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# Bifurcation Analysis Of Applied Dynamical Models

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# Declaration

I certify that this thesis, submitted for the degree of Master of Mathematics to the department of Mathematics at Birzeit University, is of my own research except where otherwise acknowledged, and that this thesis (or any part of it ) has not been submitted for a higher degree to any other university or institution.

Amal El-Barmeel Signature........
September 13, 2014

# Abstract

Bifurcation appears in models in different fields such as biology, economics, ...etc. In this thesis we study the bifurcation in discrete-time dynamical system in one and two dimensions, we consider the sufficient conditions for the existence of the different types of bifurcation. We study bifurcation in logistic competition and predator-prey models. Also we draw bifurcation diagrams using Matlab 7.12.

*To mum and dad .*

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# Chapter 1

# Introduction

In many scientific fields researchers need to study difference or differential equations that contain parameters, so it is important to study the behavior of these equations as the value of parameter varies. This study focuses on the concept of bifurcation . In this thesis we mainly consider bifurcation in discrete dynamical systems (difference equations) in one and two dimensions. Bifurcation is classified according to the change of stability of the fixed point. We investigate these types and give their sufficient conditions.

The main goal of this thesis is to study the bifurcation in some population models. We study the competition model in which two species compete for some limited food source or in some way inhibits each others growth. The most well-known competition model has been studied by Lotka and Volterra in which two species $N_1$ and $N_2$ having logistic growth in the absence of the other. In our study we consider the model

$$x_{n+1} = \frac{ax_n(1-x_n)}{1+cy_n}$$

(1.1)

$$y_{n+1} = \frac{by_n(1-y_n)}{1+dx_n}$$

where $a, b > 0$, and $c, d \in (0, 1)$. The parameters $a$ and $b$ are known as intrinsic growth rates of species $x$ and $y$, the parameters $c$ and $d$ are known as the competition parameters of $x$ and $y$.

This model was studied by Guzowska, Luis and Elaydi (2011) [2]. We find that this model has three kinds of fixed points: extinction, exclusion and coexistence fixed point. We study the stability of these fixed points, and also compute the invariant center manifold, which play a center role in studying

stability and bifurcation.
Also we study the predator-prey model, in which the growth rate of one population is decreased and the other is increased. The predator-prey model is given by

$$\begin{aligned} N_{t+1} &= N_t + rN_t(1 - N_t) - aN_tP_t \\ P_{t+1} &= P_t + aP_t(N_t - P_t) \end{aligned} \tag{1.2}$$

Where $N_t$ and $P_t$ denote prey and predator densities respectively, while $r$ and $a$ are positive constants. This model was introduced recently by Canan Çelik and Oktay Duman, [1]. We discuss the stability of the fixed points and investigate the parametric space where the bifurcation happens. Furthermore we consider the predator-prey model with Allee effect.
Allee effect is phenomenon that took a large consideration from ecologists. It describes a positive interaction among individuals at low population sizes and this interaction may be critical for survival and reproduction [11]. Çelik and Duman studied predator-prey system with Allee effect on prey population:

$$\begin{aligned} N_{t+1} &= N_t + rN_t(1 - N_t)\frac{N_t}{\mu + N_t} - aN_tP_t \\ \\ P_{t+1} &= P_t + aP_t(N_t - P_t) \end{aligned} \tag{1.3}$$

Where the $\frac{N_t}{\mu+N_t}$ is taken as Allee effect and $\mu$ is Allee constant. Results concerning stability and bifurcation for this model were clarified. We depend in our analysis on trace-determinate plane and Jury test.
Finally, we make a generalization of Beverton-Holt model and consider discrete dynamical population model which is used to model a single species:

$$f(x_n) = \frac{ax_n(1 - x_n)}{1 + cx_n} \tag{1.4}$$

where $x_n \in [0, 1]$, and $c \in (0, 1)$, $a > 1$.
Moreover we use matlab program to plot bifurcation diagrams and the region of attraction of the previous models.

# Chapter 2

# Preliminaries

## 2.1 Introduction

In this chapter we summarize briefly the basic concepts and theories in one and two dimensional discrete dynamical systems, which enables us to understand the subsequent chapters in this thesis. Most of definitions and theorems are taken directly from [7], [12], and [4]. The interested reader can see the details in these references or other books on dynamical systems. First, we define the dynamical system. Then we introduce the notations of fixed points, hyperbolic and non- hyperbolic fixed points, also we investigate the stability criteria of fixed points of one and two dimensional maps of discrete dynamical systems.

## 2.2 Definition of a dynamical systems

Dynamical systems occupied considerable attention in many areas such as economics, social sciences, physics, engineering, ..., etc, since it can predict the future state of the system if the present state and the laws governing its evolution is known, hence the concept of dynamical system includes:

1. *State space:* the set of all states of the system which is characterized by a point of the set $X$.
2. *Time*
3. *Evolution operator*: the evolution law that determines the state $x_t$ of the system at time $t$, provided the initial state $x_0$ is known.

Now we are able to give a formal definition of dynamical system:

**Definition 1.** *[4, p.7]*

*A dynamical system is a triple $\{T, X, \varphi^t\}$, where $T$ is a time set, $X$ is a state space, and $\varphi^t : X \to X$ is a family of evolution operators parametrized by $t \in T$.*

Time in dynamical systems may be continuous, in which the law of evolution is defined by a differential equation, such as

$$\frac{dx}{dt} = f(x)$$

Where $X$ is the state space, and $f : X \to X$.
Or the time may be discrete in such case the law of evolution takes the form of a map or (difference equation):

$$x_{n+1} = f(x_n)$$

This equation gives us information about how the variable $x_n$ changes as time changes from $n$ to $n + 1$.
In this thesis we consider discrete dynamical systems of one and two dimensional maps, i.e the state space is in $\mathbb{R}$, or in $\mathbb{R}^2$.

## 2.3 Stability of one dimensional maps

In this section we present the main feature of one dimensional maps in discrete dynamical systems. In order to have a clear look of a dynamical system we must clarify some concepts.

**Definition 2.** *[7, p.2] Consider a map $f : \mathbb{R} \to \mathbb{R}$, then the orbit $\mathcal{O}(x_0)$ of a point $x_0 \in \mathbb{R}$ is defined to be the set of points*

$$\mathcal{O}(x_0) = \{x_0, f(x_0), f^2(x_0), f^3(x_0), \dots\}$$

Thus the orbit is a subset of state space $X$. If the evolution operator maps a point into itself, thus the orbit consists only of one point such a point is called "fixed", or "equilibrium" point.

**Definition 3.** *[7, p.15] A point $x^\star \in X$ is said to be a fixed point "equilibrium", if $f(x^\star) = x^\star$.*

In other words, at the fixed point the state of the system dose not change in time. Moreover, to find the fixed points of a system, we must solve the equation $f(x) = x$. For example, the fixed points of $f(x) = x^3$ are the solutions of the equation $x^3 - x = 0$. Hence there are three fixed points $-1, 0, 1$ for this map. In other words, a fixed point of a map $f$ is a point where the line $y = x$ intersects the curve of $y = f(x)$, as in figure 2.1.

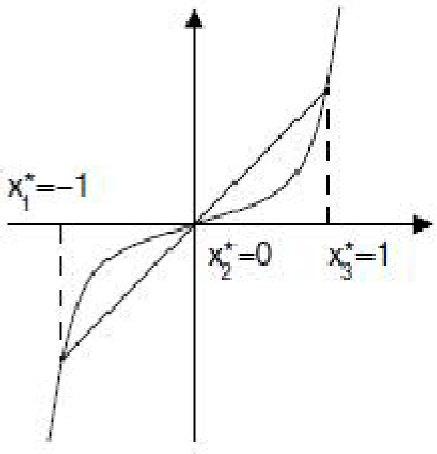



Figure 2.1: The fixed points of $f(x) = x^3$.

### 2.3.1 Stability and the Cobweb diagram

The main objective in dynamical systems is to study the behavior of orbits near fixed points. To investigate the behavior of a system near a fixed point, we define stability and unstability of fixed points as follows :

**Definition 4.** *[7, p.19] Let $f : X \to X$ be a map and $x^\star$ be a fixed point of $f$, where $X$ is an interval in $\mathbb{R}$.*

1. *The fixed point $x^*$ is said to be stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $x_0 \in X$ with $|x_0 - x^*| < \delta$ we have $|f^n(x_0) - x^*| < \epsilon$ for all $n \in \mathbb{Z}^+$ (see figure 2.2). Otherwise the fixed point $x^*$ will be called unstable(see figure 2.3).*

2. *The fixed point $x^*$ is said to be attracting if there exists $\eta > 0$ such that $|x_0 - x^*| < \eta$ implies that $\lim_{n\to\infty} f^n(x_0) = x^*$.*

3. *The fixed point $x^*$ is asymptotically stable if it is both stable and attracting (see figure 2.4).*

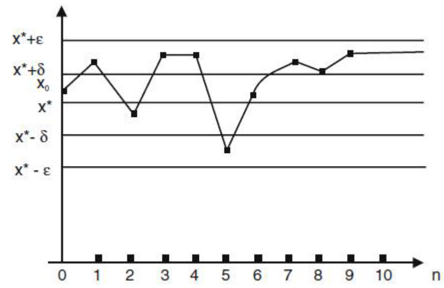



Figure 2.2: Stable fixed point $x^*$.

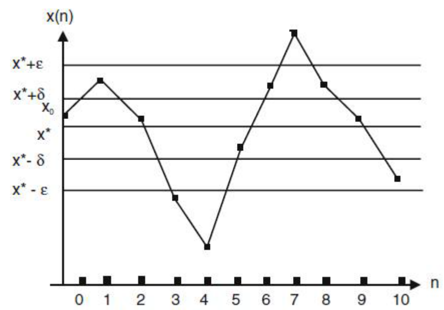



Figure 2.3: Unstable fixed point $x^*$.

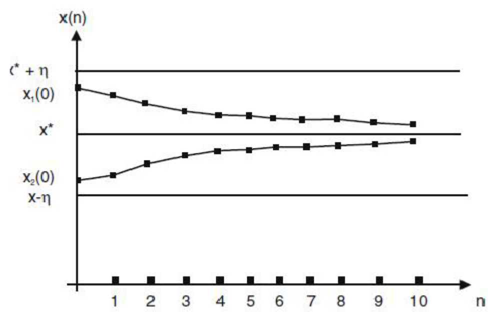



Figure 2.4: Asymptotically stable fixed point $x^*$.

**Cobweb diagram:**

Cobweb diagram allows us to iterate the function graphically, so we can determine the stability of the fixed point.
Draw the curve $y = f(x)$ and the line $y = x$ in the $xy$- plane, we start on the $x$-axis at initial point $x_0$, it is mapped to a new point $x_1$ which we find by drawing a vertical line from $x_0$ to the curve $f(x)$, to determine $x_1$ on $x$-axis, we move horizontally to the curve $y = x$, the $x$-coordinate will be $x_1$ to find the next point on the orbit $x_2$ by once again drawing a vertical line to the curve $y = f(x)$. Continuing this process will give us the point in the orbit of $x_0$ (see figure 2.5).

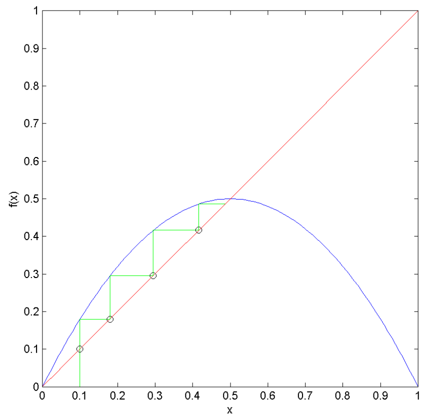



Figure 2.5: The Cobweb diagram.
The fixed point $x^* = 0.5$ is asyptotically stable, while $x^* = 0$ is unstable.

### 2.3.2 Criteria for stability

Fixed points are divided into two types: hyperbolic and non hyperbolic.

**Definition 5.** *Let $x^*$ be a fixed point for the system*

$$x_{n+1} = f(x_n) \tag{2.1}$$

*this fixed point is said to be hyperbolic if $|f'(x^*)| \neq 1$. Otherwise it is non hyperbolic.*

Now we will summarize the stability criteria for hyperbolic and non hyperbolic fixed points.

**Theorem 2.3.1.** *[7, p.25] Let $x^*$ be a hyperbolic fixed point of system(2.1), then the following statements hold:*

1. *If $|f'(x^*)| < 1$, then $x^*$ is asymptotically stable.*
2. *If $|f'(x^*)| > 1$, then $x^*$ is unstable.*

**Theorem 2.3.2.** *[7, p.28] Let $x^*$ be a fixed point of the system (2.1) such that $f'(x^*) = 1$, then the following statements hold:*

1. *If $f''(x^*) \neq 0$, then $x^*$ is unstable.*
2. *If $f''(x^*) = 0$ and $f'''(x^*) > 0$, then $x^*$ is unstable.*
3. *If $f''(x^*) = 0$ and $f'''(x^*) < 0$, then $x^*$ is asymptotically stable.*

Before we establish the stability criteria for non hyperbolic fixed points when $f'(x^*) = -1$, we need to introduce the definition of Schwarzian derivative.

**Definition 6.** *[7, p.30] The Schwarzian derivative ($Sf$) of a function $f$ is defined by*

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2$$

*If $f'(x^*) = -1$, then*

$$Sf(x^*) = -f'''(x^*) - \frac{3}{2}\left[f''(x^*)\right]^2$$

**Theorem 2.3.3.** *[7, p.31] Let $x^*$ be a fixed point of a map $f$ such that $f'(x^*) = -1$, then the following statements hold:*

1. *If $Sf(x^*) < 0$ then, $x^*$ is asymptotically stable.*
2. *If $Sf(x^*) > 0$ then, $x^*$ is unstable.*

### 2.3.3 Periodic points and their stability

Another type of orbits is a periodic orbit, in which any point return to itself after a certain period of time.

**Definition 7.** *[7, p.36] $\bar{x}$ is said to be a periodic point of $f$ with period $k$ if $f^k(\bar{x}) = \bar{x}$ for some positive integer $k$.*

The orbit of $k$-periodic point is

$$\mathcal{O}(\bar{x}) = \{\bar{x}, f(\bar{x}), f^2(\bar{x}), \ldots, f^{k-1}(\bar{x})\}$$

Now we will introduce the stability criteria of $k$-periodic points, note that the $k$-periodic point $\bar{x}$ is a fixed point for the map $f^k$, hence the study of the stability of $k$-periodic points of $f$, reduces to studying the stability of fixed points of $f^k$. Thus the following theorem holds:

**Theorem 2.3.4.** *[7, 37] Let $\bar{x}$ be a $k$- periodic point of $f$, then:*

1. *$\bar{x}$ is asymptotically stable if*
$$|f'(\bar{x}_1)f'(f(\bar{x}_2))\ldots f'(f^{k-1}(\bar{x}_k))| < 1$$
.

2. *$\bar{x}$ is unstable if*
$$|f'(\bar{x}_1)f'(f(\bar{x}_2))\ldots f'(f^{k-1}(\bar{x}_k))| > 1$$
.

## 2.4 Stability of two dimensional maps

### 2.4.1 Stability notation

Consider the discrete dynamical system

$$X_{k+1} = f(X_k), \qquad X \in \mathbb{R}^2 \tag{2.2}$$

Note that $X^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ is a fixed point of $f$ if $f(X^*) = X^*$.

Now we will give the required stability definitions of the fixed point $X^*$ of (2.2).

**Theorem 2.4.1.** *[7, p.195]*

1. *$X^*$ is stable if given $\epsilon > 0$ there exist $\delta > 0$ such that $|X - X^*| < \delta$ then, $|f^n(X) - X^*| < \epsilon$ for all $n \in \mathbb{Z}^+$, otherwise it is unstable.*

2. *$X^*$ ia attracting if there exist $\nu > 0$ such that $|X - X^*| < \nu$ then, $\lim_{\to\infty} f^n(X) = X^*$.*

3. *$X^*$ is asymptotically stable if it is both stable and attracting.*

### 2.4.2 Stability of linear system

A linear two dimensional system can be written as

$$X_{n+1} = AX_n \tag{2.3}$$

Where $A$ is $2 \times 2$ matrix. Hence the orbit of $X$ is given by

$$\{X, AX, A^2X, \dots, A^nX, \dots\}$$

By iteration, we may conclude that $X_n = A^nX_0$ is the solution of equation (2.3), where $X_0 = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$.

Obviously, the point $(0, 0)$ is a fixed point of the linear system (2.3). Now our main objective is to investigate the stability of the origin. The following theorem gives us complete information about the stability of the fixed point $X^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. For the matrix $A$ the spectral radius $\rho(A)$ is defined by

$$\rho(A) := \max\{|\lambda| : \lambda \quad \text{is eigenvalue of} \quad A\}$$

**Theorem 2.4.2.** *[7, p.198] Consider the linear system* (2.3) *then the following statements hold:*

1. *If $\rho(A) < 1$, then the origin is asymptotically stable.*
2. *If $\rho(A) > 1$, then the origin is unstable.*
3. *If $\rho(A) = 1$, then the origin is unstable if the Jordan form of $A$ is*

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

*(i.e $A$ has a single eigenvalue $\lambda$ with a single eigenvector. )*
*Otherwise the origin is stable.*

Now we will introduce another way to investigate the stability of $(0, 0)$ in linear system (2.3), namely trace-determinate plane. Recall that for matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $tr\ A = a_{11} + a_{22}$, and $detA = a_{11}a_{22} - a_{12}a_{21}$.

**Theorem 2.4.3.** *[7, p.203] let $A$ be a $2 \times 2$ matrix, then the origin is asymptotically stable if and only if*

$$|trA| - 1 < detA < 1.$$

### 2.4.3 Stability analysis of nonlinear discrete systems

Consider the nonlinear discrete system

$$X_{n+1} = f(X_n), \qquad X \in \mathbb{R}^2, \quad f \in C^r(R^2, R^2), r \geq 5 \tag{2.4}$$

The stability property of nonlinear system can be identified through the dynamical properties associated with linearized system. Suppose $X^*$ is a fixed point of system (2.4), such that $f(X^*) = X^*$. Take the Taylor expansion of $f$ around $X^*$, hence

$$f(X_n) = f(X^*) + Df(X^*)(X_n - X^*) + o(|X_n - X^*|^2) \tag{2.5}$$

Now we make the change of variables $U_n = X_n - X^*$, hence equation (2.5) becomes

$$f(U_n + X^*) - X^* = Df(X^*)U_n + o(|U_n|^2) \tag{2.6}$$

Let $g(U_n) = f(U_n + X^*) - X^*$ and $Df(X^*) = J$ in equation (2.6) we get

$$g(U_n) = JU_n + o(|U_n|^2) \tag{2.7}$$

Note that $g(0) = f(X^*) - X^* = 0$, hence 0 is a fixed point of $g$ if and only if $X^*$ is a fixed point of $f$. Also $g^n(U_n) \to 0$ if and only if $f^n(X_n) = f^n(U_n + X^*) \to X^*$. Hence, 0 is asymptotically stable under $g$ if and only if $X^*$ is asymptotically stable under $f$.
Notice that $o(|U_n|^2)$ is very small and can be neglected, hence we can approximate the nonlinear system (2.4) by the linear system

$$g(U_n) = JU_n$$

Where

$$J = Df(X^*) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(X^*) & \frac{\partial f_1}{\partial x_2}(X^*) \\ \\ \frac{\partial f_2}{\partial x_1}(X^*) & \frac{\partial f_2}{\partial x_2}(X^*) \end{pmatrix}$$

**Theorem 2.4.4.** *[12, p.15] Let $X^*$ be a fixed point of the system*

$$X_{n+1} = f(X_n), \qquad X_n \in \mathbb{R}^2$$

*Where $f$ is differentiable function, let $J$ be a Jacobian matrix of the above system, such that*

$$J = Df(X^*) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(X^*) & \frac{\partial f_1}{\partial x_2}(X^*) \\ \\ \frac{\partial f_2}{\partial x_1}(X^*) & \frac{\partial f_2}{\partial x_2}(X^*) \end{pmatrix}$$

*with eigenvalues $\lambda_1$ and $\lambda_2$, then :*

1. *$X^*$ is asymptotically stable if all of the eigenvalues of $J$ have modulus strictly less than one.*

2. *$X^*$ is unstable if $J$ has some eigenvalues with modulus greater than one.*

**Definition 8.** *A fixed point $X^*$ is hyperbolic if none of the eigenvalues of $J = Df(X^*)$ have modulus equal to one.*

### 2.4.4 Invariant manifolds

Now we will present the definition of invariant manifolds in linear and non-linear systems, and how their dynamics determine the dynamics of the orbits near the fixed point.

**Definition 9.** *[12, p.28] Let $S \subset \mathbb{R}^n$ be a set, then $S$ is said to be invariant under the map $X \to f(X)$, if for any $X_0 \in S$ then $f^n(X_0) \in S$ for all $n$.*

Consider the linear dynamical system

$$X_{n+1} = AX_n, \qquad X \in \mathbb{R}^n$$

Let $\mathrm{E}^s$, $\mathrm{E}^u$ and $\mathrm{E}^c$ be the (generalized) real eigenspace of $A$ associated with eigenvalues of $A$, such that

$$\begin{aligned}
\mathrm{E}^s &= span\{e_1, \ldots, e_s\} \\
\mathrm{E}^u &= span\{e_{s+1}, \ldots, e_{s+u}\} \\
\mathrm{E}^c &= span\{e_{s+u+1}, \ldots, e_{s+u+c}\}
\end{aligned}$$

where $e_1, \ldots, e_s$ are the (generalized) eigenvectors of $A$ corresponding to the eigenvalues of $A$ having modulus less than one, and $e_{s+1}, \ldots, e_{s+u}$ are the (generalized) eigenvectors of A corresponding to the eigenvalues of $A$ having modulus greater than one, and $e_{s+u+1}, \ldots, e_{s+u+c}$ are the (generalized) eigen-vectors of $A$ corresponding to the eigenvalues of A having modulus equal to one. Each of these spaces is invariant and represents stable, unstable and center spaces, respectively. Moreover, the orbits starting in $\mathrm{E}^s$ approach the origin as $n \to +\infty$, orbits starting in $\mathrm{E}^u$ approach the origin as $n \to -\infty$. We want to generalize these notions to the case of nonlinear system, thus the invariant manifold will correspond to nonlinear eigenspaces.

**Definition 10.** *A subset $S \subset \mathbb{R}^n$ is called as a k-manifold if it can be represented as the graph of a smooth function defined on the k-dimensional sub-space of $\mathbb{R}^n$.*

In nonlinear systems, any fixed point has an invariant manifold corresponding to eigenspaces associated with linearized system. There will be three invariant manifolds, namely, stable manifolds ($W^s_{loc}(X^*)$), unstable manifolds ($W^u_{loc}(X^*)$) and center manifolds ($W^c_{loc}(X^*)$). They are tangent to the associated linear eigenspaces at the fixed point $X^*$. We define the above invariant manifolds as follows :

$$W^s_{loc}(X^*) = \{X \in \mathbb{R}^n : f^n(X) \to X^* \text{ as } n \to \infty\}$$
$$W^u_{loc}(X^*) = \{X \in \mathbb{R}^n : f^n(X) \to X^* \text{ as } n \to -\infty\}$$

Center manifold will be studied in chapter four.

# Chapter 3

# Bifurcation of one-dimensional maps

## 3.1 Introduction

Consider the discrete-time dynamical system

$$x \longmapsto f(x,\mu), \qquad x \in \mathbb{R}^1, \mu \in \mathbb{R}^1 \tag{3.1}$$

Let $(\hat{x},\hat{\mu})$ be a hyperbolic fixed point, then a small change in the parameter $\hat{\mu}$, keeps the type of fixed points and its stability unchanged. Now if the point $(\hat{x},\hat{\mu})$ is non-hyperbolic fixed point, in this case, for $\mu$ very close to $\hat{\mu}$ new dynamics can be created, for example a new fixed point or new periodic orbit will appear. It seems that a qualitative change occurs when the system passes through a non-hyperbolic fixed point.

