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Qualitative behavior of some difference equations

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الإهداء

- بداية أقدم عملي المتواضع خالصا لوجه الله الكريم وأرجو منه القبول.
- إلى معلم البشرية الأول...حبيبنا ورسولنا...صلوا عليه.
- إلى المخلصين الذين قدموا أرواحهم لنصرة دينهم ووطنهم.
- إلى من أوصاني الله ببرهم.. وأسأل الله أن يحفظهم.. إلى أغلى الحبايب.
- إلى شريك حياتي ورفيق عمري..جزاك الله خيراً..
- إلى زينة حياتي وأمل عمري.. ضحى وعمر.. رعاهما الله.
- إلى كل أهلي وأحبائي..وأخص منهم أخي وأخواتي.
- إلى أساتذتي الكرام من لم يبخلوا علينا بكل ما لديهم.
- إلى أهل الحب وأهل الإخاء وأهل الوفاء..جمعنا الله على منابر من نور.

الملخص

في هذا البحث سنقوم بدراسة السلوك النوعي لبعض المعادلات التفاضلية المنفصلة، وسندعم نتائجنا بأمثلة عددية أجريت باستخدام MATLAB 6.5 وسيكون تركيزنا على إيجاد الفترة غير المختلفة، والحلول الدورية، وتحليل أنصاف الدورات، والثبات الشامل لجميع الحلول الموجبة لهذه المعادلات.

سندرس بشكل رئيسي الحلول الموجبة للمعادلتين التاليتين:

- المعادلة التفاضلية المنفصلة الأولى هي:

$$x_{n+1} = \mathrm{A}x_n + \frac{\beta x_n + \gamma\, x_{n-\mathrm{k}}}{\mathrm{B}x_n + \mathrm{C}x_{n-\mathrm{k}}} \,, n = 0,1,2, \dots \dots \dots.$$

وذلك عندما تكون القيم الابتدائية $x_0\,, x_{-1}, \dots., x_{-\mathrm{k}}$ هي أعداد حقيقية موجبة، والمعاملات $\mathrm{A}, \beta, \gamma, \mathrm{B}, \mathrm{C}$ هي ثوابت موجبة، بينما k هي عدد صحيح موجب.

- المعادلة التفاضلية المنفصلة الثانية هي:

$$x_{n+1} = \mathrm{A}x_n + \frac{p x_n + x_{n-\mathrm{k}}}{q + x_{n-\mathrm{k}}} \,, n = 0,1,2, \dots \dots \dots.$$

وذلك عندما تكون القيم الابتدائية $x_0\,, x_{-1}, \dots., x_{-\mathrm{k}}$ هي أعداد حقيقية موجبة، والمعاملات A, p, q هي ثوابت موجبة، بينما k هي عدد صحيح موجبة.

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Abstract

In this thesis we will study the qualitative behavior of some difference equations, and we will support our results by numerical discussion using MATLAB 6.5. Our concentration is on invariant intervals, periodic solutions, semicycle analysis, and the global asymptotic stability of all positive solutions of these equations.

We mainly study the positive solutions of the following two difference equations:
The first difference equation is

$$x_{n+1} = Ax_n + \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}} \qquad , n = 0, 1, 2, ... \tag{1}$$

where the initial conditions $x_{-k}, \cdots, x_{-1}, x_0$ are arbitrary positive real numbers and the coefficients A, β, γ, B, C are positive constants, while k is a positive integer number.

The second difference equation is

$$x_{n+1} = Ax_n + \frac{px_n + x_{n-k}}{q + x_{n-k}} \qquad , n = 0, 1, 2, ... \tag{2}$$

where the initial conditions $x_{-k}, \cdots, x_{-1}, x_0$ are arbitrary positive real numbers and the coefficients A, p, q are positive constants, while k is a positive integer number.

Introduction

The function $f(x) = 2x$ is a rule that assigns to each number x a number twice as large. This is a simple mathematical model. We might imagine that x denotes the population of bacteria in a laboratory culture and that $f(x)$ denotes the population one hour later. Then the rule expresses the fact that the population doubles every hour. If the culture has an initial population of $10,000$ bacteria, then after one hour there will be $f(10,000) = 20,000$ bacteria, after two hours there will be $f(f(10,000)) = 40,000$ bacteria, and so on.

A **dynamical system** consists of a set of possible states, together with a rule that determines the present state in terms of past states. In the previous paragraph, we discussed a simple dynamical system whose states are population levels, that change with time under the rule $x_n = f(x_{n-1}) = 2x_{n-1}$. Here the variable n stands for time, and x_n designates the population at time n. We will require that the rule be deterministic, which means that we can determine the present state (population, for example) uniquely from the past states.

We will emphasize two types of dynamical systems. If the rule is applied at discrete times, it is called a **discrete-time dynamical system (Map)**. A discrete-time system takes the current state as input and updates the situation by producing a new state as output. By the state of the system, we mean whatever information is needed so that the rule may be applied. In the first example above, the state is the population size. The rule replaces the current population with the population one hour later.

The other important type of dynamical system is essentially the limit of discrete systems with smaller and smaller updating times. The governing rule in that case becomes a set of differential equations, and the term **continuous-time dynamical system** is sometimes used.

This thesis includes two main parts: The first one is a background about dynamical systems, and the second part deals with some discrete-time dynamical system (difference equations). Part one includes chapters 1, 2 and 3, whereas part two includes chapters 4 and 5.

Chapter 1 gives an introduction to dynamical systems, while Chapter 2 is a primary chapter because it gives all preliminary results which are used in the thesis, it deals with the stability and linearized stability, semi-cycle analysis, criterion for the asymptotic stability, and the global asymptotic stability. Chapter 3 gives an idea about linear and nonlinear dynamical systems.

In chapter 4 we study the qualitative behavior of the solutions of the difference equation

$$x_{n+1} = Ax_n + \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}} \qquad , n = 0, 1, 2, ... \tag{3}$$

where the initial conditions x_{-k}, $\cdots$, x_{-1}, x_0 are arbitrary positive real numbers and the coefficients A, β, γ, B, C are positive constants, while k is a positive integer number.

Our concentration is on invariant intervals, semicycle analysis, and the global asymptotic stability of all positive solutions of Eq.(3).

E.M.E Zayed et al.[16] have investigated the global stability of Eq.(3), but their results are not accurate, and our aim is to correct these results.

Chapter 5 discusses the difference equation

$$x_{n+1} = Ax_n + \frac{px_n + x_{n-k}}{q + x_{n-k}} \qquad , n = 0, 1, 2, ... \tag{4}$$

where the initial conditions x_{-k}, $\cdots$, x_{-1}, x_0 are arbitrary positive real numbers and the coefficients A, p, q are positive constants, while k is a positive integer number.

At the end of Chapters 4 and 5 we give numerical discussion using MATLAB 6.5 which supports our theoretical results, and the codes are included in the thesis.

Chapter 1

An Introduction to Dynamical Systems

1.1 What is a dynamical system?

A dynamical system is a function which is doing the same thing over and over again, it predicts what you are going to do next. Mathematically a dynamical system has two parts: a **state vector** which describes exactly the state of some real or hypothetical system, and a **function** (i.e., a rule) which tells us, given the current state, what the state of the system will be in the next instant of time.

1.1.1 State Vectors

Dynamical systems can be described by numbers. For example, a ball tossed straight up can be described using two numbers: its height h above the ground and its (upward) velocity v. The pair of numbers (h, v) is a vector which completely describes the state of the ball and hence is called the state vector of the system, and we present it as:

$$\begin{bmatrix} h \\ v \end{bmatrix}$$

It may be possible to describe the state of a system by a single number. For example, consider a bank account opened with \$100 at 6% interest compounded annually. The state of this system at any instant in time can be described by a single number: the balance in the account. In this case, the state vector has just one component.

On the other hand, some dynamical systems require many numbers to describe them. For example, a dynamical system modeling global weather might have millions of variables accounting for temperature, pressure, wind speed, etc. at points all around the world. Although extremely complex, the state of the system is simply a list of numbers as a vector.

1.1.2 The next instant: discrete time

The second part of a dynamical system is a rule which tells us how the system changes over time. In other words, if we are given the current state of the system, the rule tells us the state of the system in the next instant. In the case of the bank account described above, the next instant will be one year later, since interest is paid only annually; time is discrete, it is easy to write down the rule which takes us from the state of the system at one instant to the state of the system in the next instant:

$$x(k+1) = 1.06x(k)$$

$$x(0) = x_0 = 100$$

The state of the bank account in all future years can now be computed. We see that $x(1) = 1.06x(0) = 1.06 \times 100 = 106$, and then $x(2) = 1.06x(1) = 1.06 \times 106$, and so

$$x(k) = (1.06)^k \times 100$$

or more generally,

$$x(k) = (1.06)^k \times x_0$$

A larger context

Let us put this example into a broader context which is applicable to all discrete time dynamical systems. We have a state vector $x \in \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}^n$ for which

$$x(k+1) = f(x(k))$$

Once we are given that $x(0) = x_0$ and that $x(k+1) = f(x(k))$, we can compute all values of $x(k)$, as follows:

$$x(1) = f(x(0)) = f(x_0)$$

$$x(2) = f(x(1)) = f(f(x_0))$$

$$x(3) = f(x(2)) = f(f(f(x_0)))$$

$$x(4) = f(x(3)) = f(f(f(f(x_0))))$$

$$\vdots$$

$$x(k) = f(x(k-1)) = f(f(\cdots(f(x_0))\cdots))$$

where in the last line we have f applied k times to x_0, and is written as

$$f^k(x) = \underbrace{f(f(f(\cdots f(x)\cdots)))}_{k\text{times}}$$

1.1.3 The next instant: continuous time

Many systems are better described with time progressing smoothly. Consider our earlier example of a ball thrown straight up. Its instantaneous status is given by its state vector

$$x = \begin{bmatrix} h \\ v \end{bmatrix}$$

However, it does not make sense to ask what its state will be in the next instant of time, there is no next instant since time advances continuously. We reflect this differently by using the letter t (rather than k) to denote time.

If the ball has (upward) velocity v, then we know that $\frac{dh}{dt} = h'(t) = v(t)$ and $\frac{dv}{dt} = v'(t) = -g$. The change in the system can thus be described by

$$\begin{bmatrix} h'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} h(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

using that

$$x(t) = \begin{bmatrix} h \\ v \end{bmatrix}$$

we have

$$x' = f(x) \tag{1.1}$$

where $f(x) = Ax + b$,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

Indeed, Eq.(1.1) is the form for all continuous time dynamical systems. A continuous time dynamical system has a state vector $x(t) \in \mathbb{R}^n$ and we are given a function $f : \mathbb{R}^n \to \mathbb{R}^n$ which specifies how quickly each component of $x(t)$ is changing.

Returning to the example at hand, suppose the ball starts at height h_0 and with upward velocity v_0, i.e.,

$$x_0 = \begin{bmatrix} h_0 \\ v_0 \end{bmatrix}$$

We claim that the equations

$$h(t) = h_0 + v_0 t - \tfrac{1}{2} g t^2$$

and

$$v(t) = v_0 - gt$$

describe the motion of the ball, and this can be verified easily.

1.2 What we want; what we can get

The notion of a dynamical system can be useful in modeling many different kinds of phenomena. Once we have created a model, we would like to use it to make predictions. Given a dynamical system either of the discrete form $x(k+1) = f(x(k))$ or of the continuous sort $x' = f(x)$, and an initial value x_0, we would very much like to know the value of $x(k)$ [or, $x(t)$] for all values of k [or t]. In some rare instances, this is possible. For example, when f is a linear function. Unfortunately, it is all too common that the dynamical system in which we are interested does not yield an analytic solution, so one option is using numerical methods. However, we can also determine the qualitative nature of the solution which we will focus on in this thesis.

Chapter 2

Preliminary Results

2.1 Introduction

In this chapter we present some definitions and state some known results which will be useful in the subsequent chapters. For further details and additional references see [1], [3], [4], [7] and [8].

2.2 Definitions of Stability and Linearized Stability

Definition 1 *([6]): A difference equation of order* $(k+1)$ *is an equation of the form*

$$x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k}) \qquad , n = 0, 1, 2, ... \tag{2.1}$$

where f *is a continuous function which maps some set* J^{k+1} *in to* J*. The set* J *is usually an interval of real numbers, or a union of intervals, but it may even be a discrete set such as the set of integers* $\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$*.*

A solution of Eq.(2.1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ which satisfies Eq.(2.1) for all $n \geq 0$. If we prescribe a set of $(k+1)$ initial conditions

$$x_{-k}, x_{-k+1}, \cdots, x_0 \in J$$

then

$$x_1 = f(x_0, x_{-1}, \cdots, x_{-k})$$

$$x_2 = f(x_1, x_0, \cdots, x_{-k+1})$$

$$\vdots$$

and so the solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.1) exists for all $n \geq -k$ and is uniquely determined by the initial conditions.

Definition 2 *[14]: A point $\bar{x}$ is called an equilibrium point of Eq.(2.1) if*

$$\bar{x} = f(\bar{x}, \bar{x}, \cdots, \bar{x})$$

That is, $x_n = \bar{x}$ for $n \geq k$ is a solution of Eq.(2.1), or equivalently, $\bar{x}$ is a fixed point of f.

Definition 3 *([14]): Let $\bar{x}$ be an equilibrium point of Eq.(2.1) and assume that I is some interval of real numbers.*

(a) The equilibrium $\bar{x}$ of Eq.(2.1) is called ***locally stable*** *(or stable) if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $x_{-k}, \cdots, x_{-1}, x_0 \in I$ and*

$$|x_{-k} - \bar{x}| + \cdots + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta$$

then

$$|x_n - \bar{x}| < \epsilon \text{ for all } n \geq -k.$$

Figure(2.1) shows a stable equilibrium point of a difference equation of the first order.

(b) The equilibrium $\bar{x}$ of Eq.(2.1) is called ***locally asymptotically stable*** *(or asymptotically stable) if it is stable and if there exist $\gamma > 0$ such that if $x_{-k}, \cdots, x_{-1}, x_0 \in I$ and*

$$|x_{-k} - \bar{x}| + \cdots + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta$$

then

$$\lim_{n \to \infty} x_n = \bar{x}$$

Figure(2.2) shows an asymptotically stable equilibrium point of a difference equation of the first order.

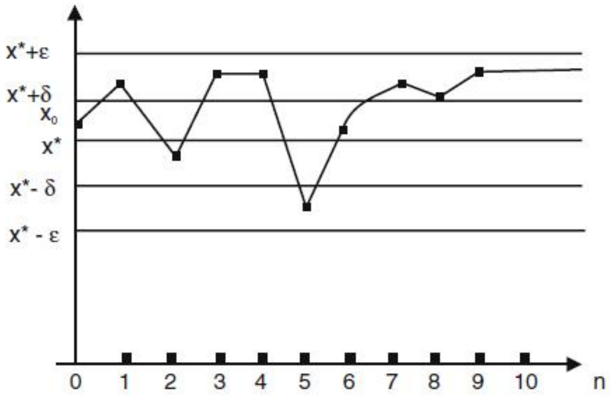

Figure 2.1: Stable equilibrium point x^* of a first order difference equation

(c) *The equilibrium $\bar{x}$ of Eq.(2.1) is called a* ***global attractor*** *if for every $x_{-k}, \cdots, x_0 \in I$, we have*

$$\lim_{n\to\infty} x_n = \bar{x}$$

(d) *The equilibrium $\bar{x}$ of Eq.(2.1) is called a* ***globally asymptotically stable*** *(or globally stable) if it is stable and is a global attractor. Figure(2.3) shows a globally stable equilibrium point of a first order difference equation.*

(e) *The equilibrium $\bar{x}$ of Eq.(2.1) is called* ***unstable*** *if it is not stable. Figure(2.4) shows an unstable equilibrium point of a first order difference equation.*

(f) *The equilibrium $\bar{x}$ of Eq.(2.1) is called a* ***repeller*** *(or a source) if there exist $r > 0$ such that if $x_{-k}, \cdots, x_{-1}, x_0 \in I$ and*

$$|x_{-k} - \bar{x}| + \cdots + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < r$$

then there exists $N \geq 1$ such that

$$|x_N - \bar{x}| \geq r$$

Clearly, a repeller is an unstable equilibrium point.

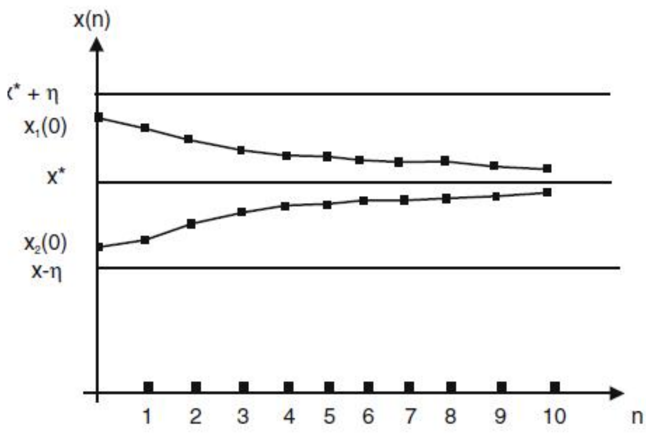

Figure 2.2: Asymptotically stable equilibrium point x^* of a first order difference equation

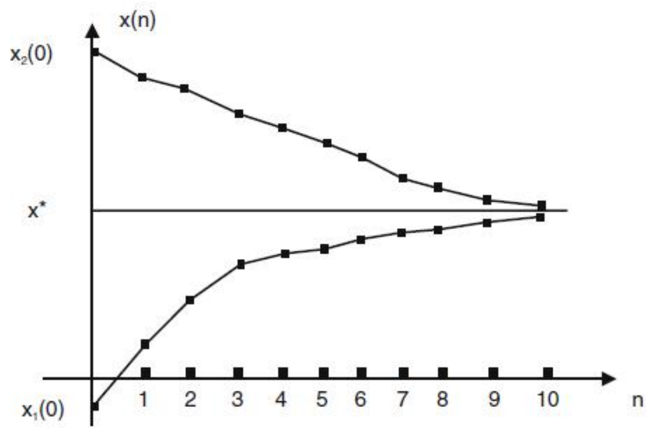

Figure 2.3: Globally asymptotically stable equilibrium point x^* of a first order difference equation

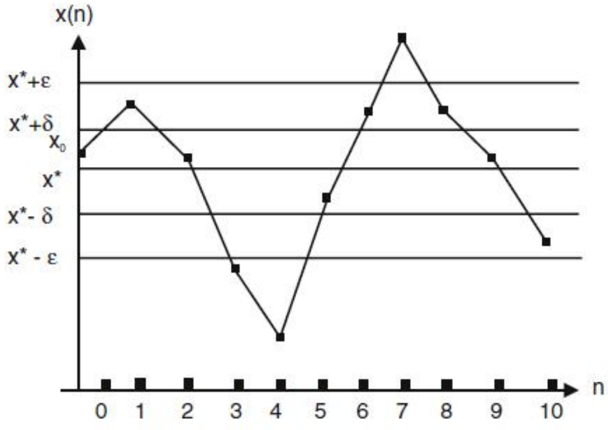

Figure 2.4: Unstable equilibrium point x^* of a first order difference equation

Suppose f is continuously differentiable in some open neighborhood of $\bar{x}$. Let

$$p_i = \frac{\partial f}{\partial u_i}(\bar{x}, \bar{x}, \cdots, \bar{x}) \qquad \text{for } i = 0,\ 1,\ \cdots,\ k$$

denote the partial derivative of $f(u_0, u_1, \cdots, u_k)$ with respect to u_i evaluated at the equilibrium point $\bar{x}$ of Eq.(2.1)

Then the equation

$$z_{n+1} = p_0 z_n + p_1 z_{n-1} + \cdots + p_k z_{n-k} \qquad n = 0, 1, \cdots \tag{2.2}$$

is called the linearized equation of Eq.(2.1) about the equilibrium point $\bar{x}$, and the characteristic equation of Eq.(2.1) about the equilibrium point $\bar{x}$ is:

$$\lambda^{k+1} - p_0 \lambda^k - \cdots - p_{k-1}\lambda - p_k = 0 \tag{2.3}$$

Now we will give a primary theorem which is applicable to the difference equation of the following form which is a special case of Eq.(2.1).

$$x_{n+1} = f(x_n, x_{n-k}) \qquad , n = 0, 1, 2, ... \tag{2.4}$$

Let

$$\rho_0 = \frac{\partial f}{\partial u_0}(\bar{x}, \bar{x})$$

and

$$\rho_1 = \frac{\partial f}{\partial u_1}(\bar{x}, \bar{x})$$

The linearized equation of Eq.(2.4) about the equilibrium point $\bar{x}$ is

$$z_{n+1} = \rho_0 z_n + \rho_1 z_{n-k}. \tag{2.5}$$

Theorem 2.2.1 *([16])*
Assume that $\rho_0, \rho_1 \in \mathbb{R}$ and $k \in \{1, 2, \cdots\}$. Then

$$|\rho_0| + |\rho_1| < 1 \tag{2.6}$$

is a sufficient condition for the asymptotic stability of the difference equation (2.4). Suppose in addition that one of the following two cases holds:

1. *k is an odd integer and $\rho_1 > 0$.*

2. *k is an even integer and $\rho_0 \rho_1 > 0$.*

Then (2.6) is also a necessary condition for the asymptotic stability of Eq.(2.4).

