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THE NUCLEUS-NUCLEUS POTENTIAL †

H R. JAQAMAN and A Z MEKJIAN

Department of Physics, Rutgers University, New Brunswick, New Jersey 08903

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Abstract: Using the generator coordinate method to investigate the validity of the use of a potential to describe heavy ion elastic scattering, it is found that, under certain conditions, an effective potential exists. These conditions are generally satisfied by heavy ions, especially for the potential's tail. The potential obtained includes a kinetic energy contribution. The potential is further investigated using a density-dependent delta interaction.

1. Introduction

The generator coordinate method (GCM)^{1,2} offers one of the most practical frameworks for the description of nuclear reactions between heavy ions. There have been numerous investigations of this method in recent years [see e.g., refs. ^{3-6,8}], and references cited in ref. ²]. In this case one starts from the Griffin-Hill-Wheeler (GHW) equation and solves for wave functions in the continuum. These solutions have to be carried out numerically [see e.g. refs. ^{3,4}] since the GHW equation is an integral equation; this prevents one from gaining insight into the nature of the solution, especially since one is accustomed to solving a Schrödinger-type differential equation. One would also like to find a connection between the solutions to the GHW equation and the phenomenological optical potentials obtained by fitting the data in heavy ion scattering. Apart from these theoretical interests, the numerical integration and solution of the GHW equation meets with some technical difficulties⁴). Furthermore, most calculations using the GCM have so far been limited to very light ions (mostly α - α), and although probably not out of the question, its application to heavier ions would entail a huge amount of calculation that has been the main obstacle so far. In this paper we will attempt to transform the GHW equation into a differential (Schrödinger-type) equation that will allow us to get a better insight into the processes taking place and to obtain the solutions more easily. In particular, this method gives us an effective potential for the interaction between two heavy ions that can be compared with phenomenological optical potentials.

In sect. 2 we review the GCM as it has been applied to heavy ion reactions, and then we use it in sect. 3 to get the differential equation for the wave function of relative motion and consequently an expression for the nucleus-nucleus potential and the condition of its validity. This potential is then further investigated in sect. 4, where

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the α - ^{16}O potential is calculated using a density-dependent force, and then used to calculate cluster states in ^{20}Ne . The α - ^{16}O example is chosen partly because it is easy to get analytic expressions for it and partly because of the availability of other calculations. In general, however, these calculations have to be done numerically, and the expressions given in subsection 4.2 pay special attention to this. The ^{16}O - ^{16}O potential is also discussed in subsection 4.2.

2. Review of GCM for reactions

In the GCM we start with a wave function of the form (\mathcal{A} is the antisymmetrizing operator):

$$\Phi(x, \alpha) = \mathcal{A} \left[\prod_{i=1}^{A_1+A_2} \phi_i(r_i, \alpha_i) \right] = \det \{ \phi_i(r_i, \alpha_i) \}, \quad (1)$$

that has $A_1 + A_2$ nucleons distributed into a number λ of potential wells (here two), each well characterized by a set α_j of parameters. The final many-body wave function is the linear combination

$$\Psi(x) = \int \Phi(x, \alpha) f(\alpha) d\alpha \quad (2)$$

Applying the Ritz variational principle to Ψ , we get the generator coordinate (GHW) wave equation

$$\int [H(\alpha, \alpha') - EI(\alpha, \alpha')] f(\alpha') d\alpha' = 0, \quad (3a)$$

where

$$\begin{Bmatrix} I(\alpha, \alpha') \\ H(\alpha, \alpha') \end{Bmatrix} = \langle \Phi(x, \alpha) | \begin{Bmatrix} 1 \\ H \end{Bmatrix} | \Phi(x, \alpha') \rangle \quad (3b)$$

These involve integrations over the $A_1 + A_2$ single-particle coordinates $\{r_i\}$ in the system. Although the use of the Ritz variational principle is acceptable for bound state solutions, it does not necessarily follow that it can be used for continuum solutions. However, the use of the GHW equation can be justified for these solutions with or without the use of some variational principle^{4,5}). It is also known that the GCM breaks down for reactions between two unequal fragments^{4,6}); therefore we will limit ourselves to two equal fragments. In the case of two unequal fragments, the wave function has a center-of-mass (c.m.) spuriousity in it (because the total c.m. motion does not drop out) but its effect would be smaller for heavy ions than for light ions, so that our results are expected to also apply fairly well to two unequal heavy ions.

The sets of parameters α_j characterize the positions of the wells plus any other properties such as their deformation. In the present work we are only interested in the positions of the wells and so we need two parameters α_1 and α_2 to describe these positions that would coincide with the expectation values of the c.m. of the two ions.

Since the total c.m. motion drops out, we actually need one parameter $\alpha = \alpha_1 - \alpha_2$. The GHW generating function for a system of two identical clusters is then

$$\begin{aligned}\Phi(\mathbf{x}; \alpha_1, \alpha_2) &= \mathcal{A}[\Phi^{(d)}(\mathbf{x}; \alpha_1, \alpha_2)], \\ \Phi^{(d)}(\mathbf{x}; \alpha_1, \alpha_2) &= \Phi_1(\mathbf{x}_1 - \alpha_1)\Phi_2(\mathbf{x}_2 - \alpha_2), \\ \Phi_\lambda(\mathbf{x}_\lambda - \alpha_\lambda) &= \mathcal{A} \left[\prod_{i=1}^{A_\lambda} \phi_i(\mathbf{r}_i - \alpha_\lambda) \right] = \det \{ \phi_i(\mathbf{r}_i - \alpha_\lambda) \}, \\ \lambda &= 1, 2.\end{aligned}\tag{1'}$$

The wave function (1') can be written

$$\Phi(\mathbf{x}, \alpha_1, \alpha_2) = \phi_{\text{c.m.}}(\mathbf{R}_{\text{c.m.}}) \mathcal{A}[G(\mathbf{r} - \alpha) \psi_1^{\text{int}} \psi_2^{\text{int}}],\tag{4}$$

where $G(\mathbf{r} - \alpha)$ is a function peaked at $\mathbf{r} = \alpha$, and $\psi_1^{\text{int}}, \psi_2^{\text{int}}$ are the internal wave functions for the two nuclei and depend only on internal coordinates. It can be easily shown that

$$g(\mathbf{r}) = \int d\alpha G(\mathbf{r} - \alpha) f(\alpha),\tag{5}$$

may be regarded as the wave function of relative motion between the two ions by comparing (2) with the resonating group wave function using the form (4) for Φ .