A change in the parameter causes a qualitative change in the dynamical system, and nature of its fixed points. This process is called bifurcation. This sudden change may happen in the number or nature of the fixed and periodic points, fixed points may appear or disappear, change their stability or even break a part into periodic points.

We will characterize and analyse the types of bifurcation of one-dimensional maps, in discrete-dynamical systems.

## 3.2 The saddle-node bifurcation

Saddle-node bifurcation is associated with a non hyperbolic fixed point $(\hat{x}, \hat{\mu})$, namely, $\frac{\partial f}{\partial x}(\hat{x}, \hat{\mu}) = 1$, in which two fixed points collide and annihilate each other, one of the fixed points is unstable, while the other is stable.

**Remark 3.2.1.** *[4, p.114]* This bifurcation has several names, as fold (or tangent bifurcation), turning point.

**Example 3.2.2.** Consider the map

$$f(x, \mu) = \mu + x + x^2, \qquad x \in \mathbb{R}^1, \ \ \mu \in \mathbb{R} \tag{3.2}$$

It is clear that the point $(0, 0)$ is non hyperbolic fixed point i.e

$$\begin{aligned} f(0,0) &= 0 \\ \frac{\partial f}{\partial x}(0,0) &= 1 \end{aligned} \tag{3.3}$$

For $\mu \neq 0$ the fixed points of the map are found by solving the equation

$$f(x, \mu) - x = \mu + x^2 = 0$$

for $\mu < 0$, there are two different branches of fixed points $x = \pm\sqrt{-\mu}$, for $\mu > 0$ the system has no fixed points. Now, we would like to check the behavior of the system near the point $(0, 0)$. Observe that $f'_\mu(x) = 1 + 2x$. Thus $|f'_\mu(-\sqrt{-\mu})| = |1 - 2\sqrt{-\mu}| < 1$ if and only if $-1 < 1 - 2\sqrt{-\mu} < 1$. Solving the latter inequality for $\mu$, so $-2 < -2\sqrt{-\mu} < 0$, we obtain $-1 < \mu < 0$. This implies that the branch $x = -\sqrt{-\mu}$ is asymptotically stable when $-1 < \mu < 0$. Furthermore, $|f'_\mu(\sqrt{-\mu})| = |1 + 2\sqrt{-\mu}| > 1$, and hence the branch $\sqrt{-\mu}$ is unstable for all $\mu < 0$. Thus when the value of the parameter varies from negative to positive, the two branches (stable, unstable) collide at $\mu = 0$ then disappear, this causes the saddle-node bifurcation.

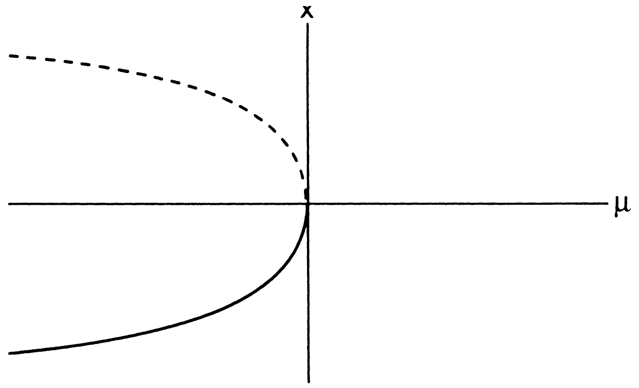



Figure 3.1: Saddle-node bifurcation of $f(x) = x + \mu + x^2$.

The stable branch $-\sqrt{-\mu}$, and unstable branch $\sqrt{-\mu}$ collide at the origin, this causes the saddle-node bifurcation.

**Remark 3.2.3.** We treat the map

$$f(x, \mu) = \mu + x - x^2$$

in the same way as the previous one, but the two branches $x = \pm\sqrt{\mu}$ appear when $\mu > 0$. And $|f'(\sqrt{\mu})| = |1 - 2\sqrt{\mu}| < 1$, if and only if $0 < \mu < 1$, while $|f'(-\sqrt{\mu})| = |1 + 2\sqrt{\mu}| > 1$. Hence in contrast, to the map(3.2), the upper branch is stable, and the lower is unstable as shown in the following fiqure:

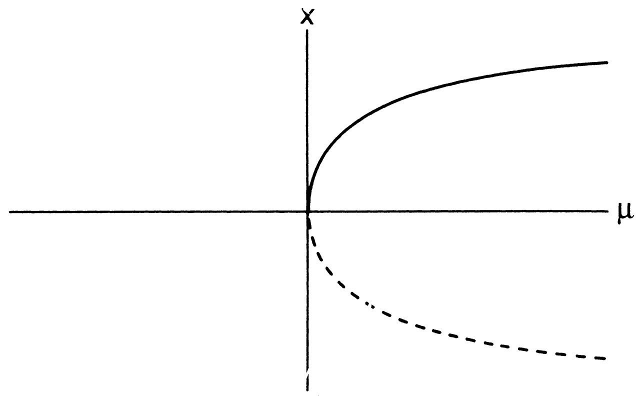



Figure 3.2: Saddle-node bifurcation of $f(x) = x + \mu - x^2$ .

**Theorem 3.2.4.** ***(The Saddle-node Bifurcation)***. *[7, P.86]*
*Suppose that $f_\mu(x) \equiv f(x,\mu)$ is a $C^2$ one-parameter family of one dimensional maps, and $\hat{x}$ is a fixed point of $f_{\hat{\mu}}$, with $\frac{\partial f}{\partial x}(\hat{x},\hat{\mu}) = 1$. Assume further that*

$$A = \frac{\partial f}{\partial \mu}(\hat{x},\hat{\mu}) \neq 0 \qquad \text{and} \qquad B = \frac{\partial^2 f}{\partial x^2}(\hat{x},\hat{\mu}) \neq 0$$

*Then there exists an interval $I$ around $\hat{x}$ and a $C^1$ map $\mu = p(x)$, where $p : I \longrightarrow \mathbb{R}$ such that $p(\hat{x}) = \hat{\mu}$, and $f(x,p(x)) = x$. Moreover, if $AB > 0$, the fixed point exists for $\mu < \hat{\mu}$, and, if $AB < 0$, the fixed point exists for $\mu > \hat{\mu}$.*

Before we prove the previous theorem, we state the implicit function theorem which we need in our proof.

**Theorem 3.2.5.** ***(The Implicit Function Theorem)*** *[7, p.86]*
*Suppose that $G : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a $C^1$ map in both variables such that for some $(\hat{x},\hat{\mu}) \in \mathbb{R}\times\mathbb{R}, G(\hat{x},\hat{\mu}) = 0$ and $\frac{\partial G}{\partial \mu}(\hat{x},\hat{\mu}) \neq 0$. Then, there exists an open interval $J$ around $\hat{\mu}$, an open interval $I$ around $\hat{x}$, and a $C^1$ map $\mu = p(\hat{x})$, where $p : I \to J$ such that*

1. *$p(\hat{x}) = \hat{\mu}$.*
2. *$G(x,p(x)) = 0$, for all $x \in I$.*

*Proof.* *of theorem* 3.2.4:
Let $G(x,\mu) = f(x,\mu) - x$. It is clear that the map $G$ satisfies the conditions of the implicit theorem, because

$$G(\hat{x},\hat{\mu}) = f(\hat{x},\hat{\mu}) - \hat{x} = 0$$
$$\frac{\partial G}{\partial \mu}(\hat{x},\hat{\mu}) = \frac{\partial f}{\partial \mu}(\hat{x},\hat{\mu}) \neq 0$$

Then there exists an open interval $I$ around $\hat{x}$ and a $C^1$ map $\mu = p(x)$ on $I$ such that $G(x,p(x)) = 0$. This implies that

$$f(x,p(x)) = x \tag{3.4}$$

Now differentiating equation (3.4) with respect to $x$, we have

$$\frac{\partial f}{\partial \mu}(\hat{x},\hat{\mu})p'(\hat{x}) + \frac{\partial f}{\partial x}(\hat{x},\hat{\mu}) = 1$$

By using the hypothesis of the theorem we have $p'(\hat{x}) = 0$, hence $\hat{x}$ is a critical point for $p(x)$. Now take the second derivative of (3.4) with respect to $x$, so we have

$$\frac{\partial^2 f}{\partial x^2}(\hat{x},\hat{\mu}) + \frac{\partial^2 f}{\partial \mu^2}(\hat{x},\hat{\mu})\left(p'(\hat{x})\right)^2 + 2\frac{\partial^2 f}{\partial \mu \partial x}(\hat{x},\hat{\mu})p'(\hat{x}) + \frac{\partial f}{\partial \mu}(\hat{x},\hat{\mu})p''(\hat{x}) = 0$$

but $p'(\hat{x}) = 0$, so we have

$$p''(\hat{x}) = -\frac{\frac{\partial^2 f}{\partial x^2}(\hat{x},\hat{\mu})}{\frac{\partial f}{\partial \mu}(\hat{x},\hat{\mu})} = -\frac{B}{A} \tag{3.5}$$

So if $AB > 0$ then $p''(\hat{x}) < 0$ hence the curve of $p(x)$ is concave downward at $x = \hat{x}$ and the fixed points exist for $\mu < \hat{\mu}$, the situation is reversed when $AB < 0$. [7, p.87] □

The above theorem specifies the conditions of the saddle-node bifurcation. In summery, a saddle-node bifurcation happens at $(\hat{x},\hat{\mu})$ if the following conditions occur

(a) $f(\hat{x},\hat{\mu}) = \hat{x}$

(b) $\frac{\partial f}{\partial x}(\hat{x},\hat{\mu}) = 1$

(c) $\frac{\partial f}{\partial \mu}(\hat{x},\hat{\mu}) \neq 0$

(d) $\frac{\partial^2 f}{\partial x^2}(\hat{x},\hat{\mu}) \neq 0$

We will now discuss another type of bifurcation that appears when $\frac{\partial f}{\partial x}(\hat{x},\hat{\mu}) = 1$.

## 3.3 The transcritical bifurcation

A transcritical bifurcation is a kind of local bifurcation of dynamical systems, where a fixed point interchanges its stability with another fixed point as the parameter is varied.

**Example 3.3.1.** Consider the map

$$f(x,\mu) = x + \mu x - x^2, \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1 \tag{3.6}$$

Notice that the equation (3.6) has two fixed points, $\hat{x}_1 = 0$ and $\hat{x}_2 = \mu$. It is clear that $(0,0)$ is non hyperbolic fixed point for equation (3.6), since

$$f(0,0) = 0$$
$$\frac{\partial f}{\partial x}(0,0) = 1$$

The curves of two fixed points passing through the bifurcation point "$(\hat{x},\hat{\mu}) = (0,0)$". We can determine the region of stability for each curve. Observe that $f'(x,\mu) = 1 + \mu - 2x$, and also $|f'(\hat{x}_1)| = |1+\mu| < 1$, if and only if $-2 < \mu < 0$. Hence the branch $x = 0$ is asymptotically stable when $-2 < \mu < 0$. Moreover, $|f'(\hat{x}_2)| = |1-\mu| < 1$, this implies that the branch $x = \mu$ is asymptotically stable when $0 < \mu < 2$. Hence, it is clear that the two curves intersect at the bifurcation point $(0,0)$ where the stability of these two curves is exchanged (i.e the curve $x = 0$ when it crosses the point $(0,0)$ goes from stable to unstable situation, while the other curve goes from unstable to stable situation). As the following figure shows:

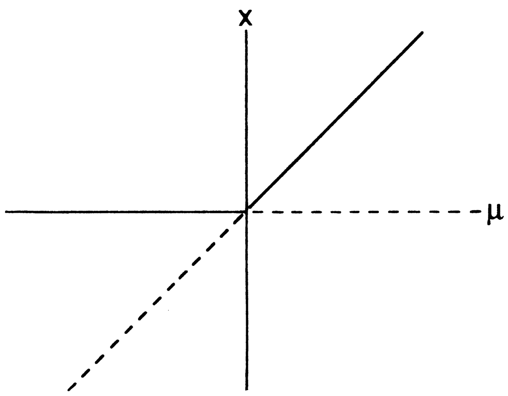



Figure 3.3: The transcritical bifurcation of $f(x) = x + \mu x - x^2$ .
An exchange of stability happen at the origin, between the branch $x = 0$ and $x = \mu$.

We want to find conditions which cause transcritical bifurcation at $(\hat{x},\hat{\mu})$.

**Theorem 3.3.2.** *[12, p.507]*

*Consider the map*

$$x \longmapsto f(x,\mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1 \tag{3.7}$$

*where $f$ is $C^r, (r \geq 2)$ and having a non hyperbolic fixed point at $(0,0)$ i.e.*

$$f(0,0) = 0$$
$$\frac{\partial f}{\partial x}(0,0) = 1$$

*and*

$$\frac{\partial f}{\partial \mu}(0,0) = 0$$

(3.8)
$$\frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0$$

*then $f$ undergoes a transcritical bifurcation at $(0,0)$.*

*Proof.* The fixed point of (3.7) is given by

(3.9)
$$h(x.\mu) = f(x,\mu) - x = 0$$

and let

$$h(x,\mu) = xH(x,\mu) = x(F(x,\mu) - 1)$$

where

(3.10)
$$H(x,\mu) = \begin{cases} \frac{h(x,\mu)}{x}, & x \neq 0 \\ \frac{\partial h}{\partial x}(0,\mu), & x = 0 \end{cases}$$

hence,

(3.11)
$$F(x,\mu) = \begin{cases} \frac{f(x,\mu)}{x}, & x \neq 0 \\ \frac{\partial f}{\partial x}(0,\mu), & x = 0 \end{cases}$$

Note that the curve $x = 0$ is a curve of fixed point. It is clear that

$$H(0,0) = F(0,0) - 1 = \frac{\partial f}{\partial x}(0,0) - 1 = 0$$

and

$$\frac{\partial H}{\partial \mu}(0,0) = \frac{\partial F}{\partial \mu}(0,0) = \frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0$$

thus by the implicit function theorem there exists $\mu(x)$ such that

$$H(x,\mu(x)) = F(x,\mu(x)) - 1 = 0 \tag{3.12}$$

Now differentiate (3.12) with respect to $x$ so we have

$$\frac{dH}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \mu}\frac{d\mu}{dx} = 0$$

thus

$$\frac{d\mu}{dx}(0) = \frac{-\frac{\partial F}{\partial x}(0,0)}{\frac{\partial F}{\partial \mu}(0,0)} = \frac{-\frac{\partial^2 f}{\partial x^2}(0,0)}{\frac{\partial^2 f}{\partial x \partial \mu}(0,0)} \neq 0$$

We can notice that $\mu(x)$ is a curve of fixed points for the system, since

$$F(x,\mu(x)) - 1 = 0$$

so

$$\frac{f(x,\mu(x))}{x} - 1 = 0$$

hence

$$f(x,\mu(x)) = x$$

And since $\frac{d\mu}{dx}(0) \neq 0$, this implies that $\mu(x)$ does not coincide with $x = 0$ and exists on both sides of $\mu = 0$.

□

## 3.4 The pitchfork bifurcation

**Example 3.4.1.** Consider the map

$$f(x,\mu) = x + \mu x - x^3, \qquad x \in \mathbb{R}^1 \quad \mu \in \mathbb{R}^1 \tag{3.13}$$

it is clear that

$$f(0,0) = 0$$
$$\frac{\partial f}{\partial x}(0,0) = 1$$

so the point $(0, 0)$ is non hyperbolic fixed point for the map (3.13). The equation of fixed points is given by

$$h(x, \mu) = f(x, \mu) - x = \mu x - x^3 = 0 \tag{3.14}$$

there are two curves of fixed points

$$x = 0$$

and

$$\mu = x^2$$

the curve $x = 0$, is stable when $-2 < \mu < 0$, while the curve $x^2 = \mu$ is stable when $0 < \mu < 1$, these branches exist at the right side of $\mu = 0$. In this case we have supercritical pitchfork bifurcation (see figure 3.4). Another case is subcritical bifurcation (see figure 3.5) where the normal form is

$$f(x, \mu) = x + \mu x + x^3$$

in this case $x = 0$ is stable for $-2 < \mu < 0$, and also there are two unstable fixed points $x = \pm\sqrt{-\mu}$ when $\mu < 0$.

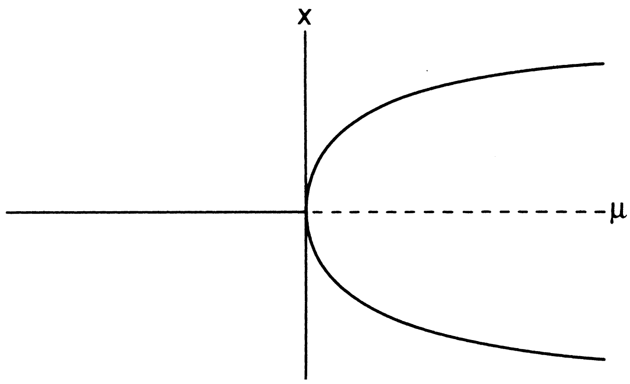



Figure 3.4: Supercritical pitchfork bifurcation.

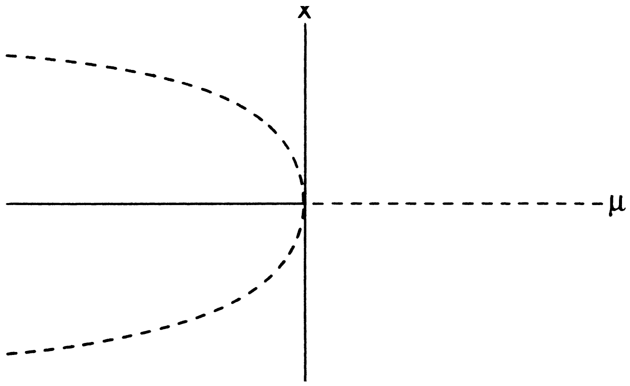



Figure 3.5: Subcritical pitchfork bifurcation.

To illustrate the conditions that cause this bifurcation, we have to work as the previous bifurcation.

**Theorem 3.4.2.** *[12, p.510] Consider a map*

$$x \longmapsto f(x,\mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1, \quad f \in C^r \quad (r \geq 3) \tag{3.15}$$

*with a non hyperbolic fixed point* $(0,0)$ *i.e.*

$$\begin{aligned} f(0,0) &= 0 \\ \frac{\partial f}{\partial x}(0,0) &= 1 \end{aligned} \tag{3.16}$$

*and*

$$\begin{aligned} \frac{\partial f}{\partial \mu}(0,0) &= 0 \\ \frac{\partial^2 f}{\partial x^2}(0,0) &= 0 \\ \frac{\partial^3 f}{\partial x^3}(0,0) &\neq 0 \\ \frac{\partial^2 f}{\partial x \partial \mu}(0,0) &\neq 0 \end{aligned} \tag{3.17}$$

*then* $f$ *undergoes a pitchfork bifurcation at* $(0,0)$

*Proof.* The fixed points of (3.15) are given by

$$f(x,\mu) - x = h(x,\mu) = 0$$

and let

$$h(x,\mu) = xH(x,\mu) = x(F(x,\mu) - 1)$$

where

$$H(x,\mu) = \begin{cases} \frac{h(x,\mu)}{x}, & x \neq 0 \\ \frac{\partial h}{\partial x}(0,\mu), & x = 0 \end{cases} \tag{3.18}$$

hence

$$F(x,\mu) = \begin{cases} \frac{f(x,\mu)}{x}, & x \neq 0 \\ \frac{\partial f}{\partial x}(0,\mu), & x = 0 \end{cases} \tag{3.19}$$

it is clear that

$$H(0,0) = 0$$

and

$$\frac{\partial H}{\partial \mu}(0,0) = \frac{\partial F}{\partial \mu}(0,0) = \frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0$$

thus by the implicit function theorem there exists $\mu(x)$ such that

$$H(x,\mu(x)) = F(x,\mu(x)) - 1 = 0 \tag{3.20}$$

Now differentiate (3.20) with respect to $x$ at $(0,0)$, and we have

$$\frac{dH}{dx}(0,0) = \frac{\partial F}{\partial x}(0,0) + \frac{\partial F}{\partial \mu}(0,0)\frac{d\mu}{dx}(0) = 0 \tag{3.21}$$

thus

$$\frac{d\mu}{dx}(0) = \frac{-\frac{\partial F}{\partial x}(0,0)}{\frac{\partial F}{\partial \mu}(0,0)} = \frac{-\frac{\partial^2 f}{\partial x^2}(0,0)}{\frac{\partial^2 f}{\partial \mu \partial x}(0,0)} = 0 \tag{3.22}$$

thus the point $(0,0)$ is a critical point of the curve $\mu(x)$.
Differentiate (3.21) with respect to $x$. This yields

$$\frac{\partial^2 F}{\partial x^2} + \frac{d^2\mu}{dx^2}\frac{\partial F}{\partial \mu} + 2\frac{\partial^2 F}{\partial \mu \partial x}\frac{d\mu}{dx} + \frac{\partial^2 F}{\partial u^2}\left(\frac{d\mu}{dx}\right)^2 = 0$$

but $\frac{d\mu}{dx}(0) = 0$, so

$$\frac{\partial^2 F}{\partial x^2}(0,0) + \frac{d^2\mu}{dx^2}(0)\frac{\partial F}{\partial \mu}(0,0) = 0$$

thus

$$\frac{d^2\mu}{dx^2}(0) = \frac{-\frac{\partial^2 F}{\partial x^2}(0,0)}{\frac{\partial F}{\partial \mu}(0,0)} = \frac{-\frac{\partial^3 f}{\partial x^3}(0,0)}{\frac{\partial^2 f}{\partial \mu \partial x}(0,0)} \neq 0$$

this means that $\mu(x)$ exists at the right side of $\mu = 0$ if

$$\frac{-\frac{\partial^3 f}{\partial x^3}(0,0)}{\frac{\partial^2 f}{\partial \mu \partial x}(0,0)} > 0$$

and at the left side of $\mu = 0$ otherwise. It is obvious that $x = 0$ is a curve of fixed points of $f$. This shows the characteristics of the curves of fixed points associated with pitchfork bifurcation. □

## 3.5 The period-doubling bifurcation

Now we will ask the following question" what is the type of bifurcation that happens at the non hyperbolic fixed point $(\hat{x}, \hat{\mu})$, with $\frac{\partial f}{\partial x}(\hat{x}, \hat{\mu}) = -1$?" This will be answered after studying this example.

**Example 3.5.1.** [12, p.513-515] Consider the one - dimensional map

$$f(x,\mu) = -x - \mu x + x^3, \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1 \tag{3.23}$$

it is clear that $(0,0)$ is a non hyperbolic fixed point and satisfy

$$\begin{aligned} &f(0,0) = 0 \\ &\frac{\partial f}{\partial x}(0,0) = -1 \end{aligned} \tag{3.24}$$

The fixed points of (3.23), are the solutions of the equation

$$f(x,\mu) - x = 0 \tag{3.25}$$

thus the map (3.23) has two curves of fixed points

$$x = 0$$

and

$$x^2 = \mu + 2$$

We will now check the stability of two curves. The curve $x = 0$ is stable when $-2 < \mu < 0$, and unstable anywhere else, while the curve $x^2 = \mu + 2$,

is unstable when $\mu > -2$, and does not exist when $\mu < -2$. We also see that only $x = 0$ passes through the bifurcation point $(0, 0)$, and nor conditions of previous bifurcation is satisfied.
We take the second iteration of (3.23), so :

$$f^2(x, \mu) = x + \mu(2 + \mu)x - 2x^3 + \mathcal{O}(x^4) \tag{3.26}$$

and it is easy to show that

$$\begin{aligned} &f^2(0,0) = 0 \\ &\frac{\partial f^2}{\partial x}(0,0) = 1 \\ &\frac{\partial f^2}{\partial \mu}(0,0) = 0 \\ &\frac{\partial^2 f^2}{\partial x^2}(0,0) = 0 \\ &\frac{\partial^3 f^2}{\partial x^3}(0,0) \neq 0 \\ &\frac{\partial^2 f^2}{\partial x \partial \mu}(0,0) \neq 0 \end{aligned} \tag{3.27}$$

thus the second iteration of $f$ undergoes a pitchfork bifurcation at a non hyperbolic fixed point $(0, 0)$. And since the fixed points of $f^2(x, \mu)$ are not the fixed points for $f(x, \mu)$, they must be period two points of $f(x, \mu)$. We say that $f(x, \mu)$ undergoes a period-doubling bifurcation at $(0, 0)$.

