On Some Types of Supplemented Modules

Prepared By

Khitam Salameh

Master in Mathematics, Birzeit University, 2013

Supervised By

Dr.Khaled Altakhman

Mathematics Department, Birzeit University

Program of Mathematics
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Khitam Salameh

This thesis was defended successfully on May 16th, 2013

And approved by:

Committee Members	Signature

1-Dr.Khaled Al-Takhman   Head Of Committee ....
2-Dr.Hasan Yousef   Internal Examiner ....
3-Professor Mohammad Saleh Internal Examiner ....

Birzeit University
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Declaration

I certify that this thesis, submitted for the degree of Master of Mathematics to the Department of Mathematics in Birzeit University, is of my own research except where otherwise acknowledged, and that this thesis(or any part of it) has not been submitted for a higher degree to any other university or institution.

Khitam Salameh   Signature ....
May 16th, 2013

الملخص

الهدف الرئيس لهذه الرسالة هو دراسة بعض أنواع الموديلات بما فيها: الموديلات الداعمة، الموديلات الداعمة الضعيفة، الموديلات الداعمة لبعض موديلاتها الجزئية، الموديلات الداعمة لكل نوع بالمضافات المباشرة .

الدراسة تقدم مميزات الأنواع المختلفة وعدد من الامثله وتناقش الدراسة بالتفصيل انغلاق كل نوع من الأنواع الأنفة بالذكر بالنسبة للموديلات الجزئية ولقسمه الموديلات والاقترانات الموديوليه.

ABSTRACT

The main purpose of this project is to study some classes of modules namely supplemented, weakly supplemented, $\oplus$-supplemented, cofinitely supplemented, cofinitely weak supplemented and $\oplus$-cofinitely supplemented modules. Characterization, examples and the closeness of each class of these kinds under submodules, direct summands, quotients, small covers, and homomorphic images will, in detail, be considered.

Keywords: supplemented, weakly supplemented, cofinitely(weak) supplemented, $\oplus$-(cofinitely) supplemented.

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Chapter 1

INTRODUCTION

It is well known that supplements or addition complements of a submodule of a given module, need not exist
[8]. As an attempt to study which module provides a supplement of each of its submodules, the notion of the supplemented modules arises. Wisbauer calls a module M supplemented if every submodule has a supplement in M. In [14], the basic properties of supplemented modules are given.

In a series of papers, Zoschinger has obtained detailed information about supplemented and related modules. Supplemented modules are also discussed in [12].

The class of weakly supplemented modules was defined and its properties studied in [11]

In [1], the authors defined cofinitely supplemented modules and obtained a characterization and some of the properties of this new class.

In 2003 and through their paper, R. Alizade and E.Büyükaşik, defined the class of cofinitely weak supplemented modules, characterization, and properties were obtained [2].

Zoschinger called a module M ⊕-supplemented if every submodule of M has a supplement that is a direct summand. The class of these modules was studied by many authors [6, 7, 9, 12].

[4] and [10] independently, called a module M ⊕-cofinitely supplemented, if every cofinite submodule of M has a supplement that is a direct summand. This notion was also studied in [13, 15].

In this thesis, we study some classes of modules including supplemented modules, weakly supplemented, ⊕-supplemented, cofinitely supplemented, cofinitely weak supplemented and ⊕-cofinitely supplemented. Characterization and closeness of each class under submodules, direct summands, quotients, small covers, and homomorphic images will be considered.

Chapter 2

Preliminaries

In this chapter we will give the basics about modules. We will give the definitions and results which we will use in this thesis. We begin with modules and submodules.

2.1 Modules, Submodules, Sum and Intersection of Submodules and Direct Summands

The notion of an R-module can be considered as a generalization of the notion of a vector space where scalars are allowed to be taken from a ring R instead of a field.

Although modules are in fact considered as a pair (M, λ) where M is an additive abelian group and λ is a ring homomorphism from R to the ring of endomorphisms of M, we prefer to begin with more common and simple definition. All rings considered in this work are with unity.

Definition 2.1.1. *Let R be a ring. A right R-module is an additive abelian group M together with a mapping $M \times R \to M$, which we call a scalar multiplication, denoted by*

$$(m, r) \mapsto mr$$

such that the following properties hold: for all $m, n \in M$ and $r, s \in R$;

1. $(m + n)r = mr + nr,$

2. $m(r+s) = mr + ms$,

3. $m(rs) = (mr)s$.

If, in addition for every $m \in M$ we have $m.1 = m$, M is called a unitary right R-module. If M is a right module, we denote it by M_R.

Left R-module are defined in an analogous way. For commutative rings, the two notions of right and left R-module coincide. In our work all modules will be unitary right R-modules.

Example 2.1.2. *Here is a list of some elementary examples of modules:*

1. Every vector space over a field F is an F-module.

2. Every abelian group is a $\mathbf{Z}$-module, where $\mathbf{Z}$ is the set of integers.

3. Every ring R is a module over itself.

In studying mathematical structures, the substructures generally play an important role.

Definition 2.1.3. *Let M be an R-module. A subset N of M is called a submodule of M, notationally $N \subseteq M$ if N is a (right) R-module with respect to the restriction of the addition and scalar multiplication of M to N.*

The reader must be aware because in our work we use the notion $N \subseteq M$ for a submodule relationship, not just for a set-theoretic inclusion, unless otherwise stated. Further we denote

$$N \subsetneqq M \text{ if } N \text{ is a proper submodule of} M$$

Lemma 2.1.4. *The Submodule Criterion. ([8], Lemma 2.2.2) Let M be an R-module. If N is a subset of M and $N \neq \emptyset$, then the following are equivalent:*

1. $N \subseteq M$.

2. *N is a subgroup of the additive group M and for all $n \in N$ and all $r \in R$ we have $nr \in N$.*

3. *For all $n_1, n_2 \in N, n_1 + n_2 \in N$ (with respect to addition in M) and for all $n \in N$ and all $r \in R$, we have $nr \in N$.*

Example 2.1.5. *Here are some elementary examples of submodules:*

1. *Every module M posses the trivial submodules 0 and M, where 0 is the submodule which contains only the zero element of M.*

2. *Let M be an arbitrary module and let $m \in M$ then*

$$mR := \{mr | r \in R\}$$

is a submodule of M called the submodule generated by m.

3. *If M_K is a vector space over the field K, then the submodules are so called (linear) subspaces.*

4. *Submodules of Z as a Z-module are $nZ, n \in Z$.*

We will give now some important definitions.

Definition 2.1.6. *Let M be an R-module.*

1. *M is called cyclic if there exists $m \in M$ such that $M = mR$.*

2. *M is called simple if $M \neq 0$ and 0 and M are the only submodules of M.*

3. *A submodule $L \subseteq M$ is called a maximal submodule of M, if $L \subsetneq M$ and for every $N \subseteq M$ such that $L \subsetneq N, N = M$.*

The following Lemma completely characterize simple modules.

Lemma 2.1.7. *Characterization of simple modules.[[8], Lemma 2.2.4]. An R-module M is simple if and only if $M \neq 0$ and for every $0 \neq m \in M, mR = M$.*

The following Lemma gives a characterization of maximal submodules.

Lemma 2.1.8 ([8], Lemma 2.3.10)**.** *Let $L \subsetneq M$. Then the following are equivalent:*

1. *L is a maximal submodule of M.*

2. *For every $m \in M$ such that $m \notin L$, $mR + L = M$.*

Example 2.1.9. 1. *The maximal submodules of Z_Z are exactly the prime ideals pZ, $p =$prime integer.*

2. *Q_Z has no maximal submodules.*

Now we turn our attention to the operations on submodules, sum and intersection.

Lemma 2.1.10 ([8], Lemma 2.3.1). *Let Γ be a set of submodules of a module M, then*

$$\bigcap_{N \in \Gamma} N = \{m \in M | m \in N \text{ for all } N \in \Gamma\}$$

is a submodule of M.

Corollary 2.1.11 ([8], §2.3). *$\bigcap_{N \in \Gamma} N$ is the largest submodule of M which is contained in all $N \in \Gamma$.*

Lemma 2.1.12 ([8], Lemma 2.3.2). *Let X be a subset of the R-module M. Then*

$$L = \begin{cases} \left\{\sum_{j=1}^{n} x_j r_j | x_j \in X, r_j \in R, \text{ and } n \in \mathbb{N}\right\}, & \text{if } X \neq \emptyset \\ \\ 0, & \text{if } X = \emptyset \end{cases}$$

is a submodule of M.

The module defined in the previous Lemma is called *the submodule of M generated by X* . This submodule, which, if $X \neq \emptyset$, is characterized as the smallest submodule of M that contains X.

We will now define the *generating set of a module, finitely generated modules, and cyclic modules*.

Definition 2.1.13. *Let M be an R-module.*

1. *A nonempty subset X of M is called a* generating set *of M if*

$$M = \left\{ \sum_{j=1}^{n} x_j r_j \mid x_j \in X, r_j \in R, n \in \mathbb{N} \right\}$$

2. *M is called* finitely generated *if M has a finite generating set X. Moreover if $X = \{x_1, x_2, ..., x_k\}$, we write $M = x_1 R + x_2 R + ... + x_k R$.*

3. *M is called* cyclic *if there exists a single generating element $x \in M$ such that $M = xR$*

Example 2.1.14. *Here are some examples to clarify the previous definitions:*

1. *If R is a ring, then $\{1\}$ is a generating set of R_R.*

2. *Q_Z is not finitely generated.*

Finitely generated modules have the following interesting property.

Corollary 2.1.15 ([8], Corollary 2.3.12)**.** *Every finitely generated module $M \neq 0$ has a maximal submodule.*

We are now ready to define the sum of submodules.

Definition 2.1.16. *Let $\Gamma = \{N_i \mid i \in I\}$ be a set of submodules $N_i \subset M$, then*

$$\sum_{i \in I} N_i = \begin{cases} \left\{ \sum_{j \in J} n_j \mid n_j \in N_j, J \subset I \text{ and } J \text{ is finite} \right\}, & \text{if } \Gamma \neq \emptyset, \\ \\ 0, & \text{if } \Gamma = \emptyset \end{cases}$$

is called the sum of submodules *$\{N_i \mid i \in I\}$.*

While $\bigcap_{N_i \in \Gamma} N_i$ is the largest submodule of M contained in all $N_i \in \Gamma$, $\sum_{N_i \in \Gamma} N_i$ is the smallest submodule of M which contains all $N_i \in \Gamma$.

These constructions posses some important properties, first of which is the Modular Law.

Lemma 2.1.17. *Modular Law([8], Lemma 2.3.15) If N, L, K are submodules of M, $L \subseteq N$, then*

$$N \cap (L + K) = L + (N \cap K).$$

Lemma 2.1.18 ([14], §41.2). *Let N, K, L be submodules of an R-module M, then*

$$N \cap (L + K) \subset L \cap (N + K) + K \cap (L + N).$$

Proof. For $n \in N$ and $n \in L + K, n = l + k$. Rearranging the last equality, we have $l = n - k \in L \cap (N + K)$ and similarly $k = n - l \in K \cap (N + L)$. So for $n = l + k, n \in L \cap (N + K) + K \cap (L + N)$ □

Finitely generated modules written as sum of submodules, have this fascinating characterization.

Theorem 2.1.19 ([8], Theorem 2.3.13)). *An R-module M is finitely generated if and only if there is in every set $\{N_i | i \in I\}$ of submodules $N_i \subset M$ with*

$$\sum_{i \in I} N_i = M$$

a finite subset $\{N_i | i \in I_0\}$ $(i.e. I_0 \subset I$ and I_0 is finite) such that

$$\sum_{i \in I_0} N_i = M.$$

Now we define the internal direct sum.

Definition 2.1.20. *An R-module M is called the* internal direct sum *of the set $\{N_i | i \in I\}$ of submodules $N_i \subset M$ in symbols, $M = \bigoplus_{i \in I} N_i$, if*

1. *$M = \sum_{i \in I} N_i$, and,*

2. *For every $j \in I, N_j \cap (\sum_{i \neq j} N_i) = 0$*

In the case of a finite index set, say $I = \{1, ..., k\}$, M is written as $M = N_1 \oplus N_2 \oplus ... \oplus N_k$.
The previous definition is equivalent to: For every $m \in M$ the representation $m = \sum_{i \in I_0} n_i$ with $n_i \in N_i, I_0$ a finite subset of I, is unique.
Also, for every $j \in I$, we have $M = N_j \oplus \sum_{i \neq j} N_i$

Strongly related to the notion of internal direct sum, is the notion of *direct summand*, which we introduce now.

Definition 2.1.21. *A submodule* $N \subseteq M$ *is called a* direct summand of M *if there exists* $L \subseteq M$ *such that* $M = N \oplus L$

From this definition one can deduce:

1. 0 and M are trivial direct summands of M.

2. In Z_Z, 0 and Z itself are the only direct summands.

2.2 Factor Modules and Module Homomorphism

In this section we will introduce two concepts which will play an important role in our research.

Definition 2.2.1. *Let* M *be an* R-module and N *be a submodule of* M. *Then the set of cosets*

$$M/N = \{m + N | m \in M\}$$

is a right R-module if we define the addition and scalar multiplication as

$$(m_1 + N) + (m_2 + N) = (m_1 + m_2) + N, (m + N)r = mr + N.$$

This new module is called the factor module of M modulo N.

The next Theorem characterize the submodules of factor modules.

Theorem 2.2.2. *Correspondence Theorem. ([3], proposition 2.9). Let* K *be a submodule of an* R-module M. *Then there is an isomorphism between the set of submodules of* M/K *and the set of submodules of* M *which contains* K. *That is, the submodules of* M/K *have the form* N/K *where* N *is a submodule of* M *which contains* K.

As a direct consequence of the previous Theorem, we present this corollary.

Corollary 2.2.3. *([3],proposition 2.9.) A factor module* M/K *is simple if and only if* K *is a maximal submodule of* M.

The Correspondence Theorem induces the following two important assertions ([8], §9.1):

(a) The maximal submodules of M/K are the factor modules N/K with N maximal in M and $K \subseteq N$.

(b) If $\{N_i | i \in I\}$ is a family of submodules of M, and for every $i \in I, K \subseteq N_i$, then we have:
$$\bigcap(N_i/K) = (\bigcap N_i)/K.$$

Now we define module homomorphism.

Definition 2.2.4. *Let M, N be R-modules. A function $f : M \rightarrow N$ is called an* R-module homomorphism *if, for all $m_1, m_2 \in M$ and for all $r \in R$,*

(i) $f(m_1 + m_2) = f(m_1) + f(m_2)$;

(ii) $f(m_1 r) = f(m_1)r$

Special names are given to homomorphism which satisfy certain properties. An onto homomorphism is called an *epimorphism*, and a one-to-one homomorphism is called a *monomorphism*. A one-to-one and onto module homomorphism is called an *isomorphism*. If there is an isomorphism between two modules M and N we say that M and N are *isomorphic* and denote it by $M \cong N$.