One of the first technical difficulties in the GCM is that many functions $g(\mathbf{r})$ cannot be represented by eq (5) with a well-behaved function $f(\alpha)$; $f(\alpha)$ does not always exist although $g(\mathbf{r})$ does⁴). It would be desirable therefore to transform the problem into an equation governing $g(\mathbf{r})$ with no mention of $f(\alpha)$.

All calculations done so far have utilized harmonic oscillator (h.o.) wave functions for the $\phi_i(\mathbf{r}_i - \alpha_\lambda)$ in (1) or (1') above. For an oscillator potential with a single-nucleon parameter b , the c.m. motion of a nucleus factors out of a completely antisymmetrized shell-model wave function as a Gaussian⁷),

$$\psi_{\text{s.m.}} = \psi^{\text{int}} \exp[-A(\mathbf{x}_\lambda - \alpha_\lambda)^2/b^2],\tag{6}$$

provided all shells are closed, except at most one. If the two nuclei have the same b , as is the case for identical nuclei, this leads to

$$G(\mathbf{r} - \alpha) = (2u/\pi)^{\frac{3}{2}} \exp[-u(\mathbf{r} - \alpha)^2],\tag{7}$$

where

$$u = \frac{A_1 A_2}{2(A_1 + A_2)} \frac{1}{b^2} \left(= \frac{A}{8b^2} \quad \text{for } A_1 = A_2 = \frac{1}{2}A \right).$$

Even in investigations of the GCM where no explicit use of h.o. wave functions was made, the expression (7) for G had to be assumed for any progress to be achieved. This is actually a very plausible assumption since the c.m. motion is expected to factor out. Eqs. (6) and (7) reflect the uncertainty in the position of the ions, and they are the price we have to pay for trying to localize the c.m. We note that as the nucleus

becomes heavier it will oscillate less about its mean c.m. and so it becomes more appropriate to assume that the c.m. fluctuation factors out as a Gaussian that gets sharper as A increases and improvements due to GCM become less important ⁹⁾

The fact that $G(\mathbf{r}-\boldsymbol{\alpha})$ depends on $\mathbf{r}-\boldsymbol{\alpha}$ and is peaked at $\mathbf{r}-\boldsymbol{\alpha} = 0$ has been utilized by Yukawa ⁸⁾ to obtain another form for the GHW equation. Yukawa was able to isolate the kinetic energy of relative motion operator in the equation, which then becomes [see Wong ²⁾]

$$\left[\frac{-\hbar^2}{2\mu} \nabla_{\boldsymbol{\alpha}}^2 - E \right] \int I(\boldsymbol{\alpha}, \boldsymbol{\alpha}') f(\boldsymbol{\alpha}') d\boldsymbol{\alpha}' + \int V(\boldsymbol{\alpha}, \boldsymbol{\alpha}') f(\boldsymbol{\alpha}') d\boldsymbol{\alpha}' = 0, \quad (8a)$$

where

$$V(\boldsymbol{\alpha}, \boldsymbol{\alpha}') = \langle \Phi^{(d)}(\mathbf{x}, \boldsymbol{\alpha}) | V_{12} | \Phi(\mathbf{x}, \boldsymbol{\alpha}') \rangle, \quad (8b)$$

E is the energy of relative motion, and $V_{12} = \sum_{j \in 2}^{i \in 1} v(\mathbf{r}_i - \mathbf{r}_j)$ with v the nucleon-nucleon force. This alternative form for the GHW equation will be more useful for the present work.

3. Differential equation for $g(\boldsymbol{\alpha})$

Using the expression (7) for G we can write the kernels (3b) and (8b) as (we use K to denote I , H or V)

$$K(\boldsymbol{\alpha}, \boldsymbol{\alpha}') = \int \exp[-u(\mathbf{r}-\boldsymbol{\alpha})^2] k(\mathbf{r}) \exp[-u(\mathbf{r}-\boldsymbol{\alpha}')^2] d\mathbf{r}, \quad (9)$$

where $k(\mathbf{r})$, apart from a multiplicative constant factor, is the resonating group method kernel, and it does not depend on either $\boldsymbol{\alpha}$ or $\boldsymbol{\alpha}'$. This is to be compared with

$$K(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = \int \exp[-2u(\mathbf{r}-\boldsymbol{\alpha})^2] k(\mathbf{r}) d\mathbf{r} \quad (10)$$

Noting that

$$(\mathbf{r}-\boldsymbol{\alpha}')^2 = (\mathbf{r}-\boldsymbol{\alpha} + \boldsymbol{\alpha}-\boldsymbol{\alpha}')^2 = (\mathbf{r}-\boldsymbol{\alpha})^2 + (\boldsymbol{\alpha}-\boldsymbol{\alpha}')^2 + 2(\mathbf{r}-\boldsymbol{\alpha}) \cdot (\boldsymbol{\alpha}-\boldsymbol{\alpha}'),$$

we can write

$$K(\boldsymbol{\alpha}, \boldsymbol{\alpha}') = \exp[-u(\boldsymbol{\alpha}-\boldsymbol{\alpha}')^2] \int \exp[-2u(\mathbf{r}-\boldsymbol{\alpha})^2] \exp[-2u(\mathbf{r}-\boldsymbol{\alpha}) \cdot (\boldsymbol{\alpha}-\boldsymbol{\alpha}')] k(\mathbf{r}) d\mathbf{r} \quad (11)$$

Because of the presence of $\exp[-u(\boldsymbol{\alpha}-\boldsymbol{\alpha}')^2]$ and $\exp[-2u(\mathbf{r}-\boldsymbol{\alpha})^2]$, the contribution of $\exp[-2u(\mathbf{r}-\boldsymbol{\alpha}) \cdot (\boldsymbol{\alpha}-\boldsymbol{\alpha}')] is mainly in the region $(\mathbf{r}-\boldsymbol{\alpha}) \cdot (\boldsymbol{\alpha}-\boldsymbol{\alpha}') \approx 0$ so that we can write (up to second order)$

$$\exp[-2u(\mathbf{r}-\boldsymbol{\alpha}) \cdot (\boldsymbol{\alpha}-\boldsymbol{\alpha}')] \approx 1 - 2u(\mathbf{r}-\boldsymbol{\alpha}) \cdot (\boldsymbol{\alpha}-\boldsymbol{\alpha}') + 2u^2[(\mathbf{r}-\boldsymbol{\alpha}) \cdot (\boldsymbol{\alpha}-\boldsymbol{\alpha}')]^2.$$