We now seek the conditions for the map (3.23) to undergoes a period-doubling bifurcation.

**Theorem 3.5.2.** *[7, p87-88] Suppose that*

*(a)* $f_\mu(\hat{x}) = \hat{x}$*, for all* $\mu$ *in an interval around* $\hat{\mu}$*.*

*(b)* $f'_{\hat{\mu}}(\hat{x}) = -1$

*(c)* $\frac{\partial^2 f^2}{\partial \mu \partial x}(\hat{x}, \hat{\mu}) \neq 0$

*Then there is an interval* $I$ *about* $\hat{x}$ *and a function* $p : I \longrightarrow \mathbb{R}$ *such that* $f_{p(x)}(x) \neq x$*, but* $f^2_{p(x)} = x$*.*

*Proof.* Let

$$B(x,\mu) = \begin{cases} \frac{G(x,\mu)}{x-\hat{x}} & \text{if } x \neq \hat{x} \\ \frac{\partial G}{\partial x}(\hat{x},\hat{\mu}) & \text{if } x = \hat{x} \end{cases}$$

where

$$G(x,\mu) = f_\mu^2(x) - x$$

It is clear that

$$\begin{aligned} B(\hat{x},\hat{\mu}) &= \frac{\partial G}{\partial x}(\hat{x},\hat{\mu}) \\ &= (\frac{\partial f}{\partial x}(\hat{x},\hat{\mu}))^2 - 1 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial B}{\partial \mu}(\hat{x},\hat{\mu}) &= \frac{\partial^2 G}{\partial \mu \partial x}(\hat{x},\hat{\mu}) \\ &= \frac{\partial^2 f^2}{\partial x \partial \mu}(\hat{x},\hat{\mu}) \\ &\neq 0 \end{aligned}$$

Hence by the implicit function theorem there exists a $C^1$ map $p(x)$ defined on an interval $I$ about $\hat{x}$, such that $B(x,p(x)) = 0$. Thus $\frac{G(x,p(x))}{x-\hat{x}} = 0, x \neq \hat{x}$. Consequently, $f^2(x,p(x)) = x$, □

**Example 3.5.3** (Logistic map)**.** [7, p.43-47] Consider the one-dimensional map

$$x_{n+1} = \mu x_n(1 - x_n) \tag{3.28}$$

Where $x \in [0,1]$ and $\mu \in (0,4]$. The fixed points of (3.28) are the solutions of the equation

$$f(\hat{x},\mu) = \mu\hat{x}(1-\hat{x}) = \hat{x} \tag{3.29}$$

So we have two fixed points which are

$$\hat{x}_1 = 0 \qquad \text{and} \qquad \hat{x}_2 = \frac{\mu - 1}{\mu}$$

Now we will illustrate the dynamical behavior of system (3.28) while parameter is varied on the interval $(0,4]$.

$0 < \mu < 1$

In this interval equation (3.28) has a unique fixed point which is $\hat{x}_1 = 0$, because $\hat{x}_2$ is negative. Also $\hat{x}_1$ is asymptotically stable (see figure 3.6a ) since $f'_\mu(x) = \mu - 2\mu x$ and $|f'_\mu(0)| < 1$

$\mu = 1$

$\hat{x}_1 = \hat{x}_2 = 0$, and $\frac{\partial f}{\partial x}(0, 1) = 1$ it follows that the point $(0, 1)$ is non hyperbolic point. Equation (3.28) undergoes at $(0, 1)$ a transcritical bifurcation since

$$\begin{aligned}
& f(0,1) = 0 \\
& \frac{\partial f}{\partial x}(0,1) = 1 \\
& \frac{\partial f}{\partial \mu}(0,1) = 0 \\
& \frac{\partial^2 f}{\partial x \partial \mu}(0,1) \neq 0 \\
& \frac{\partial^2 f}{\partial x^2}(0,1) \neq 0
\end{aligned} \tag{3.30}$$

Also, $f''(0, 1) = -2 \neq 0$, and by applying theorem 2.3.2, we may conclude that 0 is unstable. This is true if we consider negative as well as positive initial points in the neighborhood of 0. Since negative initial points are not in the domain of system (3.28), we discard them and consider only the positive initial points. Figure 3.6b tell us that the fixed point is semi-asymptotically stable frome the right, hence the fixed point $\hat{x}_1 = 0$ is asymptotically stable in the domain $[0, 1]$.

$1 < \mu < 3$

The fixed point $\hat{x_2} = \frac{\mu-1}{\mu}$ is asymptotically stable (see figure 3.6c ) because
$|f'_\mu(\hat{x}_2)| = |2 - \mu| < 1$, while $\hat{x}_1$ is unstable.

$\mu = 3$

We have $f'_3(\hat{x}_2) = f'_3(\frac{2}{3}) = -1$, thus the fixed point, $\frac{2}{3}$ is non-hyperbolic, and it is asymptotically stable ( see figure 3.6d ) because the Schwarzian derivative $(Sf_3(\frac{2}{3})) < 0$. If we check conditions (3.27) at the point $(x, \mu) = (\frac{2}{3}, 3)$ we conclude that the logistic map undergoes a period-doubling bifurcation at this point (see figure 3.7).

$\mu > 3$

The fixed point $\hat{x}_2$ is unstable (see figure 3.6e), and 2-periodic cycle will appear. To find the 2-periodic cycle we must solve the equation $f_\mu^2(x) = x$ or

$$\mu^2 x(1-x)[1-\mu x(1-x)] - x = 0$$

Discarding the fixed points $0$ and $\frac{\mu-1}{\mu}$, we get the 2-periodic cycle $\{\bar{x}_1, \bar{x}_2\}$ where

$$\bar{x}_1 = \frac{(1+\mu) - \sqrt{(\mu-3)(\mu+1)}}{2\mu}$$
$$\bar{x}_2 = \frac{(1+\mu) + \sqrt{(\mu-3)(\mu+1)}}{2\mu}$$

This 2-periodic cycle is asymptotically stable when $|f'_\mu(\bar{x}_1) f'_\mu(\bar{x}_2)| < 1$ or

$$-1 < \mu^2(1-2\bar{x}_1)(1-2\bar{x}_2) < 1$$

substituting the value of $\bar{x}_1$ and $\bar{x}_2$ leads to the follwing inequality

$$-1 < -\mu^2 + 2\mu + 4 < 1$$

solving the above inequality yields that the 2-periodic cycle is asymptotically stable when

$$3 < \mu < 1 + \sqrt{6}$$

$\mu = 1 + \sqrt{6}$

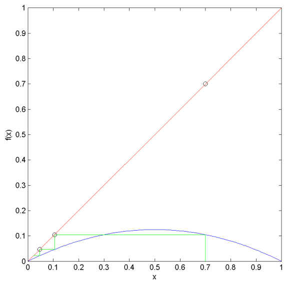
We have $f'_\mu(\bar{x}_1) f'_\mu(\bar{x}_2) = -1$, also $Sf_\mu^2(\bar{x}_1) < 0$ and $Sf_\mu^2(\bar{x}_1) < 0$, thus the periodic cycle is asymptotically stable, and it is unstable when $\mu > 1 + \sqrt{6}$. Since the 2-periodic cycle looses its stability, 4-periodic cycle will appear. It follows that the map $f_\mu^2(x)$ undergoes a period-doubling bifurcation when $\mu = 1 + \sqrt{6}$.

This process of double bifurcation continues indefinitely and produces a sequence $\{\mu_n\}_{n=1}^{\infty}$, and the ratio $\frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}$ approaches a constant called Fiegenbaum number, $\delta$ where

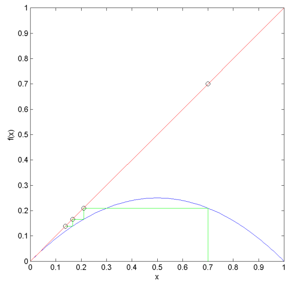
$$\delta = \lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} \approx 4.669201609\ldots \tag{3.31}$$

By using formula (3.31) we have

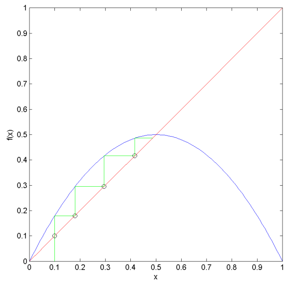
$$\mu_{n+1} = \mu_n + \frac{\mu_n - \mu_{n-1}}{\delta}$$



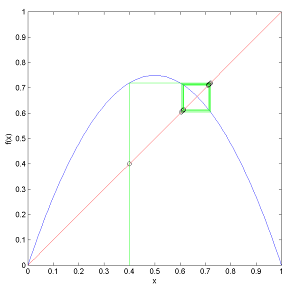
(a) $0 < \mu < 1$, the fixed point $x^* = 0$ is asymptotically stable



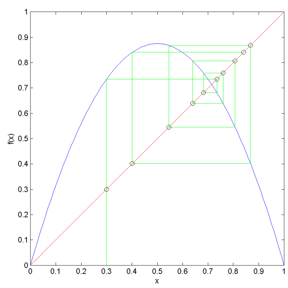
(b) $\mu = 1$, the fixed point $x^* = 0$ is asymptotically stable



(c) $1 < \mu < 3$, 0 is unstable, and $x_2^*$ is asymptotically stable



(d) $\mu = 3$, and $x_2^*$ is asymptotically stable, and two periodic cycle appears.



(e) $\mu > 3$, and $x_2^*$ is unstable.

Figure 3.6: The Cobweb diagram of logistic map

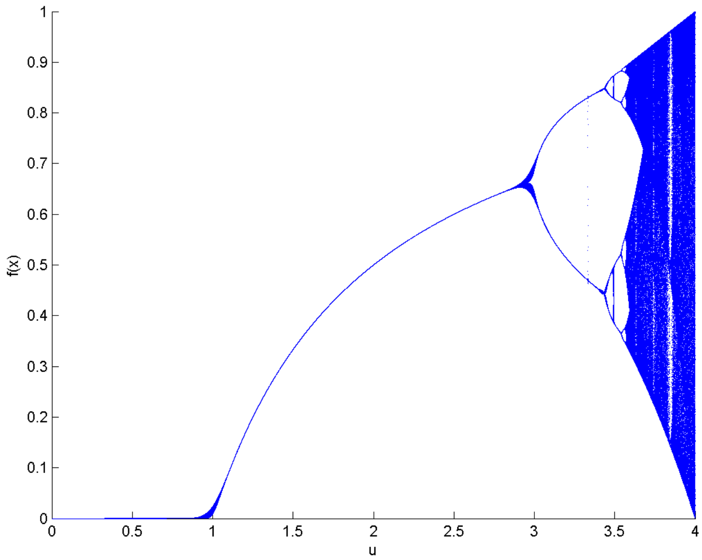



Figure 3.7: The bifurcation diagram of logistic map
An exchange of stability occures at $\mu = 1$ between $x_1^* = 0$ and $x_2^* = \frac{\mu-1}{\mu}$,(transcritical bifurcation). Also when $\mu = 3$ period doubling bifurcation occures.

# Chapter 4

# Bifurcation of two-dimensional maps

## 4.1 Introduction

Consider the map

$$(4.1) \qquad x \longmapsto f(x,\mu), \qquad x \in \mathbb{R}^2, \quad \mu \in \mathbb{R}^1, \quad f \in C^r, r \geq 5.$$

There are three different cases to consider for the non hyperbolic fixed point $(\hat{x}, \hat{\mu})$ of (4.1). We will now illustrate these different cases, and the type of bifurcation associated with each case. Let $J$ be the Jacobian matrix ($J = Df_x(\hat{x}, \hat{\mu})$).

1. $J$ has one real eigenvalue equals to 1, and the other eigenvalue is off, or in the unit circle.
2. $J$ has one real eigenvalue equals to $-1$, and the other eigenvalue is off, or in the unit circle.
3. $J$ has two complex conjugate eigenvalues with modulus equal 1. [7, p242]

## 4.2 Center manifolds

Here we will study bifurcation of system (4.1). In this case, the center manifold theorem is used to reduce the system (4.1) to one dimensional map $\tilde{f}_\mu$,

defined on the center manifold.
Consider the linear system

$$X \longrightarrow AX \qquad X \in \mathbb{R}^n$$

Where $A$ is $n \times n$ matrix. We mentioned before that each system has invariant subspaces $\mathrm{E}^s$, $\mathrm{E}^u$ and $\mathrm{E}^c$. Note that the orbits starting in $\mathrm{E}^s$ decay to zero as $n \to \infty$, orbits starting in $\mathrm{E}^u$ become unbounded as $n \to \infty$ and orbits starting in $\mathrm{E}^c$ neither grow nor decay as $n \to \infty$. If we suppose that $\mathrm{E}^u = \emptyset$ then we find that any orbit will rapidly decay to $\mathrm{E}^c$. Thus if we are interested in long-time behavior we need only to investigate the system restricted to $\mathrm{E}^c$. Similar type of reduction can be applied to study the stability of non-hyperbolic fixed points of nonlinear maps. There will be an invariant center manifold passing through the fixed point to which the system could be restricted in order to study the behavior of the system in the neighborhood of the fixed point. [12, p.245]
We state the center manifold theorem without proving it.

**Theorem 4.2.1.** *[12, p258]*
*Consider the following system*

$$\begin{aligned} x_{n+1} &= Ax_n + f(x_n, y_n) \\ y_{n+1} &= By_n + g(x_n, y_n) \end{aligned} \tag{4.2}$$

*where*

$$\begin{aligned} f(0,0) &= 0 \qquad & Df(0,0) &= 0 \\ g(0,0) &= 0 \qquad & Dg(0,0) &= 0 \end{aligned} \tag{4.3}$$

*Suppose that the system has $(0,0)$ as a fixed point, and $A$ is $c \times c$ matrix with eigenvalues of modulus one, and $B$ is $s \times s$ matrix with eigenvalues of modulus less than one. There exist a $C^r$ center manifold for the system (4.2), which can be locally represented as a graph as follows*

$$W^c = \{(x,y) \in R^c \times R^s | y = h(x), |x| < \delta, h(0) = 0, Dh(0) = 0\} \tag{4.4}$$

*for $\delta$ sufficiently small. Moreover, the dynamics of (4.2) restricted to the center manifold is, for $x$ sufficiently small, given by the c-dimensional map*

$$x \to Ax + f(x, h(x)), x \in R^c. \tag{4.5}$$

Notice that $h(0) = 0$ and $Dh(0) = 0$ imply that $W^c(0)$ is tangent to $\mathrm{E}^c$ at $(0,0)$. The dynamics of equation (4.5) determine the dynamics of the system

(4.2). If the point $(\hat{x}, \hat{y})$ is the fixed point for the system (4.2), such that $(\hat{x}, \hat{y}) \neq (0, 0)$, then we make a change of variables in system (4.2), so we can shift the point $(\hat{x}, \hat{y})$ to $(0, 0)$.
In studying the bifurcation of two dimensional maps, we have the following cases

(a) If $J$ has real eigenvalue equal to 1, then one of three different kinds of bifurcations can happen (saddle-node, transcritical, or pitchfork bifurcation).

(b) If $J$ has an eigenvalue equal to -1, then we have a period-doubling bifurcation.

(c) If $J$ has a pair of conjugate complex eigenvalues with modulus equal to one, we have a Neimark- Sacker bifurcation. [7, p.249]

**Remark 4.2.2.** [7, p.242] In the first two cases we apply the center manifold theorem to reduce to one-dimensional map. While Neimark-Sacker bifurcation has no analogue in one dimension. To distinguish any kind will happen we must check the conditions which had been studied in one dimension to the map $\check{f}_\mu$.

**Remark 4.2.3.** [7, p.243] To find the curve $y = h(x)$, we substitute $y = h(x)$ in the system (4.2), so we have

$$\begin{aligned} x(n+1) =& Ax(n) + f(x(n), h(x(n)) \\ y(n+1) =& Bh(x(n)) + g(x(n), h(x(n)) \\ =& h(Ax(n) + f(x(n), h(x(n)))) \end{aligned} \tag{4.6}$$

This leads to the functional equation

$$F(h(x)) = h[Ax + f(x, h(x))] - Bh(x) - g(x, h(x)) = 0 \tag{4.7}$$

We approximate the solution of (4.7) by power series, thus we write $h(x)$ as

$$h(x) = ax^2 + bx^3 + O(x^4) \tag{4.8}$$

**Example 4.2.4.** *[7, p.242]* Consider the system

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} xy \\ x^2 \end{pmatrix} \tag{4.9}$$

The origin is obviously a fixed point for (4.9) which is non-hyperbolic, the center manifold for system (4.9) can locally be represented as follow

$$W^c = \{(x, y) \in \mathbb{R}^2 : y = h(x), |x| < \delta, h(0) = h'(0) = 0\}$$

for $\delta$ sufficiently small. We assume that $h(x)$ has the form

$$h(x) = ax^2 + bx^3 + \mathcal{O}(x^4) \tag{4.10}$$

We recall from (4.7) that the equation for the center manifold is given by

$$F(h(x)) = h[Ax + f(x, h(x))] - Bh(x) - g(x, h(x)) = 0 \tag{4.11}$$

where, in this example, we have

$$\begin{aligned} A &= -1 & B &= -\frac{1}{2} \\ f(x, y) &= xy & g(x, y) &= x^2 \end{aligned}$$

hence equation (4.11) becomes

$$h(-x + xh(x)) + \frac{1}{2}h(x) - x^2 = 0 \tag{4.12}$$

Substituting (4.10) into (4.12), this yields

$$ax^2 - bx^3 + \frac{1}{2}(ax^2 + bx^3 + \mathcal{O}(x^4)) - x^2 = 0 \tag{4.13}$$

Hence

$$\begin{aligned} \frac{3}{2}a - 1 = 0 \quad &\text{or} \quad a = \frac{2}{3} \\ -\frac{1}{2}b = 0 \quad &\text{or} \quad b = 0 \end{aligned}$$

Consequently $h(x) = \frac{2}{3}x^2 + \mathcal{O}(x^4)$ and the map $f$ on the center manifold is given by

$$x \mapsto -x + \frac{2}{3}x^2 + \mathcal{O}(x^5)$$

## 4.3 The Neimark-Sacker bifurcation

We now focus our attention on the case when the Jacobian matrix has two complex conjugate eigenvalues with modulus equal one. We illustrate this case by the following example

**Example 4.3.1.** [7, p.250] Consider the map

$$F_\mu \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(1+\mu - x_1^2 - x_2^2\right) \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{4.14}$$

Where $\beta = \beta(\mu)$ is a smooth function of the parameter $\mu$ and $0 < \beta(0) < \pi$. It is clear that the origin is fixed point. The Jacobian of the system given in (4.14) is given by

$$JF_u = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \tag{4.15}$$

Where

$$\frac{\partial f_1}{\partial x_1} = (1+\mu)cos\beta - (3x_1^2 + x_2^2)cos\beta + 2x_1x_2sin\beta$$

$$\frac{\partial f_1}{\partial x_2} = -(1+\mu)sin\beta + (x_1^2 + 3x_2^2)sin\beta - 2x_1x_2cos\beta$$

$$\frac{\partial f_2}{\partial x_1} = (1+\mu)sin\beta - (3x_1^2 + x_2^2)sin\beta - 2x_1x_2cos\beta$$

$$\frac{\partial f_2}{\partial x_2} = (1+\mu)cos\beta - (x_1^2 + 3x_2^2)cos\beta - 2x_1x_2sin\beta$$

The Jacobian matrix evaluated at the fixed point $(x_1, x_2) = (0, 0)$ is given by

$$J = (1+\mu) \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \tag{4.16}$$

Now we find the eigenvalues of $J$ by solving the characteristic equation which is

$$(1+\mu)^2((cos\beta - \lambda)^2 + sin^2\beta) = 0$$

thus

$$\lambda^2 - 2\lambda cos\beta + 1 = 0$$

Hence the eigenvalues of $J$ are $\lambda_{1,2} = (1+\mu)e^{\pm i\beta}$, and their modulus is $|\lambda_{1,2}| = |1+\mu|$, hence at $\mu = 0$, we have $|\lambda_{1,2}| = 1$. Thus the two complex conjugate eigenvalues lie on the unit circle. It is clear that we have a Neimark-Sacker bifurcation. Also the origin is asymptotically stable when $-2 < \mu < 0$. To simplify the idea, we will invert the coordinates to the polar. Substitute in equation (4.14) the relations $x_1(n) = r(n)\cos\theta(n)$ *and* $x_2(n) = r(n)\sin\theta(n)$, we get

$$\begin{aligned} x_1(n+1) &= \big((1+\mu)r - r^3\big)\, cos(\theta+\beta) \\ x_2(n+1) &= \big((1+\mu)r - r^3\big)\, sin(\theta+\beta) \end{aligned}$$

and note that $r(n+1) = \sqrt{x_1^2(n+1) + x_2^2(n+1)}$ *and* $\theta(n+1) = \tan^{-1}\frac{x_2(n+1)}{x_1(n+1)}$ hence we have

$$\begin{aligned} r(n+1) &= (1+\mu)r(n) - r^3(n) \\ \theta(n+1) &= \theta(n) + \beta \end{aligned} \tag{4.17}$$

Bifurcation of the system (4.17) as $\mu$ passes through 0, can be easily determined. We can see that $\theta$ in the system (4.17) is independent of $\mu$, also this equation describes the rotation which depends on $\theta$ and $\beta$. Furthermore, the first equation in (4.17) defines a one dimensional map, whose fixed point is $r = 0$. This fixed point is stable for $-2 < \mu < 0$ and unstable for $\mu > 0$. At $\mu = 0$ the stability of the origin can be determined by cobweb diagram of $r \longmapsto r - r^3$, which shows that the origin is asymptotically stable ( see figure 4.1). Also the stability of the origin when $\mu = 0$ can be determined by taking the third derivative of the $r$-map at $r = 0$, which is less than zero, hence the origin is asymptotically stable. Moreover the one dimensional $r$-map of the system (4.17) has one additional stable fixed point $\hat{r} = \sqrt{\mu}$ for $0 < \mu < 1$. Thus the origin is surrounded for small $\mu > 0$ by closed invariant curve of radius $\hat{r} = \sqrt{\mu}$, and all the orbits starting outside or inside the closed invariant curve expected at the origin tend to the curve. This is a supercritical Neimark-Sacker bifurcation.

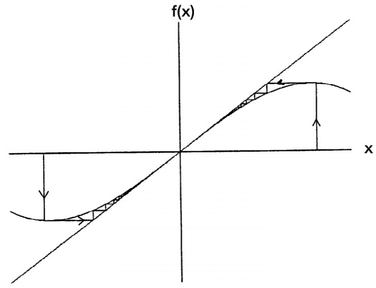



Figure 4.1: The cobweb diagram of the map $r \mapsto r - r^3$
The diaigram shows that when $\mu = 0$ then the fixed point $x = 0$ is asymptotically stable.

**Remark 4.3.2.** [7, p.250-252] and [4, p.125-127] For the system

$$F_\mu \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(1 + \mu + x_1^2 + x_2^2\right) \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{4.18}$$

The polar form associated with this system is

$$\begin{aligned} r(n+1) &= (1+\mu)r(n) + r^3(n) \\ \theta(n+1) &= \theta(n) + \beta \end{aligned}$$

This model can be analyzed as the previous one, but here we have a sub-critical Neimark-Sacker bifurcation. But here the fixed point $\hat{r} = \sqrt{-\mu}$ is unstable closed invariant curve for $\mu < 0$, which disappears when $\mu$ vary from negative to positive value.