We will present now some important examples of homomorphisms:

1. The *inclusion*(monomorphism) of a submodule $K \subset M, i : K \rightarrow M$ defined via
$$i(k) = k \in M \quad (k \in K)$$

2. The *natural (canonical) epimorphism* π of a module M onto the factor module M/K where $K \subseteq M, \pi : M \rightarrow M/K$ defined via
$$\pi(m) = m + K$$

   The homomorphism π is used in the following to mean the canonical(natural) epimorphism but with changing notation for domain and codomain.

3. Let K be a direct summand of M, so $M = K \oplus L$ for some $L \subset M$, then
$$P_K(k + l) = k \qquad (k \in K, l \in L)$$

defines an epimorphism

$$P_K : M \to K$$

called the *projection of M on K along L* , moreover, $\mathrm{Ker}P_K = L$
([3], proposition 5.4)

Let $f : M \to N$ be a homomorphism. For $K \subset M, L \subset N$, we define

$$
\begin{array}{lll}
\text{The image of } K = & f(K) = & \{f(k)|k \in K\} \\
\text{The inverse image of } L = & f^{-1}(L) = & \{m \in M|f(m) \in L\}
\end{array}
$$

These are readily seen to be submodules of N and M, respectively. In particular we have:

(a) $\mathrm{Im}\, f = f(M)$ is a submodule of N, and for every $K \subseteq M, f(K) \subseteq \mathrm{Im}f$.

(b) $\mathrm{Ker}f = f^{-1}(0)$ is a submodule of M, and for every $L \subseteq N$, $\mathrm{Ker}f \subseteq f^{-1}(L)$.

Still more is given by the following Lemma.

Lemma 2.2.5 ([8], Lemma 3.1.8). *Let $f : M \to N$ be a homomorphism. Then we have*

1. $K \subseteq M \Rightarrow f^{-1}(f(K)) = K + \mathrm{Ker}f$.

2. $L \subseteq N \Rightarrow f(f^{-1}(L)) = L \cap \mathrm{Im}f$.

3. *Let also $g : N \to T$ be a homomorphism. Then*

$$\mathrm{Ker}(gf) = f^{-1}(\mathrm{Ker}\, g) \text{ and } \mathrm{Im}(gf) = g(\mathrm{Im}\, f).$$

Image and inverse image of sum and intersection of submodules is the content of the following Lemma.

Lemma 2.2.6 ([8], Lemma 3.1.10). *Let $f : M \to N$ be a given homomorphism, with a set $\{M_i|i \in I\}$ of submodules of M and a set $\{N_j|j \in J\}$ of submodules of N. Then we have*

(a) $f(\sum_{i \in I} M_i) = \sum_{i \in I} f(M_i), \qquad f^{-1}(\bigcap_{j \in J} N_j) = \bigcap_{j \in J} f^{-1}(N_j).$

(b) $f^{-1}(\sum_{j \in J} N_j) \supseteq \sum_{j \in J} f^{-1}(N_j), \qquad f(\bigcap_{i \in I} M_i) \subseteq \bigcap_{i \in I} f(M_i).$

In our work yet a special case of the previous Lemma is also needed, which is given as an exercise in ([8]). Here we present the exercise and prove it.

Lemma 2.2.7. *([8], Exercise(3) §3)*

(a) *If $f : M \to N$ is a homomorphism such that $K \subseteq M$ and $L \subseteq N$, then*
$$f^{-1}(f(K) + L) = K + f^{-1}(L).$$

(b) *If $f : M \to N$ is a homomorphism such that $K \subseteq M$ and $L \subseteq N$, then*
$$f(f^{-1}(L) \cap K) = L \cap f(K).$$

Proof. (a) Using Lemma 2.2.6 (a) we have
$$f(K + f^{-1}(L)) = f(K) + (L \cap \mathrm{Im} f) = \mathrm{Im} f \cap (f(K) + L) = f(f^{-1}(f(K) + L)).$$

Taking the inverse image of the two equal sets we have
$$f^{-1}(f(K + f^{-1}(L))) = K + f^{-1}(L) + \mathrm{Ker} f = K + f^{-1}(L).$$

On the other hand

$$f^{-1}(f(f^{-1}(f(K) + L))) = f^{-1}(f(K) + L) + \mathrm{Ker} f = f^{-1}(f(K) + L).$$

So
$$f^{-1}(f(K) + L) = K + f^{-1}(L).$$

(b) Using Lemma 2.2.6 (a) we have
$$f^{-1}(L \cap f(K)) = f^{-1}(L) \cap f^{-1}(f(K)) = f^{-1}(L) \cap (K + \mathrm{Ker} f) = (f^{-1}(L) \cap K) + \mathrm{Ker} f,$$

as $\mathrm{Ker} f \subseteq f^{-1}(L)$ and using the Modular Law. Now taking the image of the two equal sets we have
$$f(f^{-1}(L \cap f(K))) = L \cap f(K) \cap \mathrm{Im} f = L \cap f(K),$$

on the other hand
$$f(f^{-1}(L) \cap K + \mathrm{Ker} f) = f(f^{-1}(L) \cap K).$$

So we have
$$f(f^{-1}(L) \cap K) = L \cap f(K).$$

$\square$

Theorem 2.2.8. *Isomorphism Theorems ([3], corollary 3.7) Let M, N be R-modules.*

1. *If $f : M \to N$ is an epimorphism with $Kerf = K$, then there is a unique isomorphism*

$$\eta : M/K \to N$$

   such that

$$\eta(m + K) = f(m) \qquad for\ all\ m \in M.$$

2. *If K and L are submodules of M such that $K \subseteq L$ then*

$$(M/K)/(L/K) \cong M/L.$$

3. *If H and K are submodules of M, then*

$$(H + K)/K \cong H/(H \cap K).$$

   In particular: If K is a direct summand of M, i.e., $M = K \oplus H$ then, $M/K = (H + K)/K \cong H/(H \cap K) \cong H$.

Lemma 2.2.9. *Let $f : M \to N$ be a module homomorphism, and L be a generating set of M. Then*

1. *$f(L)$ is a generating set of Imf, and so*

2. *If M is finitely generated, then Imf is also finitely generated.*

Using the canonical epimorphism $\pi : M \to M/N$, and the previous Lemma(2), we easily see that ***factor modules of finitely generated modules are also finitely generated***.

Toward understanding of the factor module Q/Z, we need more definitions and results.

Let M be a Z-module. M is called a *torsion module(torsion group)* if, for every $a \in M$, there exists a nonzero $n \in N$ with $na = 0$.

M is called a *p-torsion module (p-group)*, for a prime number p, if, for every $a \in M$, there exists $k \in N$ with $p^k a = 0$.

The *torsion submodule of M* is defined as

$$t(M) = \{a \in M | na = 0 \text{ for some } n \in N\},$$

the *p-component of M* is

$$M_p = \{a \in M | p^k a = 0 \text{ for some } k \in N\}.$$

If $t(M) = 0$, then M is called *torsion free*. Recall that we use the notation $Z_n = Z/nZ$ for $n \in N$.

Torsion modules over Z have an important property as revealed by the following Theorem.

Theorem 2.2.10. *([14], §15.10).*

1. *Every torsion module M over Z is a direct sum of its p-components: $M = \bigoplus\{M_p | p \text{ a prime number}\}$.*

2. *The p-component of Q/Z is denoted by Z_{p^∞} (prüfer group) and*

$$Q/Z = \bigoplus\{Z_{p^\infty} | p \text{ a prime number}\}.$$

2.3 Radical of module, Small submodules and Small Homomorphism

In this section we will introduce some concepts of crucial role in our work, and study their properties.

Definition 2.3.1. *Let M be an R-module.*

1. *A submodule K of M is called* small *or* superfluous *, denoted by $K \ll M$, if for every submodule N of M, $K + N = M$ implies $N = M$.*

2. *An epimorphism $f : M \to N$ is called* small *if $Kerf \ll M$. In this case, M is called a* small cover *of N.*

Obviously $K \ll M$ if and only if the canonical map $\pi : M \to M/K$ is a small epimorphism.

Example 2.3.2. *1. Every finitely generated submodule of Q_Z is small in $Q_Z([8], \S5.1)$.*

2. *Every nontrivial submodule of Z_{p^∞} is small([3], $\S5$).*

3. *In Z_Z there is no nonzero small submodules.*

The following Lemma gives some important properties of small submodules and small epimorphisms.

Lemma 2.3.3. *[[14], $\S19.3$] Let K, L, N and M be R-modules.*

1. *If $f : M \to N$ and $g : N \to L$ are two epimorphisms, then gf is small if and only if f and g are small epimorphisms.*

2. *If $K \subset L \subseteq M$, then $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.*

3. *If $K_1, ..., K_n$ are small submodules of M, then $K_1 + ... + K_n$ is also small in M.*

4. *For $K \ll M$ and $f : M \to N$ we get $f(K) \ll N$. In particular, if $K \ll L \subset M$ then $K \ll M$ (Consider the inclusion map $i : L \to M$).*

5. *If $K \subseteq L \subseteq M$ and L is a direct summand in M, then $K \ll M$ if and only if $K \ll L$.*

The previous Lemma shows that the image of a small submodule is again small. Now we will show that the inverse image of a small submodule under a small epimorphism, is also small.

Lemma 2.3.4. *[[8]] Let $f : M \to N$ be a small epimorphism and $L \ll N$ then $f^{-1}(L) \ll M$.*

Proof. Assume $f^{-1}(L) + X = M$ for some submodule $X \subseteq M$. Now $N = f(f^{-1}(L)+X) = f(f^{-1}(L))+f(X)$. As f is an epimorphism, $f(f^{-1}(L)) = L$ $\Rightarrow L + f(X) = N \Rightarrow f(X) = N \Rightarrow f^{-1}(f(X)) = X + \mathrm{Ker}f = M$. But $\mathrm{Ker}f \ll M$ so $X = M$ hence $f^{-1}(L) \ll M$. $\qquad\qquad\square$

The following result will prove useful in our work.

Lemma 2.3.5 ([14], §19.6)**.** *Let K be a small submodule of an R-module M. Then M is finitely generated if and only if M/K is finitely generated.*

Now we define the *radical* of a module.

Definition 2.3.6. *Let M be an R-module. we define the* radical *of M as the intersection of all maximal submodules of M. We denote the radical of M by $Rad(M)$. If M has no maximal submodules we set $Rad(M) = M$.*

Now we list some basic properties of the radical.

Proposition 2.3.7 ([14], §21.5)**.** *For an R-module M, we have*

$$Rad(M) = \sum \{K \subseteq M | K \ll M\}$$

It follows from the definition:

1. $Rad\,(Z_Z) = 0$ since, we know, 0 is the only small submodule(ideal) in Z.

2. $Rad(Q_Z) = Q$, since for every $q \in Q, qZ$ is small in Q. This is equivalent to saying that Q has no maximal submodules.

Proposition 2.3.8 ([14], §21.6)**.** *Let M be an R-module.*

1. *For a homomorphism $f : M \rightarrow N$, we have*

   (i) $f(Rad(M)) \subseteq Rad(N)$,

   (ii) $Rad(M/Rad(M)) = 0$,

   (iii) $f(Rad(M)) = Rad(f(M)) \qquad$ if $Kerf \subseteq Rad(M)$.

2. *If $M = \bigoplus_I M_i$, then*

   (i) $Rad\,(M) = \bigoplus_I Rad(M_i)$ and

   (ii) $M/Rad(M) \cong \bigoplus_I M_i/Rad(M_i)$.

Using the radical of a module, the following Theorem present further characterization for finitely generated modules.

Theorem 2.3.9. *([8], Theorem 9.4.1). Let M be an R-module, then M is finitely generated if and only if we have:*

(a) *$Rad(M)$ is small in M, and*

(b) *$M/Rad(M)$ is finitely generated.*

Chapter 3

Supplemented Modules

In this chapter we will investigate the properties of supplemented modules, and give some examples on this class of modules. Throughout R will be a ring with unity, and all modules will be unitary right R-modules. We start with characteristics of a *supplement submodule*.

3.1 Supplements and Their Properties

Before giving the definitions, let us talk about the motivation for studying *supplemented modules.*

In module theory, decomposition of a module into direct sum of submodules, if possible, is a very important subject and many areas of module theory are related to this. In general, a submodule need not be a direct summand, and as an attempt to generalize the concept of direct summand, the notion of supplement submodules and related concepts arise in the literature.

Definition 3.1.1. *Let N be a submodule of an R-module M. A submodule $K \subseteq M$ is called a* supplement *or* addition complement *of N in M if K is minimal in the set of submodules $L \subseteq M$ with $N + L = M$. A submodule $K \subseteq M$ is called a* supplement*, if it is a supplement of some submodule of M.*

The following Lemma provides a criterion to check when a submodule is a supplement.

Lemma 3.1.2 ([12], Lemma 4.5)**.** *Let N be a submodule of the R-module M. A submodule K is a supplement of N in M if and only if $M = N + K$ and $N \cap K \ll K$.*

Proof. If K is a supplement of N and $X \subseteq K$ with $N \cap K + X = K$, then we have $M = N + K = N + (N \cap K) + X = N + X$, hence $X = K$ by the minimality of K. Thus $N \cap K \ll K$.

On the other hand, let $M = N + K$ and $N \cap K \ll K$. For $X \subseteq K$ with $X + N = M$, we have $K = K \cap N + X$ (modular law), thus $X = K$. Hence K is minimal in the desired sense. $\square$

Observe that every direct summand satisfy the criterion for a supplement as desired. For if N is a direct summand of M, say $M = N \oplus K$ for some $K \subseteq M, N \cap K = 0 \ll N$, which by the previous Lemma, means N is a supplement of K.

Properties of supplements are given by the following Lemma.

Lemma 3.1.3 ([14], §41.1)**.** *Let N, K be submodules of the R-module M. Assume K is a supplement of N in M. Then:*

1. *If $L + K = M$ for some $L \subseteq N$ then K is a supplement of L.*

2. *If M is finitely generated, then K is also finitely generated.*

3. *If N is a maximal submodule of M, then K is cyclic, and $N \cap K = Rad(K)$ is a (the unique) maximal submodule of K.*

4. *If $L \ll M$, then K is a supplement of $N + L$.*

5. *For $L \ll M$ we have $K \cap L \ll K$ and so $Rad(K) = K \cap Rad(M)$.*

6. *For $L \subseteq N, (K + L)/L$ is a supplement of N/L in M/L.*

Proof.   1. Let $L + K = M$ for some $L \subseteq N$. As $L \cap K \subseteq N \cap K \ll K, K$ is a supplement of L in M by Lemma 3.1.2.

  2. Let M be finitely generated. Since $N + K = M$, there is a finitely generated submodule $X \subseteq K$ with $N + X = M$. By minimality of K this means $X = K$.

