eq. (1.1) is globally asymptotically stable and thus get as a corollary the boundedness and persistence of solutions.

We say that a solution $\{y_n\}$ of a difference equation $y_{n+1}=f(y_n,y_{n-1},\ldots,y_{n-k})$ is *bounded and persists* if there exist positive constants $P$ and $Q$ such that

$$P\leqslant x_n\leqslant Q,\quad \text{for } n=-1,0,\ldots$$

*A positive semi-cycle* of a solution $\{y_n\}$ of Eq. (1.1) consists of a "string" of terms $\{y_l,y_{l+1},\ldots,y_m\}$, all greater than or equal to the equilibrium $\bar{y}$, with $l\geqslant -k$ and $m\leqslant\infty$ and such that

$$\text{either } l=-k \quad \text{or} \quad l>-k \text{ and } y_{l-1}<\bar{y},$$

and

$$\text{either } m=\infty \quad \text{or} \quad m<\infty \text{ and } y_{m+1}<\bar{y}.$$

*A negative semi-cycle* of a solution $\{y_n\}$ of Eq. (1.1) consists of a "string" of terms $\{y_l,y_{l+1},\ldots,y_m\}$, all less than the equilibrium $\bar{y}$, with $l\geqslant -k$ and $m\leqslant\infty$ and such that

$$\text{either } l=-k \quad \text{or} \quad l>-k \text{ and } y_{l-1}\geqslant\bar{y},$$

and

$$\text{either } m=\infty \quad \text{or} \quad m<\infty \text{ and } y_{m+1}\geqslant\bar{y}.$$

The first semi-cycle of a solution starts with the term $y_{-k}$ and is positive if $y_{-k}\geqslant\bar{y}$ and negative if $y_{-k}<\bar{y}$.

A solution $\{y_n\}$ of Eq. (1.1) is called *nonoscillatory* if there exists $N\geqslant -k$ such that $y_n>\bar{y}$ for all $n\geqslant N$ or $y_n<\bar{y}$ for all $n\leqslant N$.

And a solution $\{y_n\}$ is called *oscillatory* if it is not nonoscillatory.

**Theorem 2.1.** *Eq.* (1.1) *has no solution of prime period* 2 *if* $A\neq -1$ *or* $k$ *is even.*

**Proof.** If $k$ is even then $\Phi=\Psi=A+1$ in which case $p\neq 2$.

If $n$ is odd, then $\Phi=A+\frac{\Phi}{\Psi}$ and $\Psi=A+\frac{\Psi}{\Phi}$. It follows that $\frac{\Phi}{\Psi}=\Phi-A$ and $\frac{\Psi}{\Phi}=\Psi-A$. Multiplying the last two equations, we get $(\Psi-A)(\Phi-A)=1$. Thus, $\Phi\neq A$ and $\Psi\neq A$.

Moreover, we conclude that $\Psi=\frac{1}{\Phi-A}+A$. But on the other hand, we have $\frac{1}{\Psi}-\frac{1}{\Phi}=\frac{1}{\Psi^2}-\frac{1}{\Phi^2}$. Therefore, we get $\frac{1}{\Psi}+\frac{1}{\Phi}=-1$. Solving for $\Psi$, we get $\Psi=\frac{-\Phi}{1+\Phi}$.

The last two equations lead to $A=-1$. We conclude that the period 2 solution takes the form $\ldots,\Phi,\frac{-\Phi}{1+\Phi},\Phi,\frac{-\Phi}{1+\Phi},\ldots$ This completes the proof. □

Notice that the solution oscillates about the steady state $y=0$ when $A=-1$. Every semi-cycle is of length one.

Now we find a global asymptotic stability result for the general case $k\in\{2,3,4,\ldots\}$.