This gives

$$\int K(\alpha, \alpha') f(\alpha') d\alpha' \approx \left\{ \frac{7}{4} K(\alpha, \alpha) + \frac{1}{4u} [\nabla_\alpha K(\alpha, \alpha)] \cdot \nabla_\alpha + \frac{1}{8u} K(\alpha, \alpha) \nabla_\alpha^2 \right. \\ \left. + \frac{1}{16u} [\nabla_\alpha^2 K(\alpha, \alpha)] + \frac{1}{32u^2} \sum_{i,j=x,y,z} \left[\frac{\partial^2}{\partial \alpha_i \partial \alpha_j} K(\alpha, \alpha) \right] \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} \right\} g(\alpha), \quad (12)$$

where differential operators contained within square brackets operate only on the functions inside those brackets. Substituting (12) back into the GHW eq (8a) we get

$$\left\{ \left(\frac{-\hbar^2}{2\mu} \nabla_\alpha^2 - E \right) \left(\frac{7}{4} I(\alpha, \alpha) + \frac{1}{4u} [\nabla_\alpha I(\alpha, \alpha)] \cdot \nabla_\alpha + \frac{1}{8u} I(\alpha, \alpha) \nabla_\alpha^2 + \frac{1}{16u} [\nabla_\alpha^2 I(\alpha, \alpha)] \right) \right. \\ \left. + \left(\frac{7}{4} V(\alpha, \alpha) + \frac{1}{4u} [\nabla_\alpha V(\alpha, \alpha)] \cdot \nabla_\alpha + \frac{1}{8u} V(\alpha, \alpha) \nabla_\alpha^2 + \frac{1}{16u} [\nabla_\alpha^2 V(\alpha, \alpha)] \right) \right\} g(\alpha) = 0,$$

where we have neglected the last term in (12) above. Such a term would introduce an anisotropy into the effective mass. The above equation can be reduced to an equation in Ig (arguments are dropped from now on):

$$\left\{ \frac{-\hbar^2}{2\mu} \nabla^2 - E \right\} \left\{ 1 + \frac{\nabla^2}{14u} - \frac{1}{28u} \frac{[\nabla^2 I]}{I} \right\} Ig = - \left\{ \frac{V}{I} + \frac{1}{7u} \left[\nabla \frac{V}{I} \right] \cdot \nabla - \frac{1}{7u} \frac{[\nabla V]}{I} \cdot \frac{[\nabla I]}{I} \right. \\ \left. + \frac{1}{14u} \frac{V}{I} \nabla^2 - \frac{1}{14u} \frac{V}{I} \frac{[\nabla^2 I]}{I} + \frac{1}{7u} \frac{V}{I} \left[\frac{\nabla I}{I} \right]^2 + \frac{1}{28u} \frac{[\nabla^2 V]}{I} \right\} Ig. \quad (13)$$

This can be put in the form

$$\left\{ \left(\frac{-\hbar^2}{2\mu} + \frac{1}{14u} \left(\frac{-\hbar^2}{2\mu} \nabla^2 + \frac{V}{I} + \frac{\hbar^2}{4\mu} \frac{[\nabla^2 I]}{I} - E \right) \right) \nabla^2 + V_{\text{eff}} + [\nabla U] \cdot \nabla - E \right\} Ig = 0, \quad (14)$$

where

$$V_{\text{eff}} = \frac{V}{I} + \frac{1}{14u} \left[\nabla^2 \left(\frac{V}{I} + \frac{\hbar^2}{4\mu} \frac{[\nabla^2 I]}{I} \right) \right] - \frac{1}{28u} \frac{[\nabla^2 V]}{I} + \frac{E}{28u} \frac{[\nabla^2 I]}{I}, \\ U = \frac{1}{7u} \frac{V}{I} + \frac{\hbar^2}{28\mu u} \frac{[\nabla^2 I]}{I}.$$

It is observed that (14) is a fourth-order differential equation or, in other words, it includes a velocity-dependent effective mass in addition to its radial dependence. On the other hand, the ordinary Schrödinger equation is of second order, and if we want a description of the scattering by a potential to be acceptable, we should try to find a second-order differential equation

$$\left\{ - \frac{\hbar^2}{2\mu} \nabla^2 + V_1 - E \right\} Ig = 0, \quad (15)$$

whose solutions satisfy (14). The potential V_1 is to be found by substituting (15) in (14) and requiring self-consistency [i.e., we should get (15) back]. The choice $V_1 = (V/I) + (\hbar^2/4\mu)([\nabla^2 I]/I)$ is found to satisfy the greater part of this condition: it exactly cancels out the velocity and radial dependence of the effective mass and the velocity dependent potential U , and would approximately satisfy the third condition that arises: $V_1 = V_{\text{eff}} - [\nabla^2 V_1]/14\mu$. Our V_1 satisfies instead

$$V_{\text{eff}} - \frac{[\nabla^2 V_1]}{14\mu} = V_1 + \frac{1}{I} \left\{ \left(\frac{E}{28u} - \frac{\hbar^2}{4\mu} \right) [\nabla^2 I] - \frac{[\nabla^2 V]}{28u} \right\}.$$

Therefore we can conclude that (15) exists, at least to a good approximation, if

$$\left| \left(\frac{E}{28u} - \frac{\hbar^2}{4\mu} \right) \nabla^2 I - \frac{\nabla^2 V}{28u} \right| \ll \left| V + \frac{\hbar^2}{4\mu} \nabla^2 I \right|. \quad (16)$$

Numerical calculations were performed to check the extent to which eq. (16) is satisfied for the case of α - α scattering using the Volkov force no. 2 [ref. 25]. Table 1 shows the values of d such that the left-hand side of (16) is $< 10\%$ and $< 20\%$ of the

TABLE I
The values of d as explained in the text

	E (MeV)	0	50	100	150	200	250
α - α	10 %	2.9	2.85	2.65	4.4	4.75	4.95
	20 %	2.75	2.55	2.60	2.7	4.15	4.5
2 nuclei of mass 100	5 %	2.6	4.4	4.55	4.6	4.65	4.7
	10 %	1.7	1.2	4.45	4.55	4.6	4.6
	20 %	1.2	1.0	0.85	4.45	4.5	4.55