3. By 2.2.8(3), we have $M/N \cong K/(N \cap K)$. So $K/(N \cap K)$ is simple, i.e. $K = xR + (N \cap K)$ for some $x \in K$, but $N \cap K \ll K$ so we have $K = xR$(i.e. K is cyclic).
   Also since $K/(K \cap N) \cong M/N$, $N \cap K$ is a maximal submodule of K and $N \cap K \supseteq \operatorname{Rad}(K)$. But $N \cap K \ll K$ so $N \cap K \subseteq \operatorname{Rad}(K)$ and hence $N \cap K = \operatorname{Rad}(K)$.

4. If $L \ll M$ and $X \subseteq K$ with $N + L + X = M$ then, as $L \ll M$, $N + X = M$ hence $X = K$. Trivially $M = N + L + K$.

5. Let $L \ll M$ and $X \subseteq K$ with $(L \cap K) + X = K$. Then $M = N + K = N + (L \cap K) + X = X + N(L \cap K \subseteq L \ll M)$, hence $X = K$ by the minimality of K, i.e. $L \cap K \ll K$. This yields $K \cap \operatorname{Rad}(M) \subseteq \operatorname{Rad}(K)$. By 2.3.3(4), $\operatorname{Rad}(K) \subseteq K \cap \operatorname{Rad}(M)$ always holds, so we get the desired equality.

6. For $L \subseteq N$, we have $N \cap (K + L) = (N \cap K) + L$(Modular Law), and $(N/L) \cap [(K + L)/L] = [(N \cap K) + L]/L$. Since $N \cap K \ll K$, it follows that $[(N \cap K) + L]/L \ll (K + L)/L$(image of small submodule, see Lemma 2.3.3(4).
   Also $(N/L) + [(K + L)/L] = M/L$, whence by Lemma 3.1.2, $(K + L)/L$ is a supplement of N/L in M/L.

$\square$

3.2 Characterization and Properties of Supplemented modules

As supplement submodules need not exist, e.g., no nontrivial submodules of $\mathbb{Z}_{\mathbb{Z}}$ has a supplement, different kinds of modules relative to supplements were defined and studied in the literature.

Definition 3.2.1. *An R-module M is called* supplemented *if every submodule of M has a supplement in M.*

Before we begin the basic part in our work, let us note that different terminology for types of supplemented modules, is used by different authors. For example, supplemented modules in [12] are called amply supplemented modules in [14].

As a corollary for Lemma 3.1.3 (6), we can begin with the following immediate result which exhibit the first property of a supplemented module.

Corollary 3.2.2. *Every factor module of a supplemented module is supplemented .*

Proof. For $K \subseteq L \subseteq M$ with M supplemented, L has a supplement N in M, we get by Lemma 3.1.3 (6), $(N + K)/K$ is a supplement of L/K in M/K. $\square$

As a consequence of the previous Corollary, we present a fascinating property of supplemented modules. An R-module M is called *semisimple* if M is the sum of simple submodules, or equivalently, if every submodule of M is a direct summand.

Corollary 3.2.3 ([14], §41.2(3)(ii))**.** *Let M be a supplemented module then $M/Rad(M)$ is semisimple.*

Proof. Let $L/\mathrm{Rad}(M) \subseteq M/\mathrm{Rad}(M)$, then by the previous Corollary, there exists a submodule $N/\mathrm{Rad}(M)$ such that $L/\mathrm{Rad}(M) + N/\mathrm{Rad}(M) = M/\mathrm{Rad}(M)$ and $L/\mathrm{Rad}(M) \cap N/\mathrm{Rad}(M) \ll M/\mathrm{Rad}(M)$. But since $\mathrm{Rad}(M/\mathrm{Rad}(M)) = 0$, then $L/\mathrm{Rad}(M) \cap N/\mathrm{Rad}(M) = 0$. So $L/\mathrm{Rad}(M)$ is a direct summand of $M/\mathrm{Rad}(M)$. $\square$

Motivated by the fact that homomorphic image of small submodule is again small, we study homomorphic images of supplemented modules. First consider this Lemma.

Lemma 3.2.4. *If $f : M \rightarrow N$ is a homomorphism and a submodule L containing $\mathrm{Ker} f$ has a supplement in M, then $f(L)$ has a supplement in $f(M)$.*

Proof. If K is a supplement of L, then $f(M) = f(L + K) = f(L) + f(K)$ and since $L \cap K \ll K$ we have, $f(L \cap K) \ll f(K)$ by Lemma 2.3.3(4). As $\mathrm{Ker} f \subseteq L$,

$$f(L \cap K) = f((L + \mathrm{Ker} f) \cap K) = f(f^{-1}f(L) \cap K) = f(L) \cap f(K).$$

So $f(K)$ is a supplement of $f(L)$ in $f(M)$. $\square$

Corollary 3.2.5. *A homomorphic image of a supplemented module is supplemented.*

Proof. Let $f : M \to N$ be a homomorphism, and M be a supplemented module. Suppose X a submodule of $f(M)$ then $f^{-1}(X) \subset M$ so it has K as a supplement in M, hence by the previous Lemma $f(K)$ is a supplement of $f(f^{-1}(X) = X$. $\square$

A module M is called a small cover of N if there exit a small epimorphism $f : M \to N$, i.e., $\operatorname{Ker} f \ll M$. We will see below that the inverse image of a supplemented module under small epimorphism is supplemented.

Lemma 3.2.6. *If $f : M \to N$ is a small epimorphism then a submodule L of M has a supplement in M if and only if $f(L)$ has a supplement in N.*

Proof. If K is a supplement of L in M, then by Lemma 3.1.3(4), K is a supplement of $L + \operatorname{Ker} f$ as well. And by Lemma 3.2.4, $f(L) = f(L + \operatorname{Ker} f)$ has a supplement in N.
Now let $f(L)$ has a supplement T in N i.e $f(L) + T = N$ and $f(L) \cap T \ll T$ then
$$f^{-1}(N) = M = L + f^{-1}(T).$$
But
$$f^{-1}(f(L) \cap T) = (L + \operatorname{Ker} f) \cap f^{-1}(T) \ll f^{-1}(T),$$
by Lemma 2.3.4. As $L \cap f^{-1}(T) \subset (L + \operatorname{Ker} f) \cap f^{-1}(T) \ll f^{-1}(T)$, $f^{-1}(T)$ is a supplement of L. $\square$

Corollary 3.2.7. *A small cover of a supplemented module is supplemented.*

Proof. Let $f : M \to N$ be a small epimorphism, N be supplemented, and assume $L \subset M$ then $f(L) \subset N$, so it has a supplement in N. Now by the previous Lemma, L has also a supplement in M. $\square$

Supplements of supplemented module inherit this property by the following corollary.

Corollary 3.2.8. *Every supplement submodule of a supplemented module is supplemented.*

Proof. Let V be a supplement of U, with M supplemented then

$$M/U = (U+V)/U \cong V/(U \cap V) \Rightarrow V/(U \cap V) \text{ is supplemented,}$$

since M/U is supplemented. Now the canonical map $\pi : V \to V/(U \cap V)$ is a small epimorphism Since $U \cap V \ll V$. i.e, V is a small cover of a supplemented module, hence by the previous Corollary V is supplemented. $\square$

Corollary 3.2.9. *Every direct summand of a supplemented module is supplemented.*

Proof. Every direct summand is a supplement, so by the previous Corollary, is supplemented. $\square$

In order to show that a finite sum of supplemented modules is supplemented, we need the following two results.

Lemma 3.2.10 ([6], Lemma 1.3)**.** *Let N and L be submodules of M, such that $N + L$ has a supplement H in M and $N \cap (H + L)$ has a supplement G in N, then $H + G$ is a supplement of L in M.*

Proof. Let H be a supplement of $N + L$ in M and let G be a supplement of $N \cap (H + L)$ in N. Then

$$\begin{aligned} M &= N + L + H \quad \text{and} \quad && H \cap (N + L) \ll H, \quad && \text{and} \\ N &= N \cap (H + L) + G \quad \text{and} \quad && N \cap G \cap (H + L) = (H + L) \cap G \ll G \end{aligned}$$

Now we have

$$\begin{aligned} (H + G) \cap L \ &\subseteq\ H \cap (L + G) + G \cap (L + H) \\ &\subseteq\ H \cap (L + N) + G \cap (L + H) \\ &\ll\ H + G, \qquad \text{and} \end{aligned}$$

$$H + G + L = N \cap (H + L) + H + L + G = N + H + L = M.$$

So $H + G$ is a supplement of L in M. $\square$

We state now ([14], 41.2(1)) and prove it using the general previous lemma.

Proposition 3.2.11. *Let M_1, U be submodules of M, with M_1 supplemented. If there is a supplement of $M_1 + U$ in M, then U has a supplement in M.*

Proof. Let H be a supplement of M_1+U in M, and since M_1 is supplemented, let G be a supplement of $M_1 \cap (H + U)$ in M_1, then by the previous Lemma $H + G$ is a supplement of U in M. $\qquad\square$

Now we are ready to prove that the class of supplemented modules is closed under finite sums.

Corollary 3.2.12. *Every finite (direct) sum of supplemented modules is supplemented.*

Proof. It suffices by induction to show that if $M = M_1 + M_2$ with M_1, M_2 supplemented, then M is supplemented.
Let $U \subseteq M$, then $M_1 + M_2 + U = M$. Now $M_1 + M_2 + U$ trivially has a supplement in M, and as M_1 is supplemented, then by the previous Proposition $M_2 + U$ has a supplement. Since M_2 is supplemented, applying the previous Proposition once more, we have U has a supplement in M. $\qquad\square$

We will present here some examples:

Example 3.2.13.   *1. Z_Z is not a supplemented module, since every nonzero submodule of Z is not small.*

   2. A Z-module M is supplemented if and only if M is a torsion module and for every prime the submodule M_p is a direct sum of an artinian module and a module with bounded order.[[6], § 1].
   Q_Z is not torsion, so it is not supplemented. Also Q/Z is not supplemented, for, if Q/Z is a supplemented module and consider the small epimorphism $\pi : Q \to Q/Z$. Then it follows by Corollary 3.2.7, that Q is supplemented, contradicting our argument above about Q_Z.

   3. We call a nonzero R-module M hollow if every proper submodule is small in M. If M has a largest submodule, i.e., a proper submodule which contains all other proper submodules, then M is called a local module.

Every local module is hollow, for if M is local with largest submodule $L \subsetneq M$ and K, N are proper submodules of M, hence contained in L, then $K + N \subseteq L \neq M$ which implies that any proper submodule of M is small.

Note that every hollow (local) module is supplemented, in fact M is a supplement of every proper submodule in M; for every $K \subsetneq M$, we have:

$$K + M = M \qquad and \ K \cap M = K \ll M,$$

which characterizes M as a supplement for every proper submodule of M.

4. *An infinite sum of supplemented modules needs not be supplemented. Consider $Q/Z = \bigoplus\{Z_{p^\infty} | p \text{ a prime number}\}$. Each p-component Z_{p^∞} of Q/Z is hollow, hence is supplemented, but by 2) Q/Z is not supplemented.*

In fact we will return to local modules for further inspection in the following chapter.

The following Lemma, which proves to be of great usefulness in our work, tells about the nature of a supplement of a maximal submodule.

Lemma 3.2.14 ([14], §41.1(3)). *A supplement of a maximal submodule of an R-module M is local.*

Proof. Let U be a maximal submodule of the R-module M. Assume V is a supplement of U in M. Now

$$M = U + V \text{ and } U \cap V \ll V \Rightarrow V/(U \cap V) \cong M/U,$$

so $U \cap V$ is a maximal submodule of V, hence $\operatorname{Rad}(V) \subseteq U \cap V$ and since $U \cap V \ll V$, then $U \cap V \subseteq \operatorname{Rad}(V) \Rightarrow \operatorname{Rad}(V) = U \cap V$ and $\operatorname{Rad}(V) \ll V$.

Let $L \subseteq V, L \neq V$ such that $L \nsubseteq \operatorname{Rad}(V)$, then $\operatorname{Rad}(V) + L = V$ by maximality of $\operatorname{Rad}(V)$. But $\operatorname{Rad}(V) \ll V \Rightarrow L = V$, a contradiction. So we have for any proper submodule of V, is a submodule of $\operatorname{Rad}(V)$, whence $\operatorname{Rad}(V)$ is a largest submodule of V, i.e., V is local. $\square$

We end this section with a characterization of finitely generated supplemented modules. If $M = \sum_{\Lambda} M_\lambda$, then this sum is called *irredundant* if, for every $\lambda_0 \in \Lambda, \sum_{\lambda \neq \lambda_0} M_\lambda \neq M$ holds.

Recall by [14], 21.6(7), if $M/\mathrm{Rad}(M)$ is semisimple and $\mathrm{Rad}(M) \ll M$ then every proper submodule of M is contained in a maximal submodule.

Theorem 3.2.15 ([14], §41.6).   *1. For a finitely generated module M, the following are equivalent:*

  (a) M is a supplemented module.

  (b) Every maximal submodule of M has a supplement in M.

  (c) M is a sum of hollow modules.

  (d) M is an irredundant (finite) sum of local submodules.

  2. If M is supplemented and $\mathrm{Rad}(M) \ll M$ then M is an irredundant sum of local modules.

Proof.   1. $a) \Rightarrow b)$ Trivial.

$b) \Rightarrow c)$ Let H be the sum of all hollow submodules of M and assume $H \neq M$. Since M is finitely generated then there is a maximal submodule $N \subseteq M$ with $H \subseteq N$, and a supplement L of N in M. By Lemma 3.2.14, L is local (hollow) and we have $L \subseteq H \subseteq N$, contradiction to $L + N = M$. So we must have $H = M$.

$c) \Rightarrow d)$ Let $M = \sum_{\Lambda} L_\lambda$ with hollow submodules $L_\lambda \subseteq M$ then $M/\mathrm{Rad}(M) = \sum_{\Lambda}(L_\lambda + \mathrm{Rad}(M))/\mathrm{Rad}(M)$. Since $\mathrm{Rad}(L_\lambda) \subseteq L_\lambda \cap \mathrm{Rad}(M) \subseteq \mathrm{Rad}(M) \ll M$ and $(L_\lambda + \mathrm{Rad}(M))/\mathrm{Rad}(M) \cong L_\lambda/(L_\lambda \cap \mathrm{Rad}(M))$, these factors are simple or zero, we obtain a representation, $M/\mathrm{Rad}(M) = \oplus_{\acute{\Lambda}}(L_\lambda + \mathrm{Rad}(M))/\mathrm{Rad}(M)$, and(since $\mathrm{Rad}(M) \ll M$) an irredundant sum $M = \sum_{\acute{\Lambda}} L_\lambda$ with local modules $L_\lambda, \lambda \in \acute{\Lambda} \subset \Lambda$.

$d) \Rightarrow a)$ Let $M = \sum_{i=1}^{n} L_i$ with L_i is local for each $1 \leq i \leq n$. Being a local module, each L_i is supplemented, and by Corollary 3.2.12, M is supplemented.