## 4.4 The trace-determinant plane

We note that the eigenvalues of the Jacobian matrix play a basic role in determining the type of bifurcation in the plane. We now introduce an important result in trace-determinate plane which illustrates the bifurcation of two dimensional maps.

**Theorem 4.4.1.** *[7, p.249]*
*Consider the map*

$$x \longmapsto f(x,\mu), \qquad x \in \mathbb{R}^2, \quad \mu \in \mathbb{R}^1 \tag{4.19}$$

*and let $J$ be the Jacobian matrix, where $J = Df_\mu(\hat{x}, \hat{\mu})$. The following are true*

(a) *If* $trJ - 1 = detJ$*, then the system*(4.19) *undergoes a saddle-node bifurcation.*

(b) *If* $-trJ - 1 = detJ$*, then the system* (4.19) *undergoes a period-doubling bifurcation.*

(c) *If* $|trJ| - 1 < detJ$ *and* $detJ = 1$*, then the system* (4.19) *undergoes a Neimark -Sacker bifurcation.*

*Proof.* Let $J = Df_\mu(\hat{x}, \hat{\mu})$ be the Jacobian matrix of system (4.19). We know that the characteristic equation of $J$ is given by

$$\lambda^2 - (trJ)\lambda + (detJ) = 0$$

thus the eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}[trJ \pm \sqrt{(trJ)^2 - 4detJ}] \tag{4.20}$$

(a) let $trJ - 1 = detJ$, then $(trJ)^2 - 4detJ = (trJ - 2)^2 > 0$, this implies that the eigenvalues are real numbers. Then substitute the value of $detJ$ in (4.20), we get

$$\lambda_{1,2} = \begin{cases} 1 \\ detJ \end{cases} \tag{4.21}$$

thus the map (4.19) undergoes a saddle-node bifurcation.

(b) Similarly, if $-trJ - 1 = detJ$ then $(trJ)^2 - 4detJ = (trJ + 2)^2 > 0$, so we have two real eigenvalues, and consequently

$$\lambda_{1,2} = \begin{cases} -1 \\ -detJ \end{cases} \tag{4.22}$$

thus we have a period - doubling bifurcation.

(c) If $|trJ| - 1 < detJ$ and $detJ = 1$, then $(trJ)^2 - 4detJ < 0$. Hence we have two complex conjugates eigenvalues

$$\lambda_{1,2} = \frac{1}{2}[trJ \pm i\sqrt{4detJ - (trJ)^2}]$$

thus

$$
\begin{aligned}
|\lambda_{1,2}| &= \left|\frac{1}{2}[trJ \pm i\sqrt{4detJ - (trJ)^2}]\right| \\
&= \frac{1}{2}\sqrt{(trJ)^2 + 4detJ - (trJ)^2} \\
&= \sqrt{detJ} \\
&= 1
\end{aligned} \tag{4.23}
$$

So we have a Neimark-Sacker bifurcation .

□

This theorem shows that the trace-determinant plane has three critical regions which are $D = T - 1, D = -T - 1$, and $D = 1$ , we illustrate the theorem by the following example.

**Example 4.4.2.** Consider the following linear system

$$
\begin{aligned}
x_1(n+1) &= 2x_1(n) + 3x_2(n) \\
x_2(n+1) &= (1+\mu)x_1(n) + 4x_2(n)
\end{aligned} \tag{4.24}
$$

The Jacobian matrix associated with the previous system is

$$
J = \begin{pmatrix} 2 & 3 \\ 1+\mu & 4 \end{pmatrix} \tag{4.25}
$$

as $\mu$ varies, the trace of the Jacobian matrix "$trJ$" is always 6, while the determinant of the matrix "$detJ$" is always $5-3\mu$. We are moving vertically in the $(T-D)$ plane along the line $T = 6$. Now if $D < -T-1$ which occurs if $\mu > 4$ the associated eigenvalues in this case are $\lambda_1 > 1$ and $\lambda_2 < -1$, thus we have a source fixed point. When $D = -T - 1$ i.e $D = -7$, $\mu$ takes a critical value which is 4, and here we have period-doubling bifurcation. When $-T - 1 < D < 1$ this happens when $\frac{4}{3} < \mu < 4$. In this case the eigenvalues are $\lambda_1 > 1, 0 > \lambda_2 > -1$, so the fixed point in this region is saddle. When $1 < D < T - 1$, which is associated with $0 < \mu < \frac{4}{3}$, then $\lambda_1 > 1$ and $0 < \lambda_2 < 1$, thus the fixed point is saddle. Moreover when $T - 1 < D < \frac{T^2}{4}$, that is $\frac{-4}{3} < \mu < 0$, then $\lambda_{1,2} > 1$, so we have a source fixed point. A saddle-node bifurcation happens when $D = T - 1$, at the critical value $\mu = 0$. When $D > \frac{T^2}{4}$ and $\mu < \frac{-4}{3}$ we have a conjugate complex eigenvalues with $|\lambda| > 1$ so the fixed point is spiral source.

# Chapter 5

# Bifurcation of logistic competition model

## 5.1 Introduction

We studied before the dynamic behavior of the logistic population model which depends on the assumption that there is no inter-specific competition between species. Here we introduce a new model which considers the inter-specific competition between two different species, in which each species affect negatively the growth of the other, [2].
The logistic competition model is given by

$$(5.1) \qquad \begin{aligned} x_{n+1} &= \frac{ax_n(1-x_n)}{1+cy_n} \\ y_{n+1} &= \frac{by_n(1-y_n)}{1+dx_n} \end{aligned}$$

where $a, b > 0$, and $c, d \in (0,1)$. The parameters $a$ and $b$ are known as intrinsic growth rates of species $x$ and $y$, the parameters $c$ and $d$ are known as the competition parameters of $x$ and $y$.
The map associated with the system (5.1) is given by

$$(5.2) \qquad F(x,y) = \left(\frac{ax(1-x)}{1+cy}, \frac{by(1-y)}{1+dx}\right)$$

With two assumptions

1. $x$ and $y$ are in $[0,1]$
2. $a$ and $b$ are in $(0,4]$

It is clear that $F$ maps $[0,1]\times[0,1]$ into $[0,1]\times[0,1]$ because the maximum of the $x$-component is $\frac{a}{4}$, which occurs at $x=\frac{1}{2}$ and $y=0$, and the maximum of the $y$-component is $\frac{b}{4}$ and it occurs at $x=0$ and $y=\frac{1}{2}$.

## 5.2 Fixed points

To find the fixed points of the map $F$, we must solve the following system of equations

$$\frac{ax(1-x)}{1+cy}=x$$

$$\frac{by(1-y)}{1+dx}=y$$

It is clear that $(0,0)$ is a solution for the previous system. Now we take the case where $x\neq 0$ and $y=0$, this leads to $x=ax(1-x)$, so $x=\frac{a-1}{a}$, and if we take $x=0$ and $y\neq 0$ , so $y=by(1-y)$, this leads to $y=\frac{b-1}{b}$. And if we take $x\neq 0$ and $y\neq 0$, then we must solve the following system of equations

$$\begin{aligned} a(1-x) &= 1+cy \\ b(1-y) &= 1+dx \end{aligned}$$

So

$$y=\frac{a-ax-1}{c}$$

substitute the value of $y$ in the above system, hence

$$x=\frac{-cb+ab-b+c}{ab-cd}$$

and

$$y=\frac{-da+ab-a+d}{ab-cd}$$

We find that the map $F$ has one extinction fixed point $(0,0)$, one coexistence fixed point

$$(\hat{x}, \hat{y}) = \left( \frac{-cb + ab - b + c}{ab - cd}, \frac{-da + ab - a + d}{ab - cd} \right)$$

and two exclusion fixed points $\left(\frac{a-1}{a}, 0\right)$, $\left(0, \frac{b-1}{b}\right)$.

**Lemma 5.2.1.** *If $a = 1$, then for all $n \in \mathbb{Z}^+$,*

$$x_n \leq x_0 \prod_{i=0}^{n-1} (1 - x_i), \qquad n \in \mathbb{Z}^+ \tag{5.3}$$

*Proof.* Depending on the fact that $x_{n+1} \leq x_n(1 - x_n)$, we prove by induction that for all $n \in \mathbb{Z}^+$

$$x_n \leq x_0 \prod_{i=0}^{n-1} (1 - x_i)$$

When $n = 1$, we have the following inequality

$$x_1 \leq x_0(1 - x_0)$$

which is true, so (5.3) is true for $n = 1$.
Let $k \in \mathbb{Z}^+$ be given and suppose (5.3) is true for $n = k$. Then

$$\begin{aligned} x_{k+1} &\leq x_k(1 - x_k) \\ &\leq x_0 \prod_{i=0}^{k-1} (1 - x_i)(1 - x_k) \qquad \text{(by induction hypothesis)} \\ &\leq x_0 \prod_{i=0}^{k} (1 - x_i) \end{aligned}$$

Thus (5.3) holds for $n = k + 1$, so (5.3) is true for all $n \in \mathbb{Z}^+$.

□

**Lemma 5.2.2.** *If $a = 1$, then the inequality,*

$$\prod_{i=0}^{n-1} (1 - x_i) \leq (1 - x_n)^n \tag{5.4}$$

*holds for all $n \in \mathbb{Z}^+$.*

*Proof.* From $1-x_i \leq 1-x_{i+1}$, and by induction we will show that inequality (5.4) holds for all $n \in \mathbb{Z}^+$.When $n = 1$ we have $(1 - x_0) \leq (1 - x_1)$, so (5.4) is true for $n = 1$. Let $k \in \mathbb{Z}^+$ be given and suppose that (5.4) is true for $n = k$. Then

$$\begin{aligned}
\prod_{i=0}^{k}(1-x_i) &= \prod_{i=0}^{k-1}(1-x_i)(1-x_k) \\
&\leq (1-x_k)^k(1-x_k) \qquad \text{(by induction hypothesis)} \\
&= (1-x_k)^{k+1}.
\end{aligned}$$

□

## 5.3 Stability analysis

### 5.3.1 Stability of the extinction fixed point

Now we investigate the stability of the fixed point $(0, 0)$.

**Lemma 5.3.1.** *Let $(x_n, y_n)$ denotes the solution of the logistic competition model (5.1) with initial condition $(x_0, y_0) \in (0, 1) \times (0, 1)$. If $a \in (0, 1]$ then $\lim_{n\to\infty} x_n = 0$, if $b \in (0, 1]$ then $\lim_{n\to\infty} y_n = 0$. Moreover if $a, b \in (0, 1]$ then $\lim_{n\to\infty}(x_n, y_n) = (0, 0)$.*

*Proof.* From the system (5.1) it is clear that

$$0 \leq x_{n+1} = \frac{ax_n(1-x_n)}{1+cy_n} \leq ax_n - ax_n^2 \leq ax_n$$

hence

$$x_{n+1} \leq ax_n$$

by simple iteration we find that

$$x_n \leq a^n x_0$$

So if $a \in (0, 1)$ then $\lim_{n\to\infty} x_n = 0, \quad \forall n \in \mathbb{Z}$. When $a = 1$ then $x_{n+1} < x_n$, this is a decreasing sequence which is bounded, hence it converges as $n \to \infty$. Let $L$ be the limit of this sequence where

$$0 \leq L \leq x_n < 1 \qquad \forall n \in \mathbb{Z}$$

By using the inequalities (5.3) and (5.4), we find that

$$x_n \leq \frac{x_0}{1-x_n}\prod_{i=0}^{n}(1-x_i) \leq \frac{x_0}{1-x_n}(1-x_n)^{n+1} \leq (1-x_n)^n$$

but we have $L \leq x_n < 1$ which implies that $0 < (1-x_n) \leq 1-L$, so

$$x_n \leq (1-x_n)^n \leq (1-L)^n$$

thus

$$\lim_{n\to\infty} x_n = 0$$

We can use the same argument to show that $\lim_{n\to\infty} y_n = 0$ when $b \in (0,1]$

□

This lemma gives us the sufficient condition for the stability of the extinction fixed point. We can say that the fixed point $(0,0)$ is asymptotically stable if and only if $a \in (0,1]$ and $b \in (0,1]$.
Note that the Jacobian matrix of $F$ is given by

$$JF(x,y) = \begin{pmatrix} \frac{a(1-2x)}{1+cy} & \frac{-acx(1-x)}{(1+cy)^2} \\ \\ \frac{-bdy(1-y)}{(1+dx)^2} & \frac{b(1-2y)}{1+dx} \end{pmatrix}$$

The value of the Jacobian matrix at the fixed point $(0,0)$ is given by

$$J_0 = JF(0,0) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

It is clear that the eigenvalues of $J_0$ are $\lambda_1 = a$ and $\lambda_2 = b$. And we know that the fixed point is asymptotically stable if and only if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. This result is consistent with lemma 5.3.1 which states that the fixed point $(0,0)$ is asymptotically stable if $0 < a \leq 1$ and $0 < b \leq 1$. Thus, it is unstable when $a > 1$ or $b > 1$.

### 5.3.2 Stability of two exclusion fixed points

We now study the stability of exclusion fixed point $\left(\frac{a-1}{a}, 0\right)$.

**Lemma 5.3.2.** *The fixed point* $\left(\frac{a-1}{a}, 0\right)$, *is asymptotically stable if* $1 < a < 3$ *and* $b < 1 + d\left(\frac{a-1}{a}\right)$

*Proof.* The Jacobian matrix at this point is

$$J_a = JF\left(\frac{a-1}{a}, 0\right) = \begin{pmatrix} 2-a & \frac{-c(a-1)}{a} \\ 0 & \frac{ab}{ad+a-d} \end{pmatrix}$$

The eigenvalues of $J_a$ are $\lambda_1 = 2-a$ and $\lambda_2 = \frac{ab}{ad+a-d}$. This fixed point is asymptotically stable if and only if $|\lambda_1| < 1$ and $|\lambda_2| < 1$ this is equivalent to

$$1 < a < 3$$

and

$$-1 < \frac{ab}{ad+a-d} < 1$$

which is equivalent to

$$-ad - a + d < ab < ad + a - d$$

so

$$\frac{d(1-a)-a}{a} < b < \frac{d(a-1)+a}{a}$$

which implies that

$$-1 + \frac{d(1-a)}{a} < b < 1 + \frac{d(a-1)}{a}$$

but

$$-\frac{2}{3} < \frac{1-a}{a} < 0$$

hence

$$-1 + \frac{d(1-a)}{a} < 0$$

so we have

$$0 < b < 1 + \frac{d(a-1)}{a}$$

$\square$

Note that if $a = 1$ then the exclusion fixed point $\left(\frac{a-1}{a}, 0\right)$ is the extinction fixed point $(0, 0)$.

**Theorem 5.3.3.** *If $a = 3$ and $b < 1 + \frac{2}{3}d$, then the exclusion fixed point $\left(\frac{a-1}{a}, 0\right) = \left(\frac{2}{3}, 0\right)$ is asymptotically stable.*

*Proof.* If $a = 3$ and $b < 1 + \frac{2}{3}d$, then $\lambda_1 = -1$ and $\lambda_2 < 1$, thus the fixed point $\left(\frac{a-1}{a}, 0\right) = \left(\frac{2}{3}, 0\right)$ is non hyperbolic, and to check its stability we apply the center manifold theorem. First we shift the point $(\frac{a-1}{a}, 0)$ to the origin, to do this we assume the following transformation

$$\begin{aligned} u &= x - \left(\frac{a-1}{a}\right) \\ v &= y \end{aligned}$$

So the system (5.1) becomes

$$(5.5)\qquad \begin{aligned} f(u,v) = u_{n+1} &= \frac{a(u_n + \frac{a-1}{a})(1 - (u_n + \frac{a-1}{a}))}{1 + cv_n} - \frac{a-1}{a} \\ g(u,v) = v_{n+1} &= \frac{bv_n(1 - v_n)}{1 + d\left(u_n + \frac{a-1}{a}\right)} \end{aligned}$$

Note that

$$\frac{\partial f}{\partial u} = \frac{1}{1 + cv}\left(a - 2a\left(u + \frac{a-1}{a}\right)\right)$$

$$\frac{\partial f}{\partial v} = \frac{-c(au + a - 1)\left(1 - \left(u + \frac{a-1}{a}\right)\right)}{(1 + cv)^2}$$

$$\frac{\partial g}{\partial u} = \frac{-bdv(1 - v)}{\left(1 + d\left(u_n + \frac{a-1}{a}\right)\right)^2}$$

$$\frac{\partial g}{\partial v} = \frac{\left(1 + d\left(u_n + \frac{a-1}{a}\right)\right)(b - 2bv)}{\left(1 + d\left(u_n + \frac{a-1}{a}\right)\right)^2}$$

Hence the Jacobian matrix of the system (5.5) is

$$\tilde{J}F(u,v) = \begin{pmatrix} \frac{-2au-a+2}{1+cv} & \frac{c(au+a-1)(au-1)}{a(1+cv)^2} \\ \\ \frac{a^2bdv(v-1)}{(a+adu+ad-d)^2} & \frac{ab(1-2v)}{a+adu+ad-d} \end{pmatrix}$$

The Jacobian matrix of system (5.5) at $(0,0)$ is

$$\tilde{J}F(0,0) = \begin{pmatrix} 2-a & \frac{-c(a-1)}{a} \\ \\ 0 & \frac{ab}{a+ad-d} \end{pmatrix}$$

When $a = 3$, we have

$$\tilde{J}F(0,0) = \begin{pmatrix} -1 & \frac{-2c}{3} \\ \\ 0 & \frac{3b}{3+2d} \end{pmatrix}$$

And system (5.5) becomes

$$u_{n+1} = \frac{1}{1+cv}\left(3\left(u_n + \frac{2}{3}\right)\left(1 - \left(u_n + \frac{2}{3}\right)\right)\right) - \frac{2}{3}$$

$$= \frac{1}{3}\left(\frac{(3u_n+2)(1-3u_n)}{(1+cv_n)} - 2\right)$$

$$= \frac{-3u_n - 9u_n^2 - 2cv_n}{3(1+cv_n)}$$

and

$$v_{n+1} = \frac{3bv_n(1-v_n)}{3+d(3u_n+2)}$$

Now we will add and subtract the linear relations $u_n + \frac{2}{3}cv_n$ and $\frac{3b}{3+2d}v_n$ from $u_{n+1}$ and $v_{n+1}$ respectively, so

$$u_{n+1} = -u_n - \frac{2}{3}cv_n + \frac{-3u_n - 9u_n^2 - 2cv_n}{3(1+cv_n)} + u_n + \frac{2}{3}cv_n$$

$$= -u_n - \frac{2}{3}cv_n + \frac{1}{3}\frac{-3u_n - 9u_n^2 - 2cv_n + (3u_n + 2cv_n)(1+cv_n)}{1+cv_n}$$

$$= -u_n - \frac{2}{3}cv_n + \frac{1}{3}\frac{3cu_nv_n - 9u_n^2 + 2c^2v_n^2}{1+cv_n}$$

And

$$v_{n+1} = \frac{3b}{3+2d}v_n + \frac{3bv_n - 3bv_n^2}{3+3du_n+2d} - \frac{3b}{3+2d}v_n$$

$$= \frac{3b}{3+2d}v_n + \frac{(3+2d)(3bv_n - 3bv_n^2) - 3bv_n(3+3du_n+2d)}{(3+3du_n+2d)(3+2d)}$$

$$= \frac{3b}{3+2d}v_n + \frac{-3bv_n(3du_n+3v_n+2dv_n)}{9+9du_n+12d+6d^2u_n+4d^2}$$

Hence when $a = 3$, system (5.5) can be written as

$$u_{n+1} = -u_n - \frac{2}{3}cv_n + \tilde{f}(u_n, v_n)$$

(5.6)

$$v_{n+1} = \frac{3b}{3+2d}v_n + \tilde{g}(u_n, v_n)$$

Where

$$\tilde{f}(u_n, v_n) = \frac{3cu_nv_n + 2c^2v_n^2 - 9u_n^2}{3(1+cv_n)}$$

and

$$\tilde{g}(u_n, v_n) = -3bv_n\frac{3du_n+3v_n+2dv_n}{9+9du_n+12d+6d^2u_n+4d^2}$$

Consider the center manifold $v = h(u)$. Let us assume that the map $h(u)$ takes the form

$$h(u) = \alpha u^2 + \beta u^3 + \mathcal{O}(u^4), \qquad \alpha, \beta \in \mathbb{R}$$

We must find the two constants $\alpha$ and $\beta$, since $h(u)$ satisfies the center manifold theorem, so it satisfies the following equation

$$h(-u - \frac{2}{3}ch(u) + \tilde{f}(u, h(u))) - \frac{3b}{3+2d}h(u) - \tilde{g}(u, h(u)) = 0$$

After some calculation we have that $\alpha = 0$ and $\beta = 0$, which implies that $h(u) = 0$. The dynamics restricted to the center manifold are given by the map

$$u \mapsto -u - \frac{2}{3}ch(u) + \tilde{f}(u, h(u))$$

Thus on the center manifold $h(u) = v = 0$, we have the following map

$$Q(u) = -u - 3u^2$$

and since $\frac{\partial Q}{\partial u}(0) = -1$, we must find the Schwarzian derivative of $Q$ when $u = 0$, where

$$SQ(0) = -Q'''(0) - \frac{3}{2}(Q''(0))^2 = -54 < 0$$

so the fixed point $(\frac{2}{3}, 0)$ is asymptotically stable. □

Now we will check the stability of $(\frac{a-1}{a}, 0)$, when $1 < a < 3$ and $b = 1+d(\frac{a-1}{a})$.

**Theorem 5.3.4.** *The fixed point* $(\frac{a-1}{a}, 0)$ *is unstable when* $1 < a < 3$ *and* $b = 1 + d(\frac{a-1}{a})$.