The results for mass 100 are true for separations $r \geq d$ except $r \approx 4.2$ fm where the right-hand side of eq. (16) vanishes. However, this is directly related to the use of the α - α analytic expression and not relevant to the two $A = 100$ nuclei. For both cases only separations ≤ 5 fm were investigated and there are always subintervals in $r < d$ where the 5%, 10% and/or 20% requirements are satisfied also.

right-hand side for separation distances $> d$. The left-hand side is not less than 5% of the right-hand side for almost all separation distances. It is thus obvious that (16) is not well satisfied for the α - α system especially since α - α scattering is sensitive to the inner region of the potential. Also shown in table 1 are similar results obtained from the same analytic expression used for the α - α case but with larger values of μ and u appropriate for a system of two nuclei of mass 100. It is apparent that up to an energy ≈ 100 MeV eq. (16) is much better satisfied for this case, especially in the tail region of the potential; this region is all-important in heavy ion scattering. This agrees with what is expected from examining eq. (16) itself where, for larger μ and u , all the terms would become smaller compared to V .

We can now make the following further observations:

(a) The above equations [(13), (14) and (15)] are for $I(\alpha, \alpha)g(\alpha)$ and not for $g(\alpha)$, but since $I(\alpha, \alpha) \rightarrow 1$ for large α (the only region where $g(\alpha)$ has to be known) we can regard $I(\alpha, \alpha)g(\alpha)$ as our wave function. This ambiguity in the wave function at short distances is a reflection of the fact that the kinetic energy operator is not unique at short distances (with a complementary ambiguity in the effective potential) and is a consequence of antisymmetrization [see ref. ²) p. 337]. It is also intimately connected with the removal of velocity-dependent potentials ¹³) so that $I(\alpha, \alpha)$ here plays a role similar to that of F in ref. ¹³). This can be seen from (15) which, when changed to an equation for $g(\alpha)$ alone, would get a new term added to the potential plus a velocity-dependent term.

(b) The potential V_1 can be looked upon as equivalent to the real part of an optical potential and can be compared with phenomenological optical potentials. It consists of two parts $V_1 = (V/I) + (\hbar^2/4\mu)(\nabla^2 I/I)$, the first part is the potential that one would get in a Heitler-London approximation ¹⁴) and the second part has a form that suggests it is a kinetic energy contribution to the potential for relative motion between the two nuclei. There has been some discussion ¹⁵) by Zint and Mosel of such a contribution, but whereas these authors predict a repulsive contribution (based on the Born-Oppenheimer approximation), the term $(\hbar^2/4\mu)(\nabla^2 I)/I$ is not repulsive everywhere. In fact it is attractive for part of the tail region, although it is negligible there.

(c) There have been some objections ¹⁰) against the interpretation of V_1 (or similar terms) as being equivalent to the real part of an optical potential. These objections depend mainly on model calculations that show a strong variation of μ_{eff} in the interaction region ¹¹). We have a few observations about this point: (i) The calculations in ref. ¹¹) do not include the velocity dependence of the effective mass. This dependence [as those authors observe ¹¹)] is important for heavy ion scattering and neglecting it would amount in eq. (14) above to omitting the $\nabla^2 \nabla^2$ term which in turn would make it impossible to get to eq. (15) where the effective mass is a constant. (ii) Those calculations were done using the cranking formula whose validity is questionable for heavy ion reactions. This formula applies to adiabatic processes where it is safe to assume that the velocity of the collective motion (in the present case this would be the relative motion between the two heavy ions) is small compared with the Fermi motion ¹²). This is why the cranking formula has a perturbation-theoretical character. (iii) There is no reason why effective masses obtained in different models or by different processes should be the same, since these masses have no physical significance by themselves, and are not necessarily defined in the same way in all models. The effective masses obtained in this work do not show any of the strong r -dependence or the order of magnitude deviation from the reduced mass μ that was found in ref. ¹¹). Even if we include the higher-order terms that were omitted in getting to (12) and (13) no major changes in the above conclusions are expected, since the important terms are all in (13) already. However, it

could still be the case that the objections raised in ref. ¹⁰) are still valid where a potential description is not possible [i.e., where (16) is not satisfied].

4. The real part of the optical model potential

4.1 GENERAL DISCUSSION

In sect. 3 it was remarked that V_1 can be regarded as the real part of an optical potential for the elastic scattering of heavy ions and that the term V/I in V_1 corresponds to the Heitler-London approximation. This approximation amounts to putting A_1 nucleons in one well to form one nucleus and A_2 nucleons in another well, at a distance α from the first, and then calculate the interaction between the two nuclei. This is to be contrasted with the two-center shell model (which is equivalent to the molecular-orbital method) in which each nucleon is supposed to be in the field of both wells.

Denoting the nucleons in nucleus 1 by the numbers $1 \rightarrow A_1$, and in nucleus 2 by $A_1 + 1 \rightarrow A_1 + A_2$, we define

$$S_{ij} = \int \phi_i^*(r_1 - \alpha_{\lambda_i}) \phi_j(r_1 - \alpha_{\lambda_j}) dr_1, \quad \lambda_i, \lambda_j = 1, 2;$$

$$\langle ij|v|kl \rangle = \int \phi_i^*(r_1 - \alpha_{\lambda_i}) \phi_j^*(r_2 - \alpha_{\lambda_j}) v(\alpha + r_2 - r_1) \phi_k(r_1 - \alpha_{\lambda_k}) \phi_l(r_2 - \alpha_{\lambda_l}) dr_1 dr_2,$$

$$i \leq A_1, \quad j > A_1, \quad \alpha = \alpha_{\lambda_i} - \alpha_{\lambda_j},$$

and define $D_{ij,kl}$ as the determinant formed from $|S_{pq}|$ by omitting the i th and j th rows and the k th and l th columns, multiplied by the parity of the transformation that brings the i th and j th rows to the positions of the first and second rows, and the k th and l th columns to the positions of the first and second columns. The two vertical lines in $|S_{pq}|$ denote the determinant of the matrix S_{pq} . The notation here is similar to that of Slater ¹⁴) but not identical. In what follows $i \leq A_1$, $A_1 < j \leq A_1 + A_2$ in all expressions. With these definitions we have ¹⁴)

$$\frac{V(\alpha, \alpha)}{I(\alpha, \alpha)} = \frac{\langle \Phi^{(d)}(\mathbf{x}, \alpha) | V_{12} | \Phi(\mathbf{x}, \alpha) \rangle}{\langle \Phi^{(d)}(\mathbf{x}, \alpha) | \Phi(\mathbf{x}, \alpha) \rangle} = \frac{\sum_{ijkl} \langle ij|v|kl \rangle D_{ij,kl}}{|S_{pq}|}.$$

Some terms vanish, namely those corresponding to exchanges between nucleons belonging to the same nucleus, and we get

$$\frac{V}{I} = \frac{1}{|S_{pq}|} \sum_{i,j} \{ [\langle ij|v|ij \rangle - \langle ij|v|ji \rangle] D_{ij,ij} + \sum_{kl} \langle ij|v|kl \rangle D_{ij,kl} \}, \quad (17)$$

for at least one of $k, l \neq i, j$.