  2. Let M be a supplemented module, then $M/\mathrm{Rad}(M)$ is semisimple by Corollary 3.2.3. Assume H is the sum of all local submodules of M, and suppose $H \neq M$ then there exists a maximal submodule $N \subseteq M$ with $H \subseteq N$ and a supplement L of N in M. By Lemma 3.2.14, L is local and so $L \subseteq H \subseteq N$, implying $N + L = N \neq M$. So we must have $H = M$.

$\square$

Chapter 4

Types of Supplemented Modules

4.1 Cofinitely Supplemented Modules

In this section we study a new special type of supplemented modules. A module which has a supplement for special types of its submodules. Similar properties are investigated in detail to reveal whether this new class share common properties with the class of supplemented modules.

The first part of this section is devoted to studying special type of submodules of a given module; the cofinite submodules and their properties.

We begin with the definition.

Definition 4.1.1. *A submodule N of an R-module M is called* cofinite *if the factor module M/N is finitely generated.*

The cofinite submodule has an important property to which we will refer several times in the sequel, as shown by the following Remark,

Remark: If N is cofinite then every submodule containing N is also cofinite due to the Isomorphism Theorem(2);

$$\text{If } N \subseteq L \subseteq M, \text{ then } M/L \cong (M/N)/(L/N),$$

and since the latter module is finitely generated, being a factor module of the finitely generated module M/N, its homomorphic(isomorphic) image, M/L,

share this property.

The most important examples of cofinite submodules in the R-module M, are the maximal submodules. This is because if $L \subset M$ is a maximal submodule, then M/L is simple, i.e., M/L is generated by each of its elements by the characterization of simple modules.

Recall a supplement K of L in M means $M = K + L$ and $L \cap K \ll K$. We have showed that a supplement of a maximal submodule is local, in the following we investigate in the nature of supplements of cofinite submodules.

It was shown that supplements in a finitely generated module are again finitely generated(see Lemma 3.1.3(2)). Now a stronger result concerning supplements of cofinite submodules, is shown by the following Lemma.

Lemma 4.1.2 ([2], §2). *Let K, L be submodules of M, if K is cofinite and L is a supplement of K, then L is finitely generated.*

Proof. Assume K is a cofinite submodule of a module M, and let L be a supplement of K, then

$$L/(L \cap K) \cong (L + K)/K = M/K,$$

so $L \cap K$ is a cofinite submodule of L which implies $L = x_1 R + x_2 R + ... + x_n R + K \cap L$. But $L \cap K \ll L$ so $L = x_1 R + x_2 R + ... + x_n R$. i.e, L is finitely generated. $\qquad \square$

Recall, by [14], M is called Local if M has a largest submodule i.e a proper submodule which contains all other proper submodules.

Before we proceed it is reasonably important to investigate local modules. The following Lemma reveals some important properties of local modules.

Lemma 4.1.3. *Let M be an R-module. If M is a local module, then*

1. *$Rad(M)$ is the largest submodule of M and $Rad(M) \ll M$.*

2. *M is supplemented.*

3. *M is finitely generated, specifically, cyclic.*

Proof. 1. Assume M with a largest proper submodule L. For $K \subsetneqq M$, $K \subseteq L$ so $K + L = L \neq M$ hence $L \ll M$ i.e., $L \subseteq \mathrm{Rad}(M)$. But L is maximal in M because:

for $m \notin L$, if $L + mR \neq M$ then $L + mR \subseteq L \Rightarrow mR \subseteq L \Rightarrow m \in L$.

Now $\mathrm{Rad}(M) \subseteq L \Rightarrow L = \mathrm{Rad}(M)$.

2. By (1), for every $K \subsetneqq M$, $K \subseteq L = \mathrm{Rad}(M) \ll M \Rightarrow K \ll M$. Clearly $K + M = M$ and $K \cap M = K \ll M$, so by the characterization of a supplement Lemma 3.1.2, M is a supplement of K.

3. We have shown that $\mathrm{Rad}(M)$ is a maximal submodule of M and hence $M/\mathrm{Rad}(M)$ is simple, i.e., $M/\mathrm{Rad}(M)$ is finitely generated(cyclic), so by Theorem 2.3.9, M is finitely generated. $\qquad\square$

Local modules exhibit examples for modules in which every submodule is cofinite because it is itself finitely generated by the previous Lemma.

The following Lemma deals with sum involving local summands.

Lemma 4.1.4. *([1], Lemma 2.9). Let $L_i(1 \leq i \leq n)$ be a finite collection of local submodules of a module M and let N be a submodule of M such that $N + L_1 + ... + L_n$ has a supplement K in M. Then there exists a(possibly empty)subset I of $(1, ..., n)$ such that $K + \sum_{i \in I} L_i$ is a supplement of N in M.*

Proof. Suppose first $n = 1$. Consider the submodule $H = L_1 \cap (N + K)$ of L_1. If $H = L_1$, then 0 is a supplement of H in L_1, and by Lemma 3.2.10, $K = K + 0$ is a supplement of N in M. If $H \neq L_1$ then, since L_1 is local, L_1 is a supplement of H in L_1 and in this case $K + L_1$ is a supplement of N in M, again by Lemma 3.2.10. This proves the result when $n = 1$.

Suppose $n > 1$. By induction on n, there exists a subset J of $\{2, ..., n\}$ such that $K + \sum_{i \in J} L_i$ is a supplement of $N + L_1$ in M. Now the case $n = 1$ shows that either $K + \sum_{i \in J} L_i$ or $K + L_1 + \sum_{i \in J} L_i$ is a supplement of N in M. $\qquad\square$

In the following we investigate homomorphic image and the inverse image of a cofinite submodules.

Lemma 4.1.5. *Let $f : M \to N$ be an epimorphism. Then*

(a) *Let X be a cofinite submodule of M. Then $f(X)$ is a cofinite submodule of N.*

(b) *Let Y be a cofinite submodule of N. Then $f^{-1}(Y)$ is a cofinite submodule of M.*

Proof. Recall by the Isomorphism Theorem (1), $N \cong M/\mathrm{Ker} f$

(a) Let X be a cofinite submodule of M, then

$$N/f(X) \cong (M/\mathrm{Ker} f)/(f^{-1}(f(X))/\mathrm{Ker} f) \cong M/f^{-1}(f(X)) = M/(X + \mathrm{Ker} f).$$

But $X + \mathrm{Ker}\ f$ is cofinite in M by containing the cofinite submodule X, so $f(X)$ is a cofinite submodule of N.

(b) Let Y be a cofinite submodule of N, then

$$M/f^{-1}(Y) \cong (M/\mathrm{Ker}\ f)/(f^{-1}(Y)/\mathrm{Ker}\ f) \cong N/f(f^{-1}(Y)) = N/Y,$$

the last equality holds since f is an epimorphism. Since Y is cofinite in N, then $f^{-1}(Y)$ is a cofinite submodule of M.

$\square$

We will now begin our basic part in this section by the following Definition.

Definition 4.1.6. *An R-module M is called* cofinitely supplemented *if every cofinite submodule of M has a supplement in M.*

Clearly supplemented modules are cofinitely supplemented. Also finitely generated cofinitely supplemented modules are supplemented, for then every submodule of it is cofinite.In general, it is not true that every cofinitely supplemented is supplemented. Since the Z-module Q of rational numbers has no proper cofinite submodule, Q is cofinitely supplemented but is not supplemented as mentioned earlier.

Lemma 4.1.7 ([1], Lemma 2.1)**.** *Every factor module of cofinitely supplemented module is cofinitely supplemented as well.*

Proof. Let M be a cofinitely supplemented module, and let N be any submodule of M. Assume L/N is a cofinite submodule of M/N and $N \subseteq L$. Using the isomorphism theorem,

$$M/L \cong (M/N)/(L/N)$$

and since the latter module is finitely generated, we have M/L is finitely generated. Being a cofinite submodule of M, L has a supplement $K \subseteq M$ in M, that is $K + L = M$ and $L \cap K \ll K$. Now

$$\pi(L \cap K) = (L \cap K + N)/N = (L/N) \cap ((K+N)/N) \ll \pi(K) = (K+N)/N$$

(Where π is the canonical map). Hence by the characterization of a supplement, $(K + N)/N$ is a supplement of L/N in M/N. $\square$

In fact Lemma 3.2.4 enables us to go beyond the previous Lemma, specifically the following Lemma shows that every homomorphic image of a cofinitely supplemented module is again cofinitely supplemented.

Lemma 4.1.8. *A homomorphic image of a cofinitely supplemented module is cofinitely supplemented.*

Proof. Let $f : M \to N$ be a homomorphism with M cofinitely supplemented. Suppose Y a cofinite submodule of $f(M)$, then

$$M/f^{-1}(Y) \cong (M/\mathrm{Ker}f)/(f^{-1}(Y)/\mathrm{Ker}f) \cong f(M)/Y.$$

So $f^{-1}(Y)$ is a cofinite submodule of M containing $\mathrm{Ker}f$, and since M is cofinitely supplemented, $f^{-1}(Y)$ has a supplement in M, and by lemma 3.2.4, $f(f^{-1}(Y)) = Y$ has a supplement in $f(M)$. $\square$

The following Lemma deals with small covers of cofinitely supplemented modules.

Lemma 4.1.9. *A small cover of a cofinitely supplemented module is cofinitely supplemented.*

Proof. Let $f : M \to N$ be a small epimorphism, with N cofinitely supplemented module. Assume K to be a cofinite submodule of M, then by Lemma 4.1.5 (a), $f(K)$ is a cofinite submodule of N. So $f(K)$ has a supplement in N, and by Lemma 3.2.6, K has a supplement in M. So M is cofinitely supplemented. $\square$

We show now that an arbitrary sum of cofinitely supplemented modules is again cofinitely supplemented. To this end, all we need is the following Lemma, which is easily proved using Lemma 3.2.10.

Lemma 4.1.10 ([1], Lemma 2.2). *Let N and L be submodules of a module M such that N is cofinite and L is cofinitely supplemented, and $N + L$ has a supplement in M, then N has a supplement in M.*

Proof. Let K be a supplement of $N + L$ in M. Note that

$$\frac{L}{L \cap (N + K)} \cong \frac{L + N + K}{N + K} = \frac{M}{N + K} \cong \frac{M/N}{(N + K)/N}.$$

So that $L \cap (N + K)$ is cofinite in L, and since L is cofinitely supplemented, there exists a supplement H of $L \cap (N + K)$ in L, so by Lemma 3.2.10, N has $K + H$ as a supplement in M. $\square$

Now we are ready to present this Lemma.

Lemma 4.1.11 ([1], Lemma 2.3). *Let $M_i (i \in I)$ be any collection of cofinitely supplemented submodules of a module M, then $\sum_{i \in I} M_i$ is a cofinitely supplemented submodule of M.*

Proof. Let $N = \sum_{i \in I} M_i$, and let L be a cofinite submodule N. Because N/L is finitely generated, there exists a finitely generated submodule H of N such that $N = L + H$. There exists a finite subset J of I such that $H \subseteq \sum_{i \in J} M_i$ and hence $N = L + \sum_{i \in J} M_i = L + M_1 + \sum_{i=2} M_i$.

Since M_1 is cofinitely supplemented and $L + \sum_{i=2} M_i$ is cofinite, by the Remark following the definition, the previous Lemma is valid, and $L + \sum_{i=2} M_i$ has a supplement in N.

By repeated use of the previous Lemma we deduce that L has a supplement in N. So it follows that N is cofinitely supplemented. $\square$

Simply we are led to this Cororollary

Corollary 4.1.12. *Any direct sum of cofinitely supplemented modules is cofinitely supplemented.*

Following [1], we characterize when a module is cofinitely supplemented.

Let M be any module. Then $\mathrm{Loc}(M)$ will denote the sum of all local submodules of M and $\mathrm{Cof}(M)$ the sum of all cofinitely supplemented submodules of M, note that 0 is a local submodule and also a cofinitely supplemented submodule of M. By Lemma 4.1.11 , $\mathrm{Cof}(M)$ is the unique maximal cofinitely supplemented submodule of M. And by the Lemma 4.1.3 , $\mathrm{Loc}(M)$ is the sum of all finitely generated(cofinitely) supplemented submodule of M. Thus $\mathrm{Loc}(M) \subseteq \mathrm{Cof}(M)$.

We are ready now to give a characterization of cofinitely supplemented modules.

Theorem 4.1.13 ([1], Theorem 2.8)**.** *Let R be any ring. The following statements are equivalent for an R-module M*

1. *M is cofinitely supplemented*

2. *Every maximal submodule of M has a supplement in M.*

3. *The module $M/Loc(M)$ doesn't contain a maximal submodule*

4. *The module $M/Cof(M)$ doesn't contain a maximal submodule*

Proof. 1) $\Rightarrow$ 2): Trivial, since every maximal submodule of M is cofinite

2) $\Rightarrow$ 3): Let K be a maximal submodule of M. There exists $L \subseteq M$ such that $M = K + L$ and $K \cap L \ll L$, by 2) and the characterization of a supplement. Now by Lemma 3.2.14, L is local, and hence $L \subseteq \mathrm{Loc}(M)$. Since $K + L = M$ then $\mathrm{Loc}(M)$ is not a submodule of K, otherwise $K + L = K \neq M$. Hence $M/\mathrm{Loc}(M)$ has not a maximal submodule.

3) $\Rightarrow$ 4): Assume $L/\mathrm{Cof}(M)$ is maximal in $M/\mathrm{Cof}(M)$. Since $\mathrm{Loc}(M) \subseteq \mathrm{Cof}(M) \subset L$, then $L/\mathrm{Loc}(M)$ is a maximal submodule of $M/\mathrm{Loc}(M)$ contradicting (3) .

4) $\Rightarrow$ 1) : Let N be a cofinite submodule of M, then $N + Cof(M)$ is cofinite submodule of M by the Remark following the definition. Now $N + \mathrm{Cof}(M)$ is a submodule of M containing $\mathrm{Cof}(M)$ so by (4) we must have $M = N + \mathrm{Cof}(M)$. Since M/N is finitely generated it follows that $M = N + K_1 + K_2 + ... + K_n$ for some positive integer n, and cofinitely

supplemented submodules $K_i, 1 \leq i \leq n$. By repeated use of Lemma 4.1.10, N has a supplement in M. So M is cofinitely supplemented. $\square$

4.2 Weakly Supplemented Modules

In this section we will introduce a generalization of supplemented modules, the notion of weakly supplemented modules. Properties and relation to other classes of supplemented modules, are the purposes of this section.

Definition 4.2.1. *A submodule $N \subseteq M$ is called a* weak supplement of L in M *if* $N + L = M$ *and* $N \cap L \ll M$.

Because of the symmetry of the definition, one can say, N has L as a weak supplement in M. Now we can state the following definition.