The following well-known result, called the Linearized Stability Theorem, is very useful in determining the local stability character of the equilibrium point $\bar{x}$ of Eq.(2.1).

Theorem 2.2.2 *([6])*
Suppose f is a continuously differentiable function defined on some open neighborhood of $\bar{x}$. Then the following statements are true:

1. *If all the roots of Eq.(2.3) have absolute value less than one, then the equilibrium point $\bar{x}$ of Eq.(2.1) is locally asymptotically stable.*

2. *If at least one of the roots of Eq.(2.3) has absolute value greater than one, then the equilibrium point $\bar{x}$ of Eq.(2.1) is unstable.*

3. *If all the roots of Eq.(2.3) have absolute value greater than one, then the equilibrium point $\bar{x}$ of Eq.(2.1) is a source.*

The equilibrium point $\bar{x}$ of Eq.(2.1) is called *hyperbolic* if no root of Eq.(2.3) has absolute value equal to one. If there exists a root of Eq.(2.3) with absolute value equal to one, then $\bar{x}$ is called non-hyperbolic.

The equilibrium point $\bar{x}$ of Eq.(2.1) is called a *sink* if every root of Eq.(2.3) has absolute value less than one. Thus a sink is locally asymptotically stable, but the converse need not be true.

The equilibrium point $\bar{x}$ of Eq.(2.1) is called a *saddle point equilibrium point* if it is hyperbolic, and if in addition, there exists a root of Eq.(2.3) with absolute value less than one and another root of Eq.(2.3) with absolute value greater than one. In particular, a saddle point equilibrium point is unstable.

A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.1) is called periodic with period p (or a period p solution) if there exists an integer $p \geq 1$ such that

$$x_{n+p} = x_n \qquad \forall n \geq -k \tag{2.7}$$

We say that the solution is periodic with prime period p if p is the smallest positive integer for which Eq.(2.7) holds. In this case a p-tuple

$$(x_{n+1}, x_{n+2}, \cdots, x_{n+p})$$

of any p consecutive values of the solution is called a p-cycle of Eq.(2.1).

A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.1) is called eventually periodic with period p if there exists an integer $N \geq -k$ such that $\{x_n\}_{n=N}^{\infty}$ is periodic with period p ; that is

$$x_{n+p} = x_n \qquad \forall n \geq N$$

The **orbit** of x under f is the set of points $\{x, f(x), f^2(x), \cdots\}$. The starting point x for the orbit is called **the initial value** of the orbit. For example, the orbit of $x = 0.01$ under the function $g(x) = 2x(1-x)$ is $\{0.01, 0.0198, 0.0388, \cdots\}$

2.3 Semi-cycle Analysis

Assume that $\bar{x}$ is an equilibrium point of Eq.(2.1), and let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(2.1). A **positive semi-cycle** of $\{x_n\}_{n=-k}^{\infty}$ consists of "a string" of terms $\{x_l, x_{l+1}, \cdots, x_m\}$ all greater than or equal to $\bar{x}$, with $l \geq -k$ and $m \leq \infty$ such that

$$\text{either} \qquad l = -k \qquad \text{or} \qquad l > -k \quad \text{and} \quad x_{l-1} < \bar{x}$$

and

$$\text{either} \qquad m = \infty \qquad \text{or} \qquad m < \infty \quad \text{and} \quad x_{m+1} < \bar{x}$$

A **negative semi-cycle** of $\{x_n\}_{n=-k}^{\infty}$ consists of "a string" of terms $\{x_l, x_{l+1}, \cdots, x_m\}$ all less than $\bar{x}$, with $l \geq -k$ and $m \leq \infty$ such that

$$\text{either} \qquad l = -k \qquad \text{or} \qquad l > -k \quad \text{and} \quad x_{l-1} \geq \bar{x}$$

and

$$\text{either} \qquad m = \infty \qquad \text{or} \qquad m < \infty \quad \text{and} \quad x_{m+1} \geq \bar{x}$$

A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.1) is called *non-oscillatory* about $\bar{x}$ if there exists $N \geq -k$ such that either

$$x_n > \bar{x} \qquad \text{for all } n \geq N$$

or

$$x_n < \bar{x} \qquad \text{for all } n \geq N$$

Otherwise, $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.1) is called oscillatory about $\bar{x}$.

A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.1) is called *strictly oscillatory* about $\bar{x}$ if for every $N \geq -k$ there exists $m, n \geq N$ such that $x_m < \bar{x}$ and $x_n > \bar{x}$.

We say that a positive solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.1) *persists* (or is persistent) if there exists a positive constant m such that

$$m \leq x_n \qquad \text{for all } n \geq -k$$

Eq.(2.1) is said to be *permanent* if there exist positive real numbers m and M such that for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.1) there exists an integer $N \geq -k$ (which depends upon the initial conditions $x_{-k}, x_{-k+1}, \cdots, x_{-1}, x_0$) such that

$$m \leq x_n \leq M \qquad \text{for all } n \geq N$$

2.4 Criterion for the Asymptotic Stability

In this section we give a simple but powerful criterion for the asymptotic stability of equilibrium points. The following theorem is our main tool in this section.

Theorem 2.4.1 *([4])*
Let $\overline{x}$ be an equilibrium point of the difference equation

$$x(n+1) = f(x(n)) \tag{2.8}$$

where f is continuously differentiable at $\overline{x}$. The following statements then hold:

(i) If $|f'(\overline{x})| < 1$, then $\overline{x}$ is asymptotically stable.

(ii) If $|f'(\overline{x})| > 1$, then $\overline{x}$ is unstable.

Remark: In the literature of dynamical systems, the equilibrium point $\overline{x}$ is said to be hyperbolic if $|f'(\overline{x})| \neq 1$.

Observe that Theorem (2.4.1) does not address the non-hyperbolic case where $|f'(\overline{x})| = 1$. Further analysis is needed here to determine the stability of the equilibrium point $\overline{x}$. Our first discussion will address the case where $f'(\overline{x}) = 1$.

Theorem 2.4.2 *([4])*
Suppose that for an equilibrium point $\overline{x}$ of (2.8), $f'(\overline{x}) = 1$. The following statements then hold:

1. *If $f''(\overline{x}) \neq 0$, then $\overline{x}$ is unstable.*
2. *If $f''(\overline{x}) = 0$, and $f'''(\overline{x}) > 0$, then $\overline{x}$ is unstable.*
3. *If $f''(\overline{x}) = 0$, and $f'''(\overline{x}) < 0$, then $\overline{x}$ is asymptotically stable.*

Now we will investigate the case $f'(\overline{x}) = -1$. But before doing so, we need to introduce the notion of the Schwarzian derivative of a function f:

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

Note that if $f'(\overline{x}) = -1$, then

$$Sf(\overline{x}) = -f'''(\overline{x}) - \frac{3}{2}(f''(\overline{x}))^2$$

Theorem 2.4.3 *[4]*
Suppose that for an equilibrium point $\overline{x}$ *of (2.8),* $f'(\overline{x}) = -1$*. The following statements then hold:*

1. *If* $Sf(\overline{x}) < 0$*, then* $\overline{x}$ *is asymptotically stable.*

2. *If* $Sf(\overline{x}) > 0$*, then* $\overline{x}$ *is unstable.*

2.5 Global Asymptotic Stability

Unfortunately when we need to establish the global attractivity of the positive equilibrium of a difference equation, we face the problem that there are not enough results in the literature to cover all various cases. In this section we mention the primary theorems in investigating the global stability of the positive equilibrium of Eq.(2.4).

Theorem 2.5.1 *([10])*

Let $[a, b]$ *be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(x, y)$ *is non-decreasing in* $x \in [a, b]$ *for each* $y \in [a, b]$*, and* $f(x, y)$ *is non-increasing in* $y \in [a, b]$ *for each* $x \in [a, b]$*.*

(b) If $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$f(m, M) = m \qquad \text{and} \qquad f(M, m) = M$$

then $m = M$*.*

Then Eq.(2.4) has a unique equilibrium $\overline{x} \in [a, b]$ *and every solution of Eq.(2.4) converges to* $\overline{x}$*.*

Theorem 2.5.2 *([10])*

Let $[a, b]$ *be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(x, y)$ *is non-increasing in* $x \in [a, b]$ *for each* $y \in [a, b]$, *and* $f(x, y)$ *is non-decreasing in* $y \in [a, b]$ *for each* $x \in [a, b]$.

(b) The difference equation Eq.(2.4) has no solutions of prime period two in $[a, b]$.

Then Eq.(2.4) has a unique equilibrium $\overline{x} \in [a, b]$ *and every solution of Eq.(2.4) converges to* $\overline{x}$.

Theorem 2.5.3 *([10])*

Let $[a, b]$ *be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(x, y)$ *is non-increasing in each of its arguments.*

(b) If $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$f(m, m) = M \qquad \text{and} \qquad f(M, M) = m$$

then $m = M$.

Then Eq.(2.4) has a unique equilibrium $\overline{x} \in [a, b]$ *and every solution of Eq.(2.4) converges to* $\overline{x}$.

Theorem 2.5.4 *([10])*

Let $[a, b]$ *be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(x, y)$ *is non-decreasing in each of its arguments.*

(b) The equation

$$f(x,x) = x \text{ has a unique positive solution.}$$

Then Eq.(2.4) has a unique equilibrium $\overline{x} \in [a,b]$ and every solution of Eq.(2.4) converges to $\overline{x}$.

Theorem 2.5.5 *([10])*

Let $I \subseteq [0,\infty)$ be some interval and assume that $f \in C[I \times I, (0,\infty)]$ satisfies the following conditions:

(i) $f(x,y)$ is non-decreasing in each of its arguments.

(ii) Eq.(2.4) has a unique positive point $\overline{x} \in I$ and the function $f(x,x)$ satisfies the ***negative feedback condition****:*

$$(x-\overline{x})(f(x,x)-x) < 0 \quad \text{for every} \qquad x \in I - \{\overline{x}\}$$

Then every positive solution of Eq.(2.4) with initial conditions in I converges to $\overline{x}$.

The following result extends Theorems (2.5.1), (2.5.3) and (2.5.4) to be applicable to Eq.(2.1).

Theorem 2.5.6 *([6])*

Let $F : [a,b]^{k+1} \to [a,b]$ be a continuous function, where k is a positive integer, and where $[a,b]$ is an interval of real numbers and consider the difference equation (2.1). Suppose that F satisfies the following conditions:

(i) For each integer i with $1 \leq i \leq k+1$, the function $F(z_1, z_2, \cdots, z_{k+1})$ is weakly monotonic in z_i for fixed z_1, z_2, $\cdots$, z_{i-1}, z_{i+1}, $\cdots$, z_{k+1}.

(ii) If (m, M) is a solution of the system

$$m = F(m_1, m_2, \cdots, m_{k+1}) \qquad \text{and} \qquad M = F(M_1, M_2, \cdots, M_{k+1})$$

then $m = M$, *where for each* $i = 1, 2, \cdots, k+1$, *we set*

$$m_i = \begin{cases} m & \text{if } F \text{ non-decreasing in } z_i \\ M & \text{if } F \text{ non-increasing in } z_i \end{cases}$$

and

$$M_i = \begin{cases} M & \text{if } F \text{ non-decreasing in } z_i \\ m & \text{if } F \text{ non-increasing in } z_i \end{cases}$$

Then there exists exactly one equilibrium point $\overline{x}$ *of the difference equation (2.1), and every solution of Eq.(2.1) converges to* $\overline{x}$.

Chapter 3

Linear and Nonlinear Dynamical Systems

3.1 Linear Dynamical Systems

In Chapter 1 we introduced discrete time dynamical system which has the form $x(k+1) = f(x(k))$. The function $f : \mathbb{R}^n \to \mathbb{R}^n$ might be quite simple or terribly complicated.

In this chapter we study dynamical systems in which the function f is particularly nice: we assume f is linear. We begin with the case when f is a function of one variable (i.e.,$f(x) = ax + b$), where a and b are constants, then we deal with the general case when f is a function of several variables (i.e.,$f(x) = Ax + b$), where A is an $n \times n$ matrix and b is a fixed n-vector.

3.1.1 One dimension

We begin by considering the discrete time dynamical systems in which $f(x) = ax + b$, so:

$$x(k+1) = ax(k) + b$$

We discuss this case first analytically (by equations) and then geometrically (with graphs).

Analytical Discussion

1. If $b = 0$

$$x(k+1) = ax(k).$$

   It is very clear that for any k we have simply that $x(k) = a^k x_0$.

   (a) If $|a| < 1$, then $a^k \to 0$ as $k \to \infty$ and so $x(k) \to 0$.

   (b) If $|a| > 1$, then $a^k \to \infty$ as $k \to \infty$. Thus unless $x_0 = 0$, we have $x(k) \to \infty$.

   (c) If $|a| = 1$

      (i) If $a = 1$, then we have just that $x(0) = x(1) = x(2) = x(3) = \cdots$, i.e., $x(k) = x_0$

      (ii) If $a = -1$, then $x(0) = -x(1) = x(2) = -x(3) = \cdots$, that is, $x(k)$ alternates between x_0 and $-x_0$ forever.

2. If $b \neq 0$

$$x(k+1) = ax(k) + b.$$

   We notice that:
   $x(0) = x_0$,
   $x(1) = ax(0) + b = ax_0 + b$,
   $x(2) = ax(1) + b = a(ax_0 + b) + b = a^2x_0 + ab + b$,
   $x(3) = ax(2) + b = a(a^2x_0 + ab + b) + b = a^3x_0 + a^2b + ab + b$,
   $x(4) = ax(3) + b = a(a^3x_0 + a^2b + ab + b) + b = a^4x_0 + a^3b + a^2b + ab + b$.
   We conclude that

$$x(k) = a^k x_0 + (a^{k-1} + a^{k-2} + \cdots + a + 1)b \tag{3.1}$$

   We can simplify Eq.(3.1) by noticing that

$$a^{k-1} + a^{k-2} + \cdots + a + 1$$

   is a geometric series which equals

$$\frac{a^k - 1}{a - 1}$$

provided that $a \neq 1$. If $a = 1$, the series $a^{k-1} + a^{k-2} + \cdots + a + 1$ simply equals k. Thus

$$x(k) = \begin{cases} a^k x_0 + (\frac{a^k-1}{a-1})b & \text{when} \quad a \neq 1; \\ x_0 + kb & \text{when} \quad a = 1. \end{cases}$$

Now we will find the equilibrium point $\bar{x}$ such that $f(\bar{x}) = \bar{x}$

$$f(x) = ax + b$$

$$\bar{x} = a\bar{x} + b$$

$$(1-a)\bar{x} = b$$

$$\bar{x} = \frac{b}{1-a}$$

Depending on the values of $x(k)$ which is given above we will analyze the behavior of solution as $k \to \infty$ for the cases $|a| < 1$, $|a| > 1$, and $|a| = 1$.

1. If $|a| < 1$,

   then $a^k \to 0$ as $k \to \infty$, and so

   $$\left[x(k) = a^k x_0 + (\frac{a^k-1}{a-1})b\right] \to 0 + (\frac{-1}{a-1})b$$

   and so

   $$x(k) \to \tfrac{b}{1-a} = \bar{x} \qquad \text{as } k \to \infty$$

   Thus $\bar{x}$ is a stable or an attractive fixed point of the dynamical system because the system is attracted to it.

2. If $|a| > 1$,

   then $a^k \to \infty$ as $k \to \infty$

   $$\left[x(k) = a^k x_0 + (\frac{a^k-1}{a-1})b = a^k(x_0 - \frac{b}{1-a}) + \frac{b}{1-a}\right]$$

   (i) If $x_0 = \bar{x}$, then $x(k) = \frac{b}{1-a} = \bar{x}$ for all time.

(ii) If $x_0 \neq \frac{b}{1-a}$, then $|x(k)| \to \infty$ as $k \to \infty$

3. $|a| = 1$.

   (a) If $a = 1$, then $x(k) = x_0 + kb$, and so if $b \neq 0$, then $|x(k)| \to \infty$, but if $(b = 0)$, then $x(k) = x_0$ regardless of the value of x_0.

   (b) If $a = -1$, then
   $x(0) = x_0$
   $x(1) = -x_0 + b$
   $x(2) = -(-x_0 + b) + b = x_0$
   $x(3) = -x_0 + b$
   $x(4) = x_0$
   $\vdots$

   Thus $x(k)$ oscillates between two values, x_0 and $b - x_0$. But if $x_0 = b - x_0$, i.e., $x_0 = b/2 = b/(1-(-1)) = \bar{x}$, then $x(k) = \bar{x}$.

Graphical Discussion

Before we start in the geometrical discussion, we will introduce some important concepts.

The first concept is that:
The equilibrium point is the x-coordinate of the point where the graph of f intersects the diagonal line $y = x$. For example, there are three equilibrium points for the equation

$$x(n+1) = x^3(n)$$

To find these equilibrium points, we let $f(\bar{x}) = \bar{x}$, and solve for $\bar{x}$. Hence, there are three equilibrium points: $\{-1, 0, 1\}$, and this can be shown graphically as in Fig.(3.1).

The second concept is that:
The Stair Step (Cobweb) Diagram is an important graphical method for analyzing the stability of equilibrium points. Since $x(n+1) = f(x(n))$, we may draw a graph of f in the $(x(n), x(n+1))$ plane. Then, given $x(0) = x_0$, we pinpoint the value $x(1)$ by drawing a vertical line through x_0 so that it also intersects the graph of f at $(x_0, x(1))$. Next, draw

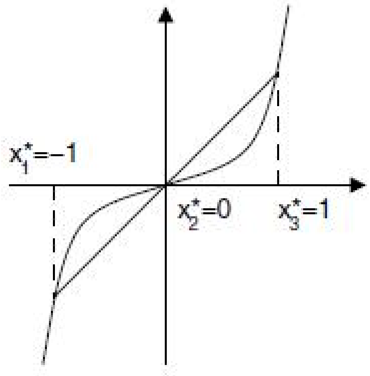

Figure 3.1: Fixed points of $f(x) = x^3$.

a horizontal line from $(x_0, x(1))$ to meet the diagonal line $y = x$ at the point $(x(1), x(1))$. A vertical line drawn from the point $(x(1), x(1))$ will meet the graph of f at the point $(x(1), x(2))$. Continuing this process, one may find $x(n)$ for all $n > 0$.