In the important tail region, where there is no substantial overlap between the two ions, the above expression simplifies considerably since all the exchange terms be-

come negligible, $S_{pq} \rightarrow \delta_{pq}$, $|S_{pq}| \rightarrow 1$, $D_{ij,ij} \rightarrow 1$ and

$$V_1 \rightarrow \frac{V}{I} \rightarrow \sum_{ij} \langle ij|v|ij \rangle = \int \rho_1(\mathbf{r}_1)\rho_2(\mathbf{r}_2)v(\boldsymbol{\alpha} + \mathbf{r}_2 - \mathbf{r}_1)d\mathbf{r}_1 d\mathbf{r}_2, \quad (18)$$

i.e., we end up with a folding model for the potential that has been previously introduced, more or less, phenomenologically^{16,17)}

It is also worthwhile to study (17) assuming it applies for all separation distances between two identical ions. As $\alpha \rightarrow 0$, $|S_{pq}| \rightarrow 0$ and, if we use as we should, a nucleon-nucleon force v that allows for nuclear saturation [a density-dependent force^{17,18)} for instance], this will lead to a steeply repulsive core which prevents the two ions from completely overlapping, as expected by the Pauli principle. This repulsion can be related to non-physical states excluded by the Pauli principle, the so-called null states^{2,8)}

For the sake of comparison, we define also an adiabatic limit for V/I , considering (17) as discussed above to be some sort of a non-adiabatic or sudden limit for the potential. Actually, the GCM, on which (17) is based, is intimately connected to a sudden-approximation picture of the interaction between the two nuclei: freeze the internal wave function of each nucleus, calculate the kernels and then allow the nucleons to interact. In the adiabatic limit we assume that the two nuclei are at rest with respect to each other at a certain distance, and allow their internal wave functions to readjust as required by the exclusion principle, the wave functions of the nucleons in one nucleus can be expanded in terms of only the unoccupied eigenstates of the Hamiltonian of the other nucleus. Then $S_{ij} = \delta_{ij}$, $I = 1$, and

$$\begin{aligned} V_1(\boldsymbol{\alpha}) &= \sum_{ij} [\langle ij|v|ij \rangle - \langle ij|v|ji \rangle] \\ &= \int \rho_1(\mathbf{r}_1)\rho_2(\mathbf{r}_2)v(\boldsymbol{\alpha} + \mathbf{r}_2 - \mathbf{r}_1)d\mathbf{r}_1 d\mathbf{r}_2 - \sum_{ij} \langle ij|v|ji \rangle \end{aligned} \quad (19)$$

For $\boldsymbol{\alpha} = 0$ and a zero-range force the exchange contribution is $\frac{1}{2}$ the direct contribution to V_1 [since $\langle ij|v|ji \rangle = \langle ij|v|ij \rangle$ for $\boldsymbol{\alpha} = 0$ and i, j having same spin and isospin] and we get

$$V_1(0) = \frac{3}{2} \int \rho_1(\mathbf{r}_1)\rho_2(\mathbf{r}_2)v(\mathbf{r}_2 - \mathbf{r}_1)d\mathbf{r}_1 d\mathbf{r}_2. \quad (20)$$

As α increases the exchange integrals become less and less important until they become negligible in the tail region so that the folding expression (18) emerges again. Two points are worth noting about (19), (i) it does not include the effect of the null states and this has to be kept in mind when using the expression (19) (see subsection 4.2), and (ii) ρ_1 and ρ_2 are not free-nucleus densities, except in the tail region of the potential where there is very little overlap, but are the instantaneous densities that include the deformation that the nuclei induce in each other. These are obviously difficult to calculate and are similar to the two-center shell-model densities. As a

first approximation, however, they can be taken as the free-nucleus densities ("frozen" density approximation) Unlike (17), (19) does not have a steeply repulsive core, if any.

For v , the nucleon-nucleon force, it is customary to use one of the effective interactions, such as the Skyrme force ¹⁹), used in nuclear structure calculations. However, for scattering calculations we have to add to this force a non-hermitean term that accounts for compound nucleus formation ²⁰). Such a term would mainly contribute to the imaginary part of the potential, is difficult to calculate so that a parametrized form is used instead, and would be neglected here as far as the real part of the potential is concerned. Furthermore, all the preceding discussion has been carried out in a one-channel picture, whereas in practice a number of channels are usually open. These can be taken into account by a coupled-channels calculation ²¹) which adds another term to the elastic channel effective operator, its real part will also be considered negligible. The neglect of all the above terms is justified in the surface region by the success of such phenomenological potentials as the folding model.

The above discussion throws some light on possible sources of an energy dependence in V_1 . Among these we identify:

(i) The energy dependence of the effective interaction itself in addition to the energy dependent terms added to it in order to account for compound nucleus formation and open channels (including both the non-locality of these terms and the increase in the number of open channels with energy)

(ii) The Pauli principle whose role changes with energy.

(iii) The energy dependence that is explicitly found in the expression for V_{eff} in eq. (14), which would be felt if eq. (16) is not well satisfied

Thus it is difficult to predict the energy dependence of the potential unless it is the case that one of the above sources dominates all the others.