Definition 4.2.2. *Let M be an R-module. Then M is called* weakly supplemented *if every submodule N of M has(is) a weak supplement.*

Recall from Lemma 2.3.3(4), for $K \subseteq L \subseteq M$, if $K \ll L$ then $K \ll M$, hence every supplemented module is weakly supplemented. For an example of weakly supplemented module which is not supplemented see Example 4.3.13

Example 4.2.3 ([5], Example §17.10)**.** *Q/Z is a weakly supplemented Z-module.*

Proof. First write $M := Q/Z = \bigoplus_P M_P$ as the direct sum of prime p-component $M_P := Z_{P^\infty}$. Every submodule N of M is of the form $N = \bigoplus N_P$ where $N_P = N \cap M_P \subseteq M_P$ are the p-components of N. Since M_P is hollow either $N_P = M_P$ or $N_P \ll M_P$. Thus $N \ll M$ if and only if $N_P \neq M_P$ for all p. If N is not small in M, set $\Lambda = \{p \,|\, N_P \neq M_P\}$ and $L := \bigoplus_{P \in \Lambda} M_P$. Then $N + L = M$ and $N \cap L = \bigoplus_\Lambda N_P \ll M$. Hence L is a weak supplement of N in M. $\qquad\square$

Next we are going to show that the class of weakly supplemented modules is closed under homomorphic images, finite direct sums and small covers. Firstly we need a Lemma similar to Lemma 3.2.4.

Lemma 4.2.4 ([2], Lemma 2.4)**.** *If $f : M \to N$ is a homomorphism and a submodule L containing $Kerf$ is a weak supplement in M, then $f(L)$ is a weak supplement in $f(M)$.*

Proof. If L is a weak supplement of K in M, then $f(M) = f(L + K) = f(L) + f(K)$ and since $(L \cap K) \ll M \Rightarrow f(L \cap K) \ll f(M)$ by Lemma 2.3.3(4). But

$$f(L \cap K) = f\,[(L + \text{Ker}f) \cap K] = f(f^{-1}f(L) \cap K) = f(L) \cap f(K) \ll f(M).$$

So $f(L)$ is a weak supplement of $f(K)$ in $f(M)$.

$\qquad\square$

Corollary 4.2.5. *Every homomorphic image of a weakly supplemented module is weakly supplemented.*

Proof. Let $f : M \to N$ be a homomorphism, and $L \subseteq f(M)$. Now $f^{-1}(L) \subseteq M$ containing $\text{Ker}f$, with a weak supplement K. Now by the previous Lemma $f(f^{-1}(L)) = L$ is a weak supplement in $f(M)$, namely of $f(K)$. $\qquad\square$

Corollary 4.2.6. *Every factor module of a weakly supplemented module is weakly supplemented.*

Proof. Let $K \subseteq M$. Consider the canonical epimorphism $\pi : M \to M/K$, since M is weakly supplemented then M/K is weakly supplemented by the previous Corollary. $\qquad\square$

Corollary 4.2.7. *Every direct summand of weakly supplemented module is weakly supplemented.*

Proof. Let $M = K \oplus L$, then $K \cong M/L$ by Theorem 2.2.8(3). $\qquad\square$

Corollary 4.2.8 ([5], §17.13)**.** *A small cover of weakly supplemented is weakly supplemented.*

Proof. Let M be a small cover of a weakly supplemented module N. Then $N \cong M/K$ for some $K \ll M$. Take a submodule L of M and a weak supplement X/K of $(L+K)/K$ in M/K. Since $K \ll M$, we have, $X \cap L + K = X \cap (L+K) \ll M$, so $X \cap L \ll M$ and X is a weak supplement of L in M. Thus M is weakly supplemented. $\qquad\square$

Example 4.2.9. Q *is weakly supplemented Z-module.*

Proof. Since Q is a small cover of the weakly supplemented module Q/Z see Example 4.2.3, we now see that Q is weakly supplemented by the previous Corollary. $\qquad\square$

Remark. Q_Z offers another example of a weakly supplemented module but not supplemented.

Corollary 4.2.10. *Every supplement submodule(e.g. direct summand) of a weakly supplemented module is weakly supplemented.*

Proof. Let M be a weakly supplemented module, and V be a supplement of U in M, then $M/U \cong V/(U \cap V)$, hence the factor module $V/(U \cap V)$ is weakly supplemented by Corollary 4.2.5. Consider the canonical map, $\pi : V \to V/(U \cap V)$ is a small epimorphism since $V \cap U \ll V$ by the characterization of a supplement. Now V is weakly supplemented by the previous Corollary. $\qquad\square$

We turn now to answer the question about the closure of the class of weakly supplemented under(finite) sum. In [[5], §17.13] a positive answer was given and proven using Lemma 17.11. In this work we will use another approach. Our tool is a Lemma similar to Lemma 3.2.10.

Lemma 4.2.11. *. Let N, L be submodules of a module M such that $N + L$ has a weak supplement H in M and $N \cap (H + L)$ has a weak supplement G in N. Then $H + G$ is a weak supplement of L in M.*

Proof. $L + (H + G) = (L + H) + G + N \cap (L + H) = L + H + N = M$, as H is a weak supplement of $N + L$ and G is a weak supplement of $N \cap (H + L)$ in N. Now

$$
\begin{aligned}
L \cap (H + G) \quad &\subset \quad H \cap (L + G) + G \cap (L + H) \\
&\subseteq \quad H \cap (L + N) + G \cap N \cap (L + H) \\
&\ll \quad M
\end{aligned}
$$

since $G \cap N \cap (L + H) \ll N \subset M$.

$\square$

Corollary 4.2.12 ([5], Corollary 17.12). *If $M = M_1 + M_2$, with M_1 and M_2 weakly supplemented, then M is weakly supplemented.*

Proof. For every submodule $N \subseteq M, M_1 + (M_2 + N)$ has a trivial weak supplement in M and since M_1 is weakly supplemented, $M_1 \cap (M_2 + N + 0)$ $= M_1 \cap (M_2 + N)$ has a weak supplement in M_1. So by the previous Lemma, $M_2 + N$ has a weak supplement in M, and since M_2 is weakly supplemented, applying the previous Lemma once more we have N has a weak supplement in M. $\square$

Corollary 4.2.13. *Finite (direct) sum of finitely many weakly supplemented modules is also weakly supplemented.*

Proof. By induction and the previous Corollary. $\square$

As for finitely generated supplemented modules, we give a characterization of finitely generated weakly supplemented modules(compare with Theorem 3.2.15). Firstly we need the following Lemma.

Lemma 4.2.14 ([2], Lemma 2.15). *Let H and K be submodules of M such that K is a weak supplement of a maximal submodule L of M. If $K + H$ has a weak supplement in M, then H has a weak supplement in M.*

Proof. Let X be a weak supplement of $K + H$ in M. If $K \cap (X + H) \subseteq K \cap L \ll M$ then $X + K$ is a weak supplement of H since

$$H \cap (X + K) \subseteq X \cap (H + K) + K \cap (X + H) \ll M$$

Now suppose $K \cap (X + H) \nsubseteq K \cap L$. Since $K/(K \cap L) \cong (K + L)/L = M/L$, $K \cap L$ is a maximal submodule of K. Therefore $(K \cap L) + K \cap (X + H) = K$. Then X is a weak supplement of H since

$$H \cap X \subseteq (K + H) \cap X \ll M \text{ and}$$

$$M = X + H + K = X + H + (K \cap L) + K \cap (X + H) = X + H.$$

as $K \cap (X + H) \subseteq X + H$ and $K \cap L \ll M$. So in both cases there is a weak supplement of H in M. $\qquad\square$

Theorem 4.2.15.   *1. For a finitely generated module M the following are equivalent:*

  (a) M is a weakly supplemented module.

  (b) Every maximal submodule of M has a weak supplement in M.

  (c) M is a (finite) sum of weak supplements of maximal submodules of M.

 2. [5], § 17.9(4). If $Rad(M) \ll M$, then M is weakly supplemented if and only if $M/Rad(M)$ is semisimple.

Proof.   1. $a) \Rightarrow b)$:Trivial.

   $b) \Rightarrow c)$: Let H is the sum of all weak supplements of some maximal submodule of M. Assume $H \neq M$, then there is a maximal submodule N of M such that $H \subseteq N$ and a weak supplement K of N, i.e. $N + K = M$. But then $K \subseteq H$ and $N + K = N \neq M$. So we must have $M = H$.

   $c) \Rightarrow a)$: Let $M = \sum_{i=1}^{n} L_i$ with L_i a weak supplement of some maximal submodule of M. Assume $K \subseteq M$, so $M = K + \sum_{i=1}^{n} L_i$, by repeated use of Lemma 4.2.14, K has a weak supplement.

2. $\Rightarrow$) Assume M is a weakly supplemented module and let $L/\mathrm{Rad}(M) \subseteq M/\mathrm{Rad}(M)$. Take a weak supplement K of L in M. Then

$$\frac{K + \mathrm{Rad}(M)}{\mathrm{Rad}(M)} + \frac{L}{\mathrm{Rad}(M)} = \frac{K + L}{\mathrm{Rad}(M)} = \frac{M}{\mathrm{Rad}(M)}, \text{ and}$$

$$\frac{K + \mathrm{Rad}(M)}{\mathrm{Rad}(M)} \cap \frac{L}{\mathrm{Rad}(M)} = \frac{L \cap K + \mathrm{Rad}(M)}{\mathrm{Rad}(M)} = 0$$

$\Leftarrow$) Let $L \subseteq M$, then $(L + \mathrm{Rad}(M))/\mathrm{Rad}(M)$ is a direct summand of $M/\mathrm{Rad}(M)$. i.e. there is a submodule $K/\mathrm{Rad}(M)$ of $M/\mathrm{Rad}(M)$ such that $(L + \mathrm{Rad}(M))/\mathrm{Rad}(M) \oplus K/\mathrm{Rad}(M) = M/\mathrm{Rad}(M)$, then $L + K = M$ and $L \cap K \subseteq \mathrm{Rad}(M) \ll M$ that is K is a weak supplement of L.

$\square$

We turn now to study a new special type of supplemented modules, namely modules that have weak supplements for special types of its submodules. Properties of these modules are investigated in detail to reveal whether this new class share common properties with weakly supplemented modules.

4.3 Cofinitely Weak Supplemented Modules

In this section we will introduce another type of supplemented modules, and study its properties and its relations with the other classes so far has been defined. We begin with this Definition.

Definition 4.3.1. *An R-module M is* cofinitely weak supplemented *if every cofinite submodule of M has a weak supplement.*

Recall a weak supplement K of a submodule L in M means: $M = L + K$ and $K \cap L \ll M$.

We have showed, see Lemma 4.1.2, that a suppleme nt of a cofinite submodule is finitely generated. Now we will show that weak supplements of cofinite submodules can be regarded as finitely generated.

Lemma 4.3.2 ([2], Lemma 2.1)**.** *Let M be a module and K is a cofinte(maximal) submodule of M. If L is a weak supplement of K in M, then K has a finitely generated(cyclic) weak supplement in M that is contained in L.*

Proof. If K is cofinite, and since $M/K = (L + K)/K \cong L/(L \cap K)$, then $L/(L \cap K)$ is finitely generated . Let $L/(L \cap K)$ be generated by the elements:

$$x_1 + L \cap K, x_2 + L \cap K, ..., x_n + L \cap K.$$

Then for the finitely generated submodule $W = x_1 R + x_2 R + ... + x_n R$ of L we have

$$W + K = W + K \cap L + K = L + K = M \text{ and } W \cap K \subseteq L \cap K \ll M.$$

Therefore W is a finitely generated weak supplement of K in M which is contained in L.

If K is maximal, then $L/(L \cap K)$ is cyclic module generated by some element $x + K \cap L$ and $W = xR$ is a weak supplement of K. $\square$

Lemma 4.2.4 will enable us to show that a homomorphic image of cofinitely weak supplemented module is also cofinitely weak supplemented as stated by the following Proposition.

Proposition 4.3.3 ([2], Proposition 2.5)**.** *A homomorphic image of a cofinitely weak supplemented module is a cofinitely weak supplemented.*

Proof. Let $f : M \to N$ be a homomorphism with M cofinitely weak supplemented. Suppose Y a cofinite submodule of $f(M)$, then

$$M/f^{-1}(Y) \cong (M/\mathrm{Ker}f)/(f^{-1}(Y)/\mathrm{Ker}f) \cong f(M)/Y$$

So $f^{-1}(Y)$ is a cofinite submodule of M containing $\mathrm{Ker}f$, and since M is cofinitely weak supplemented, $f^{-1}(Y)$ is a weak supplement in M , and by Lemma 4.2.4, $f(f^{-1}(Y)) = Y$ is a weak supplement in $f(M)$. $\square$

Let $K \subseteq M$, and consider the canonical map (epimorphism) $: \pi : M \to M/K$. If M is cofinitely weak supplemented, and applying the previous Proposition, we get the following Corollary.

Corollary 4.3.4 ([2], Corollary 2.6). *Any factor module of a cofinitely weak supplemented module is a cofinitely weak supplemented.*

To show that the inverse image of a cofinitely weak supplemented module under a small epimorphism is a cofinitely weak supplemented, we need analogous result to proposition 3.1.3(4).

Proposition 4.3.5. *If K is a weak supplement of N in a module M and $T \ll M$, then K is a weak supplement of $N + T$ in M as well.*

Proof. Let $f : M \to M/N \oplus M/K$ be defined by $f(m) = (m + N, m + K)$ and $g : (M/N) \oplus (M/K) \to M/(N+T) \oplus M/K$ be defined by $g(m+N, m'+K) = (m + N + T, m' + K)$. Then f is epimorphism as $M = N + K$ and $\mathrm{Ker} f = N \cap K \ll M$ as K is a weak supplement of N in M, so f is a small epimorphism. Now

$$\mathrm{Ker} g = (N+T)/N \oplus K/K = (N+T)/N \oplus 0 = (N+T)/N \ll M/N$$

since $T \ll M$ and $(N+T)/N = \pi(T)$ where $\pi : M \to M/N$ is the canonical map. Therefore g is small epimorphism. By Lemma 2.3.3(1), gf is small, i.e., $(N+T) \cap K = \mathrm{Ker}(gf) \ll M$. Clearly $(N+T) + K = M$ so K is a weak supplement of $N + T$ in M. $\square$

The previous proposition together with the fact that the image of a weak supplement under a homomorphism is still a weak supplement, are the theory behind the following Lemma.

Lemma 4.3.6 ([2], Lemma 2.8). *If $f : M \to N$ is a small epimorphism then a submodule L of M is a weak supplement in M if and only if $f(L)$ is a weak supplement in N.*

Proof. If K is a weak supplement of L in M then by the previous Proposition, $L + \mathrm{Ker} f$ is also a weak supplement of K in M and by Lemma 4.2.4, $f(L) = f(L + \mathrm{Ker} f)$ is a weak supplement in N.

Now let $f(L)$ be a weak supplement of a submodule T in N. i.e., $N = f(L) + T$ and $f(L) \cap T \ll N$. So $M = L + f^{-1}(T)$. It follows from Lemma 2.3.4 that

$$f^{-1}(f(L) \cap T) = f^{-1}(f(L) \cap f^{-1}(T) = (L + \mathrm{Ker} f) \cap f^{-1}(T) \ll M$$

But as $L \cap f^{-1}(T) \subseteq (L + \mathrm{Ker} f) \cap f^{-1}(T) \ll M$, we have $f^{-1}(T)$ is a weak supplement of L. $\qquad\square$

Recall that a module M is called a small cover of a module N, if there exists a small epimorphism $f : M \to N$, i.e., $\mathrm{Ker} f \ll M$.