Now, let us revisit systems of the form $x(k+1) = ax(k) + b$ from a geometric point of view. Graphs will make clear why $|a| < 1$ causes the iterations to converge to $\bar{x}$, while $|a| > 1$ causes the iterates to explode.

1. $\underline{|a| < 1}$

   (a) If $0 < a < 1$

   Figure (3.2) illustrates what happens when we iterate $y = f(x) = ax + b$ with $0 < a < 1$. We chose $x(0)$ larger than $\bar{x} = \frac{b}{1-a}$. It is easy to see that the values $x(0)$, $x(1)$, $x(2)$, etc. get smaller and travel toward $\bar{x}$. The same is true if we chose $x(0) < \bar{x}$

   (b) If $-1 < a < 0$

   Figure (3.3) illustrates what happens when we iterate $y = f(x) = ax + b$ with $-1 < a < 0$. The line slopes downward, but not very steeply. We start with $x(0)$ a good bit to the right of $\bar{x}$. Observe that $x(1)$ is to the left of $\bar{x}$, but not nearly as far. Successive iterations take us to alternate sides of $\bar{x}$, but getting closer and closer, and ultimately converging to $\bar{x}$.

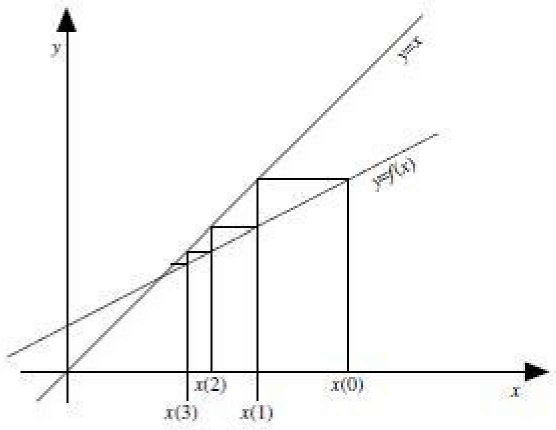

Figure 3.2: Iterating $f(x) = ax + b$ with $0 < a < 1$

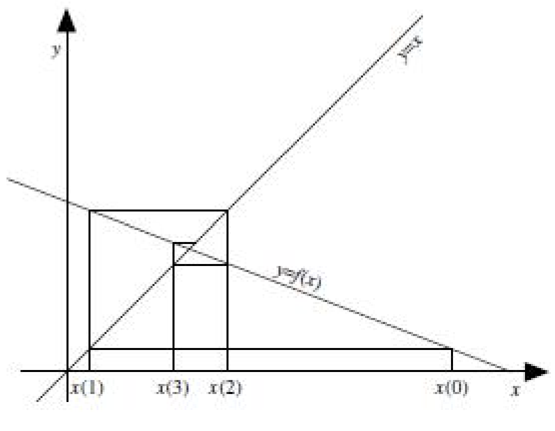

Figure 3.3: Iterating $f(x) = ax + b$ with $-1 < a < 0$.

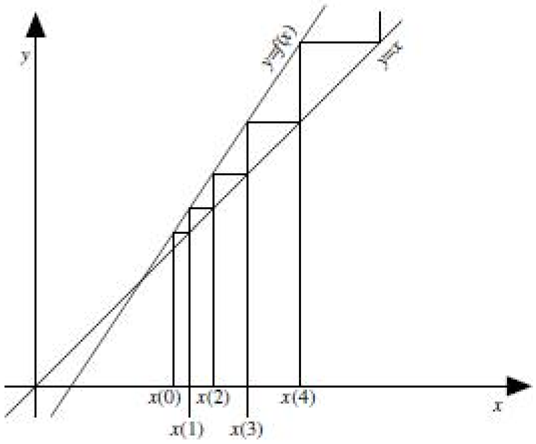

Figure 3.4: Iterating $f(x) = ax + b$ with $a > 1$.

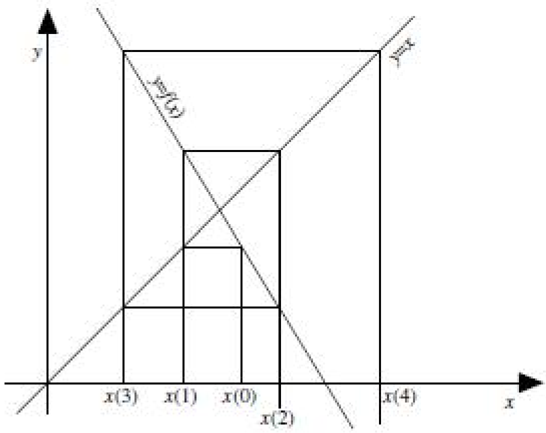

Figure 3.5: Iterating $f(x) = ax + b$ with $a < -1$.

2. $|a| > 1$

   (a) If $a > 1$

   Figure (3.4) illustrates what happens when we iterate $y = f(x) = ax + b$ with $a > 1$. The line slopes steeply upward. We start $x(0)$ just slightly greater than $\bar{x}$. Observe that $x(1)$ is now to the right of $x(0)$, and then $x(2)$ is farther right, etc. The successive iterates are going to ∞. If we chose $x(0) < \bar{x}$, then the iterates would move to $-\infty$.

   (b) If $a < -1$

   In Figure (3.5) we have $a < -1$; hence the line $y = f(x) = ax + b$ is sloped steeply downward. We begin with $x(0)$ just to the right of $\bar{x}$. Observe that $x(1) < \bar{x}$ but at a greater distance from $\bar{x}$ than $x(0)$. Next, $x(2)$ is to the right of $\bar{x}$, $x(3)$ is to the left, etc. with each at increasing distance from $\bar{x}$ and diverging to ∞.

3. $|a| = 1$

   (a) If $a = 1$

   Figure (3.6) illustrates what happens when we iterate $y = f(x) = 1x + b$ with $b \neq 0$. We see that each iteration moves the point $x(k)$ a step to the right and heads to ∞.

   (b) If $a = -1$

   Finally, Figure (3.7) considers the case $f(x) = -1x + b$. The starting value $x(0)$ is taken to be to the left of $\bar{x}$. Next, $x(1)$ is to the right of $\bar{x}$ and then $x(2)$ is back at exactly the same location as $x(0)$. In this manner, $x(0)$, $x(2)$, $x(4)$, etc. all have the same value (denoted by $x(\text{even})$ in the figure), and, likewise, $x(1) = x(3) = x(5) = \cdots$ (denoted by $x(\text{odd})$).

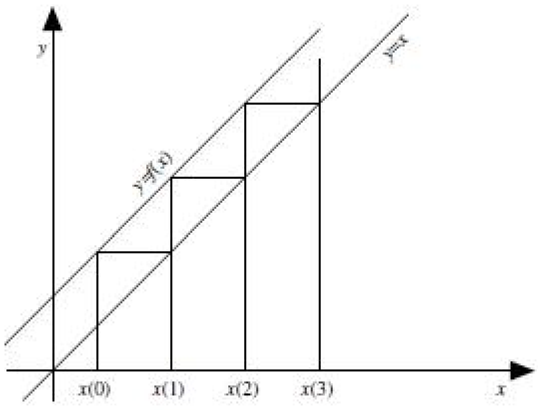

Figure 3.6: Iterating $f(x) = ax + b$ with $a = 1$ and $b \neq 0$.

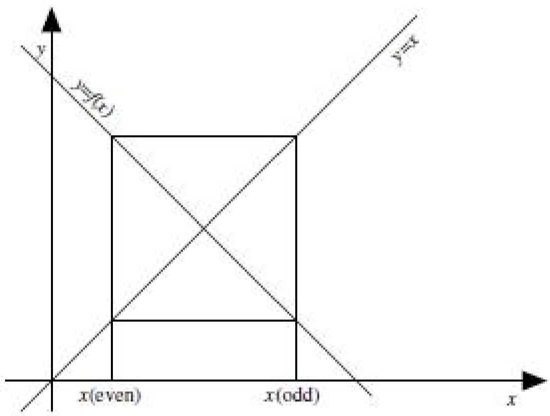

Figure 3.7: Iterating $f(x) = ax + b$ with $a = -1$.

3.1.2 Two (and more) dimensions

In this section we consider discrete linear systems in several variables. The systems, of course, have the form $x(k+1) = f(x(k))$. The state vector x is no longer a single number but rather is a vector with n components (i.e., $x \in \mathbb{R}^n$). The function $f(x) = Ax + b$, where A is an $n \times n$ matrix, and $b \in \mathbb{R}^n$ is a (constant) vector, and so we will deal with the system

$$x(k+1) = Ax(k) + b \qquad x(0) = x_0.$$

1. If $b = 0$

$$x(k+1) = Ax(k).$$

Simply

$$x(1) = Ax(0) = Ax_0$$

$$x(2) = Ax(1) = A^2x_0$$

$$x(3) = Ax(2) = A^3x_0$$

Thus

$$x(k) = A^kx_0.$$

We assume that A is diagonalizable, and that it has n linearly independent eigenvectors $v_1, \cdots, v_n$ with associated eigenvalues $\lambda_1, \cdots, \lambda_n$. Let Λ be the diagonal matrix with diagonal entries $\lambda_1, \cdots, \lambda_n$, and let S be the $n \times n$ matrix whose i^{th} column is v_i. Thus we may write $A = S\Lambda S^{-1}$. So

$$A^k = (S\Lambda S^{-1})(S\Lambda S^{-1})(S\Lambda S^{-1}) \cdots (S\Lambda S^{-1})$$

$$A^k = S\Lambda(S^{-1}S)\Lambda(S^{-1}S)\Lambda(S^{-1}S) \cdots (S^{-1}S)\Lambda S^{-1}$$

Thus

$$A^k = S\Lambda^k S^{-1}$$

Now

$$\Lambda^k = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}^k = \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

Thus to understand the behavior of A^k we need to understand the behavior of the λ_j^k's. If λ_j is a real number, then $\lambda_j \to 0$ if $|\lambda_j| < 1$, and λ_j^k explodes if $|\lambda_j| > 1$. However, the eigenvalues of A might be complex numbers, and then we need to know how λ_j^k behaves for complex λ_j. We write λ_j in its polar form: $\lambda_j = r_j e^{i\theta_j}$. Thus

$$|\lambda_j^k| = |r_j e^{ik\theta_j}| \to \begin{cases} 0 & \text{if} \quad r < 1; \\ \infty & \text{if} \quad r > 1. \end{cases}$$

Now we will return to our system

$$x(k+1) = Ax(k) \qquad \text{From which we found} \qquad x(k) = A^k x_0$$

Case 1: If all the eigenvalues of A have absolute values less than 1, then A^k tends to the zero matrix as $k \to \infty$. Thus

$$x(k) = A^k x_0 \to 0.$$

Case 2: If some eigenvalues of A have absolute values greater than 1, then entries in A^k will diverge to ∞.

We have assumed that the eigenvectors $v_1, \cdots, v_n$ are linearly independent, but we know that any family of n linearly independent vectors in $\mathbb{R}^n$ forms a basis, hence every vector (x_0 in particular) can be written uniquely as a linear combination of the v_i's. Thus

$$x(0) = x_0 = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

Multiply both sides by A

$$x(1) = Ax(0) = c_1 Av_1 + c_2 Av_2 + \cdots + c_n Av_n$$

But we have $Av_i = \lambda_i v_i$, so

$$x(1) = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + c_n \lambda_n v_n$$

In the same way we conclude that

$$x(k) = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2 + \cdots + c_n \lambda_n^k v_n$$

Now, if $|\lambda_i| > 1$, then $|\lambda_i^k| \to \infty$ as $k \to \infty$. Hence, unless $c_i = 0$, we have $|x(k)| \to \infty$.

<u>Case 3:</u> If some eigenvalues have absolute value equal to 1, and the rest have absolute value less than 1.
We again have

$$x(k) = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2 + \cdots + c_n \lambda_n^k v_n$$

The terms involving λ_i's with absolute value less than 1 disappear, but the other components neither vanish nor explode.

2. If $b \neq 0$
The form of the system is:

$$x(k+1) = Ax(k) + b$$

Let us compute the iterates $x(0)$, $x(1)$, $x(2)$, etc.

$$x(0) = x_0$$

$$x(1) = Ax(0) + b = Ax_0 + b$$

$$x(2) = Ax(1) + b = A^2 x_0 + Ab + b$$

$$x(3) = Ax(2) + b = A^3 x_0 + A^2 b + Ab + b$$

$$x(4) = Ax(3) + b = A^4 x_0 + A^3 b + A^2 b + Ab + b$$

So

$$x(k) = A^k x_0 + (A^{k-1} + A^{k-2} + \cdots + A + I)b.$$

To simplify this, observe that

$$(A^{k-1} + A^{k-2} + \cdots + A + I)(I - A) = I - A^k$$

Given that $I - A$ is invertible, we have

$$x(k) = A^k x_0 + (I - A^k)(I - A)^{-1} b.$$

<u>Case 1:</u> If the absolute values of A's eigenvalues are all less than 1 (hence $I - A$ is invertible), then A^k tends to the zero matrix, hence $x(k) \to \bar{x} = (I - A)^{-1} b$. This is a generalization of one-dimensional $\bar{x}$ which is $\frac{b}{1-a} = (1-a)^{-1} b$.

<u>Case 2:</u> If some eigenvalues have absolute values bigger than 1, then A^k blows up, and for most x_0 we have $|x(k)| \to \infty$.

3.2 Nonlinear Dynamical Systems

The general form for discrete dynamical systems is

$$x(k+1) = f(x(k))$$

We have examined the case when f is linear. Now we begin our study of more general systems in which f can be any function that is differentiable with continuous derivative.

3.2.1 Fixed points

Depending on the definition of the fixed points, finding a fixed point of the system $x(k+1) = f(x(k))$ means solving the equation $x = f(x)$. For example, suppose the system is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} (x_1(k))^2 + x_2(k) \\ x_1(k) + x_2(k) - 2 \end{bmatrix}$$

We may write this as $x(k+1) = f(x(k))$, where

$$f\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u^2 + v \\ u + v - 2 \end{bmatrix}$$

To find a point $\bar{x}$ with the property that $\bar{x} = f(\bar{x})$, we solve

$$u = u^2 + v$$

$$v = u + v - 2.$$

Therefore, the fixed point of the system is

$$\begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

3.2.2 Stability

Not all fixed points are the same. We call some stable and others unstable. We begin by illustrating these concepts with an example

Let

$$x(k+1) = f(x(k)) = [x(k)]^2.$$

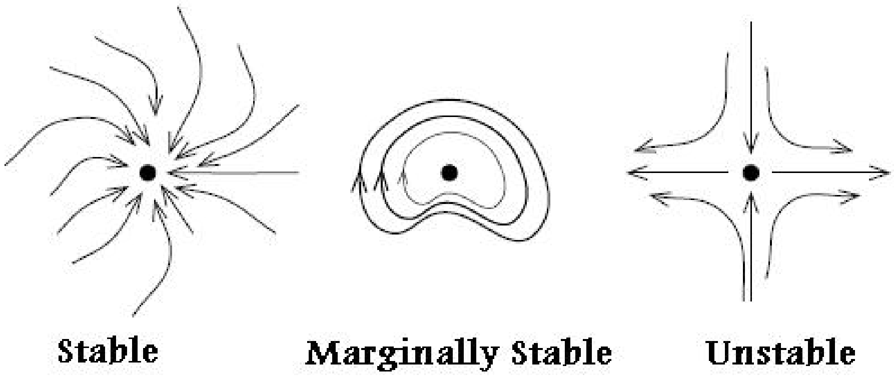

Figure 3.8: Kinds of fixed points.

The system has two fixed points: 0 and 1. First, let us start with a number which is close to 0, say 0.1. If we iterate x^2, we see

$$0.1 \mapsto 0.01 \mapsto 0.0001 \mapsto 0.0000001 \mapsto \cdots.$$

Thus if x_0 is near 0, then $x(k) \to 0$ as $k \to \infty$. For this reason 0 is a stable fixed point.

Now we will examine the other fixed point1. If we take a number near 1, say 1.1, we see

$$1.1 \mapsto 1.21 \mapsto 1.4641 \mapsto 2.1436 \mapsto 4.5950 \mapsto 21.1138 \mapsto 445.7916 \mapsto \cdots.$$

Clearly, $x(k) \to \infty$. If we take $x_0 = 0.9$, we see

$$0.9 \mapsto 0.81 \mapsto 0.6561 \mapsto 0.4305 \mapsto 0.1853 \mapsto 0.0343 \mapsto 0.0012 \mapsto \cdots.$$

Clearly $x(k) \to 0$. In any case, starting points near (but not equal to) 1 tend to iterate away from 1. We call 1 an unstable fixed point of the system.

Figure(3.8) illustrates the kinds of the fixed points. The fixed point is **stable** if all trajectories which begin near $\bar{x}$ remain near, and converge to $\bar{x}$. The fixed point is **marginally stable** (neutral) if the trajectories which begin near $\bar{x}$ stay nearby but never converge to $\bar{x}$. Finally, the fixed point is **unstable** if there are trajectories which start near $\bar{x}$ and move far away from $\bar{x}$.

Chapter 4

Qualitative behavior of the difference equation $x_{n+1} = Ax_n + \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}$

Introduction

In this chapter we will study some qualitative behavior of the solutions of the difference equation

$$x_{n+1} = Ax_n + \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}} \qquad , n = 0, 1, 2, ... \tag{4.1}$$

where the initial conditions $x_{-k}, \cdots, x_{-1}, x_0$ are arbitrary positive real numbers and the coefficients A, β, γ, B, C are positive constants, while k is a positive integer number.

Our concentration is on invariant intervals, semicycle analysis, and the global asymptotic stability of all positive solutions of Eq.(4.1).

E.M.E Zayed et al.[16] have studied Eq.(4.1), in this chapter we will give further results and correct some wrong ones, the special case of Eq.(4.1) when $A = 0$ and $k = 1$ has been studied in [10]. In [14] M.Saleh et al. have studied the global stability of Eq.(4.1) when $A = 0$ and k is a positive integer. A more general recursive sequence of the form

$$x_{n+1} = Ax_n + Bx_{n-k} + \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}} \qquad , n = 0, 1, 2, ... \tag{4.2}$$

has been studied in [17].