*Proof.* The value of the Jacobian matrix $\tilde{J}$ when $b = 1 + d(\frac{a-1}{a})$ is given by

$$\tilde{J}F(0,0) = \begin{pmatrix} 2-a & \frac{-c(a-1)}{a} \\ 0 & 1 \end{pmatrix}$$

In this case we have $|\lambda_1| = |2 - a| < 1$ and $|\lambda_2| = 1$. It is clear that the fixed point $(0, 0)$ is non hyperbolic, so to check its stability we must use the center manifold theorem. When $b = 1 + d(\frac{a-1}{a})$ system (5.5) becomes

$$u_{n+1} = \frac{a(u_n + \frac{a-1}{a})(1 - (u_n + \frac{a-1}{a}))}{1 + cv_n} - \frac{a-1}{a}$$

$$v_{n+1} = \frac{(a + (a-1)d)(1 - v_n)v_n}{a + d(au_n + (a-1))}$$

Now we will add and subtract the linear relations $(2 - a)u_n - \frac{c(a-1)}{a}v_n$ and

$v_n$ to $u_{n+1}$ and $v_{n+1}$ respectively, hence

$$
\begin{aligned}
u_{n+1} &= (2-a)u_n - \frac{c(a-1)}{a}v_n + \frac{a(u_n + \frac{a-1}{a})(1-(u_n+\frac{a-1}{a}))}{1+cv_n} - \frac{a-1}{a} \\
&- (2-a)u_n + \frac{c(a-1)}{a}v_n \\
&= (2-a)u_n - \frac{c(a-1)}{a}v_n + \frac{1}{a(1+cv_n)}[(a^2u_n + a(a-1))(1-u_n-\frac{a-1}{a})] \\
&+ \frac{1}{a(1+cv_n)}[-(a-1)(1+cv_n) - (2au_n - a^2u_n)(1+cv_n) + (acv_n - cv_n)(1+cv_n)] \\
&= (2-a)u_n - \frac{c(a-1)}{a}v_n + \frac{-a^2u_n^2 - 2acu_nv_n + a^2cu_nv_n - c^2v_n^2 + ac^2v_n^2}{a(1+cv_n)}
\end{aligned}
$$

and

$$
\begin{aligned}
v_{n+1} &= v_n + \frac{(a+(a-1)d)(1-v_n)v_n}{a+d(au_n+(a-1))} - v_n \\
&= v_n + \frac{1}{-d+a(1+d+du_n)}\big((a+ad-d)(v_n-v_n^2) - v_n(a+adu_n+ad-d)\big) \\
&= v_n + \frac{-v_n(av_n + adv_n - dv_n + adu_n)}{-d+a(1+d+du_n)}
\end{aligned}
$$

So we can write system (5.5) as

$$
(5.7) \qquad \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} (2-a) & -\frac{c(a-1)}{a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} \hat{f}(u_n, v_n) \\ \hat{g}(u_n, v_n) \end{pmatrix}
$$

where

$$\hat{f}(u_n, v_n) = \frac{-a^2u_n^2 - 2acu_nv_n + a^2cu_nv_n - c^2v_n^2 + ac^2v_n^2}{a(1+cv_n)}$$

and

$$\hat{g}(u_n, v_n) = -\frac{v_n(-dv_n + a(v_n + d(u_n + v_n)))}{-d+a(1+d+du_n)}$$

Consider the center manifold $u = h(v)$. Let us assume that the map $h$ takes the form

$$h(v) = -\frac{c}{a}v + \alpha v^2 + \beta v^3$$

and $h(v)$ satisfies the following equation

$$h(v+\hat{g}(h(v),v)) - (2-a)h(v) + \frac{c(a-1)}{a}v - \hat{f}(h(v),v) = 0$$

We find that

$$\alpha = -\frac{ac - cd + acd - c^2 d}{(-a+a^2)(a-d+ad)}$$

$$\begin{aligned}
\beta = &-\frac{2a^2c + a^3c^2 - 4acd + 4a^2cd - 6ac^2d - a^2c^2d + 2a^3c^2d}{(-1+a)^2a(a-d+ad)^2} \\
&+\frac{-a^2c^3d + 2cd^2 - 4acd^2 + 2a^2cd^2 + 6c^2d^2 - 6ac^2d^2}{(-1+a)^2a(a-d+ad)^2} \\
&+\frac{-a^2c^2d^2 + a^3c^2d^2 + 4c^3d^2 - a^2c^3d^2}{(-1+a)^2a(a-d+ad)^2}
\end{aligned} \tag{5.8}$$

Thus on the center manifold $u = h(v)$ we have the following map

$$\begin{aligned}
Q(v) &= v + \hat{g}(h(v),v) \\
&= v + \frac{dv^2 - av(v + d(-\frac{c}{a}v + \alpha v^2 + \beta v^3 + v))}{-d + a(1 + d + d(-\frac{c}{a}v + \alpha v^2 + \beta v^3))} \\
&= \frac{adv + av - dv + dv^2 - av^2 - adv^2}{-d - cdv + a + ad + ad\alpha v^2 + ad\beta v^3}
\end{aligned}$$

So that the map $Q(v)$ on the center manifold is given by

$$Q(v) = -\frac{(a-d+ad)(-1+v)v}{-d(1+cv) + a(1+d+dv^2\alpha + dv^3\beta)}$$

Since $Q'(0) = \left(\frac{-d+a+ad}{-d+a+ad}\right)^2 = 1$ and $Q''(0) \neq 0$, the fixed point $(\frac{a-1}{a}, 0)$ is unstable. $\square$

We make the same argument to check the stability of the fixed point $(0, \frac{b-1}{b})$.

**Lemma 5.3.5.** *The fixed point $(0, \frac{b-1}{b})$ is asymptotically stable if $1 < b < 3$ and $0 < a < 1 + c\left(\frac{b-1}{b}\right)$.*

*Proof.* The Jacobian evaluated at the fixed point $(0, \frac{b-1}{b})$ is given by

$$J_b = JF(0, \frac{b-1}{b}) = \begin{pmatrix} \frac{ab}{b+cb-c} & 0 \\ \frac{-d(b-1)}{b} & 2-b \end{pmatrix}$$

The eigenvalues of $J_b$ are $\lambda_1 = \frac{ab}{b+c(b-1)}$ and $\lambda_2 = 2-b$. Hence the fixed point $(0, \frac{b-1}{b})$ is asymptotically stable if $1 < b < 3$ and $0 < a < 1 + c\left(\frac{b-1}{b}\right)$. □

Now we will study the stability of $(0, \frac{b-1}{b})$ when $b = 3$ and $0 < a < 1+c\left(\frac{b-1}{b}\right)$, and also when $1 < b < 3$ and $a = 1 + c\left(\frac{b-1}{b}\right)$.

**Theorem 5.3.6.** *The fixed point $(0, \frac{b-1}{b})$ is asymptotically stable if $b = 3$ and $a < 1 + c\left(\frac{b-1}{b}\right)$.*

*Proof.* In this case $|\lambda_1| < 1$ and $\lambda_2 = -1$, so the point $(0, \frac{b-1}{b}) = (0, \frac{2}{3})$ is non hyperbolic fixed point. In order to apply center manifold theorem, we make a change of variables in system (5.1), so we can shift from the point $(0, \frac{b-1}{b})$ to $(0, 0)$. Let $u = x$ and $v = y - \left(\frac{b-1}{b}\right)$. then the new system is

$$\begin{aligned} u_{n+1} &= \frac{au_n(1-u_n)}{1 + c\left(v + \frac{b-1}{b}\right)} \\ v_{n+1} &= \frac{b\left(v_n + \frac{b-1}{b}\right)\left(1 - \left(v_n + \frac{b-1}{b}\right)\right)}{1 + du_n} - \frac{b-1}{b} \end{aligned} \tag{5.9}$$

The Jacobian matrix of system (5.9) is

$$Jf(u, v) = \begin{pmatrix} \frac{-ab(2u-1)}{b+bcv+bc-c} & \frac{-ab^2cu(1-u)}{(b+bcv+bc-c)^2} \\ \frac{d(bv+b-1)(bv-1)}{b(1+du)^2} & \frac{-(2bv-2+b)}{1+du} \end{pmatrix}$$

At $(0, 0)$, $Jf(u, v)$ has the form

$$Jf(0, 0) = \begin{pmatrix} \frac{ab}{b+c(b-1)} & 0 \\ \frac{-d(b-1)}{b} & 2-b \end{pmatrix}$$

When $b = 3$, we have

$$Jf(0, 0) = \begin{pmatrix} \frac{3a}{3+2c} & 0 \\ \frac{-2d}{3} & -1 \end{pmatrix}$$

We use the technique which is used in proving the previous theorems to write system (5.9) as

$$u_{n+1} = \frac{3a}{3+2c}u_n + \tilde{f}(u_n, v_n)$$

$$v_{n+1} = -v_n - \frac{2}{3}du_n + \tilde{g}(u_n, v_n)$$

Where

$$\tilde{f}(u_n, v_n) = \frac{-9acu_nv_n - 9au_n^2 - 6acu_n^2}{9 + 9cv_n + 12c + 6c^2v_n + 4c^2}$$

and

$$\tilde{g}(u_n, v_n) = \frac{1}{3}\frac{3du_nv_n - 9v_n^2 + 2d^2u_n^2}{1 + du_n}$$

Let us assume the center manifold $u = h(v)$ take the form

$$h(v) = \alpha v^2 + \beta v^3$$

The function $h$ must satisfy the center manifold equation

$$h\left(-v - \frac{2}{3}dh(v) + \tilde{g}(h(v), v)\right) - \frac{3a}{3+2d}h(v) - \tilde{f}(h(v), v) = 0$$

Solving this equation yields $\alpha = 0$ and $\beta = 0$. Hence $h(v) = 0$, thus on the center manifold $u = 0$, we have the following map

$$N(v) = -v - 3v^3$$

Note that $N'(0) = -1$, and the Schwarzian derivative of $N$ at $v = 0$ is

$$SN(0) = -N'''(0) - \frac{3}{2}(N''(0))^2 = -54 < 0$$

hence the fixed point $\left(0, \frac{b-1}{b}\right) = \left(0, \frac{2}{3}\right)$ is asymptotically stable. $\square$

Now we will study the stability of $\left(0, \frac{b-1}{b}\right)$ when $1 < b < 3$ and $a = 1 + \frac{c(b-1)}{b}$.

**Theorem 5.3.7.** *The fixed point* $\left(0, \frac{b-1}{b}\right)$ *is unstable when* $1 < b < 3$ *and* $a = 1 + \frac{c(b-1)}{b}$.

*Proof.* In this case we have

$$JF(0,0) = \begin{pmatrix} 1 & 0 \\ -\frac{d(b-1)}{b} & 2-b \end{pmatrix}$$

Note that $\lambda_1 = 1$ and $|\lambda_2| < 1$, so the point $\left(0, \frac{b-1}{b}\right)$ is non hyperbolic fixed point. We can write system (5.9) as

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{d(b-1)}{b} & 2-b \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} \tilde{f}(u_n, v_n) \\ \tilde{g}(u_n, v_n) \end{pmatrix}$$

where

$$\tilde{f}(u,v) = -\frac{u(-cu + b(u + c(u+v)))}{-c + b(1 + c + cv)}$$

and

$$\tilde{g}(u,v) = \frac{-b^2v^2 - 2bduv + b^2duv - d^2u^2 + bd^2u^2}{b(1+du)}$$

Let we assume the center manifold $v = h(u)$ take the form

$$h(u) = -\frac{d}{b}u + \alpha u^2 + \beta u^3$$

The map $h$ must satisfy the equation

$$h(u + \tilde{f}(u, h(u))) + \frac{d(b-1)}{b}u - (2-b)h(u) - \tilde{g}(u, h(u)) = 0$$

This leads to

$$\begin{aligned} \beta = &-\frac{2b^2d + b^3d^2 - 4bcd + 4b^2cd - 6bcd^2 - b^2cd^2 + 2b^3cd^2 - b^2cd^3}{(-1+b)^2b(b-c+bc)^2} \\ &+ \frac{2c^2d - 4bc^2d + 2b^2c^2d + 6c^2d^2 - 6bc^2d^2 - b^2c^2d^2 + b^3c^2d^2 + 4c^2d^3 - b^2c^2d^3}{(-1+b)^2b(b-c+bc)^2} \end{aligned}$$

and

$$\alpha = -\frac{bd - cd + bcd - cd^2}{(-b+b^2)(b-c+bc)}$$

Hence the map on the center manifold is given by

$$\begin{aligned} P(u) &= u + \tilde{f}(u, h(u)) \\ &= -\frac{(b-c+bc)(-1+u)u}{-c(1+du) + b(1 + c + cu^2\alpha + cu^3\beta)} \end{aligned}$$

We find that $P'(0) = 1$ and $P''(0) \neq 0$, so the fixed point $\left(0, \frac{b-1}{b}\right)$ is unstable.

$\square$

### 5.3.3 Stability of the coexistence fixed point

We will study the stability of coexistence fixed point

$$(\hat{x}, \hat{y}) = \left( \frac{-cb + ab - b + c}{ab - cd}, \frac{-da + ab - a + d}{ab - cd} \right)$$

**Lemma 5.3.8.** *The positive coexistence fixed point $(\hat{x}, \hat{y})$ exists if*

$$\frac{a-1}{c} > \frac{b-1}{b} \quad \text{and} \quad \frac{b-1}{d} > \frac{a-1}{a}$$

*where $a, b > 1$ and $c, d \in (0, 1)$.*

*Proof.* from the assumption that $a > 1$ and $b > 1$ and $c, d \in (0, 1)$, so $ab - cd > 0$, and since

$$x = \frac{b(a-1) - c(b-1)}{ab - cd} > 0$$

so

$$b(a-1) - c(b-1) > 0$$

hence

$$\frac{a-1}{c} > \frac{b-1}{b}$$

and since

$$y = \frac{a(b-1) - d(a-1)}{ab - cd} > 0$$

so

$$\frac{b-1}{d} > \frac{a-1}{a}$$

□

The Jacobian matrix at the fixed point $(\hat{x}, \hat{y})$ is given by

$$J_* = JF(\hat{x}, \hat{y}) = \begin{pmatrix} \frac{a(1-2\hat{x})}{1+c\hat{y}} & \frac{-ac\hat{x}(1-\hat{x})}{(1+c\hat{y})^2} \\ \frac{-bd\hat{y}(1-\hat{y})}{(1+d\hat{x})^2} & \frac{b(1-2\hat{y})}{1+d\hat{x}} \end{pmatrix}$$

and by noting the following relations

$$\frac{1 + c\hat{y}}{a} = 1 - \hat{x} \qquad \text{and} \qquad \frac{1 + d\hat{x}}{b} = 1 - \hat{y}$$

then $J_*$ becomes

$$J_* = JF(\hat{x},\hat{y}) = \begin{pmatrix} \frac{1-2\hat{x}}{1-\hat{x}} & \frac{-c\hat{x}}{a(1-\hat{x})} \\ \\ \frac{-d\hat{y}}{b(1-\hat{y})} & \frac{1-2\hat{y}}{1-\hat{y}} \end{pmatrix}$$

**Theorem 5.3.9.** *The coexistence fixed point* $(\hat{x},\hat{y})$ *is asymptotically stable if* $1 < a < 3$ *and* $1 < b < 3$.

*Proof.* It is difficult to find the eigenvalues of $J_*$, so we resort to Jury test, which states that the eigenvalues of $J_*$ lie inside the unit disc if and only if

$$\rho(1) > 0 \qquad \rho(-1) > 0 \qquad \rho(0) < 1$$

where $\rho(\lambda)$ is the characteristic polynomial of $J_*$, where

$$\rho(\lambda) = \left(\frac{1-2\hat{x}}{1-\hat{x}} - \lambda\right)\left(\frac{1-2\hat{y}}{1-\hat{y}} - \lambda\right) - \frac{cd}{ab}\frac{\hat{x}\hat{y}}{(1-\hat{x})(1-\hat{y})} \tag{5.10}$$

Note that the fixed point $(\hat{x},\hat{y})$ is the solution of the system

$$\begin{aligned} ax + cy &= a-1 \\ dx + by &= b-1 \end{aligned} \tag{5.11}$$

this yields

$$\hat{x} = \frac{a-1}{a} - \frac{c}{a}\hat{y} < \frac{a-1}{a}$$

and

$$\hat{y} = \frac{b-1}{b} - \frac{d}{b}\hat{x} < \frac{b-1}{b}$$

We assume that $1 < a < 3$ and $1 < b < 3$ , which implies that $\hat{x} < \frac{2}{3}$ and $\hat{y} < \frac{2}{3}$.
Now from (5.10) we have that

$$\rho(1) = \left(\frac{1-2\hat{x}}{1-\hat{x}} - 1\right)\left(\frac{1-2\hat{y}}{1-\hat{y}} - 1\right) - \frac{\frac{cd}{ab}\hat{x}\hat{y}}{(1-\hat{x})(1-\hat{y})}$$

$$= \left(\frac{-\hat{x}}{1-\hat{x}}\right)\left(\frac{-\hat{y}}{1-\hat{y}}\right) - \frac{\frac{cd}{ab}\hat{x}\hat{y}}{(1-\hat{x})(1-\hat{y})}$$

$$= \frac{\hat{x}\hat{y} - \frac{cd}{ab}\hat{x}\hat{y}}{(1-\hat{x})(1-\hat{y})}$$

but

$$(1-\hat{x})(1-\hat{y}) > 0 \qquad \text{and} \qquad \hat{x}\hat{y}\left(1-\frac{cd}{ab}\right) > 0$$

thus

$$\rho(1) > 0$$

And

$$\rho(-1) = \left(\frac{1-2\hat{x}}{1-\hat{x}}+1\right)\left(\frac{1-2\hat{y}}{1-\hat{y}}+1\right) - \frac{\frac{cd}{ab}\hat{x}\hat{y}}{(1-\hat{x})(1-\hat{y})}$$

$$= \frac{(2-3\hat{x})(2-3\hat{y}) - \frac{cd}{ab}\hat{x}\hat{y}}{(1-\hat{x})(1-\hat{y})}$$

Since $(1-\hat{x})(1-\hat{y}) > 0$ it follows that $\rho(-1) > 0$ if and only if

$$(2-3\hat{x})(2-3\hat{y}) - \frac{cd}{ab}\hat{x}\hat{y} > 0$$

which is equivalent to

$$\left(9-\frac{cd}{ab}\right)\hat{x}\hat{y} > 6(\hat{x}+\hat{y}) - 4$$

which is true under the hypothesis that $1 < a < 3$ and $1 < b < 3$. Thus $\rho(-1) > 0$.
Now we check the last condition of Jury test, $\rho(0) < 1$, but

$$\rho(0) = \frac{(1-2\hat{x})(1-2\hat{y}) - \frac{cd}{ab}\hat{x}\hat{y}}{(1-\hat{x})(1-\hat{y})}$$

The relation $\rho(0) < 1$ is equivalent to

$$\frac{(1-2\hat{x})(1-2\hat{y}) - \frac{cd}{ab}\hat{x}\hat{y}}{(1-\hat{x})(1-\hat{y})} < 1$$

Now the inequality

$$\frac{(1-2\hat{x})(1-2\hat{y}) - \frac{cd}{ab}\hat{x}\hat{y}}{(1-\hat{x})(1-\hat{y})} - 1 < 0$$

is equivalent to

$$3 - \frac{cd}{ab} < \frac{1}{\hat{x}} + \frac{1}{\hat{y}}$$

which is true under our assumption that $1 < a < 3$ and $1 < b < 3$. Thus the relation $\rho(0) < 1$ is verified, and the coexistence fixed point is asymptotically stable if $1 < a < 3$ and $1 < b < 3$. $\square$

## 5.4 Bifurcation analysis

We will explain the bifurcation in logistic competition model. We consider the saddle-node, period-doubling and Neimark-Sacker bifurcation. We use the conditions in the trace-determinant plane to determine the parameter space in which these types of bifurcation happen. We focus our attention on the bifurcation of coexistence fixed point. Firstly note that

$$det(J_*) = \frac{(1-2\hat{x})(1-2\hat{y})}{(1-\hat{x})(1-\hat{y})} - \frac{dc\hat{x}\hat{y}}{ab(1-\hat{x})(1-\hat{y})}$$

$$= \frac{1-2(\hat{x}+\hat{y})+\left(4-\frac{dc}{ab}\right)\hat{x}\hat{y}}{(1-\hat{x})(1-\hat{y})}$$

by substituting the value of $\hat{x}$ and $\hat{y}$, we have

$$\begin{aligned} det(J_*) = &\frac{-c(b-c+bc)d^2+a^3b^2(2-b+2d)-ad(3bc-b^2(4+5c+c^2)}{(ab(-(1+c)d+a(1+d))(-b(1+c)+c(1+d)))} \\ &+ \frac{-adc^2(1+d)+a^2b(2b^2(1+c)-b(4+6c+6d+5cd)+c(4+5d+d^2))}{(ab(-(1+c)d+a(1+d))(-b(1+c)+c(1+d)))} \end{aligned}$$

and

$$tr(J_*) = \frac{1-2\hat{x}}{1-\hat{x}} + \frac{1-2\hat{y}}{1-\hat{y}}$$

$$= \frac{2-3(\hat{x}+\hat{y})+4\hat{x}\hat{y}}{1-(\hat{x}+\hat{y})+\hat{x}\hat{y}}$$

this leads to

$$\begin{aligned} tr(J_*) = &\frac{a^2b(1+d)+d(b(4+7c+3c^2)-c(4+3c+3d+2cd))}{((1+c)d-a(1+d))(b(1+c)-c(1+d))} \\ &+ \frac{a(b^2(1+c)-b(4+5c+5d+6cd)+c(4+7d+3d^2))}{((1+c)d-a(1+d))(b(1+c)-c(1+d))} \end{aligned}$$

where $tr$ and $det$ denote the trace and determinant of the Jacobian matrix $J_*$.

We know that the coexistence fixed point is asymptotically stable if the following inequality is satisfied

$$|tr(J_*)| - 1 < det(J_*) < 1 \tag{5.12}$$

this is equivalent to

(i) the inequality $det(J_*) < 1$ leads to

$$\frac{-c(b-c+bc)d^2 + a^3b^2(2-b+2d) - ad(3bc - b^2(4+5c+c^2) + c^2(1+d))}{(ab(-(1+c)d + a(1+d))(-b(1+c)+c(1+d)))} + \frac{a^2b(2b^2(1+c) - b(4+6c+6d+5cd) + c(4+5d+d^2))}{(ab(-(1+c)d + a(1+d))(-b(1+c)+c(1+d)))} - 1 < 0$$

so we have

$$\frac{-c(b-c+bc)d^2 + a^3b^2(2-b+2d) + a(b-c+bc)d((3b+c+cd)}{(ab(-(1+c)d + a(1+d))(-b(1+c)+c(1+d)))} + \frac{a^2b(2b^2(1+c) + 3c(1+d) - b(3+5d+c(5+4d)))}{(ab(-(1+c)d + a(1+d))(-b(1+c)+c(1+d)))} < 0 \tag{5.13}$$

(ii) and the inequality $det(J_*) > tr(J_*) - 1$ leads to

$$\frac{(b(-1+a-c)+c)(a(-1+b-d)+d)(ab-cd)}{ab(-(1+c)d + a(1+d))(-b(1+c)+c(1+d))} < 0 \tag{5.14}$$

(iii) The inequality $det(J_*) > -tr(J_*) - 1$ is equivalent to

$$\frac{-c(b-c+bc)d^2 + a^3b^2(3-b+3d) - ad(-b^2(9+14c+5c^2) + c^2(1+d))}{ab(-(1+c)d + a(1+d))(-b(1+c)+c(1+d))} + \frac{-adbc(8+4c+4d+3cd) + a^2b(3b^2(1+c) + c(9+14d+5d^2))}{ab(-(1+c)d + a(1+d))(-b(1+c)+c(1+d))} - \frac{3a^2b^2(3+4d+4c(1+d))}{ab(-(1+c)d + a(1+d))(-b(1+c)+c(1+d))} > 0 \tag{5.15}$$

So that the above inequalities determine the stability region of the coexistence fixed point in the plane. Now the saddle-node bifurcation happens in (T-D) plane when $det(J_*) = tr(J_*) - 1$, this leads to

$$\frac{(b(-1+a-c)+c)(a(-1+b-d)+d)(ab-cd)}{ab(-(1+c)d + a(1+d))(-b(1+c)+c(1+d))} = 0 \tag{5.16}$$

since we assumed that $a, b > 1$ and $c, d \in (0,1)$, so the denominator in inequality (5.16) is nonzero, and also this assumption leads to $ab - cd > 0$, which implies that

$$(b(-1+a-c)+c) = 0 \qquad \text{or} \qquad (a(-1+b-d)+d) = 0$$

this leads to

$$b = 1 + d\left(\frac{a-1}{a}\right) \qquad \text{or} \qquad a = 1 + c\left(\frac{b-1}{b}\right)$$

Observe that when $b = 1 + d\left(\frac{a-1}{a}\right)$ the coexistence fixed point $(\hat{x}, \hat{y})$ where

$$(\hat{x}, \hat{y}) = \left(\frac{-cb + ab - b + c}{ab - cd}, \frac{-da + ab - a + d}{ab - cd}\right)$$

is equal to the exclusion fixed point $\left(\frac{a-1}{a}, 0\right)$, also when the coexistence fixed point $(\hat{x}, \hat{y})$ leaves the region of its stability to the region which is determined by

$$1 < a < 3 \qquad \text{and} \qquad b < 1 + d\left(\frac{a-1}{a}\right)$$

by crossing the curve $b = 1 + d\left(\frac{a-1}{a}\right)$ it undergoes a saddle-node bifurcation into another fixed point which is $\left(\frac{a-1}{a}, 0\right)$. The same thing will happen when coexistence fixed point crosses the curve $a = 1 + c\left(\frac{b-1}{b}\right)$ and leaves the region of its stability to region which is determined by

$$1 < b < 3 \qquad \text{and} \qquad a < 1 + c\left(\frac{b-1}{b}\right)$$

Hence when $a = 1 + c\left(\frac{b-1}{b}\right)$ the coexistence fixed point collides with the exclusion fixed point $\left(0, \frac{b-1}{b}\right)$ which causes a saddle-node bifurcation. Also the system has a period-doubling bifurcation when $det(J_*) = -tr(J_*) - 1$ this is equivalent to the equality

(5.17)

$$\frac{-c(b-c+bc)d^2 + a^3b^2(3-b+3d) - ad(-b^2(9+14c+5c^2) + c^2(1+d))}{ab(-(1+c)d + a(1+d))(-b(1+c) + c(1+d))}$$
$$+ \frac{a^2b(3b^2(1+c) + c(9+14d+5d^2) - 3b(3+4d+4c(1+d)))}{ab(-(1+c)d + a(1+d))(-b(1+c) + c(1+d))}$$
$$- \frac{adbc(8+4c+4d+3cd)}{ab(-(1+c)d + a(1+d))(-b(1+c) + c(1+d))} = 0$$

denote by $\tau$ the curve which satisfies the equality (5.17). When $a$ and $b$ passe the curve $\tau$ the coexistence fixed point undergoes a period-doubling bifurcation into a coexistence 2-period cycle. After this curve, the coexistence becomes unstable, and the system has an asymptotically stable coexistence 2-periodic cycle.