We are ready to answer the question: Is a small cover of a cofinitely weak supplemented module is again cofinitely weak supplemented. A positive answer is given by the following Corollary.

Corollary 4.3.7 ([2], Corollary 2.9). *A small cover of a cofinitely weak supplemented module is a cofinitely weak supplemented.*

Proof. Let N be a cofinitely weak supplemented module, $f : M \to N$ be a small epimorphism, and L be a cofinite submodule of M. Consider $\overline{f} : M/L \to N/f(L)$ defined by $\overline{f}(m + L) = f(m) + f(L)$, as f is an epimorphism, i.e., $f(M) = N$ we have $\overline{f}$ is an epimorphism. Since M/L is finitely generated so is $N/f(L)$. But N is cofinitely weak supplemented module so $f(L)$ is a weak supplement in N, and L is a weak supplement in M by the previous Lemma. $\qquad\square$

Recall that $\mathrm{Rad}(M)$ is the sum of all small submodules of the module M. In a module M, it is not necessary for Rad (M) to be a small submodule of M, but if Rad (M) is a small in M, then we have a characterization for a cofinitely weak supplemented module as asserted by the following Corollary.

Corollary 4.3.8 ([2], Corollary 2.10). *Suppose that M is an R-module with $Rad(M) \ll M$ then $M/Rad\ (M)$ is a cofinitely weak supplemented module if and only if M is a cofinitely weak supplemented module.*

Proof. $\Rightarrow$): By the previous Corollary.
$\Leftarrow$): By Corollary 4.3.4. $\qquad\square$

In our seek for a proof of "An arbitrary sum of cofinitely weak supplemented submodules is a cofinitely weak supplemented" we will use an analogue of Lemma 4.1.10

Lemma 4.3.9 ([2], Lemma 2.11). *Let N and L be submodules of M with N cofinitely weak supplemented and L cofinite. If $N + L$ has a weak supplement in M, then L has also a weak supplement in M.*

Proof. Let X be a weak supplement of $N + L$ in M. Then

$$N/(N \cap (X + L)) \cong (N + X + L)/(X + L) = M/(X + L).$$

But, as $X + L$ contains the cofinite submodule L, $X + L$ is cofinite, and so $N \cap (X + L)$ is a cofinite submodule of N, and hence $N \cap (X + L)$ has a weak supplement Y in N. Now Lemma 4.2.11 asserted the existence of a weak supplement of L in M, namely $X + Y$. □

The previous Lemma enables us to prove easily the following Proposition.

Proposition 4.3.10 ([2], Proposition 2.12). *An arbitrary sum of cofinitely weak supplemented modules is cofinitely weak supplemented module.*

Proof. Let $M = \sum_{i \in I} M_i$ where each submodule M_i is cofinitely weak supplemented and N is a cofinite submodule of M, then M/N is generated by some finite set $\{x_1 + N, x_2 + N, ..., x_r + N\}$ and therefore $M = x_1 R + x_2 R + ... + x_r R + N$. Since each x_i is contained in the sum $\sum_{j \in F_i} M_j$ for some finite subset F_i of I,
$x_1 R + x_2 R + ... + x_r R \subseteq \sum_{j \in F} M_j$ for some finite subset $F = \{i_1, i_2, ..., i_k\}$ of I. Then $M = N + \sum_{t=1}^{k} M_{i_t}$. Since $M = M_{i_k} + (N + \sum_{t=1}^{k-1} M_{i_t})$ has a trivial weak supplement 0 in M, and since M_{i_k} is cofinitely weak supplemented module, and $(N + \sum_{t=1}^{k-1} M_{i_t})$ is cofinite by containing the cofinite submodule N, the previous Lemma applies, and the latter submodule has a weak supplement in M. Similarly $(N + \sum_{t=1}^{k-2} M_{i_t})$ has a weak supplement in M and so on. Continuing in this way we will obtain (using the previous Lemma k times) at last that N has a weak supplement in M. □

Recall that a module M is p-torsion, if, for every $a \in M$, there exists $k \in N$ with $p^k a = 0$. For a p-torsion module M to be bounded there must exists $n \in N$ such that for all $a \in M, p^n a = 0$.
The class of cofinitely weak supplemented modules is strictly wider than the class of the weakly supplemented modules as the following example shows.

Example 4.3.11 ([2], Example 2.14)**.** *Let p be a prime integer and consider the Z-module $M = \bigoplus_{i=1}^{\infty} \langle a_i \rangle$ which is the direct sum of cyclic subgroups $\langle a_i \rangle$ of order p^i. Since each $\langle a_i \rangle$ is local and therefore is a cofinitely weak supplemented module, M is cofinitely weak supplemented module by the previous proposition. We will show that M is not weakly supplemented.*

Let $T = pM$ and suppose that T has a weak supplement L, i.e., $M = L+T$ and $N = T \cap L \ll M$. Then $N \ll E(M)$ as well, where $E(M)$ is an injective hull of M. Since the injective hull $E(N)$ of N is a direct summand of $E(M)$, $N \ll E(N)$. It is well known that if a torsion abelian group is small in its injective hull then it is bounded. Therefore N must be bounded, i.e., $p^n N = 0$ for some positive integer n. Then, as $pL \subseteq L \cap pM = L \cap T = N$,

$$p^{n+1}M = p^{n+1}T + p^n(pL) \subseteq p^{n+1}T + p^n N = P^{n+1}T$$

Therefore $p^{n+1}a_{n+2} = p^{n+1}b$ for some $b \in T = pM$. Since $b = pc$ for some $c = (m_i a_i)_{i=1}^{\infty} \in M$, we have

$$0 \neq p^{n+1}a_{n+2} = p^{n+1}(pm_{n+2}a_{n+2}) = m_{n+2}p^{n+2}a_{n+2} = 0$$

This contradiction implies that M is not a weakly supplemented module.
Remark*: This Example points out that infinite sum of weakly supplemented modules needs not be weakly supplemented. Let M be defined as above. Since each $\langle a_i \rangle$ is local and therefore is a (weakly) supplemented module, M is an infinite sum of weakly supplemented modules, but is not weakly supplemented, as the example shows.*

Now we are going to prove that a module is cofinitely weak supplemented if and only if every maximal submodule has a weak supplement. Firstly we need some notions.

For a module M, let Γ be the set of all submodules K such that K is a weak supplement of some maximal submodule of M and let $Cws(M)$ denote the sum of all submodules from Γ. As usual $Cws(M) = 0$ if $\Gamma = \emptyset$

Theorem 4.3.12 ([2], Theorem 2.16)**.** *For a module M, the following statements are equivalent.*

1. *M is cofinitely weak supplemented module.*

2. *Every maximal submodule of M has a weak supplement.*

3. *$M/Cws(M)$ has no maximal submodules.*

Proof. (1) $\Rightarrow$ (2): is obvious since every maximal submodule is cofinite.

(2) $\Rightarrow$ (3): Suppose that there is a maximal submodule $L/\mathrm{Cws}(M)$ of $M/\mathrm{Cws}(M)$. Then L is a maximal submodule of M. By (2), there is a weak supplement K of L in M. Then $K \in \Gamma$, therefore $K \subseteq \mathrm{Cws}(M) \subseteq L$. Hence $M = L + K = L$. This contradiction shows that $M/\mathrm{Cws}(M)$ has no maximal submodules.

(3) $\Rightarrow$ (1): Let N be a cofinite submodule of M. Then $N + \mathrm{Cws}(M)$ is also cofinite by the remark following the definition of cofinite submodule. If $M/(N+\mathrm{Cws}(M)) \neq 0$ then by 2.1.15 , there is a maximal submodule $L/(N+\mathrm{Cws}(M))$ of the finitely generated module $M/(N+\mathrm{Cws}(M))$. It follows that L is a maximal submodule of M and $M/\mathrm{Cws}(M)$ which contradicts (3). So we must have $M = N + \mathrm{Cws}(M)$. Now M/N is finitely generated, say by elements $x_1+N, x_2+N, ..., x_m+N$, therefore $M = N+x_1R+x_2R+...+x_mR$. Each element $x_i(i = 1, 2, ..., m)$ can be written as $x_i = n_i + c_i$, where $n_i \in N$ and $c_i \in Cws(M)$. Since each c_i is contained in the sum of finite number of submodules of Γ, $M = N + K_1 + K_2 + ... + K_n$ for some submodules $K_1, K_2, ..., K_n$ of M from Γ. Now $M = (N + K_1 + K_2 + ... + K_{n-1}) + K_n$ has a trivial weak supplement. By Lemma 4.2.14, $N + K_1 + K_2 + ... + K_{n-1}$ has a weak supplement in M. Continuing in this way (applying the Lemma 4.2.14 n times) we obtain that N has a weak supplement in M. $\qquad\square$

Recall that a module M is cofinitely supplemented if every cofinite submodule of M has a supplement in M. Clearly if M is cofinitely supplemented, then M is cofinitely weak supplemented.

The following example shows that a cofinitely weak supplemented module need not be cofinitely supplemented.

Example 4.3.13 ([11], Remark(3.3)). *Consider the ring,*

$$\mathbf{R} = \mathbf{Z}_{p,q} = \left\{ \frac{a}{b} \,|\, a, b \in Z,\ b \neq 0,\ (p, b) = 1,\ (q, b) = 1 \right\}.$$

The only maximal ideals pR and qR are weak supplements of each other because $pR+qR = R$ and $pR \cap qR = RadR \ll R_R$ as R_R is finitely generated, so by the previous Theorem $\mathbf{R}_R$ is (cofinitely) weak supplemented. Moreover, if k is any such that $(k,p) = 1$ then $pR+kqR = R$ and $kqR \subsetneqq qR$, so pR, qR are not supplements of each other. By Theorem 4.1.13, the right module $\mathbf{R}_R$

is not (cofinitely) supplemented.
Remark: R_R is finitely generated, so we can drop the term cofinitely in the
above statement.

It is known (see Lemma 3.1.3 (5)) that for supplement submodule K of a module M, Rad $(K) = K \cap$ Rad (M) and that for a weak supplemented module M the last equality implies that K is a supplement as the following Lemma shows.

Lemma 4.3.14 ([2], Lemma 2.18)**.** *Let M be an R-module and N be a cofinite submodule of M. If N has a weak supplement L in M and for every finitely generated submodule K of L, Rad $(K) = K \cap Rad$ (M), then N has a finitely generated supplement in M.*

Proof. L is a weak supplement of N, i.e., $N + L = M$ and $N \cap L \ll M$. Since N is cofinite, then by Lemma 4.3.2, N has a finitely generated weak supplement $K \subseteq L$ in M, i.e., $M = N + K$ and $N \cap K \ll M$. Then $N \cap K \subseteq \text{Rad}(M)$. Therefore $N \cap K \subseteq K \cap \text{Rad}(M) = \text{Rad}(K)$. But $\text{Rad}(K) \ll K$, since K is finitely generated. So $N \cap K \ll K$, i.e., K is a supplement of N in M. $\qquad\square$

It is obvious that every cofinitely supplemented module is a cofinitely weak supplemented.
The following Theorem gives a condition under which the converse is true.

Theorem 4.3.15 ([2], Theorem 2.19)**.** *Let M be an R-module such that for every finitely generated submodule K of M, Rad $(K) = K \cap Rad(M)$, then M is cofinitely weak supplemented if and only if M is cofinitely supplemented.*

Proof. Let N be a cofinite submodule of M. Since M is cofinitely weak supplemented, N has a weak supplement L in M and by the previous Lemma, N has a supplement. Hence M is cofinitely supplemented.
The converse statement is obvious. $\qquad\square$

Applying the previous Theorem on a finitely generated module, shows an equivalence between being weakly supplemented and being supplemented.

Corollary 4.3.16 ([2], Corollary 2.20). *Let M be a finitely generated module such that for every finitely generated submodule N of M, Rad $N = N \cap Rad(M)$. Then M is weakly supplemented if and only if M is supplemented. Furthermore in this case every finitely generated submodule of M is a supplement.*

Proof. The first statement follows from the previous Theorem, as in a finitely generated module, every submodule is cofinite. If N is finitely generated submodule, then N has a weak supplement K, therefore $N + K = M$ and $N \cap K \subseteq N \cap \text{Rad}(M) = \text{Rad}(N) \ll N$, i.e., N is a supplement of K. $\qquad\square$

The following Theorem gives a characterization of cofinitely weak supplemented modules with small Radical.

Theorem 4.3.17 ([2], Theorem 2.21). *Let M be an R-module with $Rad(M) \ll M$. Then the following statements are equivalent.*

1. *M is a cofinitely weak supplemented module.*

2. *$M/Rad(M)$ is a cofinitely weak supplemented.*

3. *Every cofinite submodule of $M/Rad(M)$ is a direct summand.*

4. *Every maximal submodule of $M/Rad(M)$ is a direct summand.*

5. *Every maximal submodule of $M/Rad(M)$ is a weak supplement.*

6. *Every maximal submodule of M is a weak supplement.*

Proof. (1) $\Rightarrow$ (2): By corollary 4.3.4.

(2) $\Rightarrow$ (3): Let $U \subset M/\text{Rad}(M)$ such that U is cofinite, then by (2), U is a weak supplement of K, i.e., $U \cap K \ll M/\text{Rad}(M)$ but since $\text{Rad}(M/\text{Rad}(M)) = 0$, i.e., the only small submodule of $M/\text{Rad}(M)$ is $0 \Rightarrow U \cap K = 0$ together with $U + K = M/\text{Rad}(M) \Rightarrow U$ is a direct summand of $M/\text{Rad}(M)$.

(3) $\Rightarrow$ (4): Obvious since every maximal submodule is cofinite.

(4) $\Rightarrow$ (5): Every direct summand U is a supplement since $U + V = M$ and $U \cap V = 0 \ll U$ and then $U \cap V = 0 \ll M$. So U is a weak supplement.

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$(5) \Rightarrow (6)$: Since $\mathrm{Rad}(M) \ll M$ and by Lemma 4.3.6, (6) follows from (5).

$(6) \Rightarrow (1)$: By Theorem 4.3.12 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

In the following section we study still a new type of supplemented modules. More definitions and theorems are given to clarify the relation between the new class and the classes we have studied till now.

4.4 ⊕-Supplemented Modules

Recall a submodule K of M is a direct summand if there exists a submodule L such that $M = L \oplus K$. i.e., $M = K + L$ and $K \cap L = 0$. Following [12] we will give the following definition of a new class which is a subclass of supplemented modules.