4.2. CALCULATIONS WITH A DENSITY-DEPENDENT INTERACTION

In a previous work ¹⁷), the present authors used a density-dependent zero-range nuclear force linear in the local density.

$$v(\mathbf{r}_1 - \mathbf{r}_2) = -\alpha_0 \delta(\mathbf{r}_1 - \mathbf{r}_2) + \beta \rho \left(\frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2) \right) \delta(\mathbf{r}_1 - \mathbf{r}_2). \quad (21)$$

Such a force can be regarded as a very simplified form of the Skyrme interaction (which itself can be used if desired), and is chosen for the ease it provides in carrying out calculations and obtaining analytic results and its ability to reproduce important nuclear effects, especially saturation. A more sophisticated force is clearly needed for accurate calculations. It was shown ¹⁷) that by choosing $\alpha_0 = 956 \text{ MeV} \cdot \text{fm}^3$ and $\beta_0 = 2175 \text{ MeV} \cdot \text{fm}^6$, (21) gives a reasonable value for the tail of the potential obtained in a folding procedure (eq. (18)). This does not provide a sensitive test of the effect of the density-dependent part of (21) because of the little overlap between the densities of the two nuclei in this case. A better test would involve the inner region of the potential where there is an appreciable overlap between the nuclei. Such a

situation is provided by cluster-model energy levels in nuclei. For this case it is reasonable to use the adiabatic limit expression (19)

Vary and Dover ¹⁶⁾ used this expression for the calculation of energy levels of α -clusters on ^{16}O in ^{20}Ne with a density-independent delta interaction, but they had to introduce a considerable renormalization of the strength of the interaction (as compared to the one used for scattering calculations) in order to obtain a quite successful reproduction of the bound and resonant energy levels in ^{20}Ne . The use of density-dependent forces explains the major part of the renormalization needed. Substituting (21) in (19) we get

$$V_1(\alpha) = \frac{3}{4} \left\{ -\alpha_0 \int \rho_1(\mathbf{r}_1) \rho_2(\mathbf{r}_1 - \alpha) d\mathbf{r}_1 + \frac{3}{2} \beta_0 \int \rho_1(\mathbf{r}_1) \rho_2(\mathbf{r}_1 - \alpha) \rho_{\text{loc}} d^3 r_1 \right\}, \quad (22)$$

where ρ_{loc} is the local density at the site of interaction, the factor $\frac{3}{4}$ simulates the effect of the Pauli principle, and the $\frac{3}{2}$ factor is due to the rearrangement effects ²²⁾ that result from using a density-dependent force. Both factors are exact at the origin and are a fairly good approximation in the region near the origin that determines the depth of the potential. They go to 1 as the separation increases.

The first problem that arises is what to use for ρ_{loc} . A first guess might be the sum of the densities ρ_1 and ρ_2 at the site of the interaction, however, this is not the case. Even where the sudden approximation is applicable, $\rho_{\text{loc}} = \rho_1 + \rho_2 + \text{interference terms}$, since we should really add the wave functions and not the densities. In addition to the interference terms, ρ_1 and ρ_2 might be distorted from the free state densities. On the other hand, we have to keep in mind that the two clusters (α and ^{16}O) are part of the nucleus ^{20}Ne which, as a first approximation, can be thought of as a uniform sphere of density ρ_0 . Neglecting edge effects, (22) reduces to

$$V_1(\alpha) = \frac{3}{4} \left\{ -\alpha_1 \int \rho_1(\mathbf{r}_1) \rho_2(\mathbf{r}_1 - \alpha) d\mathbf{r}_1 \right\}, \quad (23)$$

where, now, $\alpha_1 = \alpha_0 - \frac{3}{2} \beta_0 \rho_0$. Putting $\rho_0 \approx 0.16 \text{ fm}^{-3}$, $\alpha_0 = 956 \text{ MeV} \cdot \text{fm}^3$, $\beta_0 = 2175 \text{ MeV} \cdot \text{fm}^6$, we get. $\alpha_1 \approx 434 \text{ MeV} \cdot \text{fm}^3$ to be compared with the value used by Vary and Dover ¹⁶⁾

$$\alpha'_1 = \frac{4}{3} \frac{2\pi \hbar^2}{M} \bar{f} = \frac{4}{3} \frac{2\pi(197)^2}{940} 1.237 = 427 \text{ MeV} \cdot \text{fm}^3,$$

while the value they used for scattering calculations is $\approx \alpha_0$. So while the very close agreement between α_1 and α'_1 could be accidental, we can at least conclude that we are able to account for the magnitude of the normalizing "strength factor" \bar{f} that they determine phenomenologically. The null states are taken into consideration by making sure that the calculated wave functions have the correct number of nodes, e.g., the 0^+ state in the ground state band should have four nodes, etc. On the other hand, if we use (17) we do not do this, the afore-mentioned 0^+ state will have no nodes

The repulsive core suppresses the inner oscillations of the wave function [see ref. ^{2,3}] for a similar discussion about ^{16}O - ^{16}O scattering].

We now consider a classical-geometrical model in which the two overlapping nuclei remain segments of uniformly dense spheres whose radii are functions of the distance between the two centers of the nuclei under the assumption that the enclosed volume is a constant (incompressibility). This gives the equation (see fig. 1):

$$R_1^3 + R_2^3 - \frac{1}{8}r^3 + \frac{3}{4}r(R_1^2 + R_2^2) + \frac{3}{8r}(R_1^2 - R_2^2)^2 = 2(R_{01}^3 + R_{02}^3).$$

Examination of this equation indicates that $R_2 \rightarrow R_1 \rightarrow \{R_{01}^3 + R_{02}^3\}^{\frac{1}{3}}$ as $r \rightarrow 0$, i.e., the radii of the spheres become larger. For two identical spheres

$$\frac{R}{R_0} \approx 2^{\frac{1}{3}} - \frac{1}{4} \frac{r}{R_0} + \frac{1}{2^{\frac{1}{3}}} \frac{1}{16} \frac{r^2}{R_0^2} + \frac{1}{96} \frac{1}{2^{\frac{1}{3}}} \frac{r^3}{R_0^3} + \dots$$

For $R_{01} \gg R_{02}$, the smaller sphere undergoes a more drastic change than the larger one.

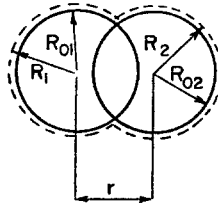


Fig. 1. Continuous line shows the original size of the nuclei, and the dotted line the sizes with the assumption of no compression.