We know that the region of stability of the two exclusion fixed points $\left(\frac{a-1}{a}, 0\right)$, $\left(0, \frac{b-1}{b}\right)$ is determined by the inequalities $1 < a < 3$, $0 < b < 1 + d\left(\frac{a-1}{a}\right)$ and $1 < b < 3$, $0 < a < 1 + c\left(\frac{b-1}{b}\right)$ respectively. We prove that when $a = 3$ the fixed point $\left(\frac{a-1}{a}, 0\right)$ is non-hyperbolic fixed point and the map on the center manifold has a first derivative which is equal to $-1$. Hence when $a = 3$ and $0 < b < 1 + d\left(\frac{a-1}{a}\right)$ the exclusion fixed point $\left(\frac{a-1}{a}, 0\right)$ undergoes a period-doubling bifurcation, the same thing happens to the fixed point $\left(0, \frac{b-1}{b}\right)$ when $b = 3$ and $0 < a < 1 + c\left(\frac{b-1}{b}\right)$.

# Chapter 6

# Analysis of discrete-time predator-prey system

## 6.1 Introduction

Now we consider the growth of two interdependent populations, one species "the prey" and the other species is "the predator", [1].

The predator- prey system is given by :

$$f(N_t, P_t) = N_{t+1} = N_t + rN_t(1 - N_t) - aN_tP_t \tag{6.1}$$
$$g(N_t, P_t) = P_{t+1} = P_t + aP_t(N_t - P_t)$$

Where $N_t$ and $P_t$ denote prey and predator densities respectively, while $r$ and $a$ are positive constants.
In the absence of predators "$P = 0$" the growth of prey population will be

$$N_{t+1} = N_t + rN_t(1 - N_t)$$

The term $(-aN_tP_t)$ represents the rate of decrease in prey species due to predation, so the prey growth rate falls as the predator population become larger. In absence of prey the growth of the predator population follows the logistic model.

We will study the dynamical behavior of predator-prey system, and its fixed points.

**Theorem 6.1.1.** *The system* (6.1) *has three fixed points which are* $(0, 0), (1.0)$ *and* $(N^*, P^*)$*, where* $N^* = P^* = \frac{r}{a+r}$.

*Proof.* The fixed points of system (6.1) satisfy the following system of equations:

$$\begin{aligned} N &= N + rN(1-N) - aNP \\ P &= P + aP(N-P) \end{aligned} \tag{6.2}$$

It is obvious that $(0,0)$ is a solution for the previous system. Now take the case when $N \neq 0$, so from the first equation we have

$$P = \frac{r(1-N)}{a}, \qquad where \quad a \neq 0 \tag{6.3}$$

Substitute (6.3) in into the second equation, we get the following equation :

$$r(1-N)\left(N - r\left(\frac{1-N}{a}\right)\right) = 0$$

This leads to two solutions

$$N = 1$$

or

$$N = \frac{r}{a+r}$$

If $N = 1$ then $P = 0$, also when $N = \frac{r}{a+r}$ then $P = \frac{r}{a+r}$. Hence system (6.1) has three fixed points : $(0,0), (1,0)$ and $\left(\frac{r}{a+r}, \frac{r}{a+r}\right)$. □

## 6.2 Stability analysis of predator-prey system

Now we will study the stability of these fixed points which is determined by the modulus of eigenvalues of the Jacobian matrix at the fixed points. Observe that the Jacobian matrix of system (6.1) is :

$$J(N,P) = \begin{pmatrix} 1 + r(1-2N) - aP & -aN \\ aP & 1 + aN - 2aP \end{pmatrix}$$

**Lemma 6.2.1.** *The fixed point* $(0,0)$ *is unstable.*

*Proof.* Note that the Jacobian matrix evaluated at the fixed point $(0,0)$ is given by

$$J(0,0) = \begin{pmatrix} 1+r & 0 \\ 0 & 1 \end{pmatrix}$$

In this case the matrix has two eigenvalues: $\lambda_1 = 1+r$ and $\lambda_2 = 1$. Since $|\lambda_1| > 1$, so $(0,0)$ is unstable, such a point is called non hyperbolic fixed point because one of the eigenvalues has a modulus equal to one. $\square$

**Lemma 6.2.2.** *If* $0 < r < 2$ *then* $(1,0)$ *is saddle fixed point.*

*Proof.* The Jacobian matrix evaluated at the fixed point $(1,0)$ is given by

$$J(1,0) = \begin{pmatrix} 1-r & -a \\ 0 & 1+a \end{pmatrix}$$

The corresponding characteristic equation is

$$\rho(\lambda) = \lambda^2 - (2 + (a-r))\lambda + (1+a)(1-r)$$

Its roots are $\lambda_1 = 1-r$ and $\lambda_2 = 1+a$, note that $|\lambda_2| < 1$ if and only if $|1-r| < 1$. This holds when $0 < r < 2$, and since $|\lambda_2| = |1+a| > 1$ for all $a > 0$ so the point $(1,0)$ is saddle fixed point. $\square$

In the next theorem we give sufficient conditions for the stability of the positive fixed point $(N^*, P^*) = \left(\frac{r}{a+r}, \frac{r}{a+r}\right)$.

**Theorem 6.2.3.** *The positive fixed point* $(N^*, P^*)$ *is asymptotically stable if* $2 - \frac{4}{r} < \frac{ar}{a+r} < 1$.

*Proof.* At $(N^*, P^*)$, the Jacobian matrix is

$$J^*(N^*, P^*) = \begin{pmatrix} 1 + r - \frac{2r^2}{a+r} - \frac{ar}{a+r} & -\frac{ar}{a+r} \\ \frac{ar}{a+r} & 1 + \frac{ar}{a+r} - \frac{2ar}{a+r} \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{r^2}{a+r} & \frac{-ar}{a+r} \\ \frac{ar}{a+r} & 1 - \frac{ar}{a+r} \end{pmatrix}$$

Note that

$$tr(J^*) = 1 - \frac{r^2}{a+r} + 1 - \frac{ar}{a+r} = 2 - r$$

and

$$det(J^*) = \left(1 - \frac{r^2}{a+r}\right)\left(1 - \frac{ar}{a+r}\right) + \left(\frac{ar}{a+r}\right)^2$$

$$=1 - \frac{ar}{a+r} - \frac{r^2}{a+r} + \frac{ar^3}{(a+r)^2} + \frac{a^2r^2}{(a+r)^2}$$

$$=1 - r + \frac{ar^2}{a+r}$$

Thus the characteristic equation is

$$\begin{aligned}\rho(\lambda) &= \lambda^2 - tr(J^*)\lambda + det(J^*)\\ &= \lambda^2 + (r-2)\lambda + 1 - r + \frac{ar^2}{a+r}\end{aligned}$$

And by using Jury conditions which say that if

$$\rho(1) > 0, \qquad \rho(-1) > 0 \qquad and \quad det(J^*) < 1. \tag{6.4}$$

then the modulus of all roots of the characteristic equation is less than one, in other words if the previous conditions hold then the fixed point is asymptotically stable. We can observe that

$$\rho(1) = \frac{ar^2}{(a+r)}$$

which is positive for any $a, r > 0$. Also

$$\rho(-1) = 4 - 2r + \frac{ar^2}{a+r}$$

it follows that $\rho(-1) > 0$ if and only if

$$\frac{ar}{a+r} > 2 - \frac{4}{r}$$

Finally $detJ^* < 1$ hold if and only if

$$1 - r + \frac{ar^2}{a+r} < 1$$

which is equivalent to

$$\frac{ar}{a+r} < 1$$

Now combining the previous inequalities, we get the following result, the fixed point $(N^*, P^*)$ is asymptotically stable if

$$2 - \frac{4}{r} < \frac{ar}{a+r} < 1$$

. $\square$

## 6.3 Stability analysis of predator-prey system with Allee effect

Now we will study the stability of predator-prey system with Allee effect. Allee effect may be caused by variety of mechanisms applicable in small population. In this part we will study Allee effect on prey population, as the following system, [1] and [10].

$$f(N_t, P_t) = N_{t+1} = N_t + rN_t(1 - N_t)\frac{N_t}{\mu + N_t} - aN_tP_t \tag{6.5}$$

$$g(N_t, p_t) = P_{t+1} = P_t + aP_t(N_t - P_t)$$

Where the parameters $a, r$ are positive, and $\mu$ is the Allee constant that satisfies the assumption

$$0 < \mu < \frac{r}{a}$$

We take the term $\frac{N_t}{\mu+N_t}$ as Allee effect.
The fixed points of system (6.5) are the solutions of the following system of equations

$$N + rN(1 - N)\frac{N}{\mu + N} - aNP = N$$

$$P + aP(N - P) = P$$

It is clear that the point $(0,0)$ is a solution of the system, for $N \neq 0$ we have the following relation

$$rN\frac{1-N}{\mu+N} = aP$$

then substitute in the second equation of the previous system, so

$$\frac{rN(1-N)}{\mu+N}\left(N - \frac{rN(1-N)}{a(\mu+N)}\right) = 0$$

This leads to two solutions $N = 1$ and $N = \frac{r-a\mu}{a+r}$ By substitution we get three fixed points $(0,0), (1,0)$ and $(N_\mu^*, P_\mu^*) = \left(\frac{r-a\mu}{a+r}, \frac{r-a\mu}{a+r}\right)$. To investigate the stability conditions for system (6.5), we find the Jacobian matrix. Note that

$$\frac{\partial f}{\partial N} = 1 + (r - 2rN)\frac{N}{\mu+N} + rN(1-N)\frac{\mu}{(\mu+N)^2} - aP$$

$$\frac{\partial f}{\partial P} = -aN$$

$$\frac{\partial g}{\partial N} = aP$$

$$\frac{\partial g}{\partial P} = 1 + aN - 2aP$$

Hence the Jacobian matrix of system (6.5) is

$$J_\mu = \begin{pmatrix} 1 + \frac{rN(1-2N)}{\mu+N} + \frac{\mu rN(1-N)}{(\mu+N)^2} - aP & -aN \\ aP & 1 + aN - 2aP \end{pmatrix}$$

The Jacobian matrix at the fixed point $(0,0)$ is

$$J_\mu(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So $(0,0)$ is non-hyperbolic fixed point since $J_\mu(0,0)$ has eigenvalues of modulus equal to one.

While the Jacobian matrix for $(1,0)$ is

$$J_\mu(1,0) = \begin{pmatrix} 1 - \frac{r}{\mu+1} & -a \\ 0 & 1 + a \end{pmatrix}$$

**Lemma 6.3.1.** *The fixed point* $(1,0)$ *is saddle if and only if* $0 < \frac{r}{\mu+1} < 2$.

*Proof.* The corresponding characteristic equation of the matrix $J_\mu(1,0)$ is

$$\rho(\lambda) = \lambda^2 - (2 + a - \frac{r}{\mu+1})\lambda + (1 - \frac{r}{\mu+1})(1+a)$$

The roots of this equation are

$$\lambda_1 = 1 - \frac{r}{\mu+1} \qquad \lambda_2 = 1 + a$$

It is clear $|\lambda_2| = |1+a| > 1$, also if $|\lambda_1| < 1$ then $(1,0)$ is saddle fixed point. This holds if and only if $-1 < 1 - \frac{r}{\mu+1} < 1$, which is equivalent to $0 < \frac{r}{\mu+1} < 2$. □

Note that if $\frac{r}{\mu+1} = 2$ then $|\lambda_1| = 1$, so the point $(1,0)$ is non hyperbolic fixed point. This case will be studied when we investigate the bifurcation in the system.
Now we study the stability of system (6.5) at the fixed point $(N_\mu^*, P_\mu^*) = \left(\frac{r-a\mu}{a+r}, \frac{r-a\mu}{a+r}\right)$. Before this note that

$$\frac{\partial f}{\partial N}(N_\mu^*, P_\mu^*) = 1 + (r - 2rN_\mu^*)\frac{N_\mu^*}{\mu + N_\mu^*} + \frac{r\mu N_\mu^*(1 - N_\mu^*)}{(\mu + N_\mu^*)^2} - aN_\mu^*$$

$$= 1 + N_\mu^* \left( \frac{r - 2rN_\mu^*}{\mu + N_\mu^*} + \frac{r\mu - r\mu N_\mu^*}{(\mu + N_\mu^*)^2} - \frac{a(\mu + N_\mu^*)}{\mu + N_\mu^*} \right)$$

$$= 1 + N_\mu^* \left( \frac{r - a\mu}{\mu + N_\mu^*} - \frac{2rN_\mu^* + aN_\mu^*}{\mu + N_\mu^*} + \frac{r\mu(1 - N_\mu^*)}{(\mu + N_\mu^*)^2} \right)$$

But

$$N_\mu^*(a + r) = r - a\mu$$

Hence

$$\frac{\partial f}{\partial N}(N_\mu^*, P_\mu^*) = 1 + N_\mu^* \left( \frac{N_\mu^*(a+r)}{\mu + N_\mu^*} - N_\mu^* \left( \frac{2r+a}{\mu + N_\mu^*} \right) + \frac{r\mu(1-N_\mu^*)}{(\mu+N_\mu^*)^2} \right)$$

$$= 1 + N_\mu^* \left( \frac{-rN_\mu^*}{\mu + N_\mu^*} + \frac{r\mu(1-N_\mu^*)}{(\mu+N_\mu^*)^2} \right)$$

$$= 1 - N_\mu^* \left( \frac{rN_\mu^*}{\mu + N_\mu^*} - \frac{r\mu(1-N_\mu^*)}{(\mu+N_\mu^*)^2} \right)$$

Let

$$\alpha_\mu = \frac{rN_\mu^*}{\mu + N_\mu^*} - \frac{r\mu(1-N_\mu^*)}{(\mu+N_\mu^*)^2}$$

So

$$\frac{\partial f}{\partial N}(N_\mu^*, P_\mu^*) = 1 - \alpha_\mu N_\mu^*$$

Also

$$\frac{\partial f}{\partial P}(N_\mu^*, P_\mu^*) = -aN_\mu^*$$

$$\frac{\partial g}{\partial N}(N_\mu^*, P_\mu^*) = aN_\mu^*$$

$$\frac{\partial g}{\partial P}(N_\mu^*, P_\mu^*) = 1 - aN_\mu^*$$

This implies that the Jacobian matrix evaluated at $(N_\mu^*, P_\mu^*)$ is

$$J_\mu(N_\mu^*, P_\mu^*) = \begin{pmatrix} 1 - \alpha_\mu N_\mu^* & -aN_\mu^* \\ aN_\mu^* & 1 - aN_\mu^* \end{pmatrix}$$

Now we back to $\alpha_\mu$ and substitute the value of $N^*_\mu$ in it so

$$\alpha_\mu = \frac{rN^*_\mu}{\mu + N^*_\mu} - \frac{r\mu(1-N^*_\mu)}{(\mu+N^*_\mu)^2}$$

$$= r\left(\frac{r-a\mu}{a+r}\right)\frac{a+r}{r\mu+r} - r\mu\left(\frac{a+a\mu}{a+r}\right)\left(\frac{a+r}{r\mu+r}\right)^2$$

$$= \frac{r-a\mu}{\mu+1} - a\mu\left(\frac{a+r}{r(\mu+1)}\right)$$

The characteristic equation associated with $J_\mu(N^*_\mu, P^*_\mu)$ is

$$\rho_\mu(\lambda) = \lambda^2 - tr(J_\mu(N^*_\mu, P^*_\mu))\lambda + det(J_\mu(N^*_\mu, P^*_\mu))$$

Where

$$tr(J_\mu(N^*_\mu, P^*_\mu)) = 1 - \alpha_\mu N^*_\mu + 1 - aN^*_\mu = 2 - (\alpha_\mu + a)N^*_\mu$$

And

$$\begin{aligned} det(J_\mu(N^*_\mu, P^*_\mu)) &= (1-aN^*_\mu)(1-\alpha_\mu N^*_\mu) + (aN^*_\mu)^2 \\ &= 1 - \alpha_\mu N^*_\mu - aN^*_\mu + a\alpha_\mu N^{*2}_\mu + (aN^*_\mu)^2 \\ &= 1 - (\alpha_\mu + a)N^*_\mu + a(\alpha_\mu + a)N^{*2}_\mu \end{aligned}$$

**Theorem 6.3.2.** *The positive fixed point* $(N^*_\mu, P^*_\mu) = \left(\frac{r-a\mu}{a+r}, \frac{r-a\mu}{a+r}\right)$ *of predator-prey system* (6.5) *is asymptotically stable if*
$2 - \frac{4r(\mu+1)}{(r-a\mu)^2} < a\left(\frac{r-a\mu}{a+r}\right) < 1.$

*Proof.* We use the Jury conditions to obtain the parametric region where $\left(\frac{r-a\mu}{a+r}, \frac{r-a\mu}{a+r}\right)$ is asymptotically stable.
First we observe that

$$\begin{aligned} \rho_\mu(1) &= 1 - tr(J_\mu(N^*_\mu, P^*_\mu)) + det(J_\mu(N^*_\mu, P^*_\mu)) \\ &= 1 - (2 - (\alpha_\mu + a)N^*_\mu) + 1 - (\alpha_\mu + a)N^*_\mu + a(\alpha_\mu + a)N^{*2}_\mu \\ &= a(\alpha_\mu + a)N^{*2}_\mu \end{aligned}$$

$\rho_\mu(1) > 0$ holds if and only if $(\alpha_\mu + a) > 0$. To prove that $\alpha_\mu + a > 0$ we assume $\varphi(\mu) = \alpha_\mu + a$, where $\mu \in \left[0, \frac{r}{a}\right]$. This leads to

$$\begin{aligned}\varphi(\mu) &= \alpha_\mu + a \\ &= a + \frac{r - a\mu}{\mu + 1} - \frac{a\mu(a + r)}{r(\mu + 1)} \\ &= \frac{ar(\mu + 1) + r^2 - a\mu r - a\mu(a + r)}{r(\mu + 1)} \\ &= \frac{r^2 + ar - a^2\mu - ar\mu}{r(\mu + 1)}\end{aligned}$$

Since

$$\begin{aligned}\varphi'(\mu) &= \frac{-r(\mu + 1)(a^2 + ar) - (r^2 + ar - a^2\mu - ar\mu)r}{r^2(\mu + 1)^2} \\ &= \frac{-a^2r - ar^2 - r^3 - ar^2}{r^2(\mu + 1)^2} \\ &= \frac{-(r^2 + 2ar + a^2)}{r(\mu + 1)^2} \\ &= -\frac{(r + a)^2}{r(\mu + 1)^2} < 0\end{aligned}$$

Hence $\varphi(\mu)$ is strictly decreasing on $[0, \frac{r}{a}]$, thus $\varphi(\mu)$ attains its minimum value at $\mu = \frac{r}{a}$, but $\varphi(\frac{r}{a}) = 0$. We conclude that $a + \alpha_\mu > 0$ for all $\mu \in [0, \frac{r}{a}]$, thus $\rho_\mu(1) > 0$ if and only if $a + \alpha_\mu > 0$.
Now we investigate the other conditions

$\rho_\mu(-1) > 0$ if and only if:

$$1 + 2 - (a + \alpha_\mu)N_\mu^* + 1 - (a + \alpha_\mu)N_\mu^* + a(a + \alpha_\mu)N_\mu^{*2} > 0$$

which is equivalent to

$$4 - 2(a + \alpha_\mu)N_\mu^* + a(a + \alpha_\mu)N_\mu^{*2} > 0$$

this lead to

$$a(a+\alpha_\mu)N_\mu^{*2} > 2(a+\alpha_\mu)N_\mu^* - 4$$

so

$$aN_\mu^* > 2 - \frac{4}{(a+\alpha_\mu)N_\mu^*}$$

And by substituting the value of $a+\alpha_\mu$ and $N_\mu^*$ we have

$\rho_\mu(-1) > 0$ if and only if :

$$a\left(\frac{r-a\mu}{r+a}\right) > 2 - 4\frac{r(\mu+1)}{r^2+ar-a^2\mu-ar\mu}\left(\frac{r+a}{r-a\mu}\right)$$

which is equialent to

$$a\left(\frac{r-a\mu}{r+a}\right) > 2 - \frac{4r(\mu+1)(r+a)}{(r(a+r)-a\mu(a+r))(r-a\mu)}$$

so

$$a\left(\frac{r-a\mu}{r+a}\right) > 2 - \frac{4r(\mu+1)}{(r-a\mu)^2}$$

To find necessary conditions for the inequality $det(J_\mu(N_\mu^*, P_\mu^*)) < 1$. We note that

$det(J_\mu(N_\mu^*, P_\mu^*)) < 1$ if and only if :

$$1 - (a+\alpha_\mu)N_\mu^* + a(a+\alpha_\mu)N_\mu^{*2} < 1$$

hence

$$a(a+\alpha_\mu)N_\mu^{*2} < (a+\alpha_\mu)N_\mu^*$$

But since $(a+\alpha_\mu) > 0$ on $\left[0, \frac{r}{a}\right]$ this implies that $aN_\mu^* < 1$, so $a\left(\frac{r-a\mu}{a+r}\right) < 1$. Thus the fixed point $\left(\frac{r-a\mu}{a+r}, \frac{r-a\mu}{a+r}\right)$ is asymptotically stable if

$$2 - \frac{4r(\mu+1)}{(r-a\mu)^2} < a\left(\frac{r-a\mu}{a+r}\right) < 1$$

$\square$

## 6.4 Bifurcation analysis

Our objective now is to find the parameter space where the bifurcation of the fixed points of the discrete predator-prey system with and without Allee effect happens.
First we will discuss the bifurcation of the positive fixed point $(N^*, P^*) = \left(\frac{r}{a+r}, \frac{r}{a+r}\right)$ of system (6.1). And by using the rules in Trace-determinate plane $(T-D)$ and theorem 4.4.1, we will determine the parameter space where each kind of bifurcation happens.
The Jacobian matrix at $(N^*, P^*)$ is

$$J^* = \begin{pmatrix} 1 - \frac{r^2}{a+r} & \frac{-ar}{a+r} \\ \frac{ar}{a+r} & 1 - \frac{ar}{a+r} \end{pmatrix}$$

where

$$tr(J^*) = 2 - r$$

and

$$det(J^*) = 1 - r + \frac{ar^2}{a+r}$$

The saddle-node bifurcation occurs when the Jacobian matrix has an eigenvalue equal to 1. This is equivalent $det(J^*) = tr(J^*) - 1$ in $(T - D)$ - plane, i.e

$$1 - r + \frac{ar^2}{a+r} = 2 - r - 1$$

Thus

$$\frac{ar^2}{a+r} = 0$$

This implies that $a = 0$ or $r = 0$, but this can not happen since $a, r > 0$.

**Theorem 6.4.1.** *The fixed point* $(N^*, P^*) = \left(\frac{r}{a+r}, \frac{r}{a+r}\right)$ *of* (6.1), *undergoes period doubling bifurcation when* $\frac{ar}{a+r} = 2 - \frac{4}{r}$.