Definition 4.4.1. *Let M be an R-module. M is called ⊕-supplemented module if every submodule has a supplement that is a direct summand of M. i.e., for $N \subseteq M$, there exist submodules K, L such that $M = K \oplus L$, $M = N + K$ and $K \cap N \ll K$.*

Recall that a nonzero module M is called hollow if every proper submodule is small in M, and is called local if the sum of all proper submodules of M is also a proper submodule of M. Note that local modules are hollow and hollow modules are ⊕-supplemented, namely each submodule N has M as a supplement since $M = N + M$ and $M \cap N = N \ll M$ and $M \oplus 0 = M$

Clearly ⊕-supplemented modules are supplemented, but the converse is not true in general (see Example 4.4.4).

The following Theorem proves the closeness of the class of ⊕-supplemented under finite direct sum of its elements.

Theorem 4.4.2 ([6], Theorem 1.4)**.** *For any ring R, any finite direct sum of ⊕-supplemented R-modules is ⊕-supplemented.*

Proof. Let n be any positive integer and let M_i be a $\oplus$-supplemented R-module for each $1 \le i \le n$. Let $M = M_1 \oplus M_2 \oplus ... \oplus M_n$. To prove that M is $\oplus$-supplemented it is sufficient by induction on n to prove that this is the case when $n = 2$. Thus suppose $n = 2$.

Let L be any submodule of M. Then $M = M_1 + M_2$ so that $M_1 + M_2 + L$ has a supplement 0 in M. Let H be a supplement of $M_2 \cap (M_1 + L + 0)$ $= M_2 \cap (M_1 + L)$ in $M_2(M_2$ is $\oplus$-supplemented) such that H is a direct summand in M_2. By Lemma 3.2.10, $H + 0 = H$ is a supplement of $M_1 + L$ in M. Let K be a supplement of $M_1 \cap (L + H)$ in M_1 such that K is a direct summand of M_1. Again applying Lemma 3.2.10, we have $H + K$ is a supplement of L in M.
Assume $M_2 = H \oplus U$ and $M_1 = K \oplus V$, so now $M = M_1 \oplus M_2 = H \oplus U \oplus K \oplus V$ $\implies H + K = H \oplus K$ is a direct summand of M.

$\square$

Since hollow(local) modules are $\oplus$-supplemented, using the previous theorem we get the following

Corollary 4.4.3. *Any finite direct sum of hollow(local) modules is $\oplus$-supplemented.*

Quotient of a $\oplus$-supplemented module is not in general $\oplus$-supplemented. In [7] some examples are given to show the previous statement. In our thesis an example is given after a while.

A commutative ring R is a valuation ring if it is a local ring and every finitely generated ideal is principal. A module M is called finitely presented if $M \cong F/K$ for some finitely generated free module F and finitely generated submodule K of F.

Example 4.4.4. *[[7], Example 2.2]Let R be a commutative ring which is not a valuation ring and let $n \ge 2$. There exists a finitely presented indecomposable module $M = R^{(n)}/K$ which cannot be generated by fewer than n elements. By Theorem 4.4.2, $R^{(n)}$ is $\oplus$-supplemented. We will show that M is not $\oplus$-supplemented. Let L be a maximal submodule of M (such a maximal submodule exists by Corollary 2.1.15). Assume there exist submodules K, H such that $M = K \oplus H, M = K + L$ and $L \cap K \ll K$. Now $M/L \cong (K + L)/L \cong K/(K \cap L)$. As $L \subsetneq M$ and M is indecomposable, we must have $K = M$, so $M/(K \cap L)$ is cyclic, and $M = m_1 R + K \cap L$, but $K \cap L \ll M$ so $M = m_1 R$, which is a contradiction.*
Remark*: Note that as $R^{(n)}$ is $\oplus$-supplemented, then $R^{(n)}$ is supplemented and*

by Corollary 3.2.2, $R^{(n)}/K$ is supplemented which is not $\oplus$-supplemented as shown above.

Nevertheless, we intend to present a proposition which deals with a special case of factor modules of a $\oplus$-supplemented modules. First we prove the following Lemma.

Lemma 4.4.5 ([7], Lemma 2.4). *Let M be a nonzero module and let N be a submodule of M such that $f(N) \subseteq N$ for each $f \in End_R(M)$. If $M = M_1 \oplus M_2$ then $N = N \cap M_1 \oplus N \cap M_2$.*

Proof. Let $\pi_i : M \to M_i (i = 1, 2)$ denote the canonical projections. Let x be an element of N. Then $x = \pi_1(x) + \pi_2(x)$. By hypothesis, $\pi_i(N) \subseteq N$ for $i = 1, 2$. Thus $\pi_i(x) \in N \cap M_i$ for $i = 1, 2$. Hence

$$N \subseteq \pi_1(N) + \pi_2(N) \subseteq N \cap M_1 \oplus N \cap M_2$$

also $N \cap M_1 \oplus N \cap M_2 \subseteq N \Rightarrow N = N \cap M_1 \oplus N \cap M_2$. $\qquad \square$

The submodules satisfying the property of the previous Lemma, their factor modules preserve the $\oplus$-supplementary as shown by the following proposition.

Proposition 4.4.6 ([7], Proposition 2.5). *Let M be a nonzero module and let N be a submodule of M such that $f(N) \subseteq N$ for each $f \in End_R(M)$. If M is $\oplus$-supplemented, then M/N is $\oplus$-supplemented. If, moreover, N is a direct summand of M, then N is also $\oplus$-supplemented.*

Proof. Suppose that M is $\oplus$-supplemented. Let L be a submodule of M which contains N. There exist submodules K and H of M such that $M = K \oplus H$ and $M = K + L$ and $L \cap K \ll K$ (M is $\oplus$-supplemented). By Lemma 3.1.3(6), $(N + K)/N$ is a supplement of L/N in M/N. Now apply the previous Lemma, to get that $N = N \cap K \oplus N \cap H$. Thus

$$
\begin{aligned}
(K + N) \cap (H + N) \ &\subseteq\ H \cap (K + N + N) + N \cap (K + N + H) \\
&\subseteq\ H \cap (K + N \cap K + N \cap H) + N \\
&\subseteq\ H \cap (K + N \cap H) + N \\
&=\ H \cap N + H \cap K + N \\
&=\ 0 + N = N
\end{aligned}
$$

It follows that $\frac{K+N}{N} \cap \frac{H+N}{N} = 0$ and $\frac{K+N}{N} + \frac{H+N}{N} = \frac{M}{N}$

So $(N + K)/N$ is a direct summand of M/N, consequently M/N is $\oplus$-supplemented.

Now suppose that N is a direct summand of M. Let L be a submodule of N. Since M is $\oplus$-supplemented, there exist submodules K, H of M such that $M = K \oplus H, M = K + L$, and $L \cap K \ll K$. Thus $N = L + N \cap K$. But by the previous Lemma $N = N \cap K \oplus N \cap H$, hence $N \cap K$ is a direct summand of N. Moreover, $L \cap (N \cap K) = L \cap K \ll K$. Then $L \cap (N \cap K)$ is small in $N \cap K$, since $N \cap K$ is a direct summand of M. Therefore $N \cap K$ is a supplement of L in N and it is a direct summand of N. So N is $\oplus$-supplemented. $\qquad\square$

In the following section we will introduce the last kind of supplemented modules in our work. Some more definitions and theorems are introduced to complete the thesis.

4.5 $\oplus$-Cofinitely Supplemented Modules

Recall that if M is an R-module, then a submodule N is called cofinite if M/N is finitely generated. we will begin by the following Definition.

Definition 4.5.1. *An R-module M is called $\oplus$-cofinitely supplemented if every cofinite submodule of M has a supplement in M which is a direct summand of M.*

Clearly $\oplus$-supplemented modules are $\oplus$-cofinitely supplemented. Since in a finitely generated module every submodule is cofinite, so finitely generated $\oplus$-cofinitely supplemented modules are $\oplus$-supplemented modules .

In general, it is not true that $\oplus$-cofinitely supplemented module is $\oplus$-supplemented:
The Z-module Q of rational numbers has not any proper cofinite submodule. Thus Q is $\oplus$-cofinitely supplemented (Q is the only cofinite submodule)

but the Z-module Q is not torsion, so it is not supplemented(whence not$\oplus$-supplemented)[see [4]].

The following Lemma gives a way to obtain $\oplus$-cofinitely supplemented modules from just cofinitely supplemented ones.

Lemma 4.5.2 ([4], Lemma 2.1). *Let M be a cofinitely supplemented module. Then $M/Rad(M)$ is $\oplus$-Cofinitely supplemented.*

Proof. By Lemma 4.1.7, we have $M/Rad(M)$ cofinitely supplemented as a factor module of the cofinitely supplemented module M.

Moreover, any cofinite submodule of $M/Rad(M)$ has the form $N/Rad(M)$ where N is cofinite submodule of M, hence there exists a submodule K of M such that $M = N + K$ and $N \cap K \ll K$, hence $N \cap K \ll M$ and so $N \cap K \subseteq Rad(M)$. Thus

$$\frac{N}{\mathrm{Rad}(M)} + \frac{K + \mathrm{Rad}(M)}{\mathrm{Rad}(M)} = \frac{N + K + \mathrm{Rad}(M)}{\mathrm{Rad}(M)} = \frac{N + K}{\mathrm{Rad}(M)} = \frac{M}{\mathrm{Rad}(M)}$$

and

$$\frac{N}{\mathrm{Rad}(M)} \cap \frac{K + \mathrm{Rad}(M)}{\mathrm{Rad}(M)} = \frac{N \cap (K + \mathrm{Rad}(M))}{\mathrm{Rad}(M)} = \frac{N \cap K + \mathrm{Rad}(M)}{\mathrm{Rad}(M)} = \frac{\mathrm{Rad}(M)}{\mathrm{Rad}(M)} = 0$$

hence
$M/\mathrm{Rad}(M) = N/\mathrm{Rad}(M) \oplus (K + \mathrm{Rad}(M))/\mathrm{Rad}(M)$, as required. $\qquad\square$

Some properties of $\oplus$-Cofinitely supplemented modules will be revealed after some more notation and definitions.

Let $\{L_\lambda\}_{\lambda \in \Lambda}$ be a family of local submodules of M such that each of them is a direct summand of M. $\mathrm{Loc}^\oplus M$ will denote the sum of L_λs for all $\lambda \in \Lambda$. That is $\mathrm{Loc}^\oplus M = \sum_{\lambda \in \Lambda} L_\lambda$. Note 0 is a local submodule of M.

Lemma 4.5.3 ([4], Lemma 2.2). *Let M be an R-module. Then every maximal submodule of M has a supplement which is a direct summand of M if and only if $M/\mathrm{Loc}^\oplus M$ does not contain a maximal submodule.*

Proof. ($\Rightarrow$) Suppose that $M/Loc^\oplus M$ contains a maximal submodule $N/\text{Loc}^\oplus M$. Then N is a maximal submodule of M. By assumption there are submodules L, H such that $M = L \oplus H$ and $M = N + L$ and $N \cap L \ll L$. L is local by Lemma 3.2.14. So $L \subseteq \text{Loc}^\oplus M \subseteq N$ which is a contradiction since then $N + L = N \neq M$.

$\Leftarrow$) Let P be a maximal submodule of M. By assumption P does not contain $\text{Loc}^\oplus M$. Hence there is a local submodule L that is a direct summand of M such that L is not a submodule of P. By maximality of P, $P + L = M$ and $P \cap L \neq L(L \nsubseteq P)$. So $P \cap L \ll L$, since L is local. $\square$

A module M is said to have *the summand sum property (SSP)* if the sum of two direct summands of M is again a direct summand of M.
The following theorem characterizes $\oplus$-cofinitely supplemented modules that have the SSP.

Theorem 4.5.4 ([4], Theorem 2.3)**.** *Let M be an R-module with (SSP), then the following statements are equivalent*

1. *M is $\oplus$-Cofinitely supplemented.*

2. *Every maximal submodule of M has a supplement that is a direct summand of M.*

3. *$M/Loc^\oplus M$ does not contain a maximal submodule.*

Proof. $(1) \Rightarrow (2)$: Every maximal submodule is cofinite.

$(2) \Rightarrow (3)$: By the previous lemma.

$(3) \Rightarrow (1)$: Let N be a cofinite submodule of M. Then $N + \text{Loc}^\oplus M$ is a cofinite submodule of M by containing the cofinite submodule N, and by (3), we have $M = N + \text{Loc}^\oplus M$. Because M/N is finitely generated, there exist local submodules $L_{\lambda_i} \in \{L_\lambda\}_{\lambda \in \Lambda}, 1 \leq i \leq n$ for some positive integer n, such that $M = N + L_{\lambda_1} + ... + L_{\lambda_n}$. Clearly $N + L_{\lambda_1} + ... + L_{\lambda_n}$ has a supplement 0 in M. By Lemma 4.1.4, there exists a subset J of $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ such that $\sum_{j \in J} L_j$ is a supplement of N in M. By hypothesis, $\sum_{j \in J} L_j$ is a direct summand of M. Thus M is $\oplus$-Cofinitely supplemented. $\square$

In the class of ⊕-supplemented modules, it has been shown that finite direct sum of ⊕-supplemented is again ⊕-supplemented. Now we will present a stronger result about ⊕-Cofinitely supplemented.

Theorem 4.5.5 ([4], Theorem 2.6). *Arbitrary direct sum of ⊕-Cofinitely supplemented R-modules is ⊕-Cofinitely supplemented.*

Proof. Let $M_i (i \in I)$ be any collection of ⊕-Cofinitely supplemented R-modules. Let $M = \bigoplus_{i \in I} M_i$ and N be a cofinite submodule of M. Then M/N is generated by some finite set $\{x_1 + N, x_2 + N, ..., x_k + N\}$ and therefore $M = x_1 R + x_2 R + ... + x_k R + N$. Since each x_i is contained in the direct sum $\bigoplus_{j \in F_i} M_j$ for some finite subset F_i of I, $x_1 R + x_2 R + ... + x_k R \subseteq \bigoplus_{j \in F} M_j$ for some finite subset $F = \{i_1, i_2, ..., i_r\}$ of I. Then $M = \bigoplus_{t=1}^{r} M_{i_t} + N$. Clearly $M = M_{i_1} + (\bigoplus_{t=2}^{r} M_{i_t} + N)$ has a trivial supplement in M. Since M_{i_1} is ⊕-Cofinitely supplemented, $M_{i_1} \cap (\bigoplus_{t=2}^{r} M_{i_t} + N)$ has a supplement S_{i_1} in M_{i_1} such that S_{i_1} is a direct summand in M_{i_1}. By Lemma 3.2.10, S_{i_1} is a supplement of $(\bigoplus_{t=2}^{r} M_{i_t} + N)$ in M. Note that since M_{i_1} is a direct summand of M, S_{i_1} is also a direct summand of M. Continuing in this way, since the set J is finite at the end we will obtain that N has a supplement $S_{i_1} + S_{i_2} + ... + S_{i_r}$ in M such that every $S_{i_t} (1 \leq t \leq r)$ is a direct summand of M_{i_t}. Since every M_{i_t} is a direct summand of M, it follows that $\sum_{t=1}^{r} S_{i_t} = \bigoplus_{t=1}^{r} S_{i_t}$ is a direct summand of M. □

As a direct consequence of the previous Theorem and the fact that every ⊕-Supplemented module is ⊕-Cofinitely supplemented, we have this corollary.