4.1 Change of variables

The change of variables $x_n = \frac{\gamma}{C} y_n$, which was used in [16], reduces Eq.(4.1) to the difference equation

$$y_{n+1} = Ay_n + \frac{py_n + y_{n-k}}{qy_n + y_{n-k}} \qquad , n = 0, 1, 2, ... \tag{4.3}$$

where $p = \frac{\beta}{\gamma}$ and $q = \frac{B}{C}$ with p, $q \in (0, \infty)$, y_{-k}, $\cdots$, y_{-1}, $y_0 \in (0, \infty)$.
Lets verify this.
Since

$$x_n = \frac{\gamma}{C} y_n$$
$$x_{n+1} = \frac{\gamma}{C} y_{n+1}$$
$$x_{n-k} = \frac{\gamma}{C} y_{n-k}$$

Substitute in the equation

$$x_{n+1} = Ax_n + \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}$$

so

$$\frac{\gamma}{C} y_{n+1} = A\frac{\gamma}{C} y_n + \frac{\beta \frac{\gamma}{C} y_n + \gamma \frac{\gamma}{C} y_{n-k}}{B\frac{\gamma}{C} y_n + C\frac{\gamma}{C} y_{n-k}}$$

and so

$$\frac{\gamma}{C} y_{n+1} = \frac{\gamma}{C} Ay_n + \frac{\frac{\gamma}{C}(\beta y_n + \gamma y_{n-k})}{\frac{\gamma}{C}(By_n + Cy_{n-k})}$$

multiply both sides by $\frac{C}{\gamma}$, so

$$y_{n+1} = Ay_n + \frac{\beta y_n + \gamma y_{n-k}}{\frac{B\gamma}{C} y_n + \gamma y_{n-k}}$$

divide the numerator and the dominator of the fraction on the righthand side by γ, we get:

$$y_{n+1} = Ay_n + \frac{\frac{\beta}{\gamma} y_n + y_{n-k}}{\frac{B}{C} y_n + y_{n-k}}$$

The substitution: $p = \frac{\beta}{\gamma}, q = \frac{B}{C}$ reduces the above equation to

$$y_{n+1} = Ay_n + \frac{py_n + y_{n-k}}{qy_n + y_{n-k}} \qquad , n = 0, 1, 2, ...$$

To avoid a degenerate situation we'll assume that: $p \neq q$.

Next we investigate the equilibrium points of Eq.(4.3) where the parameters p, q and the initial conditions $y_{-k}, \cdots, y_{-1}, y_0$ are arbitrary positive real numbers, while k is a positive integer number.

The equilibrium points of Eq.(4.3) are the positive solutions of:

$$\bar{y} = A\bar{y} + \frac{p\bar{y} + \bar{y}}{q\bar{y} + \bar{y}}$$

so

$$\bar{y} = \bar{y}(A + \frac{p+1}{(q+1)\bar{y}})$$

$$\bar{y}(1 - A - \frac{p+1}{(q+1)\bar{y}}) = 0$$

but

$$\bar{y} \neq 0$$

so

$$1 - A - \frac{p+1}{(q+1)\bar{y}} = 0$$

$$1 - A = \frac{p+1}{(q+1)\bar{y}}$$

$$\bar{y} = \frac{p+1}{(q+1)(1-A)}$$

and so if $0 < A < 1$, then the only positive equilibrium point is:

$$\bar{y} = \frac{p+1}{(q+1)(1-A)} \tag{4.4}$$

4.2 Linearization

In this section we derive the linearized equation of Eq.(4.3) and prove the result which is included in [16]. To this end, we introduce a continuous function $F : (0, \infty)^2 \to (0, \infty)$ which is defined by:

$$F(u_0, u_1) = Au_0 + \frac{pu_0 + u_1}{qu_0 + u_1}$$

now

$$\frac{\partial F}{\partial u_0} = A + \frac{(qu_0 + u_1)p - (pu_0 + u_1)q}{(qu_0 + u_1)^2}$$

$$\frac{\partial F}{\partial u_0}(\bar{y},\bar{y}) = A + \frac{(q\bar{y}+\bar{y})p - (p\bar{y}+\bar{y})q}{(q\bar{y}+\bar{y})^2}$$

$$\frac{\partial F}{\partial u_0}(\bar{y},\bar{y}) = A + \frac{\bar{y}(q+1)p - \bar{y}(p+1)q}{\bar{y}^2(q+1)^2}$$

divide the numerator and the dominator of the fraction in the righthand side by $\bar{y}$ "since $\bar{y} \neq 0$", now we have:

$$\frac{\partial F}{\partial u_0}(\bar{y},\bar{y}) = A + \frac{(q+1)p - (p+1)q}{\bar{y}(q+1)^2}$$

$$\frac{\partial F}{\partial u_0}(\bar{y},\bar{y}) = A + \frac{qp+p-pq-q}{\bar{y}(q+1)^2}$$

$$\frac{\partial F}{\partial u_0}(\bar{y},\bar{y}) = A + \frac{p-q}{\bar{y}(q+1)^2}$$

but $\bar{y} = \frac{p+1}{(q+1)(1-A)}$

so

$$\frac{\partial F}{\partial u_0}(\bar{y},\bar{y}) = A + \frac{p-q}{\frac{p+1}{(q+1)(1-A)}(q+1)^2}$$

$$\frac{\partial F}{\partial u_0}(\bar{y},\bar{y}) = A + \frac{p-q}{\frac{(p+1)(q+1)}{1-A}}$$

hence

$$\frac{\partial F}{\partial u_0}(\bar{y},\bar{y}) = A + \frac{(p-q)(1-A)}{(p+1)(q+1)} = \rho_0 \tag{4.5}$$

Also

$$\frac{\partial F}{\partial u_1} = \frac{(qu_0+u_1)\times 1 - (pu_0+u_1)\times 1}{(qu_0+u_1)^2}$$

$$\frac{\partial F}{\partial u_1} = \frac{qu_0+u_1-pu_0-u_1}{(qu_0+u_1)^2}$$

$$\frac{\partial F}{\partial u_1} = \frac{(q-p)u_0}{(qu_0+u_1)^2}$$

hence

$$\frac{\partial F}{\partial u_1}(\bar{y},\bar{y}) = \frac{(q-p)\bar{y}}{(q\bar{y}+\bar{y})^2}$$

$$\frac{\partial F}{\partial u_1}(\bar{y},\bar{y}) = \frac{(q-p)\bar{y}}{(q+1)^2\bar{y}^2}$$

divide the numerator and the dominator by $\bar{y}$

$$\frac{\partial F}{\partial u_1}(\bar{y},\bar{y}) = \frac{(q-p)}{(q+1)^2\bar{y}}$$

substitute:

$$\bar{y} = \frac{p+1}{(q+1)(1-A)}$$

so

$$\frac{\partial F}{\partial u_1}(\bar{y},\bar{y}) = \frac{(q-p)}{(q+1)^2\frac{p+1}{(q+1)(1-A)}}$$

$$\frac{\partial F}{\partial u_1}(\bar{y},\bar{y}) = \frac{(q-p)}{\frac{(p+1)(q+1)}{(1-A)}}$$

hence

$$\frac{\partial F}{\partial u_1}(\bar{y},\bar{y}) = \frac{(q-p)(1-A)}{(q+1)(p+1)} = \frac{-(p-q)(1-A)}{(q+1)(p+1)} = \rho_1 \tag{4.6}$$

Then the linearized equation of Eq.(4.3) about $\bar{y}$ is:

$$y_{n+1} - \rho_0 y_n - \rho_1 y_{n-k} = 0 \tag{4.7}$$

where ρ_0, ρ_1 are given by (4.5) and (4.6)

Let $y_n = \lambda^n$, so $y_{n+1} = \lambda^{n+1}$, and $y_{n-k} = \lambda^{n-k}$
substitute in (4.7):

$$\lambda^{n+1} - \rho_0\lambda^n - \rho_1\lambda^{n-k} = 0$$

divide both sides by λ^{n-k}:

$$\lambda^{k+1} - \rho_0\lambda^k - \rho_1 = 0$$

This equation is called the characteristic equation.

4.3 Local stability

In this section, we investigate the local stability of the positive solutions of Eq.(4.3).

Theorem 4.3.1 *([16])*

(a) Assume that $0 < p - q < \frac{1}{2}(p+1)(q+1)$ *and* $0 < A < 1$*, then the positive equilibrium point* $\bar{y}$ *of Eq.(4.3) is locally asymptotically stable.*

(b) Assume that $p < q$, $0 < A < 1$ *and* $A > \frac{(q-p)(1-A)}{(p+1)(q+1)}$*, then condition (2.6) is the necessary and sufficient condition for the asymptotic stability of the positive solutions.*

Proof:

First we prove part (a) of Theorem (4.3.1)

$$|\rho_0| + |\rho_1| = |A + \frac{(p-q)(1-A)}{(p+1)(q+1)}| + |-\frac{(p-q)(1-A)}{(p+1)(q+1)}|$$

$$= A + \frac{(p-q)(1-A)}{(p+1)(q+1)} + \frac{(p-q)(1-A)}{(p+1)(q+1)}$$

$$= A + \frac{2(p-q)(1-A)}{(p+1)(q+1)}$$

$$= \frac{A(p+1)(q+1) + 2(p-q)(1-A)}{(p+1)(q+1)}$$

From assumption

$$(p-q) < \frac{1}{2}(p+1)(q+1)$$

so

$$|\rho_0|+|\rho_1| = \frac{A(p+1)(q+1) + 2(p-q)(1-A)}{(p+1)(q+1)} < \frac{A(p+1)(q+1) + 2 \times \frac{1}{2}(p+1)(q+1)(1-A)}{(p+1)(q+1)}$$

hence

$$|\rho_0| + |\rho_1| < \frac{(p+1)(q+1)}{(p+1)(q+1)} = 1$$

$$|\rho_0| + |\rho_1| < 1$$

so $\bar{y}$ is locally asymptotically stable according to Theorem (2.2.1).

Now we will prove part (b) of Theorem (4.3.1).

$$|\rho_0| + |\rho_1| = |A + \frac{(p-q)(1-A)}{(p+1)(q+1)}| + | - \frac{(p-q)(1-A)}{(p+1)(q+1)}|$$

$$= |A - \frac{(q-p)(1-A)}{(p+1)(q+1)}| + |\frac{-(p-q)(1-A)}{(p+1)(q+1)}|$$

From assumption

$$A > \frac{(q-p)(1-A)}{(p+1)(q+1)}$$

so

$$A - \frac{(q-p)(1-A)}{(p+1)(q+1)} > 0$$

also

$$p < q \quad \rightarrow \quad (p-q) < 0 \quad \rightarrow -(p-q) > 0$$

so

$$|\rho_0| + |\rho_1| = A - \frac{(q-p)(1-A)}{(p+1)(q+1)} - \frac{(p-q)(1-A)}{(p+1)(q+1)}$$

$$= A - \frac{(q-p)(1-A)}{(p+1)(q+1)} + \frac{(q-p)(1-A)}{(p+1)(q+1)}$$

$$|\rho_0| + |\rho_1| = A < 1$$

so condition (2.6) is a sufficient condition for the asymptotic stability of the positive solutions of Eq.(4.3).

If k is an odd integer

We have:

$$\rho_1 = \frac{-(p-q)(1-A)}{(p+1)(q+1)} > 0$$

so condition (2.6) is the necessary and sufficient condition for the asymptotic stability of the positive solutions.

<u>If k is an even integer</u>

We have:

$$\rho_0\rho_1 = (A + \frac{(p-q)(1-A)}{(p+1)(q+1)}) \times (\frac{-(p-q)(1-A)}{(p+1)(q+1)})$$

$$= \frac{-A(p-q)(1-A)}{(p+1)(q+1)} - \left(\frac{(p-q)(1-A)}{(p+1)(q+1)}\right)^2$$

Using the assumption that:

$$A > \frac{(q-p)(1-A)}{(p+1)(q+1)}$$

We get

$$\rho_0\rho_1 = \frac{A(q-p)(1-A)}{(p+1)(q+1)} - \left(\frac{(p-q)(1-A)}{(p+1)(q+1)}\right)^2$$

$$> \frac{(q-p)(1-A)}{(p+1)(q+1)} \times \frac{(q-p)(1-A)}{(p+1)(q+1)} - \left(\frac{(p-q)(1-A)}{(p+1)(q+1)}\right)^2$$

so

$$\rho_0\rho_1 > \left(\frac{(p-q)(1-A)}{(p+1)(q+1)}\right)^2 - \left(\frac{(p-q)(1-A)}{(p+1)(q+1)}\right)^2 = 0$$

Consequently,

$$\rho_0\rho_1 > 0$$

. so condition (2.6) is the necessary and sufficient condition for the asymptotic stability of the positive solutions.

Thus the proof of Theorem (4.3.1) is now finished.

4.4 Periodic solutions

In this section we give necessary and sufficient conditions for Eq.(4.3) to have prime period-two solutions. Some results are included in [16], and others are correction of other results in [16].

Theorem 4.4.1 *(a) If $p > q$, then Eq.(4.3) has no positive solutions of prime period two.*

(b) If k is an even integer, then Eq.(4.3) has no positive solutions of prime period two.

(c) If k is an odd integer, then Eq.(4.3) has prime period two solutions

$$\cdots, \phi, \psi, \phi, \psi, \cdots$$

if the following condition is valid:

$$(p-1)(q-1)(A+1) < -4(qA+p)$$

where $p < 1$ and $q > 1$ while the values of ϕ and ψ are the (positive and distinct) solutions of the quadratic equation

$$t^2 - \frac{(1-p)t}{(qA+1)} + \frac{(qA+p)(1-p)}{(q-1)(A+1)(qA+1)^2} = 0$$

Remark: *Notice that the previous condition is not the one which is assumed in [16]*

$$(p-1)(q-1)(A+1) > -4(qA+p)$$

and we will prove that our condition is the right one.

Proof:

First of all, we prove part (a) in the case $p > q$. Assume for the sake of contradiction that there exists distinctive positive real numbers ϕ and ψ such that

$$\cdots,\ \phi,\ \psi,\ \phi,\ \psi,\ \cdots$$

is a prime period two solution of Eq.(4.3).

If k is odd, then $y_{n+1} = y_{n-k}$, so from Eq.(4.3) we have

$$\phi = A\psi + \frac{p\psi + \phi}{q\psi + \phi} \tag{4.8}$$

and

$$\psi = A\phi + \frac{p\phi + \psi}{q\phi + \psi} \tag{4.9}$$

From (4.8)

$$\phi = \frac{A\psi(q\psi + \phi) + (p\psi + \phi)}{q\psi + \phi}$$

so

$$\phi(q\psi + \phi) = A\psi(q\psi + \phi) + (p\psi + \phi)$$

Thus

$$q\phi\psi + \phi^2 = Aq\psi^2 + A\psi\phi + p\psi + \phi \tag{4.10}$$

From (4.9)

$$\psi(q\phi + \psi) = A\phi(q\phi + \psi) + p\phi + \psi$$

so

$$q\phi\psi + \psi^2 = Aq\phi^2 + A\psi\phi + p\phi + \psi \tag{4.11}$$

By subtracting (4.11) from (4.10) we get

$$\phi^2 - \psi^2 = Aq(\psi^2 - \phi^2) + p(\psi - \phi) + (\phi - \psi)$$

$$(\phi - \psi)(\phi + \psi) = Aq(\psi - \phi)(\psi + \phi) + p(\psi - \phi) + (\phi - \psi)$$

Divide both sides by $(\psi - \phi)$ since we know that ϕ and ψ are distinct. Thus

$$-(\phi + \psi) = Aq(\psi + \phi) + p - 1$$

$$1 - p = (\psi + \phi)(Aq + 1)$$

so

$$\phi + \psi = \frac{1 - p}{qA + 1} \tag{4.12}$$

while by adding (4.10) to (4.11) we get

$$2q\phi\psi + \phi^2 + \psi^2 = Aq(\psi^2 + \phi^2) + 2A\psi\phi = p(\psi + \phi) + (\phi + \psi)$$

$$(2q - 2A)\phi\psi = (qA - 1)(\phi^2 + \psi^2) + (p + 1)(\phi + \psi)$$

Add $2(qA - 1)\phi\psi$ to both sides

$$2(q - A + qA - 1)\phi\psi = (qA - 1)(\phi^2 + \psi^2 + 2\phi\psi) + (p + 1)(\phi + \psi)$$

$$2(q - A + qA - 1)\phi\psi = (qA - 1)(\phi + \psi)^2 + (p + 1)(\phi + \psi) \tag{4.13}$$

Substitute (4.12) in (4.13)

$$2(q - A + qA - 1)\phi\psi = (qA - 1)\left(\frac{1-p}{qA+1}\right)^2 + (p+1)\left(\frac{1-p}{qA+1}\right)$$

Multiply the last term by $\frac{(qA+1)}{(qA+1)}$

$$2(q - A + qA - 1)\phi\psi = \frac{(qA-1)(1-p)^2}{(qA+1)^2} + \frac{(1+p)(1-p)(qA+1)}{(qA+1)^2}$$

$$2(q - A + qA - 1)\phi\psi = \frac{(qA-1)(1-p)^2 + (1-p^2)(qA+1)}{(qA+1)^2}$$

$$2(q - A + qA - 1)\phi\psi = \frac{(qA-1)(1-2p+p^2) + (qA+1-p^2qA-p^2)}{(qA+1)^2}$$

$$2(q - A + qA - 1)\phi\psi = \frac{(qA-2pqA+p^2qA-1+2p-p^2) + (qA+1-p^2qA-p^2)}{(qA+1)^2}$$

$$2(q - A + qA - 1)\phi\psi = \frac{qA-2pqA+p^2qA-1+2p-p^2+qA+1-p^2qA-p^2}{(qA+1)^2}$$

$$2(q - A + qA - 1)\phi\psi = \frac{2qA-2pqA+2p-2p^2}{(qA+1)^2}$$

$$2(q - A + qA - 1)\phi\psi = \frac{2(qA-pqA+p-p^2)}{(qA+1)^2}$$

$$(q - A + qA - 1)\phi\psi = \frac{(qA-pqA+p-p^2)}{(qA+1)^2}$$

$$\phi\psi\left[q(A+1) - (A+1)\right] = \frac{qA(1-p)+p(1-p)}{(qA+1)^2}$$

$$\phi\psi(A+1)(q-1) = \frac{(1-p)(qA+p)}{(qA+1)^2}$$

so

$$\phi\psi = \frac{(1-p)(qA+p)}{(qA+1)^2(A+1)(q-1)} \tag{4.14}$$

we know that ϕ and ψ are positive real numbers, so $\phi\psi$ is positive.

Thus there are two probabilities:

$$(1-p)(qA+p) > 0 \quad \text{and} \quad (qA+1)^2(A+1)(q-1) > 0 \tag{4.15}$$

or

$$(1-p)(qA+p) < 0 \quad \text{and} \quad (qA+1)^2(A+1)(q-1) < 0 \tag{4.16}$$

we know that

$$(qA+p) > 0,\ (qA+1)^2 \text{ and } (A+1) > 0$$

From (4.15)

$$(q-1) > 0 \ \rightarrow \ q > 1 \quad \text{and} \quad (1-p) > 0 \ \rightarrow \ p < 1$$

but from assumption $p > q$, so $p > q > 1$ and consequently $p > 1$ which is a contradiction. Thus (4.15) is not acceptable.

From (4.16)
$(1-p) < 0$ and we know that $\phi + \psi = \frac{1-p}{qA+1}$
since $(1-p) < 0$, we conclude that $(\phi + \psi) < 0$ which contradicts our assumption that ϕ and ψ are positive real numbers.
Thus (4.16) is not acceptable.

The proof of part (a) is now finished in the case that k is odd, the case when k is even is part(b) of the theorem.

We now start proving part (b) of the theorem.

For the sake of contradiction we assume that there exists a prime period two solution of Eq.(4.3)

$$\cdots, \phi, \psi, \phi, \psi, \cdots$$

where ϕ and ψ are distinct positive real numbers.
If k is even, then $y_n = y_{n-k}$, from Eq.(4.3) it follows that:

$$\phi = A\psi + \frac{p\psi + \psi}{q\psi + \psi} = A\psi + \frac{p+1}{q+1} \tag{4.17}$$

and

$$\psi = A\phi + \frac{p\phi + \phi}{q\phi + \phi} = A\phi + \frac{p+1}{q+1} \tag{4.18}$$

subtract Eq.(4.18) from Eq.(18)

$$\phi - \psi = A(\psi - \phi) = -A(\phi - \psi)$$

so we conclude that $A = -1$ and here is the contradiction.