*Proof.* Period-doubling bifurcation occurs when $det(J^*) = -tr(J^*) - 1$ so

$$1 - r + \frac{ar^2}{a+r} = -2 + r - 1$$

which leads to

$$\frac{ar^2}{a+r} = 2r - 4$$

hence

$$\frac{ar}{a+r} = 2 - \frac{4}{r}$$

We saw that if $\frac{ar}{a+r} > 2-\frac{4}{r}$ the fixed point $(N^*, P^*)$ is stable, and if $\frac{ar}{a+r} < 2-\frac{4}{r}$ then the fixed point $(N^*, P^*)$ is unstable. When $a$ and $r$ passes the curve $\frac{ar}{a+r} = 2-\frac{4}{r}$ the fixed point $(N^*, P^*)$ undergoes a period-doubling bifurcation into two-periodic cycle. □

**Theorem 6.4.2.** *The fixed point $(N^*, P^*)$ undergoes Neimark-Sacker bifurcation if $\frac{ar}{a+r} = 1$, where $0 < r < 4$.*

*Proof.* When the Jacobian matrix $J^*$ has a pair of complex eigenvalues of modulus 1, then system (6.1) undergoes Neimark-Sacker bifurcation at the fixed point $(N^*, P^*)$. This happens when $det(J^*) = 1$ and $-2 < tr(J^*) < 2$, which is equivalent to

$$det(J^*) = 1 - r + \frac{ar^2}{a+r} = 1$$

hence

$$\frac{ar}{a+r} = 1$$

and

$$-2 < tr(J^*) < 2$$

so

$$-2 < 2 - r < 2$$

which leads to

$$0 < r < 4$$

So when $\frac{ar}{a+r} = 1$ and $0 < r < 4$, the fixed point $(N^*, P^*)$ of system (6.1) undergoes Neimark-Sacker bifurcation. □

Now we will investigate the bifurcation scenario of the fixed points of discrete-time predator prey system with Allee effect on prey population. We begin with the fixed point $(0,0)$. We found the Jacobian matrix of system (6.5) at $(0,0)$ which is

$$J_\mu(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since the eigenvalues of $J_\mu(0,0)$ are $\lambda_1 = 1$ and $\lambda_2 = 1$, the fixed point is non hyperbolic and undergoes saddle node bifurcation.
The Jacobian matrix at the fixed point $(1,0)$ is

$$J_\mu(1,0) = \begin{pmatrix} 1 - \frac{r}{\mu+1} & -a \\ & \\ 0 & 1+a \end{pmatrix}$$

Where

$$tr(J_\mu(1,0)) = 2 + a - \frac{r}{\mu+1}$$

And

$$det(J_\mu(1,0)) = \left(1 - \frac{r}{\mu+1}\right)(1+a) = 1 + a - \frac{r}{\mu+1} - \frac{ar}{\mu+1}$$

**Theorem 6.4.3.** *The fixed point* $(1,0)$ *of system* (6.5) *undergoes period doubling bifurcation when* $\frac{r}{\mu+1} = 2$.

*Proof.* The period-doubling bifurcation occurs when the Jacobian matrix $J_\mu(1,0)$ has an eigenvalue equal to $-1$. In $T-D$ plane this occurs when $det(J_\mu(1,0)) = -tr(J_\mu(1,0)) - 1$, this lead to

$$\begin{aligned} 1 + a - \frac{r}{\mu+1} - \frac{ar}{\mu+1} &= -2 - a + \frac{r}{\mu+1} - 1 \\ 4 + 2a &= \frac{2r}{\mu+1} + \frac{ar}{\mu+1} \end{aligned}$$

and since $a + 2 > 0$, so we have $\frac{r}{\mu+1} = 2$. $\square$

We will study the bifurcation of the fixed point $(N_\mu^*, P_\mu^*) = \left(\frac{r-a\mu}{a+r}, \frac{r-a\mu}{a+r}\right)$. The Jacobian matrix at $(N_\mu^*, P_\mu^*)$ is

$$J_\mu^* = \begin{pmatrix} 1 - \alpha_\mu N_\mu^* & -aN_\mu^* \\ & \\ aN_\mu^* & 1 - aN_\mu^* \end{pmatrix}$$

Where

$$\alpha_\mu = \frac{r - a\mu}{\mu + 1} - \frac{a\mu(a + r)}{r(\mu + 1)}$$

And

$$tr(J^*_\mu) = 2 - (a + \alpha_\mu)N^*_\mu$$

Also

$$det(J^*_\mu) = 1 - (a + \alpha_\mu)N^*_\mu + a(a + \alpha_\mu)N^{*2}_\mu$$

**Theorem 6.4.4.** *The fixed point* $(N^*_\mu, P^*_\mu)$ *undergoes saddle node bifurcation when* $\mu = \frac{r}{a}$

*Proof.* Saddle-node bifurcation occurs when $det(J^*_\mu) = tr(J^*_\mu) - 1$ this leads to

$$1 - (a + \alpha_\mu)N^*_\mu + a(a + \alpha_\mu)N^{*2}_\mu = 1 - (a + \alpha_\mu)N^*_\mu$$

hence

$$a(a + \alpha_\mu)N^{*2}_\mu = 0$$

But since $a(a + \alpha_\mu) > 0$, this leads to

$$N^*_\mu = \frac{r - a\mu}{a + r} = 0$$

so

$$\mu = \frac{r}{a}$$

Observe when $\mu = \frac{r}{a}$ the fixed point $(N^*_\mu, P^*_\mu)$ equals $(0, 0)$. □

**Theorem 6.4.5.** *The fixed point*$(N^*_\mu, P^*_\mu)$*, undergoes a period-doubling bifurcation into two cycle when*

$$a\left(\frac{r - a\mu}{a + r}\right) = 2 - \frac{4r(\mu + 1)}{(r - a\mu)^2}$$

*Proof.* The fixed point$(N^*_\mu, P^*_\mu)$, undergoes a period-doubling bifurcation when

$$det(J^*_\mu) = -tr(J^*_\mu) - 1$$

so

$$1-(\alpha_\mu+a)N_\mu^*+a(\alpha_\mu+a)N_\mu^{*2}=-2+(\alpha_\mu+a)N_\mu^*-1$$

which leads to

$$a(\alpha_\mu+a)N_\mu^{*2}=2(\alpha_\mu+a)N_\mu^*-4$$

hence

$$aN^*=2-\frac{4}{(\alpha_\mu+a)N_\mu^*}$$

by substituting the value of $(\alpha_\mu+a)$ and $N_\mu^*$ we get

$$a\left(\frac{r-a\mu}{r+a}\right)=2-\frac{4r(\mu+1)}{(r-a\mu)^2}$$

□

**Theorem 6.4.6.** *The fixed point $(N_\mu^*, P_\mu^*)$, undergoes Neimark-Sacker bifurcation when*
$a\left(\frac{r-a\mu}{a+r}\right)=1$ *and* $0<\frac{(r-a\mu)^2}{r(\mu+1)}<4$.

*Proof.* Here we assume that $N^*\neq 0$, we know that Neimark-Sacker bifurcation happens in $(T-D)$-plane when $det(J_\mu^*)=1$ and $-2<tr(J_\mu^*)<2$, which is equivalent to

$$1-(\alpha_\mu+a)N_\mu^*+a(\alpha_\mu+a)N_\mu^{*2}=1$$

so

$$(\alpha_\mu+a)N_\mu^*(aN^*-1)=0$$

since $(\alpha_\mu+a)N_\mu^*>0$ so

$$a\left(\frac{r-a\mu}{a+r}\right)=1$$

and

$$-2<tr(J_\mu^*)<2$$

so

$$-2 < 2 - (\alpha_\mu + a)N^*_\mu < 2$$

hence

$$0 < (\alpha_\mu + a)N^*_\mu < 4$$

which is equivalent to

$$0 < \frac{(r - a\mu)^2}{r(\mu + 1)} < 4$$

□

## 6.5 Numerical examples

In this section we will use a numerical examples which support our disscution in the previous sections to illustrate the bifurcation diagram of the predator prey model, we use Matlab 7.12.

**Example 6.5.1.** In this example we draw the bifurcation diagram of predator-prey model without Allee effect.
In system (6.1) we fix the the parameter $a$ and we consider $r$ as bifurcation parameter. We take $a = 2$ and $0 \leq r \leq 3$, then the positive fixed point $(N^*, P^*)$ of system (6.1) is $N^*{=}P^*{=}\frac{r}{2+r}$
The region of stability of $(N^*, P^*)$ is $2 - \frac{4}{r} < \frac{2r}{2+r} < 1$. Depending on theorem 6.4.2 the fixed point $(N^*, P^*) = \left(\frac{r}{2+r}, \frac{r}{2+r}\right)$ undergoes a Neimark-Sacker bifurcation when

$$\frac{2r}{2 + r} = 1$$

which leads to the fact that the fixed point $\left(\frac{r}{2+r}, \frac{r}{2+r}\right)$ undergoes a Nimark-Sacker bifurcation when $r = 2$, (see figure 6.1 ).

**Example 6.5.2.** Here we will illustrate the bifurcation diagram of predator-prey model with Allee effect, (model (6.5) ). As the previous example we fixed $a = 2$, $\mu = .09$ and let $r$ varies. Hence the positive fixed point of system (6.5) becomes $(N^*_\mu, P^*_\mu) = \left(\frac{r-2\mu}{2+r}, \frac{r-2\mu}{2+r}\right)$.

By theorem 6.4.4 and theorem 6.4.6 model (6.5) undergoes a saddle-node bifurcation when $r = 0.18$, also it undergoes a Neimark-sacker bifurcation when $r = 2.36$ (see figure 6.2 and figure6.3 )

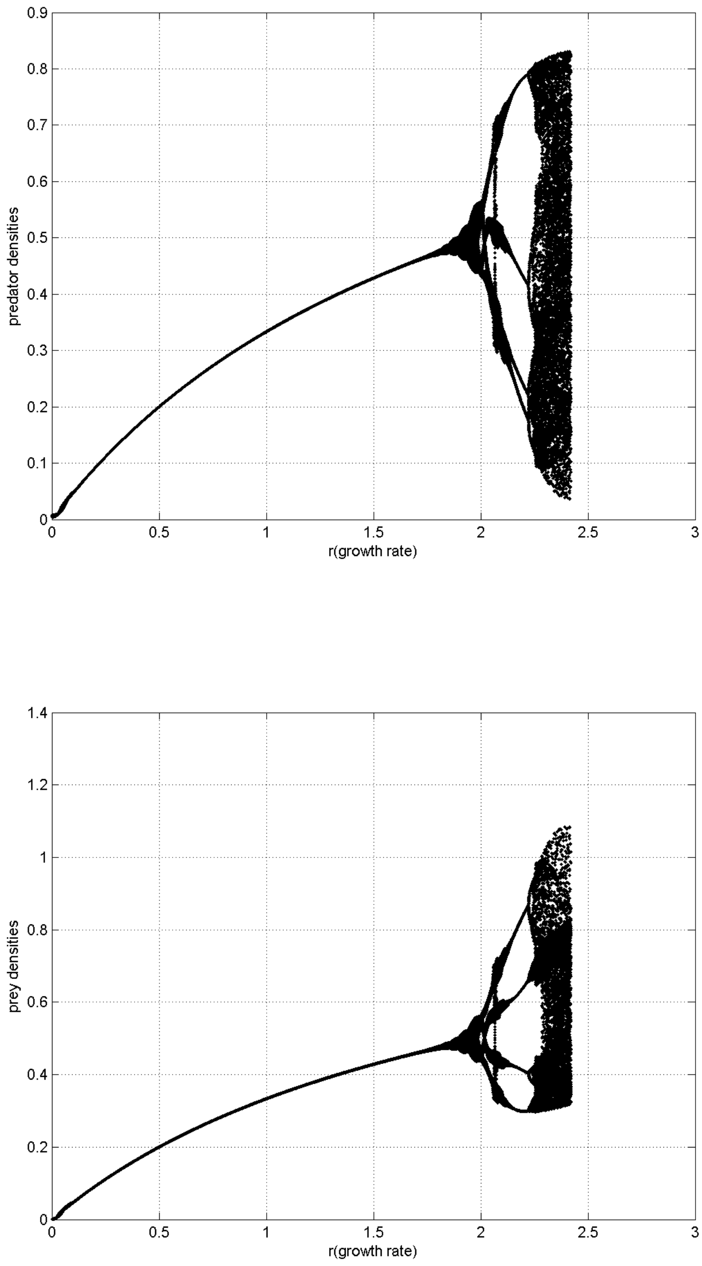
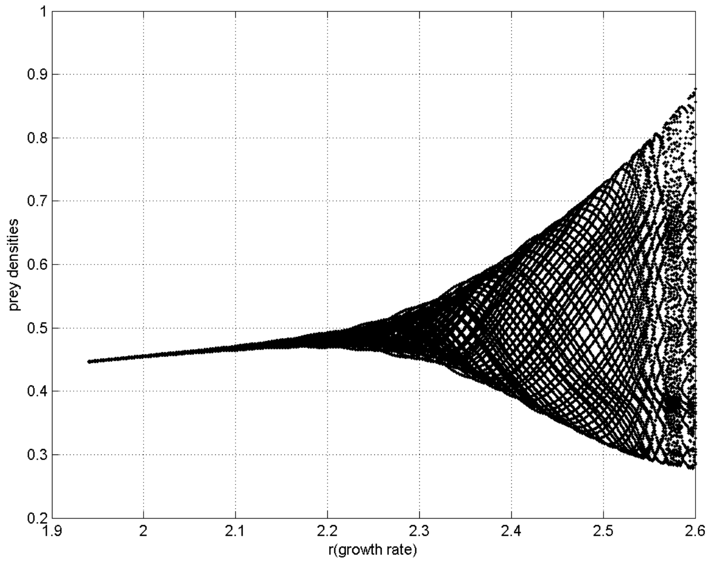



Figure 6.1: The bifurcation diagram of predator-prey ( model (6.1)). The initial conditions $N_0 = 0.3, P_0 = 0.2$, and $a = 2$ and $r$ varies from 0 to 3.



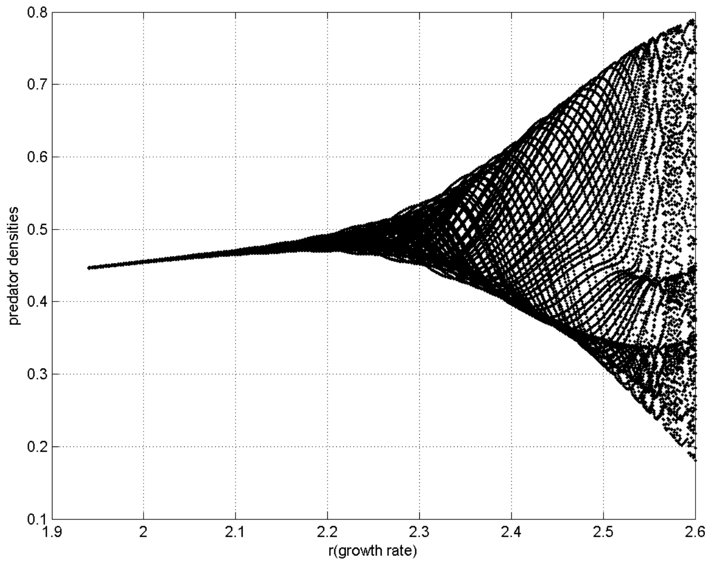
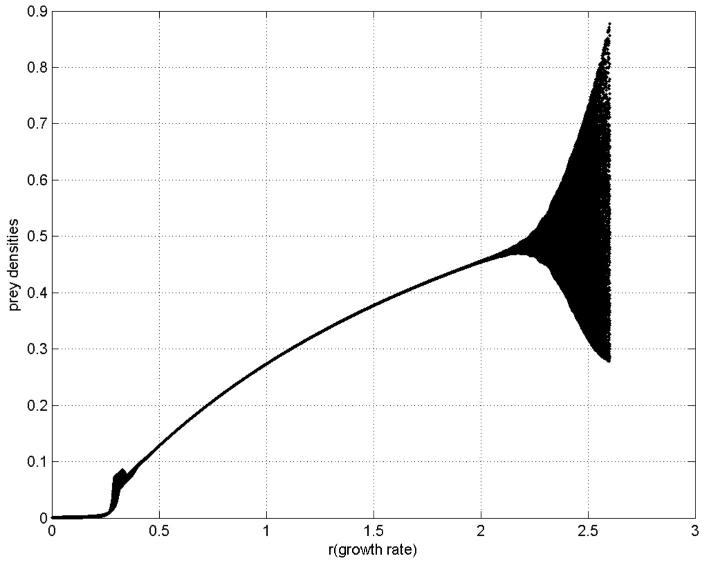




Figure 6.2: The bifurcation diagram (1) of predator-prey with Allee effect ( model (6.5)).

The initial conditions $N_0 = 0.3, P_0 = 0.2$, and $a = 2$,$u = 0.09$ and $r$ varies from 1.94 to 2.6.



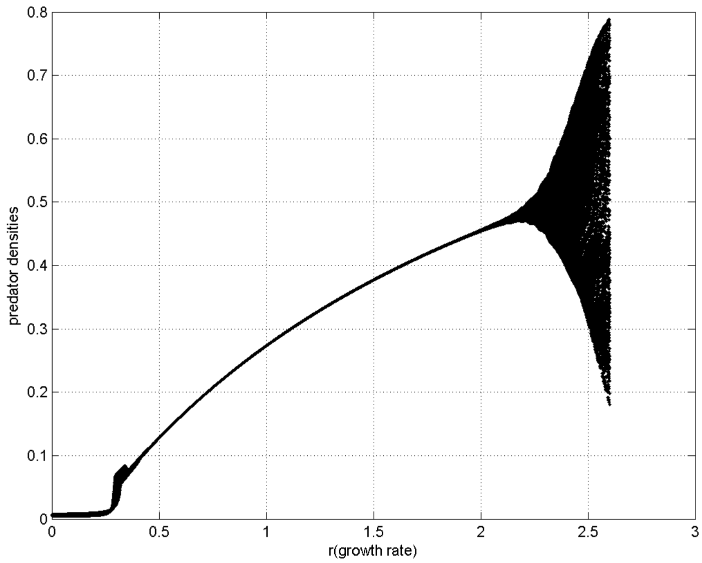




Figure 6.3: The bifurcation diagram (1) of predator-prey with Allee effect ( model (6.5)).
The initial conditions $N_0 = 0.3, P_0 = 0.2$, and $a = 2$,$u = 0.09$ and $r$ varies from 0 to 2.6.

# Chapter 7

# Bifurcation analysis a population model

## 7.1 Introduction

Consider the following model

$$x_{n+1} = f(x_n) = \frac{ax_n(1-x_n)}{1+cx_n} \tag{7.1}$$

where $x_n \in [0,1]$, and $c \in (0,1)$.
This model is a generalization of the logistic model, here we will investigate stability and bifurcation of this model. We will find also its fixed points and 2-periodic orbit, determine their stability region, and study their bifurcation, saddle-node and period-doubling bifurcation. The model (7.1) has two fixed points, $\tilde{x}_1 = 0$, and $\tilde{x}_2 = \frac{a-1}{a+c}$. To insure that $\tilde{x_2} \in [0,1]$, we assume that $a \geq 1$.

## 7.2 Stability analysis

Now we give the sufficient conditions for the stability of these fixed points.

**Theorem 7.2.1.** *For the model* (7.1)*, the fixed point* $\tilde{x}_1 = 0$ *is stable if* $0 < a < 1$ *and unstable for* $a \geq 1$*.*

*Proof.* We find the first derivative of $f(x)$ and we have

$$f'(x) = \frac{a}{(1+cx)^2}\left[1-2x-cx^2\right] \tag{7.2}$$

Notice that when $a > 1$ the fixed point $\tilde{x}_1 = 0$ is unstable since $f'(0) = a$, for $a = 1$, we have $f'(0) = 1$ hence $\tilde{x}_1$ is non-hyperbolic fixed point which requires to find the second derivative of $f(x)$, and we find that

$$f''(x) = a\left[\frac{-2c}{(1+cx)^3}\left[1-2x-cx^2\right] - \frac{2}{(1+cx)}\right] \tag{7.3}$$

Also we observe that $f''(0) = -2a(c+1) \neq 0$ since $c \in (0,1)$, hence the fixed point $\tilde{x}_1$ is unstable for all $a \geq 1$. $\square$

**Theorem 7.2.2.** *The fixed point $\tilde{x}_2 = \frac{a-1}{a+c}$ is asymptotically stable if $1 < a < \frac{(c+3)+\sqrt{(c+3)^2+4c}}{2}$*

*Proof.* The fixed point $\tilde{x}_2 = \frac{a-1}{a+c}$ is asymptotically stable if and only if

(7.4)

$$|f'(\tilde{x_2})| = \left|\frac{a}{\left[1+c\left(\frac{a-1}{a+c}\right)\right]^2}\left(1-2\left(\frac{a-1}{a+c}\right)-c\left(\frac{a-1}{a+c}\right)^2\right)\right|$$

$$= \frac{a(a+c)^2}{(a+c)^2+2c(a-1)(a+c)+c^2(a-1)^2}\left[\frac{(a+c)^2-2(a-1)(a+c)-c(a-1)^2}{(a+c)^2}\right]$$

$$= \left|a\frac{(a+c)^2-2(a-1)(a+c)-c(a-1)^2}{(a+c)^2+2c(a-1)(a+c)+c^2(a-1)^2}\right| < 1$$

hence we have the following two inequalities

$$-1 < a\frac{(a+c)^2-2(a-1)(a+c)-c(a-1)^2}{(a+c)^2+2c(a-1)(a+c)+c^2(a-1)^2} < 1 \tag{7.5}$$

The relation

$$\frac{a(a+c)^2-2a(a-1)(a+c)-ac(a-1)^2}{(a+c)^2+2c(a-1)(a+c)+c^2(a-1)^2} - 1 < 0 \tag{7.6}$$

which leads to

(7.7)

$$\frac{a(a+c)^2-2a(a-1)(a+c)-ac(a-1)^2-\left[(a+c)^2+2c(a-1)(a+c)+c^2(a-1)^2\right]}{(a+c)^2+2c(a-1)(a+c)+c^2(a-1)^2} < 0$$

And under our assumptions that $a > 1$ and $c \in (0,1)$ the denominator is always positive, we have

(7.8)
$$a(a+c)^2 - 2a(a-1)(a+c) - ac(a-1)^2 - \left[(a+c)^2 + 2c(a-1)(a+c) + c^2(a-1)^2\right] < 0$$

which is equivalent to

$$(a-1)(a+c)^2 - 2(a-1)(a+c)^2 - c(a-1)^2(a+c) < 0$$

hence

$$-(a-1)(a+c)^2 - c(a-1)^2(a+c) < 0$$

which leads to

$$-(a+c) - c(a-1) < 0$$

The last inequality is true for $a > 1$. The second inequality is

(7.9)
$$\frac{a(a+c)^2 - 2a(a-1)(a+c) - ac(a-1)^2}{(a+c)^2 + 2c(a-1)(a+c) + c^2(a-1)^2} + 1 > 0$$

So we have the following inequality

(7.10)
$$\frac{a(a+c)^2 - 2a(a-1)(a+c) - ac(a-1)^2 + (a+c)^2 + 2c(a-1)(a+c) + c^2(a-1)^2}{(a+c)^2 + 2c(a-1)(a+c) + c^2(a-1)^2} > 0$$

This leads to

$$(a+1)(a+c)^2 + (a-1)(c-a)(2(a+c) + c(a-1)) > 0$$

so

$$(a+1)(a+c)^2 > (a-1)(a-c)(a(2+c) + c)$$

which is equivelant to

$$(a+1)(a^2 + 2ac + c^2) > (a^2 - ac - a + c)(2a + ac + c)$$

hence

$$-a^3 + 4a^2c + ac^2 + 3a^2 + ac + a^2c^2 - a^3c > 0$$

so

$$-a^2 + 4ac + c^2 + 3a + c + ac^2 - a^2c > 0$$

consequently

$$a(-a + 4c + 3 + c^2 - ac) + c(c + 1) > 0$$

therefore

$$a(-a(1 + c) + c(1 + c) + 3(1 + c)) + c(c + 1) > 0$$

thus

$$a(1 + c)(-a + c + 3) + c(1 + c) > 0$$

hence we need to solve the quadratic inequality

$$a^2 - (c + 3)a - c < 0 \tag{7.11}$$

Now the solution of inequality (7.11) is given by

$$\frac{(c + 3) - \sqrt{(c + 3)^2 + 4c}}{2} < a < \frac{(c + 3) + \sqrt{(c + 3)^2 + 4c}}{2} \tag{7.12}$$

This implies that the fixed point $\tilde{x}_2$ is asymptotically stable if

$$1 < a < \frac{(c + 3) + \sqrt{(c + 3)^2 + 4c}}{2} \tag{7.13}$$

□

## 7.3 Bifurcation analysis

Now we will investigate the bifurcation of the fixed points of model (7.1).