Corollary 4.5.6 ([4], Corollary 2.7). *Any direct sum of ⊕-Supplemented modules is ⊕-Cofinitely supplemented.*

Therefore any direct sum of hollow(local) modules is ⊕-Cofinitely supplemented.

In recent years various generalizations of the notions studied in this thesis appeared in the literature. We mention here the definition of generalized supplements.

Definition 4.5.7. *Let K, L be submodules of a module M, we say that K is a generalized supplement of L in M if $M = K + L$ and $K \cap L \subseteq Rad(K)$. M is called generalized supplemented if every submodule of M has a generalized supplement in M.*

All the notions and concepts studied in this thesis have their generalized form in the same manner as generalized supplemented modules.

Chapter 5

Rings For Which Some Classes of Modules Are ⊕-Supplemented

In this chapter we study rings for which certain modules are ⊕-supplemented. It turns out that a ring R is (semi)-perfect if and only if every (finitely generated) free module is ⊕-supplemented.

Recall that an R-module M is ⊕-supplemented if for every submodule N of M there exist a summand K of M such that $M = N + K$ and $N \cap K \ll K$.

An R-module M is called lifting (or satisfies (D1)) if for every submodule N of M there are submodules K and H of M such that $M = K \oplus H, H \subseteq N$ and $N \cap K \ll K$. In fact another equivalent definition is given later on.

Every Hollow(local) module is lifting; let $N \subseteq M$ then $M = 0 \oplus M, 0 \subseteq N$ and $N \cap M = N \ll M$.

Moreover every lifting module is ⊕-supplemented, for, if $N \subseteq M$ then with $M = K \oplus H, H \subseteq N, K \cap N \ll K$ then $M = K + N$, hence by Lemma 3.1.2, K is a supplement of N which is a summand of M.

Now we are ready to give another definition of lifting module. Equivalent for the former definition, an R-module is lifting if for every $N \subseteq M$ there exists a direct summand H of M such that $H \subseteq N$ and $N/H \ll M/H$. For, assume the former definition, and let $N/H + X/H = M/H$ for some $X/H \subseteq M/H$, then $N + X = M$. Since $M = H \oplus K$, then $X = H + X \cap K$. Now $M = N + X = N + H + X \cap K = N + X \cap K$, implies, by the minimality of supplement K, that $X \cap K = K$, i.e. $K \subseteq X$ hence $H + K = M \subseteq X$. So $X = M$ and $N/H \ll M/H$.

Conversely, assume the latter definition. It remains to show that $N \cap K \ll K$. Let $K = X + N \cap K$ for some $X \subseteq K$, then $M = N + K = N + X + N \cap K = N + X$ as $N \cap K \subseteq N$. Now $N/H + (X + H)/H = M/H$, and since $N/H \ll M/H$, we must have $X + H = M$. By the minimality of a supplement(K is a supplement of H), we have $X = K$. So $K \cap N \ll K$.

5.1 Perfect and Semiperfect Rings

Let P be an R-module. If M is an R-module, then P is called M-projective in case for each epimorphism $g : M \to N$ and for each homomorphism $k : P \to N$ there is a homomorphism $h : P \to M$ such that $gh = k$. A module is said to be projective in case it is projective relative to every module M. Every free module is projective.

Every module is a homomorphic image of a free(hence projective)module. An epimorphism $f : P \to M$ with P projective is called a *projective cover of* M if $\mathrm{Ker} f \ll P$.

A ring R is right (semi)perfect if every (finitely generated)R-module has a projective cover. Also it is well known that for a ring R to be semi-perfect it suffices that every simple R-module has a projective cover.

Theorem 5.1.1 ([9], Theorem 2.1). *The following are equivalent for a ring R*

1. *R is semiperfect.*

2. *Every finitely generated free R-module is $\oplus$-supplemented.*

3. *R_R is $\oplus$-supplemented.*

4. *For every maximal right ideal A of R, there exists an idempotent $e \in R - A$ such that $A \cap eR \subseteq Rad(R)$.*

Proof. 1) $\Rightarrow$ 2) Let R be a semiperfect ring. Let M be a finitely generated free module. Let $A \subseteq M$ and P be a projective cover of M/A with respect to the small epimorphism $f : P \to M/A$. Since M is free, $M$is projective so the natural epimorphism $\pi : M \to M/A$ is factored as $\pi = fh$ with a homomorphism $h : M \to P$. Since $fh(M) = \pi(M) = M/A$ with f small epimorphism, it follows by [3], Lemma 5.15, that $h(M) = P$. But P is projective so the

epimorphism $h : M \to P$ must split i.e. $M = B \oplus \mathrm{Ker}h$ for some submodule B of M, and $P \cong M/\mathrm{Ker}h \cong B$.

Since $\pi = fh$ then $A = \mathrm{Ker}\pi = \mathrm{Ker}(fh) = h^{-1}(\mathrm{Ker}f)$. As $\mathrm{Ker}h \subseteq h^{-1}(\mathrm{Ker}f$, then $\mathrm{Ker}h \subseteq A$. So from $M = B \oplus \mathrm{Ker}h$ we have $M = A + B$. Now consider the restriction $\pi : B \to M/A$ is a product of the isomorphism $B \to P$ and the small epimorphism $f : P \to M/A$, thus $\pi|_B$ is a small epimorphism by Lemma 2.3.3(1) i.e. $\mathrm{Ker}(\pi|_B) = A \cap B \ll B$. Thus M is $\oplus$-supplemented.

$(2) \Rightarrow (3)$: Since R_R is a finitely generated(cyclic) free R-module, then by (2), R_R is $\oplus$-supplemented.

$(3) \Rightarrow (4)$: Let A be a maximal right ideal of R. By (3), there exists a direct summand K of R such that $R = A + K$ and $A \cap K \ll K$. There exists an idempotent e in R such that $K = eR$. Clearly $e \notin A$ otherwise $A + K = A \neq R$. Moreover $A \cap K \ll K \subseteq R_R$, hence $A \cap K \subseteq \mathrm{Rad}(R)$.

$(4) \Rightarrow (1)$: Let M be any simple R-module, then M is generated by any of its nonzero elements. Let $0 \neq m \in M$ and $B := \{r \in R | mr = 0\}$. B is a maximal right ideal of R by Corollary 3.9, of [3]. By (4) there exists an idempotent $e \in R - B$ such that $B \cap eR \subseteq \mathrm{Rad}(R)$. Since $eR \nsubseteq B$ and B is maximal then $R = eR + B$. As R_R is cyclic, $\mathrm{Rad}(R) \ll R$ and we have from $B \cap eR \subseteq \mathrm{Rad}(R) \ll R, B \cap eR \ll eR$, since eR is a direct summand of R. Now $eR/(B \cap eR) \cong (B + eR)/B = R/B \cong M$. So M has $eR/(B \cap eR)$ as a projective cover. $\qquad\square$

Corollary 5.1.2 ([9], Corollary 2.2). *A commutative ring R is semiperfect if and only if every cyclic R-module is $\oplus$-supplemented.*

Proof. $\Rightarrow$) Let R be a semiperfect ring, then by the previous Theorem the finitely generated free cyclic module is $\oplus$-supplemented.

$\Leftarrow$) Assume every cyclic R-module is $\oplus$-supplemented. Since R_R is a cyclic R-module, R_R is $\oplus$-supplemented, so by the previous Theorem R is semiperfect. $\qquad\square$

Theorem 5.1.3 ([9], Theorem 2.3). *Let R be any ring and let M be a finitely generated R-module such that every direct summand of M is $\oplus$-supplemented. Then M is a direct sum of cyclic modules.*

Proof. Suppose $M = m_1R + m_2R + ... + m_kR$ for some positive integer k and elements $m_i \in M (1 \leq i \leq k)$. If $k = 1$ then the result trivially holds.

Suppose that $k > 1$, and the result holds for $(k-1)$-generated modules with the stated condition. There exist submodules K, H of M such that $M = K \oplus H$, $M = m_1R + K$ and $m_1R \cap K \ll K$. Note that $H \cong M/K = (m_1R + K)/K \cong m_1R/(m_1R \cap K)$, so that H is cyclic. On the other hand $K/(m_1R \cap K) \cong (K + m_1R)/m_1R$. So that $K/(m_1R \cap K)$ is a $(k-1)$-generated module. Since $m_1R \cap K \ll K$ it follows that K is $(k-1)$-generated module. By induction step, K is a direct sum of cyclic modules. thus $M = K \oplus H$ is a direct sum of cyclic modules. $\square$

The following Corollary gives a characterization of $\oplus$-supplemented finitely generated modules.

Corollary 5.1.4 ([9], Corollary 2.6). *Let R be a ring. Then every finitely generated R-module is $\oplus$-supplemented if and only if*

(i) every cyclic R-module is $\oplus$-supplemented.

(ii) every finitely generated R-module is a direct sum of cyclic modules.

Proof. $\Rightarrow$: i) Trivial. Every cyclic module is finitely generated.

ii) : For a finitely generated module M, every direct summand is finitely generated, so the previous Theorem applies and M is a direct sum of cyclic modules.

$\Leftarrow$: By Theorem 4.4.2, any finite direct sum of $\oplus$-supplemented modules is $\oplus$-supplemented. $\square$

A commutative ring R is called *FGC* ring if every finitely generated module is a direct sum of cyclic modules.

The following Proposition gives a class of rings whose finitely generated modules are $\oplus$-supplemented.

Proposition 5.1.5 ([9], Proposition 2.8). *Let R be a commutative ring. Then the following statements are equivalent*

1. Every finitely generated R-module is $\oplus$-supplemented.

2. R is a semiperfect FGC ring.

Proof. $(1) \Leftrightarrow (2)$ By Corollaries 5.1.2 and 5.1.4. $\square$

So far we have proved that a ring R is semiperfect if and only if every finitely generated free R-module is $\oplus$-supplemented.

At the end, we will prove a similar result for perfect rings.

Theorem 5.1.6 ([9], Theorem 2.11)**.** *A ring R is right perfect if and only if every free right R-module is $\oplus$-supplemented.*

Proof. $\Rightarrow$) Let R be a perfect ring. Let M be any free right R-module. Let $A \subseteq M$ and P be a projective cover of M/A with respect to the small epimorphism $f : P \to M/A$. Since M is free, M is projective so the natural epimorphism $\pi : M \to M/A$ is factored as $\pi = fh$ with a homomorphism $h : M \to P$. Since $fh(M) = \pi(M) = M/A$ with f small epimorphism, it follows by [3], Lemma 5.15, that $h(M) = P$. But P is projective so the epimorphism $h : M \to P$ must split i.e. $M = B \oplus \mathrm{Ker}h$ for some submodule B of M, and $P \cong M/\mathrm{Ker}h \cong B$.
Since $\pi = fh$ then $A = \mathrm{Ker}\pi = \mathrm{Ker}(fh) = h^{-1}(\mathrm{Ker}f)$. As $\mathrm{Ker}h \subseteq h^{-1}(\mathrm{Ker}f$, then $\mathrm{Ker}h \subseteq A$. So from $M = B \oplus \mathrm{Ker}h$ we have $M = A + B$. Now consider the restriction $\pi : B \to M/A$ is a product of the isomorphism $B \to P$ and the small epimorphism $f : P \to M/A$, thus $\pi|_B$ is a small epimorphism by Lemma 2.3.3(1) i.e. $\mathrm{Ker}(\pi|_B) = A \cap B \ll B$. Thus M is $\oplus$-supplemented.

$\Leftarrow$) Assume (2). Let M be any R-module, then there exists an epimorphism $h : F \to M$ with F free module. Then by our assumption, for $\mathrm{Ker}h \subseteq F$, there exists a supplement K which is a direct summand of F. Now the restriction of h into $K, h|_K : K \to M$ is a small epimorphism as $M \cong F/\mathrm{Ker}h \cong K/(K \cap \mathrm{Ker}h)$ and $K \cap \mathrm{Ker}h \ll K$ since K is a supplement of $\mathrm{Ker}h$. So M has a projective cover hence R is perfect. $\qquad \square$

In fact an equivalent condition also for the previous Theorem is that the R-module $R^{(N)}$ is $\oplus$-supplemented.([9], Theorem 2.10)

It was shown in Theorem 4.4.2 that any *finite* direct sum of $\oplus$-supplemented modules is $\oplus$-supplemented, but it is not generally true that any infinite direct sum of $\oplus$-supplemented modules is $\oplus$-supplemented.

Example 5.1.7 ([4], § 1)**.** *Let R be a semiperfect ring not right perfect. Then the R-module R_R is $\oplus$-supplemented by Theorem 5.1.1 but the R-module $R^{(N)}$ is not $\oplus$-supplemented since R is not perfect.*

Supplemented modules	Relative to supplements two classes of modules are studied.	cofinitely supplemented modules
✓	closeness under homomorphic images	✓
✓	closeness under small covers	✓
✓	closeness under supplements	✓
✓	closeness under finite (direct)sum	✓
X; Example 3.2.13(4)	infinite sum	✓
supplemented module	$\longrightarrow$	cofinitely supplemented
supplemented	$\overset{}{\longleftarrow\!\!\!/}$ Q_Z	cofinitely supplemented module
A finitely generated module is supplemented	$\Longleftrightarrow$	cofinitely supplemented.

weakly Supplemented modules	Relative to weak supplements two classes of modules are studied.	cofinitely weak supplemented modules
✓	closeness under homomorphic images	✓
✓	closeness under small covers	✓
✓	closeness under supplements	✓
✓	closeness under finite (direct)sum	✓
X; Example 4.3.11	infinite sum	✓
weakly supplemented module	$\longrightarrow$	cofinitely weak supplemented
weakly supplemented	$\longleftarrow\!\!\!/$ Example 4.3.11	cofinitely weak supplemented module
Supplemented module	$\longrightarrow$	weakly supplemented
supplemented	$\longleftarrow\!\!\!/$ Q_Z Example 4.3.13	weakly supplemented

⊕-Supplemented modules	Relative to ⊕-supplements two classes of modules are studied.	⊕-cofinitely supplemented modules
✓	closeness under finite direct sum	✓
X; Example 5.1.7	infinite direct sum	✓
⊕-supplemented	$\not\longrightarrow$ Example 4.4.4	factor module is ⊕-supplemented
⊕-supplemented module	$\longrightarrow$	⊕-cofinitely supplemented
⊕-supplemented	$\longleftarrow\!\!\!/\ \ Q_Z$	⊕-cofinitely supplemented module
A finitely generated module is ⊕-supplemented	$\Longleftrightarrow$	⊕-cofinitely supplemented.
⊕-supplemented module	$\longrightarrow$	supplemented
Supplemented	$\not\longrightarrow$ Example 4.4.4	⊕-supplemented

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