The proof of part (b) is now finished.

It remains to prove part (c) of the theorem

Assume that Eq.(4.3) has prime period two solution

$$\cdots, \phi, \psi, \phi, \psi, \cdots$$

where ϕ and ψ are distinct positive real numbers.
If k is odd, then $y_{n+1} = y_{n-k}$.
We will use (4.12) and (4.14) which we found in the proof of part (1)

$$\phi + \psi = \frac{1-p}{qA+1}$$

$$\phi\psi = \frac{(1-p)(qA+p)}{(qA+1)^2(A+1)(q-1)}$$

Now consider the quadratic equation

$$t^2 - \frac{(1-p)}{(qA+1)}t + \frac{(qA+p)(1-p)}{(q-1)(A+1)(qA+1)^2} = 0$$

so ϕ and ψ are the positive and distinct solutions of the above quadratic equation, and we get

$$t = \frac{\frac{(1-p)}{(qA+1)} \mp \sqrt{\frac{(1-p)^2}{(qA+1)^2} - \frac{4(qA+p)(1-p)}{(q-1)(A+1)(qA+1)^2}}}{2}$$

$$t = \frac{\frac{(1-p)}{(qA+1)} \mp \frac{1}{qA+1}\sqrt{(1-p)^2 - \frac{4(qA+p)(1-p)}{(q-1)(A+1)}}}{2}$$

$$t = \frac{(1-p) \mp \sqrt{(1-p)^2 - \frac{4(qA+p)(1-p)}{(q-1)(A+1)}}}{2(qA+1)}$$

so

$$t = \frac{(1-p) \mp \delta}{2(qA+1)}$$

where

$$\delta = \sqrt{(1-p)^2 - \frac{4(qA+p)(1-p)}{(q-1)(A+1)}}$$

Thus, we deduce that

$$(1-p)^2 - \frac{4(qA+p)(1-p)}{(q-1)(A+1)} > 0$$

$$(1-p)^2 > \frac{4(qA+p)(1-p)}{(q-1)(A+1)}$$

From assumption $p < 1$, so $1 - p > 0$

Now divide both sides of the inequality by $(1-p)$

$$(1-p) > \frac{4(qA+p)}{(q-1)(A+1)}$$

$$(1-p)(q-1)(A+1) > 4(qA+p)$$

$$(p-1)(q-1)(A+1) < -4(qA+p)$$

The proof is now complete.

4.5 Invariant intervals

In this section we will find the invariant intervals which were not identified in [16].

Theorem 4.5.1 *Suppose that $p > q$, $0 < A < 1$, $A^2(p+p^2) + q < p$, and assume that for some $N \geq 0$*

$$y_{N-k+1}, \cdots, y_{N-1}, y_N \in \left[A+1, \tfrac{p}{q}(A+1)\right]$$

then

$$y_n \in \left[A+1, \tfrac{p}{q}(A+1)\right], \text{ for all } n > N$$

Proof:

$$y_{n+1} = Ay_n + \frac{py_n + y_{n-k}}{qy_n + y_{n-k}} \geq Ay_n + \frac{qy_n + y_{n-k}}{qy_n + y_{n-k}}$$

so

$$y_{n+1} \geq Ay_n + 1$$

but we know that for some $N > 0$, $A+1 \leq y_N \leq \frac{p}{q}(A+1)$, so

$$y_{n+1} \geq Ay_n + 1 \geq A(A+1) + 1 = A^2 + A + 1$$

$$y_{n+1} \geq A^2 + A + 1 > A + 1$$

Thus

$$\mathbf{y_{n+1} \geq A + 1}$$

Now assume

$$y_{n+1} = f(x, y) = Ax + \frac{px + y}{qx + y}$$

$f(x, y)$ is increasing in x for each fixed y, and decreasing in y for each fixed x, since:

$$\frac{\partial f}{\partial x} = A + \frac{(qx + y) \times p - (px + y) \times q}{(qx + y)^2}$$

$$\frac{\partial f}{\partial x} = A + \frac{qxp + yp - pxq - qy}{(qx + y)^2}$$

since $p > q$ we have

$$\frac{\partial f}{\partial x} = A + \frac{y(p - q)}{(qx + y)^2} > 0$$

and

$$\frac{\partial f}{\partial y} = \frac{(qx + y) \times 1 - (px + y) \times 1}{(qx + y)^2}$$

but $p > q$, so

$$\frac{\partial f}{\partial y} = \frac{qx + y - px - y}{(qx + y)^2} = \frac{(q - p)x}{(qx + y)^2} < 0$$

Return to our equation:

$$y_{n+1} = f(y_n, y_{n-k})$$

$f(y_n, y_{n-k})$ is increasing in y_n for each fixed y_{n-k}, and $f(y_n, y_{n-k})$ is decreasing in y_{n-k} for each fixed y_n.

We know that for some $N > 0$, $A + 1 \leq y_N \leq \frac{p}{q}(A+1)$

so

$$y_{n+1} = f(y_n, y_{n-k}) \leq f\left(\frac{p}{q}(A+1), y_{n-k}\right)$$

$$y_{n+1} \leq f\left(\frac{p}{q}(A+1), y_{n-k}\right) \leq f\left(\frac{p}{q}(A+1), A+1\right)$$

$$y_{n+1} \leq A\frac{p}{q}(A+1) + \frac{p\frac{p}{q}(A+1)+(A+1)}{q\frac{p}{q}(A+1)+(A+1)}$$

$$y_{n+1} \leq A^2\frac{p}{q} + \frac{Ap}{q} + \frac{(\frac{p^2}{q}+1)(A+1)}{(\frac{pq}{q}+1)(A+1)}$$

$$y_{n+1} \leq \frac{Ap}{q} + A^2\frac{p}{q} + \frac{(\frac{p^2}{q}+1)}{(\frac{pq}{q}+1)} \tag{4.19}$$

but from our assumption:

$$A^2(p+p^2) + q < p$$

$$A^2p + A^2p^2 + q < p$$

Add p^2 to both sides of the inequality.

$$A^2p + A^2p^2 + q + p^2 < p + p^2$$

$$A^2p(1+p) + q + p^2 < p + p^2$$

Divide both sides by $q(p+1)$

$$\frac{A^2p(1+p)+q+p^2}{q(p+1)} < \frac{p+p^2}{q(p+1)}$$

$$\frac{A^2p(1+p)}{q(p+1)} + \frac{q+p^2}{q(p+1)} < \frac{p(1+p)}{q(p+1)}$$

$$\frac{A^2p}{q} + \frac{q+p^2}{q(p+1)} < \frac{p}{q}$$

$$\frac{A^2p}{q} + \frac{\frac{q+p^2}{q}}{\frac{q(p+1)}{q}} < \frac{p}{q}$$

$$\frac{A^2p}{q} + \frac{\frac{p^2}{q}+1}{\frac{pq}{q}+1} < \frac{p}{q}$$

Now return to (4.19)

$$y_{n+1} \leq \frac{Ap}{q} + A^2\frac{p}{q} + \frac{(\frac{p^2}{q}+1)}{(\frac{pq}{q}+1)} \leq \frac{Ap}{q} + \frac{p}{q}$$

so

$$\mathbf{y_{n+1} \leq \frac{p}{q}(A+1)}$$

The proof is complete.

Theorem 4.5.2 *Suppose that*
$\frac{1}{q} < p < q, \quad Aq > q-p, \quad A^2 < A - q + p, \quad 0 < A < 1,$
Assume that for some $N \geq 0$

$$y_{N-k+1}, \cdots, y_{N-1}, y_N \in \left[\tfrac{1}{q}, \tfrac{A}{q-p}\right]$$

then

$$y_n \in \left[\tfrac{1}{q}, \tfrac{A}{q-p}\right], \textit{ for all } n > N$$

Proof:

First of all $\frac{A}{q-p} > \frac{1}{q}$, since from the assumption

$Aq > q-p \quad \rightarrow \quad \frac{Aq}{q-p} > 1$, since $(q-p) > 0$

so

$$\frac{A}{q-p} > \frac{1}{q}$$

and since $p < q$ we have

$$y_{n+1} = Ay_n + \frac{py_n + y_{n-k}}{qy_n + y_{n-k}} \le Ay_n + \frac{qy_n + y_{n-k}}{qy_n + y_{n-k}}$$

$$y_{n+1} \le Ay_n + \frac{qy_n + y_{n-k}}{qy_n + y_{n-k}} = Ay_n + 1$$

But we assumed that for some $N > 0$, $\frac{1}{q} \le y_N \le \frac{A}{q-p}$, so

$$y_{n+1} \le Ay_n + 1 \le A \times \frac{A}{q-p} + 1$$

and

$$y_{n+1} \le \frac{A^2}{q-p} + 1 \tag{4.20}$$

we assumed that $A^2 < A - q + p$, so

$$A^2 + q - p < A$$

Divide both sides of the inequality by $q - p$

$$\frac{A^2 + q - p}{q-p} < \frac{A}{q-p}$$

$$\frac{A^2}{q-p} + 1 < \frac{A}{q-p}$$

Return to (4.20)

$$y_{n+1} \le \frac{A^2}{q-p} + 1 < \frac{A}{q-p}$$

so

$$\mathbf{y_{n+1} \le \frac{A}{q-p}}$$

Now assume

$$y_{n+1} = f(x,y) = Ax + \frac{px + y}{qx + y}$$

$f(x,y)$ is increasing in y for each fixed x, since

$$\frac{\partial f}{\partial y} = \frac{(qx+y) \times 1 - (px+y) \times 1}{(qx+y)^2} = \frac{(q-p)x}{(qx+y)^2} > 0$$

but, in general, $f(x,y)$ does not behave monotonically in x for fixed y and this is **a mistake in [16]**, because they said in the proof of Thm.(6.2) that **"the function**

$F(x, y)$ **is decreasing in** x **for each fixed** y**, and increasing in** y **for each fixed** x**"**.
Now we will clarify this:

$$\frac{\partial f}{\partial x} = A + \frac{(qx+y) \times p - (px+y) \times q}{(qx+y)^2}$$

$$\frac{\partial f}{\partial x} = A + \frac{qxp + yp - pxq - qy}{(qx+y)^2}$$

$$\frac{\partial f}{\partial x} = A + \frac{y(p-q)}{(qx+y)^2}$$

but $p < q$, so

$$\frac{y(p-q)}{(qx+y)^2} < 0$$

1. If $|A| > |\frac{y(p-q)}{(qx+y)^2}| = \frac{y(q-p)}{(qx+y)^2}$, then $\frac{\partial f}{\partial x}$ is positive.
2. If $|A| < |\frac{y(p-q)}{(qx+y)^2}| = \frac{y(q-p)}{(qx+y)^2}$, then $\frac{\partial f}{\partial x}$ is negative.

Under our assumptions $f(x, y)$ is increasing in x for each fixed y since $y < \frac{A}{q-p}$, and $x > \frac{1}{q}$ as we will clarify now

$$y(q-p) < A \tag{4.21}$$

and

$$xq > 1 \tag{4.22}$$

From (4.21) and (4.22) we have

$$A > y(q-p) > \frac{y(q-p)}{xq}$$

This is true since $xq > 1$
Also, since $xq + y > xq$ we conclude that

$$A > \frac{y(q-p)}{xq+y}$$

but we know that $(xq+y)^2 > xq + y > xq > 1$ so

$$A > \frac{y(q-p)}{xq+y} > \frac{y(q-p)}{(xq+y)^2}$$

Consequently,

$$A > \frac{y(q-p)}{(xq+y)^2}$$

Thus under our assumptions $f(x,y)$ is increasing in both arguments.

Return to

$$y_{n+1} = f(x,y) = Ax + \frac{px+y}{qx+y}$$

$$f(x,y) \geq f\left(\frac{1}{q}, y\right) > f\left(\frac{1}{q}, \frac{1}{q}\right)$$

$$f(x,y) > f\left(\frac{1}{q}, \frac{1}{q}\right) = A \times \frac{1}{q} + \frac{p \times \frac{1}{q} + \frac{1}{q}}{q \times \frac{1}{q} + \frac{1}{q}}$$

$$f(x,y) > A \times \frac{1}{q} + \frac{(p+1) \times \frac{1}{q}}{(q+1) \times \frac{1}{q}}$$

$$f(x,y) \geq \frac{A}{q} + \frac{(p+1)}{(q+1)} > \frac{(p+1)}{(q+1)} \tag{4.23}$$

but since we assumed that

$$p > \frac{1}{q}$$

we have

$$pq > 1$$

Add (q) to both sides to get

$$pq + q > 1 + q$$

Divide both sides by $(1+q)$

$$\frac{(p+1)q}{1+q} > \frac{1+q}{1+q}$$

$$\frac{(p+1)q}{1+q} > 1$$

Divide both sides by (q)

$$\frac{(p+1)}{(1+q)} > \frac{1}{q}$$

substitute in (4.23)

$$f(x,y) \geq \frac{(p+1)}{(q+1)} > \frac{1}{q}$$

Thus

$$\mathbf{y_{n+1} \geq \frac{1}{q}}$$

The proof is now complete.

4.6 Semi-cycle analysis

Theorem 4.6.1 *[14]*
Assume that $f \in [(0,\infty)\times(0,\infty),(0,\infty)]$ is such that : $f(x,y)$ is increasing (respectively, decreasing) in x for each fixed y, and $f(x,y)$ is decreasing (respectively, increasing) in y for each fixed x. Let $\bar{x}$ be a positive equilibrium of Eq.(2.4). Then, except possibly for the first semi-cycle, every oscillatory solution of Eq.(2.4) has semi-cycle of length at least k. Furthermore, if we assume that

$f(u,u)=\bar{x}$ *for every* u
and
$f(x,y)<x$ *for every* $\bar{x}<y<x$

then $\{x_n\}$ oscillates about the equilibrium $\bar{x}$ with semi-cycles of length $k+1$ or $k+2$, except possibly for the first semi-cycle which may have length k. The extreme in each semi-cycle occurs in the first term if the semi-cycle has two terms and in the second term if the semi-cycle has three terms, and in the $k+1$ term if the semi-cycle has $k+2$ terms.

Corollary 1 *Assume that $p>q$, then except possibly for the first semi-cycle every oscillatory solution of Eq.(4.3) has semi-cycle of length at least k.*

Proof: The proof follows from Theorem (4.6.1), since under the assumption that $p>q$, $f(x,y)$ is increasing in x for each fixed y and decreasing in y for each fixed x, as we have proved in the previous section.

4.7 Global stability

Theorem 4.7.1 *Assume that*
$p>q, \quad 0<A<1, \quad (A^2P+A^2p^2)<(p-q)<\frac{1}{2}(p+1)(q+1)$ *and* $p-1<2q(1-A^2)$
, then the positive equilibrium point of Eq.(4.3) is globally asymptotically stable.

Proof:
We will apply Theorem (2.5.1) in the proof using the interval $\left[A+1,\frac{p}{q}(A+1)\right]$
Under these assumptions we have shown in part (a) of Theorem (4.3.1) that $\bar{y}$ is locally

stable. We need to show that $\bar{y}$ is a global attractor. To this end, we consider the function

$$F(x,y) = Ax + \frac{px+y}{qx+y}$$

We have shown that when $p > q$, $f(x,y)$ is increasing in x for each fixed y, and decreasing in y for each fixed x.

Suppose that $(m,M) \in \left[A+1, \frac{p}{q}(A+1)\right] \times \left[A+1, \frac{p}{q}(A+1)\right]$ is a solution of the system

$$M = F(M,m) \qquad \text{and} \qquad m = F(m,M)$$

Then we get

$$M = AM + \frac{pM+m}{qM+m}$$

and

$$m = Am + \frac{pm+M}{qm+M}$$

so

$$(1-A)M = \frac{pM+m}{qM+m}$$

and

$$(1-A)m = \frac{pm+M}{qm+M}$$

From which we have

$$q(1-A)M^2 + (1-A)mM = pM + m \tag{4.24}$$

and

$$q(1-A)m^2 + (1-A)mM = pm + M \tag{4.25}$$

Subtract (4.25) from (4.24) to get

$$q(1-A)(M^2 - m^2) = p(M-m) - (M-m)$$

$$q(1-A)(M-m)(M+m) = p(M-m) - (M-m)$$

$$(M-m)\left(q(1-A)(M+m) - p + 1\right) = 0 \tag{4.26}$$

If $q(1-A)(M+m) - p + 1 = 0$, then $m + M = \frac{p-1}{q-qA}$

But, this contradicts our assumption that $p - 1 < 2q(1-A^2)$, and we will clarify this.

$$p - 1 < 2q - 2qA^2$$

$$p-1<2q-2qA^2\overbrace{+2qA-2qA}$$

$$p-1<2Aq+2q-2qA^2-2qA$$

$$p-1<q(2A+2)-qA(2A+2)$$

$$p-1<(q-qA)(2A+2)$$

$$\frac{p-1}{q-qA}<(2A+2)$$

But we have assumed that $m>A+1$ and $M>A+1$, so $m+M>2A+2$ and here is the contradiction.

Thus, $m-M=0$, and $m=M$. According to Theorem (2.5.1) the proof is complete.

Theorem 4.7.2 *Assume that*
$\frac{1}{q}<p<q, \quad Aq>q-p, \quad A^2<A-q+p, \quad 0<A<1$ *and* $A>\frac{(q-p)(1-A)}{(p+1)(q+1)}$
, then the positive equilibrium point of Eq.(4.3) is globally asymptotically stable.

Proof:
We will apply Theorem (2.5.4) in the proof using the interval $\left[\frac{1}{q},\frac{A}{q-p}\right]$
Under these assumptions we have shown in part (b) of Theorem (4.3.1) that $\bar{y}$ is locally stable. We need to show that $\bar{y}$ is a global attractor. To this end, we consider the function

$$F(x,y)=Ax+\frac{px+y}{qx+y}$$

We have shown that $F(x,y)$ is increasing in both arguments in this interval.

Suppose that $(m,M)\in\left[\frac{1}{q},\frac{A}{q-p}\right]\times\left[\frac{1}{q},\frac{A}{q-p}\right]$ is a solution of the system

$$m=F(m,m) \quad \text{and} \quad M=F(M,M)$$

Then we get

$$m=Am+\frac{pm+m}{qm+m}$$

and

$$M=AM+\frac{pM+M}{qM+M}$$

so

$$(1-A)m=\frac{(p+1)m}{(q+1)m}$$

and

$$(1-A)M = \frac{(p+1)M}{(q+1)M}$$

Now we have

$$(1-A)(q+1)m = p+1 \tag{4.27}$$

and

$$(1-A)(q+1)M = p+1 \tag{4.28}$$

Subtract (4.28) from (4.27) to get

$$(1-A)(q+1)(m-M) = 0 \tag{4.29}$$

Thus, $m - M = 0$, and $m = M$. According to Theorem (2.5.4) the proof is complete.

Remark: The main problem in [16] was in using the global stability theorems without finding invariant intervals which led to results which are not accurate.

4.8 Numerical Discussion

In this section we give numerical examples which support the theoretical discussion in the previous sections. These examples are of the form of Eq.(4.3) with different values of p, q, A, k and different initial conditions. These examples were carried out by MATLAB 6.5.

Example 1:

Assume that Equation (4.3) holds, take $k = 1$, $A = .1$, $\beta = .001$, $\gamma = 1$, $B = 300$ and $C = 3$. So the equation will be reduced to the following

$$x_{n+1} = .1x_n + \frac{.001x_n + x_{n-1}}{300x_n + 3x_{n-1}}$$

with the initial conditions $x_0 = x_1 = .33$, and $n = 0, 1, 2, ...$
or equivalently

$$y_{n+1} = .1y_n + \frac{.001y_n + y_{n-1}}{100y_n + y_{n-1}}$$

with the initial conditions $y_0 = 1$, $y_1 = 1$, and $n = 0, 1, 2, ...$

This example supports our result in Theorem (4.4.1), since the assumptions of this theorem are existent in this example as we will clarify now

1. $(k = 1)$ is an odd integer.
2. $[(p-1)(q-1)(A+1) = -108.7911] < [-4(qA+p) = -40.004]$.
3. $(p = .001) < 1$.
4. $(q = 100) > 1$.