**Theorem 2.2.** [4] *Consider the difference equation*

$$y_{n+1}=f(y_n,y_{n-k}),\quad n=0,1,\ldots, \tag{2.1}$$

*where* $k\in\{1,2,\ldots\}$. *Let* $I=[a,b]$ *be some interval of real numbers and assume that*

$$f:[a,b]\times[a,b]\to[a,b]$$

*is a continuous function satisfying the following properties*:

(*a*) $f(u, v)$ *is nonincreasing in u and nondecreasing in v.*

(*b*) *If* $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$m = f(M, m) \quad \text{and} \quad M = f(m, M),$$

*then* $m = M$. *Then Eq.* (2.1) *has a unique equilibrium* $\bar{y}$ *and every solution of Eq.* (2.1) *converges to* $\bar{y}$.

**Theorem 2.3.** *Let* $A < -3$. *Then the unique negative equilibrium* $\bar{y} = 1 + A$ *of Eq.* (1.1) *is globally asymptotically stable.*

**Proof.** Define $f(u, v) = A + u/v$. Then the result follows directly from Theorem 2.2. □

The global stability of the difference equation implies the boundedness of the difference equation.

**Corollary 2.4.** *Let* $A < -3$. *Then every solution of Eq.* (1.1) *is bounded and persists.*

We consider the following lemma about the behavior of the semi-cycles of Eq. (1.1).

**Lemma 2.5.** *Let* $\{y_n\}$ *be a nontrivial solution of Eq.* (1.1), $A = -1$, $k \geqslant 2$. *Then every semi-cycle has at most* 2 *terms.*

**Theorem 2.6.** *Let k be odd and let*

$$y_{-k}, y_{-k+0}, \ldots, y_{-1} \leqslant A + 1, \quad 0 > y_{-k+1}, y_{-k+3}, \ldots, y_0 > A + 1.$$

*Then, the solution* $\{y_n\}_{n=-k}^{\infty}$ *is oscillatory and every semi-cycle has length one. Moreover, every term of* $\{y_n\}_{n=-k}^{\infty}$ *is strictly greater than A with the possible exception of the first* $k + 1$ *semi-cycles, no term of* $\{y_n\}_{n=1}^{\infty}$ *is ever equal to* $A + 7$.

**Proof.** Just notice that, for any $n \geqslant 1$,

$$y_{2n+1} = A + \frac{y_{2n-k}}{y_{2n}} > A + 1,$$

and

$$y_{2n} = A + \frac{y_{2n-(k+1)}}{y_{2n-1}} < A + 1.$$

The result then follows. □

## 3. Case $-3 \leqslant A < 0$

As it is noticed in Theorem 1.8, in this case the equilibrium point $A + 1$ is not even asymptotically stable. Also, it is shown in [2] that for the case $k = 1$ not every solution is bounded and thus is not even asymptotically stable. For the case $A < 1$, one might study necessary and sufficient condition on the initial conditions so that every solution is asymptotically or globally stable or even is bounded.

## 4. The case $k = 1$

DeVault et al. [3] studied the difference equation

$$y_{n+1} = A + \frac{y_n}{y_{n-1}}, \tag{4.1}$$

with $A > 0$ and strictly positive initial conditions. Now, we study the stability properties and semi-cycle behavior of this equation without positivity restrictions. Consider the equation

$$y_{n+1} = A - \frac{y_n}{y_{n-1}}, \tag{4.2}$$

where $A < 0$. Then, using the change of variables

$$y_n = -x_n.$$

Eq. (4.2) becomes

$$x_{n+1} = -A + \frac{x_n}{x_{n-1}} = \alpha + \frac{x_n}{x_{n-1}},$$

where $\alpha = -A > 0$. It follows that all the results in DeVault et al. hold for Eq. (4.2) in the following cases:

- $A < 0$, $y_{-1}, y_0 < 0$.
- $A < 0$, $y_{-1}, y_0 > 0$.
- $A > 0$, $y_{-1} > 0$, $y_0 < 0$ (without the above change of variables).

**Theorem 4.1.** *Let* $A = -1$, $y_{-1}, y_1 \in \mathbb{R}^*$, *and let* $\{y_n\}_{n=1}^{\infty}$ *be a solution of Eq.* (4.1). *Then, every positive semicycle is of length one.*

**Proof.** When $k = 1$. Let $y_n < 0$ and $y_{n+1} > 0$. Then

$$y_{n+2} = A + \frac{y_{n+1}}{y_n} < 0.$$

This completes the proof. □

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