**Theorem 7.3.1.** *The model* (7.1) *undergoes a transcritical bifurcation when* $a = 1$.

*Proof.* We observe that when $a = 1$, the system has only one non-hyperbolic fixed point $\tilde{x} = 0$, so at the point $(\tilde{x}, a) = (0, 1)$ the system undergoes a transcritical bifurcation, since it satisfies the following conditions

1. $\frac{\partial f}{\partial a}(x,a) = \frac{x(1-x)}{(1+cx)}$ then $\frac{\partial f}{\partial a}(0,1) = 0$
2. $\frac{\partial f}{\partial x \partial a}(a,x) = \frac{(1+cx)(1-2x)-c(x-x^2)}{(1+cx)^2}$ then $\frac{\partial^2 f}{\partial x \partial a}(0,1) = 1 \neq 0$
3. $\frac{\partial^2 f}{\partial x^2}(x,a) = \frac{-2ca}{(1+cx)^2}[1-2x-cx^2] - \frac{2a}{(1+cx)}$ then $\frac{\partial^2 f}{\partial x^2}(0,1) = -2c-2 \neq 0$.

At $a=1$ the stable fixed point $\tilde{x}_1 = 0$ meet with another fixed point $\tilde{x}_2 = \frac{a-1}{a+c}$. Beyond $a=1$ the first branch $x=0$ becomes unstable and the other branch $x = \frac{a-1}{a+c}$ is asymptotically stable. In other words, exchange of stability occurs at $a=1$. □

In the other case, when $a = \frac{(c+3)+\sqrt{(c+3)^2+4c}}{2}$, $f'(\tilde{x}_2) = f'(\frac{a-1}{a+c}) = -1$, hence the fixed point $\tilde{x}_2$, undergoes a period-doubling bifurcation into 2-period cycle. To find the two cycle, we find $f^2$, and solve the equation $f^2(x) = x$. Now

$$f^2(x) = f(f(x)) = a\left[\frac{ax(1-x)}{(1+cx)} - \left(\frac{ax(1-x)}{1+cx}\right)^2\right]\frac{1}{1+\frac{cax(1-x)}{1+cx}}$$

(7.14)

$$= \frac{a^2x(1-x)\left[(1+cx)-ax(1-x)\right]}{(1+cx)\left[(1+cx)+acx(1-x)\right]}$$

And

$$f^2(x) - x = \frac{a^2x(1-x)\left[(1+cx)-ax(1-x)\right]}{(1+cx)\left[(1+cx)+acx(1-x)\right]} \tag{7.15}$$

$$- \frac{x(1+cx)\left[(1+cx)+acx(1-x)\right]}{(1+cx)\left[(1+cx)+acx(1-x)\right]} = 0 \tag{7.16}$$

This is true if and only if

(7.17)

$$a^2x(1-x)\left[(1+cx)-ax(1-x)\right] - x(1+cx)\left[(1+cx)+acx(1-x)\right] = 0$$

But since $f^2(x)-x=0$ has $\tilde{x}_1 = 0$ and $\tilde{x}_2 = \frac{a-1}{a+c}$, as roots. We need to factor out $\tilde{x}_1$ and $\tilde{x}_2$, to do this we divide the left hand side of equation (7.17) by $x^2 - \left(\frac{a-1}{a+c}\right)x$, to obtain

$$Q(x) = (a^3-c^2a)x^2 + (c^2-a^2)(1+a)x + (1+a)(c+a) \tag{7.18}$$

Hence to find the two cycle we need to solve the quadratic equation

$$(a^3-c^2a)x^2 + (c^2-a^2)(1+a)x + (1+a)(c+a) = 0 \tag{7.19}$$

And since $a + c > 0$, equation (7.19) is equivalent to

$$a(a-c)x^2 - (a-c)(1+a)x + (1+a) = 0 \tag{7.20}$$

We find the two-cycle $\{\hat{x}_1, \hat{x}_2\}$ where $a \neq c$, this is true since $a \geq 1$ and $c \in (0,1)$

$$\hat{x}_1 = \frac{(a-c)(1+a) + \sqrt{(a-c)^2(1+a)^2 - 4a(a-c)(a+1)}}{2a(a-c)}$$

$$\hat{x}_2 = \frac{(a-c)(1+a) - \sqrt{(a-c)^2(1+a)^2 - 4a(a-c)(a+1)}}{2a(a-c)}$$

To check the stability of this cycle, we use the fact that the 2-period cycle is asymptotically stable if and only if $|\frac{d}{dx} f^2(\hat{x})| < 1$.By using chain rule, we can show that $\frac{d}{dx} f^2(\hat{x}) = f'(\hat{x})f'(f(\hat{x}))$. Hence to check the stability of the two cycle we apply the following condition

$$|f'(\hat{x}_1)f'(\hat{x}_2)| < 1 \tag{7.21}$$

Now substitute $\hat{x}_1$ and $\hat{x}_2$ in $f'(x)$. So we have the following two inequalities

$$-1 < \frac{a^2}{(1+c\hat{x}_1)^2(1+c\hat{x}_2)^2}\left[(1-2\hat{x}_1 - c\hat{x}_1^2)(1-2\hat{x}_2 - c\hat{x}_2^2)\right] < 1 \tag{7.22}$$

The relation

$$-1 < \frac{a^2}{(1+c\hat{x}_1)^2(1+c\hat{x}_2)^2}\left[(1-2\hat{x}_1 - c\hat{x}_1^2)(1-2\hat{x}_2 - c\hat{x}_2^2)\right] \tag{7.23}$$

leads to the following inequality

$$a^2(1-2\hat{x}_1 - c\hat{x}_1^2)(1-2\hat{x}_2 - c\hat{x}_2^2) + (1+c\hat{x}_1)^2(1+c\hat{x}_2)^2 > 0$$

consequently

$$\begin{aligned} &a^2(1 - \hat{x}_2 - c\hat{x}_2^2 - 2\hat{x}_1 + 4\hat{x}_1\hat{x}_2 + 2c\hat{x}_1\hat{x}_2^2 - c\hat{x}_1^2 + 2c\hat{x}_1^2\hat{x}_2 + c^2\hat{x}_1^2\hat{x}_2^2) \\ &+(1 + 2c\hat{x}_2 + c^2\hat{x}_2^2 + 2c\hat{x}_1 + 4c^2\hat{x}_1\hat{x}_2 + 2c^3\hat{x}_1\hat{x}_2^2 + c^2\hat{x}_1^2 + 2c^3\hat{x}_1^2\hat{x}_2 + c^4\hat{x}_1^2\hat{x}_2^2) > 0 \end{aligned}$$

thus

$$\begin{aligned} &(a^2+1) \\ &+ (2c - 2a^2)\hat{x}_2 + (c^2 - a^2c)\hat{x}_2^2 + (2c - 2a^2)\hat{x}_1 + (4a^2 + 4c^2)\hat{x}_1\hat{x}_2 \\ &(2c^3 + 2ca^2)\hat{x}_1\hat{x}_2^2 + (c^2 - ca^2)\hat{x}_1^2 + (2ca^2 + 2c^3)\hat{x}_1^2\hat{x}_2 + (c^2a^2 + c^4)\hat{x}_1^2\hat{x}_2^2 > 0 \end{aligned}$$

this is equivalent to

(7.24)
$$(a^2+1)+(2c-2a^2)(\hat{x}_1+\hat{x}_2)+(c^2-a^2c)(\hat{x}_1^2+\hat{x}_2^2)+(2a^2c+2c^3)\hat{x}_1\hat{x}_2(\hat{x}_1+\hat{x}_2)$$
$$+(a^2+c^2)\hat{x}_1\hat{x}_2(4+c^2\hat{x}_1\hat{x}_2)>0$$

Substitute the value of $\hat{x}_1$ and $\hat{x}_2$, and to make our calculation easier observe that

$$\begin{aligned}\hat{x}_1+\hat{x}_2 &= \frac{1+a}{a}\\ \hat{x}_1^2+\hat{x}_2^2 &= \frac{(a-c)(1+a)^2-2a(a+1)}{a^2(a-c)}\\ \hat{x}_1\hat{x}_2 &= \frac{a+1}{a(a-c)}\end{aligned}$$

And we have the following inequality

$$(a^2+1)+(2c-2a^2)\left(\frac{1+a}{a}\right)+(c^2-a^2c)\left(\frac{(a-c)(1+a)^2-2a(a+1)}{a^2(a-c)}\right)$$
$$+(2a^2c+2c^3)\left(\frac{(1+a)^2}{a^2(a-c)}\right)+(a^2+c^2)\left(\frac{(a+1)}{a(a-c)}\right)\left(4+c^2\left(\frac{a+1}{a(a-c)}\right)\right)>0$$

Which is equivelant to

$$\begin{aligned}&(a^2+1)+\frac{(a+1)}{a}\left[2c-2a^2-2\frac{(c^2-a^2c)}{(a-c)}+4\frac{(a^2+c^2)}{(a-c)}\right]\\ &+\frac{(1+a)^2}{a^2}\left[(c^2-a^2c)+2\frac{ca^2+c^3}{(a-c)}+\frac{c^2(a^2+c^2)}{(a-c)^2}\right]>0\end{aligned}$$

Thus

$$(a^2+1)+\frac{(a+1)}{a^2(a-c)^2}\left[a(2c-2a^2)(a-c)^2-2a(a-c)(c^2-a^2c)+4a(a^2+c^2)(a-c)\right]$$
$$+\frac{(a+1)^2}{a^2(a-c)^2}\left[(c^2-a^2c)(a-c)^2+(2ca^2+2c^3)(a-c)+c^2(a^2+c^2)\right]>0$$

Hence

(7.25)
$$(-1-c)a^6+(2+4c+2c^2)a^5+(-c^3+c^2+7c+5)a^4+(-2c-4c^2-2c^3)a^3$$
$$+(-c^2-c^3)a^2>0$$

Now factor out (1+$c$) from inequality (7.25) to obtain the following inequality

$$-a^4 + 2(c+1)a^3 + (-c^2 + 2c + 5)a^2 - 2c(c+1)a - c^2 > 0 \tag{7.26}$$

However the relation

$$\frac{a^2}{(1+c\hat{x_1})^2(1+c\hat{x_2})^2}\left[(1 - 2\hat{x_1} - c\hat{x_1}^2)(1 - 2\hat{x_2} - c\hat{x_2}^2)\right] < 1 \tag{7.27}$$

leads to the inequality

(7.28)

$$(a^2 - 1) - (2a^2 + 2c)(\hat{x}_1 + \hat{x}_2) - (ca^2 + c^2)(\hat{x}_1^2 + \hat{x}_2^2) + (2a^2c - 2c^3)\hat{x}_1\hat{x}_2(\hat{x}_1 + \hat{x}_2)$$
$$+(a^2 - c^2)(4 + c^2\hat{x}_1\hat{x}_2)\hat{x}_1\hat{x}_2 < 0$$

Substitute the value of $\hat{x}_1$ and $\hat{x}_2$, and we have the following inequality

$$(a^2 - 1) - (2a^2 + 2c)\left(\frac{1+a}{a}\right) - (ca^2 + c^2)\left(\frac{(a-c)(1+a)^2 - 2a * (1+a)}{a^2(a-c)}\right)$$
$$+ (2a^2c - 2c^3)\left(\frac{(a+1)^2}{a^2(a-c)}\right) + (a^2 - c^2)\left(4 + c^2\left(\frac{a+1}{a(a-c)}\right)\right)\left(\frac{a+1}{a(a-c)}\right) < 0$$

Thus

$$(a^2 - 1) + \frac{1+a}{a^2(a-c)}\left(a(a-c)(-2a^2 - 2c) + a(2ca^2 + 2c^2) + 4a(a^2 - c^2)\right)$$
$$\frac{(1+a)^2}{a^2(a-c)}\left((-ca^2 - c^2)(a-c) + 2a^2c - 2c^3 + c^2(a+c)\right) < 0$$

So we have

$$(-1-c)a^5 + (c^2 + 3c + 2)a^4 + (2c^2 + 5c + 3)a^3 + (c^2 + c)a^2 < 0 \tag{7.29}$$

Which is equivalent to

$$-a^3 + (c+2)a^2 + (2c+3)a + c < 0 \tag{7.30}$$

Hence when the model (7.1) satisfies (7.26) and (7.30) the 2-period cycle is asymptotically stable. Moreover when $a$ and $c$ passe the curve

$$-a^3 + (c+2)a^2 + (2c+3)a + c = 0 \tag{7.31}$$

then the 2-period cycle undergoes a saddle- node bifurcation. And when $a$ and $c$ passe the curve

$$-a^4 + 2(c+1)a^3 + (-c^2 + 2c + 5)a^2 + -2c(c+1)a - c^2a = 0 \tag{7.32}$$

the system undergoes a period-doubling bifurcation .

## 7.4 Numerical examples

In this section we will take some fixed values of the parameter $c$ in model (7.1), to view four previous result.

**Example 7.4.1.** Now in model (7.1), we will take $c = 0$, and substitute it in all previous results. The fixed points will be $\tilde{x}_1 = 0$ and $\tilde{x}_2 = \frac{a-1}{a}$. For $\tilde{x}_1 = 0$, observe that $f'(0) = a$, hence we conclude that $\tilde{x}_1 = 0$ is asymptotically stable if $0 < a < 1$, when $a = 1$ the fixed point is non-hyperbolic, and also $f''(0) = -2$, so the fixed point $\tilde{x}_1 = 0$ is unstable. The fixed point $\tilde{x}_2 = \frac{a-1}{a}$ where $a > 1$, as in the results which we proved, it is asymptotically stable when $1 < a < 3$.
Note that an exchange of stability occurs at $a = 1$ between $\tilde{x}_1 = 0$ and $\tilde{x}_2 = \frac{a-1}{a}$. Hence at the point $(0, 1)$ the system undergoes a transcritical bifurcation.
Moreover when $a = 3$, we have $f'(\tilde{x}_2) = f'(\frac{a-1}{a}) = -1$, therefore $\tilde{x}_2 = \frac{a-1}{a}$ is non-hyperbolic fixed point. To check stability of $\tilde{x}_2$, we need to compute the Schwarzian derivative. We observe that

$$\mathbf{Sf}(\tilde{x}_2) = \mathbf{Sf}\left(\frac{2}{3}\right) = -f'''\left(\frac{2}{3}\right) - \frac{3}{2}\left[f''\left(\frac{2}{3}\right)\right]^2 = -54 < 0$$

Hence the fixed point $\tilde{x}_2 = \frac{2}{3}$ is asymptotically stable. Now at the point $(\tilde{x}_2, a) = (\frac{2}{3}, 3)$ the system undergoes a period-doubling bifurcation, and we have

$$f^2(x) = a^2 x(1-x)\left[1 - ax(1-x)\right] \tag{7.33}$$

We find that the 2-periodic cycles are

$$\begin{aligned}\hat{x}_1 &= \frac{(1+a) + \sqrt{(1+a)^2 - 4(a+1)}}{2a} = \frac{(1+a) + \sqrt{(a-3)(a+1)}}{2a}\\ \hat{x}_2 &= \frac{(1+a) - \sqrt{(1+a)^2 - 4(a+1)}}{2a} = \frac{(1+a) - \sqrt{(a-3)(a+1)}}{2a}\end{aligned} \tag{7.34}$$

Clearly the 2-period cycle $\{\hat{x}_1, \hat{x}_2\}$ exists only if $a > 3$. To know whether this 2-period cycle is asymptotically stable we substitute $c = 0$ in the inequalities (7.26) and (7.30), so we have

$$-a^2 + 2a + 5 > 0 \tag{7.35}$$

and

$$-a^2 + 2a + 3 < 0 \tag{7.36}$$

Solving inequality (7.35) leads to

$$1 - \sqrt{6} < a < 1 + \sqrt{6} \tag{7.37}$$

and also the solution of inequality (7.36) leads to

$$a > 3 \quad or \quad a < -1 \tag{7.38}$$

Hence the last solutions of the two inequalities yields the cycle $\{\hat{x}_1, \hat{x}_2\}$ is asymptotically stable if

$$3 < a < 1 + \sqrt{6} \tag{7.39}$$

**Remark 7.4.2.** We can see that if we take $c = 0$ in the results related to model (7.1), we have the known results about logistic model, hence model (7.1) is a generalization of the logistic model (see figure 7.1).

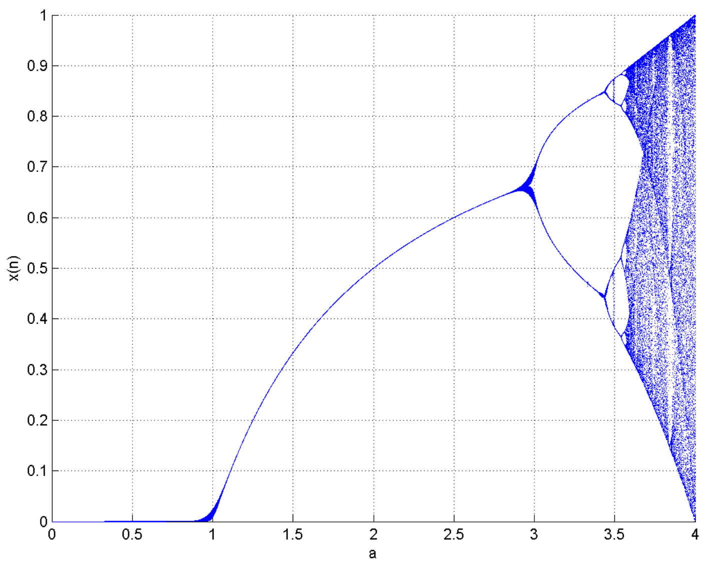



Figure 7.1: The bifurcation diagram of model (7.1), when $c = 0$.
The bifurcation diagram of model (7.1) when $c = 0$ is the same as the bifurcation diagram of logistic map

**Example 7.4.3.** If we take $c = 0.3$ then, model (7.1) has two fixed points $\tilde{x}_1 = 0$ and $\tilde{x}_2 = \frac{a-1}{a+0.3}$ . Depending on our results we have the fixed point $\tilde{x}_1 = 0$ is asymptotically stable when $0 < a < 1$. An exchange of stability between two fixed points happens when $a = 1$, this causes a transcritical bifurcation. By theorem 7.2.2 and by using some numerical calculation we note that the fixed point $\frac{a-1}{a+0.3}$ is asymptotically stable when $1 < a < 3.3885$. When $a = 3.3885$ the model (7.1) undergoes a period doubling bifurcation (see figure 7.2).

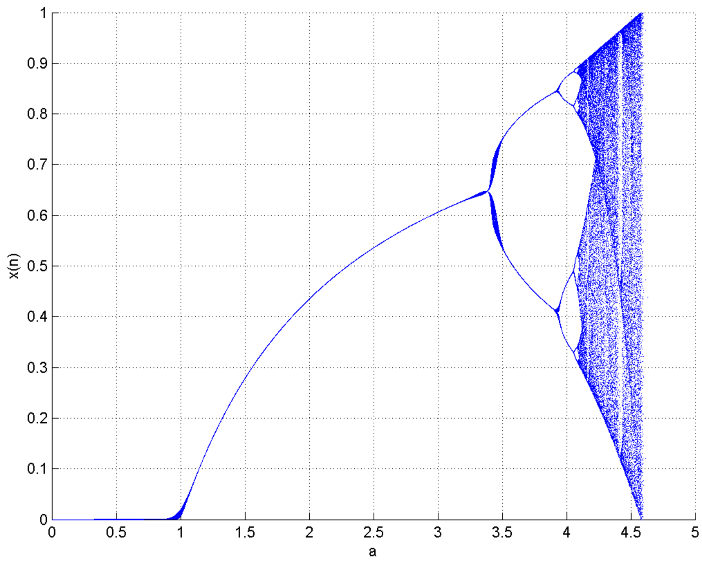



Figure 7.2: The bifurcation diagram of model (7.1), when $c = 0.3$.

# Appendix A

# The Matlab 7.12 codes

## A.1 The cobweb diagram of logistic map

```
alpha=input('alpha=')
x0=input('x0=')
N=20;      x(1) = x0;
for ic=1:N
x(ic+1) = alpha*x(ic)*(1-x(ic));
end
```

plot the map function and the line y=x

```
clf;
t = 0:0.01:1;

plot(t,alpha*(t.*(1-t))); hold on;
xlabel('x');
ylabel('f(x)');

axis('square'); axis([0 1 0 1]);
set(gca,'XTick',(0:0.1:1),'YTick',(0:0.1:1))
grid on;

fplot('1*y',[0 1],'r');
```

STEP 3: PLOT COBWEB

```
line([x(1) x(1)],[0 x(2)],'Color','g');
plot(x(1), x(1),'ko');pause
for ic=1:N-1
line([x(ic) x(ic+1)],[x(ic+1) x(ic+1)],'Color','g');pause
plot(x(ic+1), x(ic+1),'ko');pause
line([x(ic+1) x(ic+1)],[x(ic+1) x(ic+2)],'Color','g');pause
end
line([x(N) x(N+1)],[x(N+1) x(N+1)],'Color','g')
```

## A.2 The bifurcation diagram of logistic map

```
close all
clear all
avalues=0:0.0001: 4;
N=100; a=avalues; x=0.1;
X=zeros(N,length(a));
for n=1:.3*N
x=a.*x.*(1-x);
X(n,:)=x;
End
figure (9), hold on
for n=.3*N:N
x=a.*x.*(1-x);
X(n,:)=x;
plot(a,x,'.','MarkerSize',0.01)
xlabel('u');
ylabel('f(x)');

axis ([0 4 0 1])
end
hold off
```

## A.3 The bifurcation diagram of predator-prey model

```
clc; clear all;
n = input('number of iterations = ');
a=2;
r=0:0.001:3;
N(:,1)=zeros(size(r,2),1);
P(:,1)=zeros(size(r,2),1);
N(:,1)=0.3;
P(:,1)=0.2
for k=1 : size(r,2)
for i=1:130
N(k,i+1)=N(k,i)+r(k)*N(k,i)*(1-N(k,i))-a*N(k,i)*P(k,i);
P(k,i+1)=P(k,i)+a*P(k,i)*(N(k,i)-P(k,i));
end
end
s=r(1,1)*ones(1,51);
m=P(1,80:130);
for k=2 : size(r,2)
s=[s,r(1,k)*ones(1,51)];
m=[m,P(k,80:130)];
end
plot(s,m,'.k');
xlabel('r(growth rate)');
ylabel('predator densities');

grid;
zoom;
```

## A.4 The bifurcation diagram of model (7.1)

```
close all
clear all
avalues=0:0.001: 5;
c=0.3;
N=100; a=avalues; x=0.1;
```

```
X=zeros(N,length(a));
for n=1:.3*N
x=a.*x.*(1-x)./(1+c.*x);
X(n,:)=x;
end
figure (9), hold on
for n=.3*N:N
x=a.*x.*(1-x)./(1+c.*x);
x(n,:)=x;
plot(a,x,'.','MarkerSize',0.01)
axis ([0 5 0 1])
xlabel('a'),ylabel('x(n)'),grid on

end
hold off
```

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