By theory, Eq.(4.3) has prime periodic two solutions as it is obvious from Figure (4.1).

Example 2:

Assume that Eq.(4.3) holds, take $k = 3$, $A = .1$, $\beta = 50$, $\gamma = 5$, $B = 18$, $C = 2$ and $n = 0, 1, 2, ...$ So the equation will be reduced to the following

$$x_{n+1} = .1x_n + \frac{50x_n + 5x_{n-3}}{17x_n + 2x_{n-3}}$$

with the initial conditions $x_0 = 10$, $x_1 = 12.5$, $x_2 = .25$ and $x_3 = 5$
or equivalently

$$y_{n+1} = .1y_n + \frac{10y_n + y_{n-3}}{8.5y_n + y_{n-3}}$$

with the initial conditions $y_0 = 4$, $y_1 = 5$, $y_2 = .1$ and $y_3 = 2$

This example supports our result in Theorem (4.7.1), since the assumptions of this theorem are existent in this example as we will clarify now

1. $(p = 10) > (q = 8.5)$.

2. $0 < (A = .1) < 1$

3. $([p-1] = 9) < ([2q(1-A^2)] = 16.83)$

4. $([A^2P + A^2p^2] = 1.1) < ([p-q] = 1.5) < ([\frac{1}{2}(p+1)(q+1)] = 52.25)$

By theory, the positive equilibrium point $\bar{x} = \frac{p+1}{(1-A)(q+1)} = \frac{10+1}{(1-.1)(8.5+1)} = 1.286$ is globally asymptotically stable as it is obvious from Figure (4.2).

Example 3:

Assume that Eq.(4.3) holds, take $k = 2$, $A = .2$, $\beta = 19.8$, $\gamma = 2$, $B = 10$, $C = 1$ and $n = 0, 1, 2, ...$ So the equation will be reduced to the following

$$x_{n+1} = .2x_n + \frac{19.8x_n + 2x_{n-2}}{10x_n + x_{n-2}}$$

with the initial conditions $x_0 = 8$, $x_1 = 10$ and $x_2 = .2$
or equivalently

$$y_{n+1} = .2y_n + \frac{9.9y_n + y_{n-2}}{10y_n + y_{n-2}}$$

with the initial conditions $y_0 = 4$, $y_1 = 5$ and $y_2 = .1$

This example supports our result in Theorem (4.7.2), since the assumptions of this theorem are existent in this example as we will clarify now

1. $(\frac{1}{q} = .1) < (p = 9.9) < (q = 10)$

2. $0 < (A = .2) < 1$

3. $(Aq = 2) > ([q-p] = .1)$

4. $(A^2 = .04) < ([A - q + p] = .1)$

5. $(A = .2) > ([\frac{(q-p)(1-A)}{(p+1)(q+1)}] = 6.6722 \times 10^{-4})$

By theory, the positive equilibrium point $\bar{x} = \frac{p+1}{(1-A)(q+1)} = \frac{9.9+1}{(1-.2)(10+1)} = 1.2386$ is globally asymptotically stable as it is obvious from Figure (4.3).

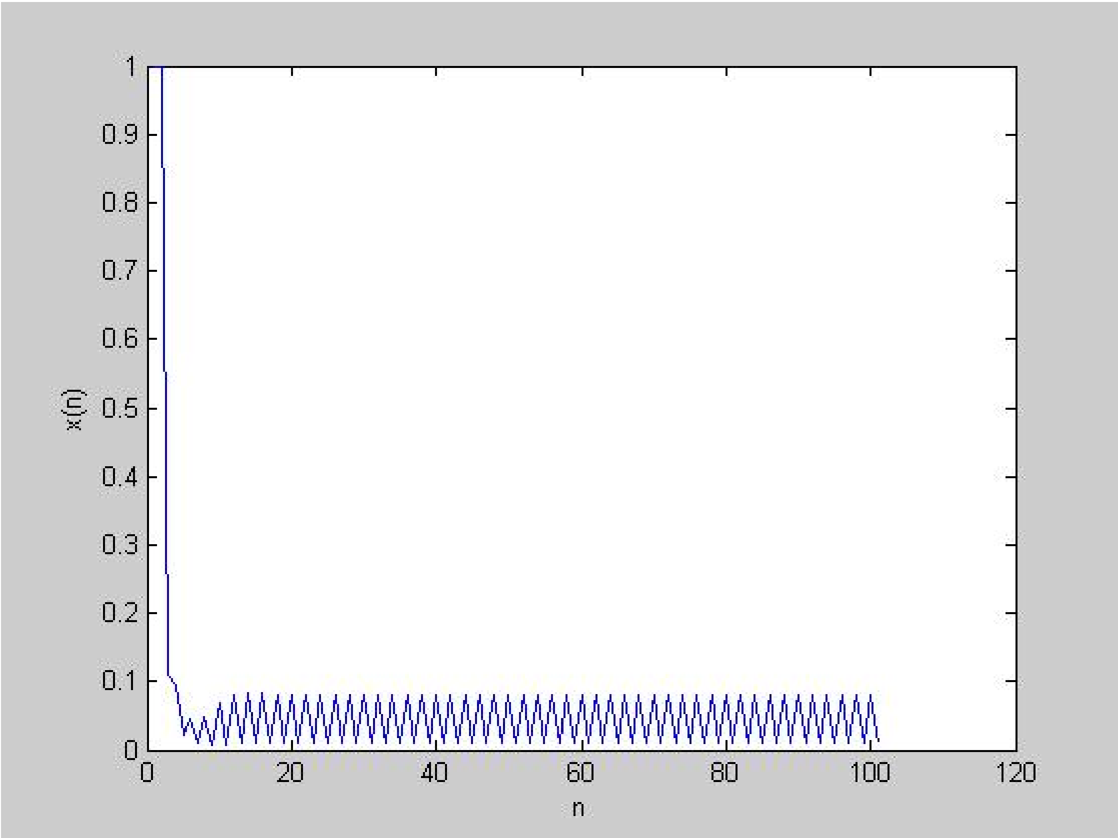

Figure 4.1: $x_{n+1} = .1x_n + \frac{.001x_n + x_{n-1}}{300x_n + 3x_{n-1}}$ has prime periodic two solutions

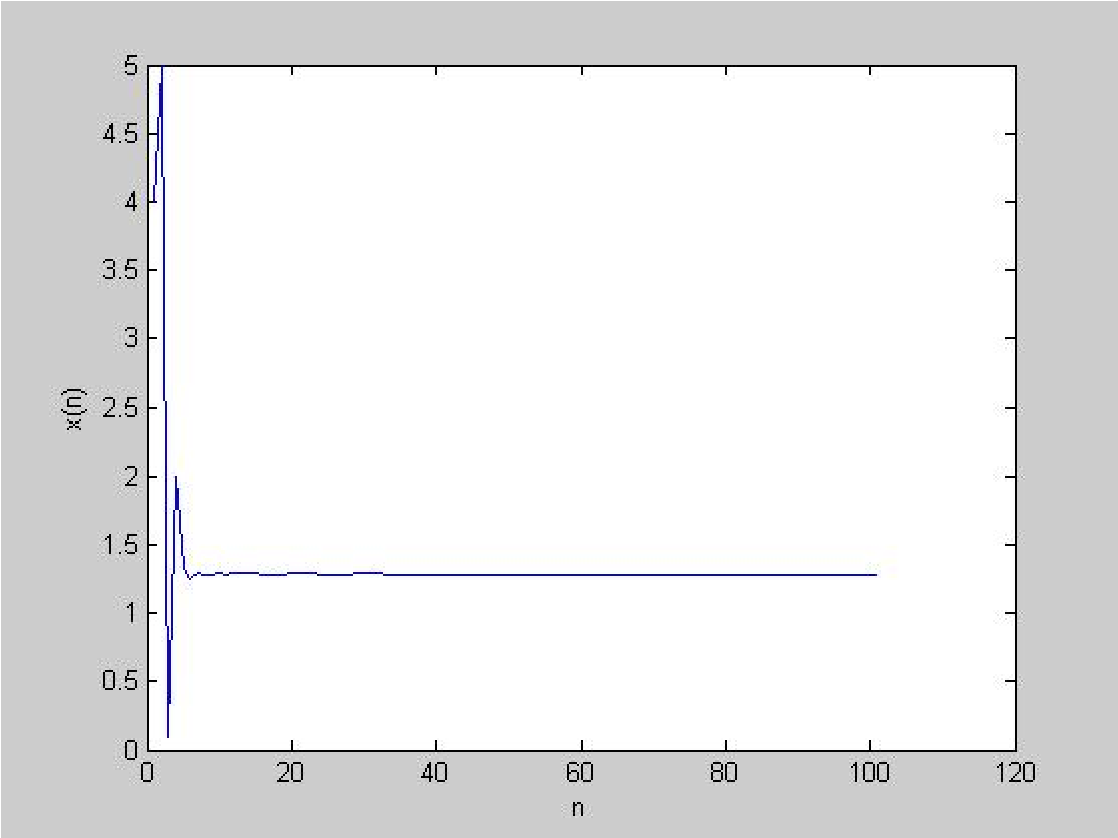

Figure 4.2: The behavior of the positive equilibrium point of $x_{n+1} = .1x_n + \frac{50x_n+5x_{n-3}}{17x_n+2x_{n-3}}$

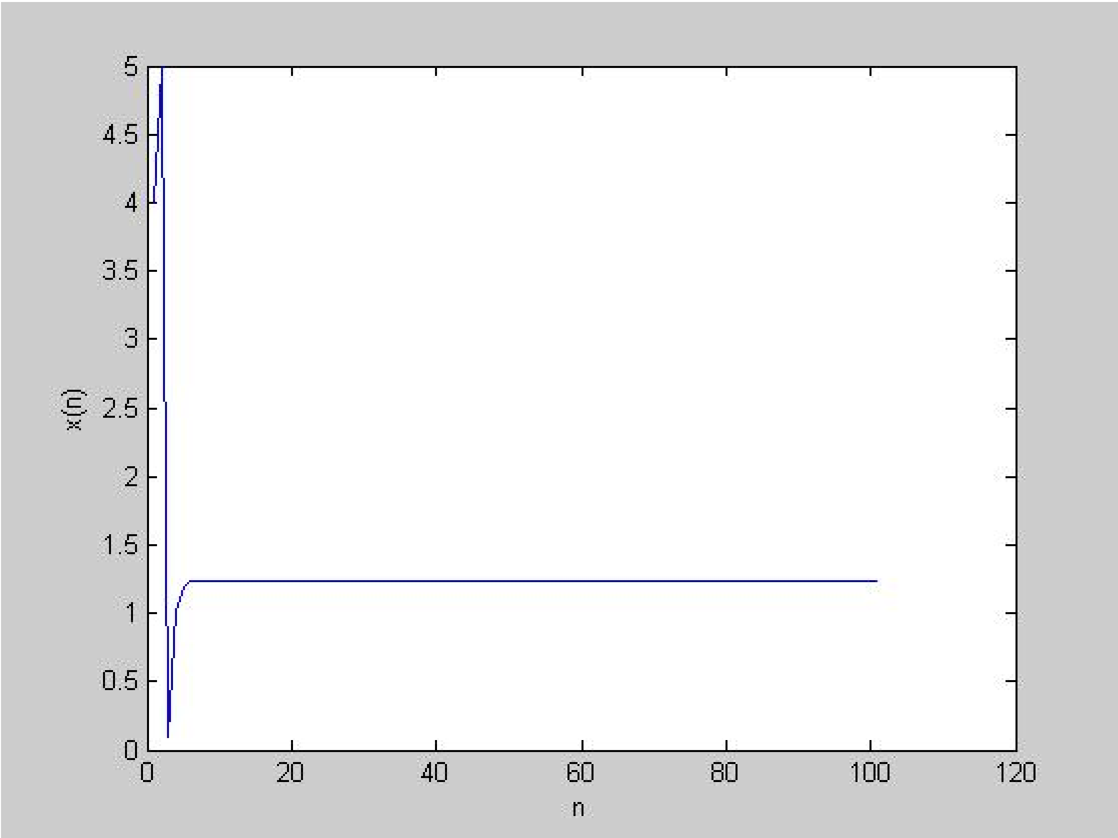

Figure 4.3: The behavior of the positive equilibrium point of $x_{n+1} = .2x_n + \frac{19.8x_n+2x_{n-2}}{10x_n+x_{n-2}}$

Chapter 5

Qualitative behavior of the difference equation $x_{n+1} = Ax_n + \frac{px_n + x_{n-k}}{q + x_{n-k}}$

Introduction

In this chapter we will study some qualitative behavior of the solutions of the difference equation

$$x_{n+1} = Ax_n + \frac{px_n + x_{n-k}}{q + x_{n-k}} \qquad , n = 0, 1, 2, ... \tag{5.1}$$

where the initial conditions x_{-k}, $\cdots$, x_{-1}, x_0 are arbitrary positive real numbers and the coefficients A, p, q are positive constants, while k is a positive integer number.

Our concentration is on invariant intervals, periodic solutions, and the global asymptotic stability of all positive solutions of Eq.(5.1).

The global stability of Eq.(5.1) for $A = 0$ has been studied in [13]. Kulenvic et al.[11] studied Eq.(5.1) when $A = 0$ and $k = 1$. A more general recursive sequence of the form

$$x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}} \qquad , n = 0, 1, 2, ... \tag{5.2}$$

has been studied in [15].

5.1 Equilibrium points

In this section we will find the equilibrium points of Eq.(5.1)

$$x_{n+1} = Ax_n + \frac{px_n + x_{n-k}}{q + x_{n-k}}$$

According to the definition of the equilibrium point we have:

$$\bar{x} = A\bar{x} + \frac{p\bar{x} + \bar{x}}{q + \bar{x}}$$

$$(1 - A)\bar{x} = \frac{(p + 1)\bar{x}}{q + \bar{x}}$$

$$\left[(1 - A) - \frac{p + 1}{q + \bar{x}}\right] \bar{x} = 0$$

so

$$\bar{x} = 0$$

or

$$(1 - A) - \frac{(p + 1)}{q + \bar{x}} = 0$$

If $(1 - A) - \frac{(p+1)}{q+\bar{x}} = 0$, then $1 - A = \frac{(p+1)}{q+\bar{x}}$

$$(1 - A)(q + \bar{x}) = p + 1$$

$$q - qA + \bar{x} - A\bar{x} = p + 1$$

$$\bar{x} = \frac{p + 1 - q + qA}{1 - A}$$

$$\bar{x} = \frac{(p - q) + (1 + qA)}{1 - A} \tag{5.3}$$

Thus $\bar{x}$ is a positive equilibrium point if one of the following two cases is valid:

1. $p > q$ and $0 < A < 1$
2. $p < q$, $(q - p) < 1 + qA$ and $0 < A < 1$

Lemma 1 *If $p > q$, $0 < A < 1$, then the positive equilibrium point satisfies the inequality $\bar{x} > \frac{q}{p}$*

Proof:

From (5.3) we deduce that:

$$\bar{x} = \frac{(p - q) + (1 + qA)}{1 - A} = \frac{p + 1 - q + qA}{1 - A} = \frac{(p + 1) - q(1 - A)}{1 - A}$$

$$\bar{x} = \frac{p + 1}{1 - A} - q > \frac{q + 1}{1 - A} - q = \frac{q + 1 - q + qA}{1 - A}$$

$$\bar{x} > \frac{1+qA}{1-A} = [1+qA]\left[\frac{1}{1-A}\right]$$

Thus

$$\bar{x} > [1+qA][1+A+A^2+\cdots] = 1+A+A^2+\cdots+qA+qA^2+qA^3+\cdots$$

but all terms on the right side are positive so

$$\bar{x} > 1 > \frac{q}{p}$$

5.2 Linearization

In this section we derive the linearized equation of Eq.(5.1) about its equilibrium points. To this end, we introduce a continuous function $F : (0,\infty)^2 \to (0,\infty)$ such that

$$F(x,y) = Ax + \tfrac{px+y}{q+y}$$

5.2.1 The linearized equation about the positive equilibrium point

$$\frac{\partial F(x,y)}{\partial x} = A + \frac{(q+y)p}{(q+y)^2} = A + \frac{p}{q+y}$$

$$\frac{\partial F(\bar{x},\bar{x})}{\partial x} = A + \frac{p}{q+\bar{x}}$$

Substituting $\bar{x}$ from (5.3) we have

$$\frac{\partial F(\bar{x},\bar{x})}{\partial x} = A + \frac{p}{q+\frac{(p-q)+(1+qA)}{1-A}}$$

$$\frac{\partial F(\bar{x},\bar{x})}{\partial x} = A + \frac{p}{\frac{q(1-A)+p-q+1+qA}{1-A}}$$

$$\frac{\partial F(\bar{x},\bar{x})}{\partial x} = A + \frac{p}{\frac{q-qA+p-q+1+qA}{1-A}}$$

hence

$$\frac{\partial F(\bar{x},\bar{x})}{\partial x} = A + \frac{p(1-A)}{p+1} = \rho_0 \tag{5.4}$$

also

$$\frac{\partial F(x,y)}{\partial y} = \frac{(q+y)\times 1-(px+y)\times 1}{(q+y)^2} = \frac{q-px}{(q+y)^2}$$

so

$$\frac{\partial F(\bar{x},\bar{x})}{\partial y} = \frac{q-p\bar{x}}{(q+\bar{x})^2}$$

Substituting $\bar{x} = \frac{(p-q)+(1+qA)}{1-A}$ we have

$$\frac{\partial F(\bar{x},\bar{x})}{\partial y} = \frac{q-p\frac{(p-q)+(1+qA)}{1-A}}{\left(q+\frac{(p-q)+(1+qA)}{1-A}\right)^2}$$

$$\frac{\partial F(\bar{x},\bar{x})}{\partial y} = \frac{\frac{q(1-A)-p(p-q+1+qA)}{1-A}}{\left(\frac{q(1-A)+p-q+1+qA}{1-A}\right)^2}$$

$$\frac{\partial F(\bar{x},\bar{x})}{\partial y} = \frac{\frac{q-qA-p^2+pq-p-qAp}{1-A}}{\frac{(q-qA+p-q+1+qA)^2}{(1-A)^2}}$$

$$\frac{\partial F(\bar{x},\bar{x})}{\partial y} = \frac{\frac{(-qA-qAp)+(q+pq)+(-p^2-p)}{1-A}}{\frac{(q-qA+p-q+1+qA)^2}{(1-A)^2}}$$

$$\frac{\partial F(\bar{x},\bar{x})}{\partial y} = \frac{-qA(1+p)+q(1+p)-p(1+p)}{(p+1)^2}(1-A)$$

$$\frac{\partial F(\bar{x},\bar{x})}{\partial y} = \frac{(p+1)(-qA+q-p)(1-A)}{(p+1)^2}$$

$$\frac{\partial F(\bar{x},\bar{x})}{\partial y} = \frac{-(qA-q+p)(1-A)}{p+1}$$

Thus

$$\frac{\partial F(\bar{x},\bar{x})}{\partial y} = \frac{-(1-A)(p-q+qA)}{p+1} = \rho_1 \tag{5.5}$$

so, the linearized equation about the positive equilibrium point is

$$z_{n+1} - \rho_0 z_n - \rho_1 z_{n-k} = 0$$

where ρ_0 and ρ_1 are given by (5.4) and (5.5).