Motivated by the above picture, we can get a fairly good approximation for the local density when the two nuclei have a large overlap by giving them larger effective radii (and a smaller central density to conserve the total mass), and then putting $\rho_{loc} = \rho_1 + \rho_2$. It is interesting that exact calculations ^{2,4}) of ρ_{loc} that include the interference effects tend to agree qualitatively with this picture, the nuclei appearing to have larger radii, in the static limit that applies when the relative motion between the two nuclei can be neglected. Harmonic oscillator densities are used for ^{16}O and α :

$$\rho_{^{16}\text{O}}(r) = 4 \left(\frac{v_1}{\pi} \right)^{\frac{3}{2}} (1 + 2v_1 r^2) e^{-v_1 r^2}, \quad (24)$$

$$\rho_{\alpha}(r) = 4 \left(\frac{v_2}{\pi} \right)^{\frac{3}{2}} e^{-v_2 r^2},$$

with $v_1 = 0.314$ and $v_2 = 0.538 \text{ fm}^{-2}$ for the free nuclei. To get larger radii, however, we have to use smaller values for v_1 and v_2 , which should be functions of the

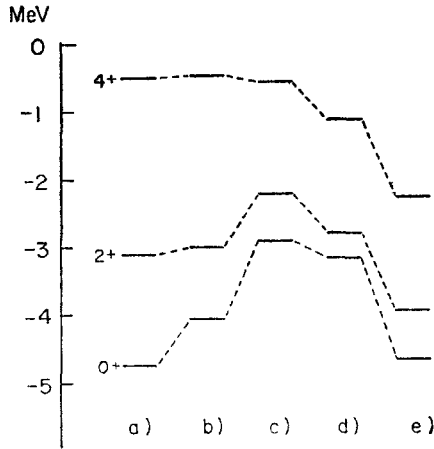


Fig 2. Energy levels for the ^{20}Ne ground-state band. (a) Experimental results. (b) Results using eq. (23) with $\alpha_1 = 417 \text{ MeV} \cdot \text{fm}^3$. (c) Results using eqs. (22) and (24) with $\rho_{loc} = \rho_1 + \rho_2$ and $\nu_1 = \nu_2 = 0.314$. (d) Same as (c) but with $\nu_1 = 0.314$ and $\nu_2 = 0.3$. (e) Same as (c) but with $\nu_1 = \nu_2 = 0.3$. Note that it is possible to fit each level by a slight readjustment of ν_1 and ν_2 and that this gives the correct depth for the potential. Energies are measured with respect to the α - ^{16}O threshold, and only bound levels are considered.

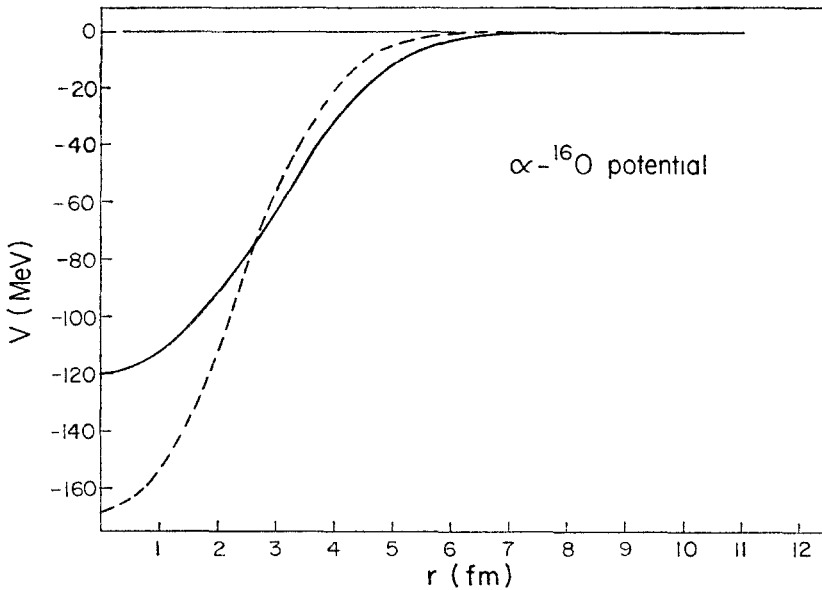


Fig. 3 The real part of the α - ^{16}O potential in the adiabatic limit. (i) continuous curve corresponds to (c) in fig. 2, and (ii) dashed curve corresponds to (b) in fig 2.

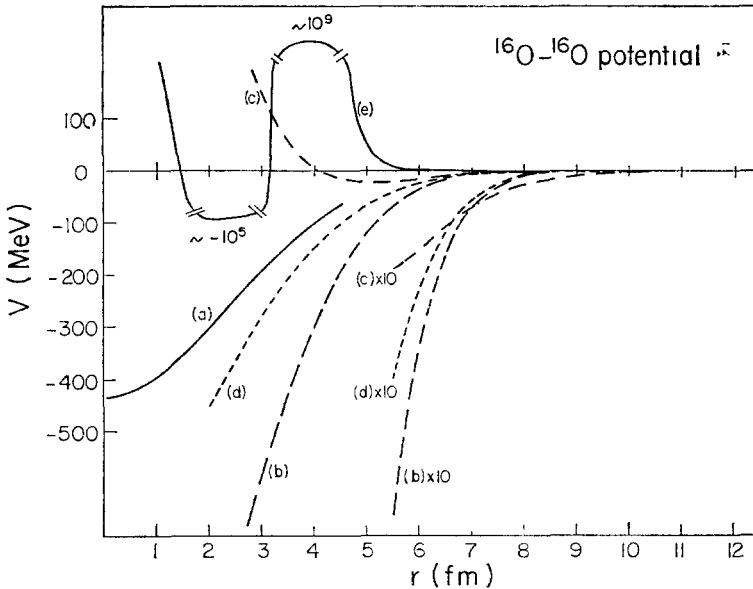


Fig. 4 The $^{16}\text{O}-^{16}\text{O}$ potential. (a), (b), and (d) are calculated in the adiabatic limit; (a) is valid near the origin, (b) and (d) are valid in the tail region [the actual potential curve in the intermediate region lying between (a) and (d)] (c) corresponds to V/I (eq (17)), 'the sudden approximation' (b), (c), and (d) are also magnified ten times for $r > 5.5$ fm to show the tail region more clearly. (e) is the $(\hbar^2/4\mu)(\nabla^2 I/I)$ correction that has to be added to (c) to give V_1 (eq (15)) The $^{16}\text{O}+^{16}\text{O}$ elastic scattering is sensitive to the potential for $r > 6.5$ fm only