5.2.2 The linearized equation about the zero equilibrium point

$$\frac{\partial F(x,y)}{\partial x} = A + \frac{(q+y)p}{(q+y)^2} = A + \frac{p}{q+y}$$

$$\frac{\partial F(0,0)}{\partial x} = A + \frac{p}{q} = \bar{\rho_0} \tag{5.6}$$

and

$$\frac{\partial F(x,y)}{\partial y} = \frac{q-px}{(q+y)^2}$$

so

$$\frac{\partial F(0,0)}{\partial y} = \frac{q}{q^2} = \frac{1}{q} = \bar{\rho_1} \tag{5.7}$$

so, the linearized equation about the zero equilibrium point is

$$z_{n+1} - \bar{\rho_0} z_n - \bar{\rho_1} z_{n-k} = 0$$

where $\bar{\rho_0}$ and $\bar{\rho_1}$ are given by (5.6) and (5.7).

5.3 Local stability

In this section we investigate the local stability of the positive solutions of Eq.(5.1)

Theorem 5.3.1 *The zero equilibrium point ($\bar{x} = 0$) is locally asymptotically stable if $p - q < -(1 + qA)$. In particular, if $p - q \geq -(1 + qA)$, then $\bar{x} = 0$ is unstable.*

Proof:

First suppose that $p - q < -(1 + qA)$

$$|\bar{\rho_0}| + |\bar{\rho_1}| = |A + \frac{p}{q}| + |\frac{1}{q}| = A + \frac{p}{q} + \frac{1}{q}$$

$$= A + \frac{p+1}{q} = \frac{Aq + p + 1}{q}$$

but from assumption

$$p - q < -1 - qA$$

so

$$p + 1 + qA < q$$

and

$$\frac{Aq + p + 1}{q} < 1$$

Thus

$$|\bar{\rho_0}| + |\bar{\rho_1}| < 1$$

and so the zero equilibrium point is locally asymptotically stable under this condition according to Theorem (2.2.1).

In particular, if $p - q \geq -(1 + qA)$, then $|\bar{\rho_0}| + |\bar{\rho_1}| = \frac{Aq+p+1}{q}$

but from assumption

$$p + 1 + qA \geq q$$

and

$$\frac{p + 1 + qA}{q} \geq 1$$

so

$$|\bar{\rho_0}| + |\bar{\rho_1}| = \frac{Aq + p + 1}{q} \geq 1$$

But in addition we have one of the following two cases holds:

1. k is an odd integer and
$$\bar{\rho_1} = \frac{1}{q} > 0$$
2. k is an even integer and
$$\bar{\rho_0}\bar{\rho_1} = (A + \frac{p}{q})(\frac{1}{q}) > 0$$

and so condition (2.6) is a necessary condition for the asymptotic stability according to Theorem(2.2.1), which is not true under this condition as we have proved previously. Thus $\bar{x} = 0$ is unstable in this case.

Theorem 5.3.2 *If $p > q$, $0 < A < 1$ and $p - q + qA < 1$, then the positive equilibrium point is locally asymptotically stable.*

Proof:

$$\begin{aligned} |\rho_0| + |\rho_1| &= |A + \frac{p(1 - A)}{p + 1}| + |\frac{-(1 - A)(p - q + qA)}{p + 1}| \\ &= A + \frac{p(1 - A)}{p + 1} + |\frac{-(1 - A)(p - q + qA)}{p + 1}| \end{aligned}$$

but we know that $|-x| = |x|$, and so

$$|\rho_0| + |\rho_1| = A + \frac{p(1-A)}{p+1} + |\frac{(1-A)(p-q+qA)}{p+1}|$$

From assumption $p > q$, so $p - q > 0$ and $p - q + qA > 0$. Consequently,

$$|\rho_0| + |\rho_1| = A + \frac{p(1-A)}{p+1} + \frac{(1-A)(p-q+qA)}{p+1}$$

since $p - q + qA < 1$ we have

$$|\rho_0|+|\rho_1| = \frac{A(p+1) + p(1-A) + (1-A)(p-q+qA)}{p+1} < \frac{A(p+1) + p(1-A) + (1-A)}{p+1}$$

and so

$$|\rho_0| + |\rho_1| < \frac{A(p+1) + (1-A)(p+1)}{p+1} = \frac{(p+1)(A+1-A)}{p+1} = 1$$

Hence,

$$|\rho_0| + |\rho_1| < 1$$

This proves that the positive equilibrium point is locally asymptotically stable under these conditions.

Theorem 5.3.3 *If $p < q$, $0 < A < 1$, $q - p < Aq$ and $p - q + qA < 1$, then the positive equilibrium point is locally asymptotically stable.*

Proof:

First we notice that under these assumptions we have a positive equilibrium point since $p < q$, $0 < A < 1$ and $q - p < Aq < Aq + 1$

Now we will check the local stability:

$$|\rho_0| + |\rho_1| = |A + \frac{p(1-A)}{p+1}| + |\frac{-(1-A)(p-q+qA)}{p+1}|$$

$$= A + \frac{p(1-A)}{p+1} + |\frac{-(1-A)(p-q+qA)}{p+1}|$$

but we know that $|-x| = |x|$, and so

$$|\rho_0| + |\rho_1| = A + \frac{p(1-A)}{p+1} + |\frac{(1-A)(p-q+qA)}{p+1}|$$

From assumption $q - p < Aq$, so $Aq + p - q > 0$. Consequently,

$$|\rho_0| + |\rho_1| = A + \frac{p(1-A)}{p+1} + \frac{(1-A)(p-q+qA)}{p+1}$$

since $p - q + qA < 1$ we have

$$|\rho_0|+|\rho_1| = \frac{A(p+1)+p(1-A)+(1-A)(p-q+qA)}{p+1} < \frac{A(p+1)+p(1-A)+(1-A)}{p+1} = 1$$

This proves that the positive equilibrium point is locally asymptotically stable under these conditions.

Theorem 5.3.4 *If $p < q$, $0 < A < 1$ and $Aq < q - p < Aq + 1$, then the positive equilibrium point is locally asymptotically stable. Furthermore, condition (2.6) can be considered as a necessary and sufficient condition for the asymptotic stability of Eq.(5.1).*

Proof:

$$|\rho_0| + |\rho_1| = |A + \frac{p(1-A)}{p+1}| + |\frac{-(1-A)(p-q+qA)}{p+1}|$$
$$= A + \frac{p(1-A)}{p+1} + |\frac{-(1-A)(p-q+qA)}{p+1}|$$
$$= A + \frac{p(1-A)}{p+1} + |\frac{(1-A)(p-q+qA)}{p+1}|$$

since $Aq < q - p$, and so $Aq - q + p < 0$ we have

$$|\rho_0| + |\rho_1| = A + \frac{p(1-A)}{p+1} - \frac{(1-A)(p-q+qA)}{p+1}$$

$$= \frac{A(p+1)+p(1-A)-(1-A)(-[-p+q-qA])}{p+1}$$
$$= \frac{A(p+1)+p(1-A)+(1-A)(q-p-qA)}{p+1}$$

but we know that $q - p < Aq + 1$, and so $q - p - Aq < 1$. Thus

$$|\rho_0| + |\rho_1| < \frac{A(p+1)+p(1-A)+(1-A)}{p+1} = 1$$

This proves that the positive equilibrium point is locally asymptotically stable. Thus, condition (2.6) is sufficient for the asymptotic stability of Eq.(5.1). In addition, we see that one of the following two cases is valid:

1. k is an odd integer, we know that $Aq < q - p$ and $Aq - q + p < 0$ so

$$\rho_1 = \frac{-(1-A)(p-q+qA)}{p+1} > 0$$

2. k is an even integer and

$$\rho_0\rho_1 = \left(A + \frac{p(1-A)}{p+1}\right)\left(\frac{-(1-A)(p-q+qA)}{p+1}\right) > 0$$

Thus, according to Theorem (2.2.1) condition (2.6) is also necessary for the asymptotic stability of Eq.(5.1).

5.4 Periodic solutions

In this section, we investigate the periodic character of the positive solutions of Eq.(5.1)

Theorem 5.4.1 *Equation(5.1) has no positive solutions of prime period two for all positive A, p, q*

Proof:

1. If k is an even integer.

   Assume for the sake of contradiction that there exists distinct positive real numbers ϕ and ψ such that

   $$\cdots, \phi, \psi, \phi, \psi, \cdots$$

   is a prime period two solution of Eq.(5.1).
   Since k is even, $x_{n-k} = x_n$. Substituting in Eq.(5.1) we get

   $$\psi = A\phi + \frac{p\phi + \phi}{q + \phi} \tag{5.8}$$

   and

   $$\phi = A\psi + \frac{p\psi + \psi}{q + \psi} \tag{5.9}$$

   From (5.8) we have

   $$\psi(q + \phi) = A\phi(q + \phi) + (p + 1)\phi$$

   so

   $$\psi q + \phi\psi = Aq\phi + A\phi^2 + p\phi + \phi \tag{5.10}$$

   From (5.9) we have

   $$\phi(q + \psi) = A\psi(q + \psi) + p\psi + \psi$$

Thus,

$$\phi q + \phi\psi = Aq\psi + A\psi^2 + p\psi + \psi \tag{5.11}$$

By subtracting (5.11) from (5.10) we have

$$q(\psi - \phi) = Aq(\phi - \psi) + A(\phi^2 - \psi^2) + p(\phi - \psi) + (\phi - \psi)$$

$$q(\psi - \phi) = -Aq(\psi - \phi) + A(\phi - \psi)(\phi + \psi) - p(\psi - \phi) - (\psi - \phi)$$

$$q(\psi - \phi) = -Aq(\psi - \phi) - A(\psi - \phi)(\psi + \phi) - p(\psi - \phi) - (\psi - \phi)$$

$$(\psi - \phi)(q + Aq + A[\psi + \phi] + p + 1) = 0$$

$$(\psi - \phi)(q[1 + A] + A[\psi + \phi] + p + 1) = 0$$

but we know that

$$(q[1 + A] + A[\psi + \phi] + p + 1) > 0$$

so

$$\psi - \phi = 0$$

Thus

$$\phi = \psi$$

which contradicts the assumption that $\phi \neq \psi$

Thus, the proof of Theorem (5.4.1) when k is even is now finished.

2. If k is an odd integer.

   Assume for the sake of contradiction that there exists distinct positive real numbers ϕ and ψ such that

   $$\cdots, \phi, \psi, \phi, \psi, \cdots$$

   is a prime period two solution of Eq.(5.1).

   Since k is odd, $x_{n-k} = x_{n+1}$. Substituting in Eq.(5.1) we get

   $$\psi = A\phi + \frac{p\phi + \psi}{q + \psi} \tag{5.12}$$

   and

   $$\phi = A\psi + \frac{p\psi + \phi}{q + \phi} \tag{5.13}$$

From (5.12) we have

$$\psi(q+\psi) = A\phi(q+\psi) + p\phi + \psi$$

so

$$q\psi + \psi^2 = Aq\phi + A\phi\psi + p\phi + \psi \tag{5.14}$$

From (5.13) we have

$$\phi(q+\phi) = A\psi(q+\phi) + p\psi + \phi$$

Thus,

$$q\phi + \phi^2 = Aq\psi + A\phi\psi + p\psi + \phi \tag{5.15}$$

By subtracting (5.15) from (5.14) we have

$$q(\psi-\phi) + (\psi^2-\phi^2) = Aq(\phi-\psi) + p(\phi-\psi) + (\psi-\phi)$$

$$q(\psi-\phi) + (\psi-\phi)(\psi+\phi) = -Aq(\psi-\phi) - p(\psi-\phi) + (\psi-\phi)$$

Divide both sides by $(\psi-\phi)$, since $\phi \neq \psi$

$$q + (\phi+\psi) = -Aq - p + 1$$

Hence,

$$\phi + \psi = -Aq - q - p + 1 \tag{5.16}$$

While by adding (5.15) to (5.14) we have

$$q(\psi+\phi) + \psi^2 + \phi^2 = Aq(\phi+\psi) + 2A\phi\psi + p(\phi+\psi) + (\phi+\psi)$$

$$\psi^2 + \phi^2 = (\phi+\psi)(Aq+p+1-q) + 2A\phi\psi$$

Add $(2\phi\psi)$ to both sides

$$\psi^2 + 2\phi\psi + \phi^2 = (\phi+\psi)(Aq+p+1-q) + (2A+2)\phi\psi$$

$$(\psi+\phi)^2 = (\phi+\psi)(Aq+p+1-q) + 2(A+1)\phi\psi$$

Now, substitute $(\phi+\psi)$ from (5.16)

$$(-Aq-q-p+1)^2 = (-Aq-q-p+1)(Aq+p+1-q) + 2(A+1)\phi\psi \tag{5.17}$$

First, we will find $(-Aq-q-p+1)^2$

$$(-Aq-q-p+1)^2 = (-Aq-q-p+1)(-Aq-q-p+1)$$

$$= A^2q^2 + Aq^2 + Aqp - Aq + Aq^2 + q^2 + qp - q + Aqp + qp + p^2 + -p - Aq - q - p + 1$$

and so

$$(-Aq - q - p + 1)^2 = A^2q^2 + 2Aq^2 + 2Aqp - 2Aq + q^2 + 2qp - 2q + p^2 - 2p + 1 \quad (5.18)$$

Now, we will find $(-Aq - q - p + 1)(Aq + p + 1 - q)$

$$(-Aq - q - p + 1)(Aq + p + 1 - q) =$$

$$-A^2q^2 - Aqp - Aq + Aq^2 - Aq^2 - qp - q + q^2 - Aqp - p^2 - p + pq + Aq + p + 1 - q$$

Thus

$$(-Aq - q - p + 1)(Aq + p + 1 - q) = -A^2q^2 - 2Aqp - 2q + q^2 - p^2 + 1 \quad (5.19)$$

Substitute (5.18) and (5.19) in (5.17).

$$A^2q^2+2Aq^2+2Aqp-2Aq+q^2+2qp-2q+p^2-2p+1 = -A^2q^2-2Aqp-2q+q^2-p^2+1+2(A+1)\phi\psi$$

$$A^2q^2+2Aq^2+2Aqp-2Aq+q^2+2qp-2q+p^2-2p+1+A^2q^2+2Aqp+2q-q^2+p^2-1 = 2(A+1)\phi\psi$$

$$2A^2q^2 + 2Aq^2 + 4Aqp - 2Aq + 2p^2 + 2qp - 2p = 2(A+1)\phi\psi$$

$$2(A^2q^2 + Aq^2 + 2Aqp - Aq + p^2 + qp - p) = 2(A+1)\phi\psi$$

$$A^2q^2 + Aq^2 + 2Aqp - Aq + p^2 + qp - p = (A+1)\phi\psi$$

$$A^2q^2 + Aq^2 + Aqp - Aq + Aqp + p^2 + pq - p = (A+1)\phi\psi$$

$$Aq(Aq + q + p - 1) + p(Aq + p + q - 1) = (A+1)\phi\psi$$

$$(Aq + p)(Aq + q + p - 1) = (A+1)\phi\psi$$

Consequently,

$$\phi\psi = \frac{(Aq + p)(Aq + q + p - 1)}{A + 1} \quad (5.20)$$

From (5.16) and (5.20) we have

$$\phi\psi(\phi + \psi) = \frac{(Aq + p)(Aq + q + p - 1)}{A + 1}(-Aq - q - p + 1)$$

$$\phi\psi(\phi + \psi) = \frac{(Aq + p)(Aq + q + p - 1)(-1)(Aq + q + p - 1)}{A + 1}$$

since $(Aq+p)$, $(A+1)$ and $(Aq+q+p-1)^2$ are positive numbers, we conclude that

$$\phi\psi(\phi+\psi) = \frac{-(Aq+p)(Aq+q+p-1)^2}{A+1} < 0$$

This contradicts our assumption that both ϕ and ψ are positive numbers, and so the proof of Theorem (5.4.1) is now complete.

5.5 Invariant intervals

Theorem 5.5.1 *Suppose that*

$$p > q, \quad 0 < A < 1, \quad p^3 + q^2 < pq(1-A), \quad p - q + qA < 1 \quad \textit{and} \quad 2q < p - pA.$$

Assume that for some $N \geq 0$

$$x_{N-k+1}, \cdots, x_{N-1}, x_N \in \left[\tfrac{q}{p}, \tfrac{p}{q}\right]$$

then

$$x_n \in \left[\tfrac{q}{p}, \tfrac{p}{q}\right], \textit{ for all } n > N$$

Proof:
Since $x_n \geq \frac{q}{p}$ we have

$$x_{n+1} = Ax_n + \frac{px_n + x_{n-k}}{q + x_{n-k}} \geq A(\frac{q}{p}) + \frac{p(\frac{q}{p}) + x_{n-k}}{q + x_{n-k}}$$

so

$$x_{n+1} \geq A(\frac{q}{p}) + \frac{q + x_{n-k}}{q + x_{n-k}} = A(\frac{q}{p}) + 1$$

substitute $1 = \frac{q}{q}$

$$x_{n+1} \geq A(\frac{q}{p}) + \frac{q}{q} = A(\frac{q}{p}) + \frac{p(\frac{q}{p})}{q}$$

now since $p > q$ we have

$$x_{n+1} \geq \frac{q}{p}\left(A + \frac{p}{q}\right) \geq \frac{q}{p}\left(\frac{p}{q}\right) \geq \frac{q}{p} \times 1$$

Thus

$$\mathbf{x_{n+1} \geq \frac{q}{p}}$$

Now assume that

$$x_{n+1} = f(x,y) = Ax + \frac{px+y}{q+y}$$

In our interval $f(x,y)$ is increasing in x for each fixed y, and decreasing in y for each fixed x since

$$\frac{\partial f}{\partial x} = A + \frac{(q+y)\times p - (px+y)\times 0}{(q+y)^2}$$

$$\frac{\partial f}{\partial x} = A + \frac{(q+y)\times p}{(q+y)^2} = A + \frac{p}{q+y} > 0$$

and

$$\frac{\partial f}{\partial y} = \frac{(q+y)\times 1 - (px+y)\times 1}{(q+y)^2}$$

$$\frac{\partial f}{\partial y} = \frac{q+y-px-y}{(q+y)^2} = \frac{q-px}{(q+y)^2}$$

so $f(x,y)$ is increasing in y for each fixed x if $\frac{q}{p} > x$, and $f(x,y)$ is decreasing in y for each fixed x if $\frac{q}{p} < x$.

Return to our equation: $x_{n+1} = f(x_n, x_{n-k})$

In our interval $f(x_n, x_{n-k})$ is increasing in (x_n) for each fixed (x_{n-k}), and decreasing in (x_{n-k}) for each fixed (x_n).

Since we assumed that for some $N > 0$, $\frac{q}{p} \le x_N \le \frac{p}{q}$, we have

$$f(x_n, x_{n-k}) \le f(\frac{p}{q}, x_{n-k}) \le f(\frac{p}{q}, \frac{q}{p})$$

and so

$$f(x_n, x_{n-k}) \le A\left(\frac{p}{q}\right) + \frac{p(\frac{p}{q}) + (\frac{q}{p})}{q + (\frac{q}{p})} \tag{5.21}$$

but we have so

$$\frac{p^3+q^2}{q} < (1-A)p$$

and since $pq + q > q$ we conclude that

$$\frac{p^3+q^2}{pq+q} < \frac{p^3+q^2}{q} < (1-A)p$$

and so

$$\frac{p^3+q^2}{pq+q} < (1-A)p$$

In another way

$$\frac{1}{p}\left(\frac{p^3+q^2}{pq+q}\right) < (1-A)$$

substitute

$$\frac{1}{p} = \frac{\frac{1}{p^2}}{\frac{1}{p}}$$

to get

$$\frac{(p^3+q^2)(\frac{1}{p^2})}{(pq+q)(\frac{1}{p})} < 1-A$$

$$\frac{\frac{p^3+q^2}{p^2}}{\frac{pq+q}{p}} < 1-A$$

Multiply both sides by $(\frac{p}{q})$

$$\frac{p}{q}\left(\frac{\frac{p^3+q^2}{p^2}}{\frac{pq+q}{p}}\right) < \frac{p}{q}(1-A)$$

so

$$\frac{p}{q}\left(\frac{\frac{p^3+q^2}{p^2}}{\frac{pq+q}{p}}\right) + \frac{p}{q}A < \frac{p}{q}$$

and

$$\frac{\frac{p}{q}(p+\frac{q^2}{p^2})}{q+\frac{q}{p}} + \frac{p}{q}A < \frac{p}{q}$$

so

$$\frac{(\frac{p}{q})p + (\frac{q}{p})(\frac{p}{q})(\frac{q}{p})}{q+\frac{q}{p}} + \frac{p}{q}A < \frac{p}{q}$$

Thus

$$\frac{p(\frac{p}{q}) + (\frac{q}{p})}{q+\frac{q}{p}} + \frac{p}{q}A < \frac{p}{q} \tag{5.22}$$

substitute (5.22) in (5.21) to get

$$x_{n+1} \leq \frac{p(\frac{p}{q}) + (\frac{q}{p})}{q+\frac{q}{p}} + \frac{p}{q}A < \frac{p}{q}$$

and so

$$\mathbf{x_{n+1}} \leq \frac{\mathbf{p}}{\mathbf{q}}$$

The proof is now complete.

Theorem 5.5.2 *Suppose that*

$$p < q, \quad 0 < A < 1, \quad q - p < Aq \quad \textit{and} \quad Aq + p^2 + \tfrac{1}{2}p < q.$$

Assume that for some $N \geq 0$

$$x_{N-k+1}, \cdots, x_{N-1}, x_N \in \left[\tfrac{q}{p}, \tfrac{2q}{p}\right]$$

then

$$x_n \in \left[\tfrac{q}{p}, \tfrac{2q}{p}\right], \textit{ for all } n > N$$

Proof:

Since $x_n \geq \frac{q}{p}$ we have

$$x_{n+1} = Ax_n + \frac{px_n + x_{n-k}}{q + x_{n-k}} \geq A(\frac{q}{p}) + \frac{p(\frac{q}{p}) + x_{n-k}}{q + x_{n-k}}$$

so

$$x_{n+1} \geq A(\frac{q}{p}) + \frac{q + x_{n-k}}{q + x_{n-k}} = A(\frac{q}{p}) + 1$$

substitute $1 = \frac{q}{q}$

$$x_{n+1} \geq A(\frac{q}{p}) + \frac{q}{q} = A(\frac{q}{p}) + \frac{q(\frac{p}{p})}{q} = A(\frac{q}{p}) + (\frac{q}{p}) \times (\frac{p}{q})$$

so

$$x_{n+1} \geq \frac{q}{p}\left(A + \frac{p}{q}\right) = \frac{q}{p}\left(\frac{Aq + p}{q}\right) \tag{5.23}$$

but from assumption $q - p < Aq$ so $Aq + p > q$ and

$$\frac{Aq + p}{q} > 1 \tag{5.24}$$

substitute (5.24) in (5.23) to get

$$x_{n+1} \geq \frac{q}{p}\left(\frac{Aq + p}{q}\right) \geq \frac{q}{p} \times 1$$

Thus,

$$\mathbf{x_{n+1} \geq \frac{q}{p}}$$

Assume that

$$x_{n+1} = f(x, y) = Ax + \frac{px + y}{q + y}$$

We have proved in the previous theorem that $f(x, y)$ is increasing in x for each fixed y, and decreasing in y for each fixed x in this interval.

We have assumed that for some $N > 0$, $\frac{q}{p} \leq x_N \leq \frac{2q}{p}$, so

$$x_{n+1} = f(x_n, x_{n-k}) \leq f(\frac{2q}{p}, \frac{q}{p})$$

$$x_{n+1} \leq f(\frac{2q}{p}, \frac{q}{p}) = A(\frac{2q}{p}) + \frac{p(\frac{2q}{p}) + \frac{q}{p}}{q + \frac{q}{p}} = A(\frac{2q}{p}) + \frac{p(\frac{2q}{p}) + (\frac{q}{p})(\frac{2}{2})}{q + \frac{q}{p}}$$

$$x_{n+1} \leq (\frac{2q}{p})A + \frac{\frac{2q}{p}(p + \frac{1}{2})}{q + \frac{q}{p}} \leq \frac{2q}{p}A + \frac{\frac{2q}{p}(p + \frac{1}{2})}{\frac{q}{p}}$$

$$x_{n+1} \leq \frac{2q}{p}\left(A + \frac{p + \frac{1}{2}}{\frac{q}{p}}\right)$$

$$x_{n+1} \leq \frac{2q}{p}\left(A + (p + \frac{1}{2})(\frac{p}{q})\right) = \frac{2q}{p}\left(A + \frac{p^2 + \frac{1}{2}p}{q}\right)$$

so

$$x_{n+1} \leq \frac{2q}{p}\left(\frac{Aq + p^2 + \frac{1}{2}p}{q}\right)$$

but from assumption $Aq + p^2 + \frac{1}{2}p < q$, and so

$$x_{n+1} \leq \frac{2q}{p}\left(\frac{Aq + p^2 + \frac{1}{2}p}{q}\right) < \frac{2q}{p}(\frac{q}{q}) = \frac{2q}{p}$$

Thus

$$\mathbf{x_{n+1} \leq \frac{2q}{p}}$$

The proof is complete.

5.6 Global stability

Theorem 5.6.1 *Assume that*

$$p > q, \quad 0 < A < 1\ , \quad (p^3 + q^2) < pq(1 - A), \quad 2q < (p - pA) \quad \textit{and} \quad p - q + qA < 1$$

Then the positive equilibrium point of Eq.(5.1) is globally asymptotically stable.

Proof: Set

$$f(x, y) = Ax + \frac{px + y}{q + y}$$

Under these assumptions, we have shown in Theorem (5.3.2) that the positive equilibrium point is locally asymptotically stable. We need to prove that $\bar{x}$ is a global attractor and we will use Theorem (2.5.1) for this.

We know that $\frac{q}{p} \leq f(x, y) \leq \frac{p}{q}$, and in this interval $f(x, y)$ is increasing in x for each fixed y, and decreasing in y for each fixed x.

Suppose that $(m, M) \in \left[\frac{q}{p}, \frac{p}{q}\right] \times \left[\frac{q}{p}, \frac{p}{q}\right]$ is a solution of the system

$$M = F(M, m) \qquad \text{and} \qquad m = F(m, M)$$

Then we get

$$M = AM + \frac{pM + m}{q + m}$$

so

$$(1 - A)M = \frac{pM + m}{q + m}$$

From which we have

$$(M - AM)(q + m) = pM + m$$

so

$$Mq + mM - AMq - AmM = pM + m \tag{5.25}$$

Also

$$m = Am + \frac{pm + M}{q + M}$$

and

$$(1 - A)m = \frac{pm + M}{q + M}$$

Thus

$$(m - Am)(q + M) = pm + M$$

and so

$$mq + mM - Amq - AmM = pm + M \tag{5.26}$$

Subtract (5.26) from (5.25) to get

$$q(M - m) - Aq(M - m) = p(M - m) + (m - M)$$

$$q(M - m) - Aq(M - m) = p(M - m) - (M - m)$$

$$(M - m)(q - qA - p + 1) = 0 \tag{5.27}$$

but from assumption $p - q + qA < 1$, and so $1 - p + q - qA > 0$. Thus, $M - m = 0$ and $m = M$.

Theorem 5.6.2 *Assume that*

$$p < q, \quad 0 < A < 1, \quad 0 < p - q + qA < 1, \quad Aq + p^2 + \tfrac{1}{2}p < q \quad \text{and}$$
$$p(p - q + 1 + qA) < 2q(1 - A)$$

Then the positive equilibrium point of Eq.(5.1) is globally asymptotically stable.

Proof: Set

$$f(x, y) = Ax + \frac{px + y}{q + y}$$

Under these assumptions, we have shown in Theorem (5.3.3) that the positive equilibrium point is locally asymptotically stable. We need to prove that $\bar{x}$ is a global attractor and we will use Theorem (2.5.1) for this.

We know that $\frac{q}{p} \le f(x, y) \le \frac{2q}{p}$, and in this interval $f(x, y)$ is increasing in x for each fixed y, and decreasing in y for each fixed x.

Suppose that $(m, M) \in \left[\frac{q}{p}, \frac{2q}{p}\right] \times \left[\frac{q}{p}, \frac{2q}{p}\right]$ is a solution of the system

$$M = F(M, m) \quad \text{and} \quad m = F(m, M)$$

Then we get

$$M = AM + \frac{pM + m}{q + m}$$

and

$$m = Am + \frac{pm + M}{q + M}$$

In the same procedure as in Theorem (5.6.1) we conclude that

$$(q - Aq - p + 1)(M - m) = 0$$

but $p - q + qA < 1$, and $0 < 1 - p + q - qA$. Thus, $M = m$.

The result is a consequence of Theorem (2.5.1).

5.7 Numerical Discussion

In this section we give numerical examples which support the theoretical discussion in the previous sections. These examples are of the form of equation (5.1) with different values of p, q, A, k and the initial conditions. The examples were carried out on MATLAB.

Example 1:

Assume that Equation (5.1) holds, take $k = 1$, $A = .01$, $p = .05$, $q = .02$ and $n = 0, 1, 2, ...$ So the equation will be reduced to the following

$$x_{n+1} = .01x_n + \frac{.05x_n + x_{n-1}}{.02 + x_{n-1}}$$

with the initial conditions $x_0 = x_1 = 1$

This example supports our result in Theorem (5.6.1), since the assumptions of this theorem are existent in this example as we will clarify now

1. $(p = .05) > (q = .02)$.
2. $0 < (A = .01) < 1$.
3. $([p^3 + q^2] = 5.25 \times 10^{-4}) < (pq[1 - A] = 9.9 \times 10^{-4})$.
4. $(2q = .04) < ([p - pA] = .0495)$.
5. $([p - q + qA] = .0302) < 1$.

By theory, the positive equilibrium point $\bar{x} = \frac{p-q+1+qA}{1-A} = \frac{.05-.02+1+.02\times.01}{1-.01} = 1.0406$ is globally asymptotically stable as it is obvious from Figure (5.1)

Example 2:

Assume that Equation (5.1) holds, take $k = 1$, $A = .81$, $p = .1$, $q = .5$ and $n = 0, 1, 2, ...$ So the equation will be reduced to the following

$$x_{n+1} = .81x_n + \frac{.1x_n + x_{n-1}}{.5 + x_{n-1}}$$

with the initial conditions $x_0 = x_1 = 1$

This example supports our result in Theorem (5.6.2), since the assumptions of this theorem are existent in this example as we will clarify now

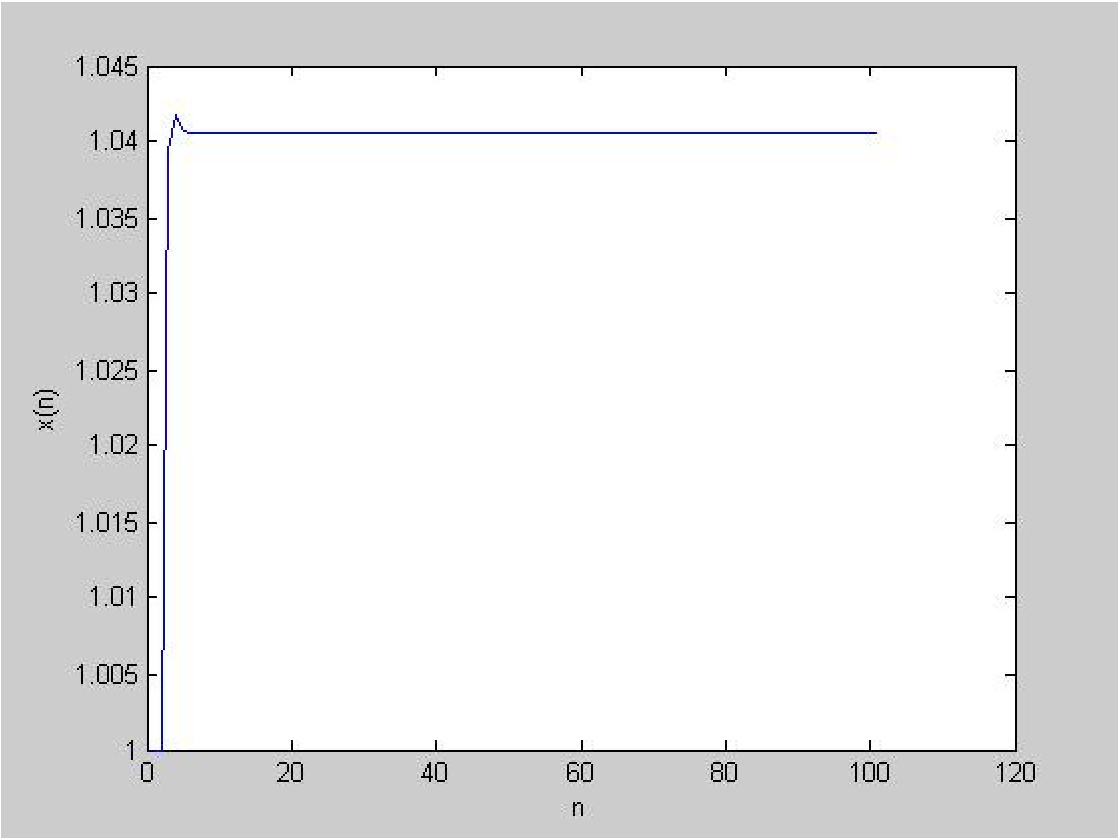

Figure 5.1: The behavior of the positive equilibrium point of $x_{n+1} = .01x_n + \frac{.05x_n + x_{n-1}}{.02 + x_{n-1}}$

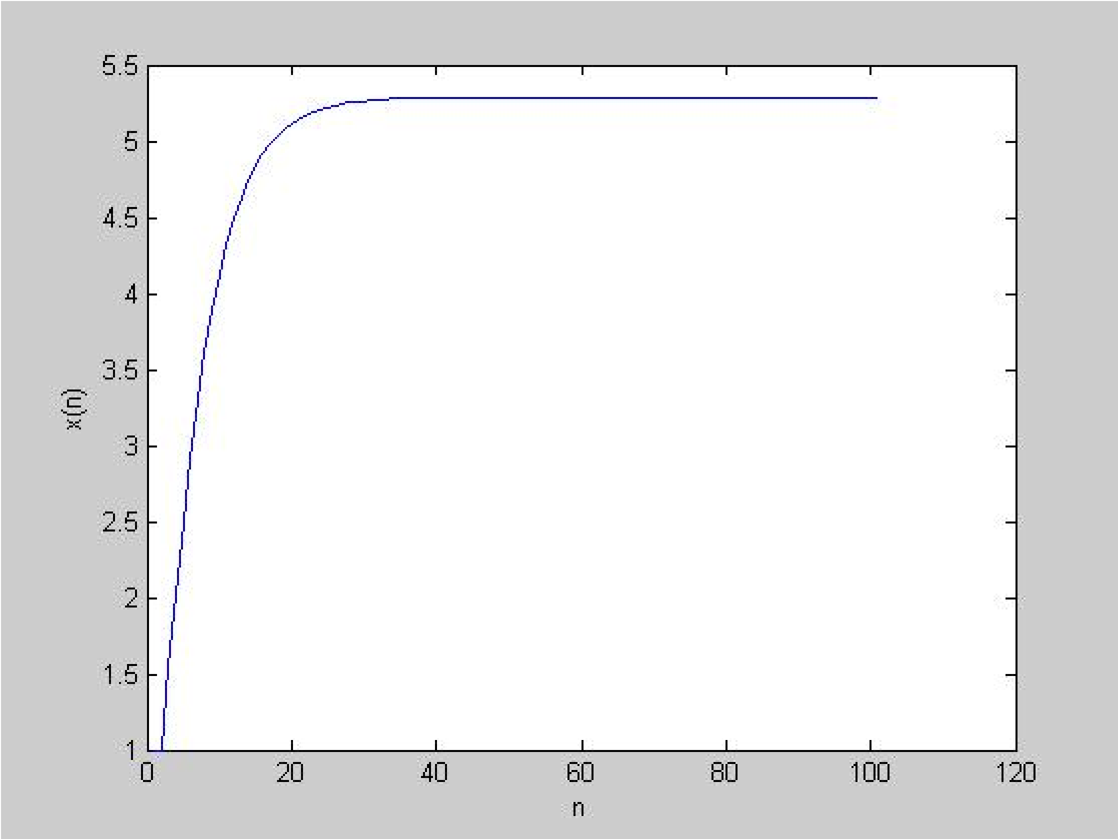

Figure 5.2: The behavior of the positive equilibrium point of $x_{n+1} = .81x_n + \frac{.1x_n + x_{n-1}}{.5 + x_{n-1}}$

1. $(p = .1) < (q = .5)$.
2. $0 < (A = .81) < 1$.
3. $0 < ([p - q + qA] = 5 \times 10^{-3}) < 1$
4. $([Aq + p^2 + .5p] = .465) < (q = .5)$
5. $([p(p - q + 1 + qA)] = .1005) < ([2q(1 - A)] = .19)$

By theory, the positive equilibrium point $\bar{x} = \frac{p-q+1+qA}{1-A} = \frac{.1-.5+1+.5\times.81}{1-.81} = 5.2895$ is globally asymptotically stable as it is obvious from Figure (5.2).

Appendix A

The MATLAB 6.5 Codes

A.1 The difference equation $x_{n+1} = Ax_n + \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}$

```
%Qualitative behavior of the difference equation
%Xn+1 = AXn + ( Beta*Xn + Gamma*Xn-k ) / ( B*Xn + C*Xn-k )

%We'am Masarweh
%1095374

%format long

Beta=input('insert the value of Beta=');
Gamma=input('insert the value of Gamma=');
'P=Beta/Gamma'
P=Beta/Gamma

B=input('insert the value of B=');
C=input('insert the value of C=');
'Q=B/C'
Q=B/C

A=input('insert the value of A=');
```

```
K=input('insert the value of K=');
for h=1:K+1
    x(h)=input('insert the value of initial value =');
    y(h)=(C/Gamma).*x(h);
end

for i = K+1 :100,
 y(i+1)=A.*y(i)+((P.*y(i)+y(i-K))/(Q.*y(i)+y(i-K)));
end
plot (y)
```

A.2 The difference equation $x_{n+1} = Ax_n + \frac{px_n + x_{n-k}}{q + x_{n-k}}$

```
%Qualitative behavior of the difference equation
%Xn+1 = AXn + ( p*Xn + Xn-k ) / ( q + Xn-k )

%We'am Masarweh
%1095374

%format long

p=input('insert the value of p=');
q=input('insert the value of q=');
A=input('insert the value of A=');
K=input('insert the value of K=');
for h=1:K+1
    x(h)=input('insert the value of initial value =');
end

for i = K+1 :100,
 x(i+1)=A.*x(i)+((p.*x(i)+x(i-K))/(q+x(i-K)));
end
plot (x)
```

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