separation distance. In practice, this is not easy to do and constant values are chosen so that the resulting potential is probably not acceptable for the surface region. Results for cluster-state calculations are shown in fig. 2. Fig. 3 gives the corresponding $\alpha-^{16}\text{O}$ potentials. The $^{16}\text{O}-^{16}\text{O}$ potential is also calculated and shown in fig. 4 where curve (a) is the potential obtained when eq (23) is used with $\alpha_1 = 420 \text{ MeV} \cdot \text{fm}^3$ (this potential is good near the origin and gives the depth of the potential), (b) is the potential obtained from eq (18) using (21) with $\alpha_0 = 956 \text{ MeV} \cdot \text{fm}^3$ and neglecting the density-dependent part since the resulting potential is good only in the tail region where this part is negligible. Curve (c) is the $^{16}\text{O}-^{16}\text{O}$ potential according to eq. (17); it is taken from the work of Yukawa⁸⁾ who uses a Volkov²⁵⁾ effective force which has a finite range, and is shown here for comparison. The longer range of (c) in the far tail region is due to the use of a finite-range force in the calculation (as opposed to the zero-range force used for (b)). The difference between (b) and (c) for $r \lesssim 7$ fm is due to the neglect of the density-dependent part of the delta interaction and the omission of exchange effects. This is obvious from curve (d) which is the same as (b) except that the density-dependent part of the force is included. Also shown in fig. 4, curve (e), is the kinetic energy term $(\hbar^2/4\mu)(\nabla^2 I/I)$ that has to be added to V/I (curve

(c)) to form the optical potential V_1 (see eq (15) and the discussion following it). It is apparent that this term modifies the potential and the final result is much shallower than without this term [cf. ref. ¹⁵]].

In the α - ^{16}O calculations, the Coulomb potential used was that due to the interaction between two spheres ²⁶) which is probably better than the sphere-point charge potential usually used. However, the energy level splitting is not sensitive to the Coulomb potential, and choosing different forms for the Coulomb potential merely shifts their position, which can be compensated for by a slight readjustment of the depth of the nuclear potential.

5. Conclusion

We have shown that the GCM for heavy ion reactions is equivalent, to a good approximation, to solving a Schrodinger equation with an effective potential whose real part has mainly a simple expression and includes a kinetic energy contribution besides the usual term that results from the interaction between the nucleons belonging to the different nuclei. This implies that a potential description of heavy ion reactions is not without a physical basis, and it explains the fact that phenomenological optical potentials have been very successful. The GCM has been a valuable tool, and this would become more apparent when calculations for heavier nuclei are performed. The simple density-dependent delta force we used has proved to be quite successful in reproducing many results. In spite of all the simplifications introduced the potential obtained still retains most of the properties of interest, and its simplicity gives a better feel for the interaction between nuclei, at least as a first step towards a comprehensive and more accurate unified microscopic description.

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References

- 1) D. J. Hill and J. A. Wheeler, Phys. Rev. **89** (1953) 1102,
J. J. Griffin and J. A. Wheeler, Phys. Rev. **108** (1957) 311
- 2) C. W. Wong, Phys. Lett. **15C** (1975), and references cited therein
- 3) N. de Takacsy, Phys. Rev. **C5** (1972) 1883
- 4) R. Beck, J. Borysowicz, D. M. Brink and M. V. Mihailovic, Nucl. Phys. **A244** (1975) 45 and 58
- 5) B. Giraud, J. LeTourneux and E. Osnes, Ann. of Phys. **89** (1975) 359
- 6) B. Giraud and J. LeTourneux, Nucl. Phys. **A240** (1975) 365
- 7) J. P. Elliott and T. H. R. Skyrme, Proc. Roy. Soc. **A232** (1955) 561
- 8) T. Yukawa, Nucl. Phys. **A186** (1972) 127, Phys. Lett. **38B** (1972) 1, Phys. Rev. **C8** (1973) 1593
- 9) T. Fliessbach, Z. Phys. **242** (1971) 287, Z. Phys. **247** (1971) 117;
D. M. Brink and Fl. Stancu, Nucl. Phys. **A243** (1975) 175
- 10) H. Friedrich, K. Langanke, A. Weiguny and R. Santo, Phys. Lett. **55B** (1975) 345

- 11) P. Lichtner, D. Drechsel, J. Maruhn and W. Greiner, *Phys. Rev. Lett.* **28** (1972) 829
- 12) L. Zamick, invited talk, Tucson Conf. on effective interactions and operators, University of Arizona, June 2, 1975 (preprint);
Y. M. Engel *et al.*, University of Oxford, Department of Theoretical Physics, preprint
- 13) F. G. Perey, in *Nuclear spectroscopy and reactions*, ed. J. Cerny (Academic Press, New York, 1974) p. 145
- 14) W. Heitler and F. London, *Z. Phys.* **44** (1927) 455;
H. Margenau, *Phys. Rev.* **59** (1941) 37;
J. C. Slater, *Quantum theory of molecules and solids* (McGraw-Hill, New York, 1963) vol. 1
- 15) P. G. Zint and U. Mosel, *Phys. Lett.* **B56** (1975) 424
- 16) J. P. Vary and C. B. Dover, *Phys. Rev. Lett.* **31** (1973) 1510,
G. W. Greenlees, G. J. Pyle and Y. C. Tang, *Phys. Rev.* **171** (1968) 1115,
D. M. Brink and N. Rowley, *Nucl. Phys.* **A219** (1974) 79
- 17) H. Jaqaman and A. Z. Mekjian, *Phys. Lett.* **56B** (1975) 237
- 18) B. Sinha, *Phys. Rev. Lett.* **33** (1974) 600
- 19) T. H. R. Skyrme, *Nucl. Phys.* **9** (1959) 615
- 20) N. Austern, *Direct nuclear reaction theories* (Wiley, New York, 1970) p. 63
- 21) H. Feshbach, *Ann. of Phys.* **5** (1958) 357; **19** (1962) 287
- 22) R. Sharp, Ph. D. Thesis, Rutgers University (unpublished)
- 23) A. Tohsaki, F. Tanabe and R. Tamagaki, *Prog. Theor. Phys.* **53** (1975) 1022
- 24) T. Fließbach, *Z. Phys.* **238** (1970) 329, **242** (1971) 287
- 25) A. B. Volkov, *Nucl. Phys.* **74** (1965) 33
- 26) T. W. Donnelly, J. Dubach and J. D. Walecka, *Nucl. Phys.* **A232** (1974) 355