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Dynamics of a k^{th} order Rational Difference Equation Using Theoretical and Computational Approaches

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الإهداء...

- إلى فلسطين يزين جيدها قدس الأقداس...
- إلى الذين صدقوا ما عاهدوا ا عليه فمنهم من قضى نحبه و منهم من ينتظر...
- إلى روح الإمام الشهيد أحمد الياسين و رفاق دربه القادة العظام...
- إلى مسك الأرض... آساد الوغى... رجال المصاحف و البندقية... شهدائنا و جرحانا...
- إلى من رؤوسهم تعانق عنان السماء رغم مر العناء... أسرانا... زوجي و إخوتي...
- إلى رمز الصبر والتفاني... إلى الغوالي... أمي و أبي و إخواني...
- إلى زهرات حياتي... شهد و رند و وجد...
- إلى طلائع النور... إلى المجد القادم...

إلى كل هؤلاء...

اهدي هذا العمل خالصا ليكون زادا على طريق العلم ووقودا لمسيرة التحرر...

سائدة أبو البهاء

شكر و تقديم...

لا يسعني هنا إلا أن أتقدم بخالص الشكر و التقدير إلى كل من ساهم في إنجاز هذه الرسالة و أخص منهم:

- د. محمد صالح.
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سائلا المولى أن يبقيهم ذخرا لمسيرة العلم والعطاء.

سائدة أبو البهاء

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By

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الملخص:

في السنوات الماضية كان للأنظمة الحركية العديد من التطبيقات في العلوم والهندسة وهذه تشمل الاهتزاز الميكانيكي، الليزر، الإيقاعات البيولوجية، دوائر التوصيل الكبيرة، وحالات تفشي الحشرة، والتذبذبات الكيميائية، وأنظمة التحكم بالجينات، وأمواج الماء، حتى أن لها دور في استخدام النظام الفوضوي لإرسال الرسائل السرية بعضها يندرج تحت عنوان التحليل الغير خطي ، خلف هذه التطبيقات هناك الكثير من الموضوعات الرياضية الغنية التي سوف نتناول أحدها في هذه الرسالة .

هذا الموضوع يركز على التكرار المداري للمعادلة الكسرية المختلفة وبالذات نحن مهتمون بالجانب التحليلي مثل التحليل حول نقطة التعادل، بالرغم من أننا نركز على الجانب التحليلي فإن للجانب الهندسي دور كبير في التعرف على صفات هذه المعادلة الكسرية المختلفة.

في هذه الرسالة تناولنا المعادلة الكسرية المختلفة التالية:

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, n = 0,1,2,...(1)$$

حيث أن المتغيرات B ، C β , γ و العوامل الأولية x(-k), …, x(-1) عبارة عن أعداد حقيقية موجبة، k={1,2,…}.

كان تركيزنا على إيجاد حل لهذه المعادلة وملاحظة شكل هذا الحل بناء على العوامل الأولية المدخلة. لذلك أوجدنا فترات الحل الممكنة, الخصائص الدورية للحل إذا وجد, وكذلك مدى ثبات الحلول الموجبة للمعادلة (1).

حتى نتمكن من التعرف أو تحقق الصفات الدورية وكذلك مدى ثبات هذه المعادلة، سنستخدم Mathlab للتعرف على تصرف هذه المعادلة، Mathlab الآن يقدر على ايجاد حلول تقريبية لهذه المعادلة المختلفة وإنتاج رسومات عالية الدقة للتعرف على شكل هذه المعادلة وحلها إذ أنه يعتبر الأساس في بناء النظرية الرياضية، إلا أنه لا يثبتها لذلك سنستخدم تقنيات عديدة ومختلفة لتساعدنا على حل هذه المعادلة وإثباتها.

هناك العديد من الأوراق العلمية والكتيبات في موضوع الأنظمة الحركية ، إلا أن هناك نواحي عدة تجعل من هذه الرسالة تتميز عن غيرها من الإصدارات :-

أولاً : هذه الرسالة هي حل للمشكلة المفتوحة التي عرضها Kulenvic وLadas في كتابهم:

[Dynamics of Second Order Rational Difference Equations: with Open Problems and Conjectures, Chapman & Hall/CRC, Boca Rataon, 2002].

ثانياً: هذه الرسالة تعالج الموضوع من المنظور الرياضي الذي يحوي على براهين لمعظم النتائج ، البراهين التي لم ترد في هذه الرسالة هي إما تركت للقارىء ، أو ذكرت في أوراق علمية أخرى ، ومع ذلك سيجد القارىء المهتم بالجوانب الحسابية النتائج جاهزة دون أن يرهق نفسه بالنظر إلى البراهين .

ثالثاً: هذه الرسالة عنيت لتكون متطلب تخرج وليس مجرد ورقة علمية وهذا يعكس على الطريقة التي عرضت فيها الرسالة ، الأفكار طرحت من خلال أمثلة عملية للتعرف على النظرية والتطبيقات للبراهين والمواضيع ذات العلاقة .

التغيير التالي:

$$x_n = \frac{\gamma}{C} y_n$$

يحول المعادلة إلى الشكل:

$$y_{n+1} = \frac{py_n + y_{n-k}}{qx_n + y_{n-k}}, n = 0,1,2,...(2)$$

حيث أن:

$$p = \frac{\beta}{\gamma}, q = \frac{B}{C}.$$

المتغيرات p,q و العوامل الولية y(-k),…,y(-1) عبارة عن أعداد حقيقية موجبة، k={1,2,…}.

كما كان متوقعا، الحالتان p>q, p<q تعطيان أنظمة حركية مختلفة.

الحالة الأولى: p>q.

- نقطة التعادل ثابتة محليا.
- لا يوجد حل دوري ثنائي.
- الحل يتذبذب حول نقطة التعادل بشبه دورة طولها k+1 أو k+2 ما عدا شبه الدورة الأولى التي يمكن أن يكون طولها k .
- الحل محصور بين القيمتين 1, p/q.
- نقطة التعادل ثابتة مطلقا إذا تحقق الشرط التالي: p<=pq+3q+1.

الحالة الثانية: p<q.

يندرج تحت هذه الحالة الحالتان الفرعيتان التاليتان:

- **K عدد زوجي.**

- نقطة التعادل ثابتة محليا.
- الحل يتذبذب حول نقطة التعادل بشبه دورة طولها k بعد شبه الدورة الأولى أو يتناهى الى نقطة التعادل.
- الحل محصور بين القيمتين 1, p/q.

❖ **K عدد فردي.**

يندرج تحت هذه الحالة الحالتان الفرعيتان التاليتان:

- $q>pq+3p+1$.
  - نقطة التعادل غير ثابتة.
  - يوجد حل دوري ثنائي.
  - الحل يتذبذب حول نقطة التعادل بشبه دورة طولها k بعد شبه الدورة الأولى أو يتناهى الى نقطة التعادل.
  - الحل محصور بين القيمتين $1, p/q$.

- $q<pq+3p+1$.
  - نقطة التعادل ثابتة محليا.
  - الحل يتذبذب حول نقطة التعادل بشبه دورة طولها k بعد شبه الدورة الأولى أو يتناهى إلى نقطة التعادل.
  - الحل محصور بين القيمتين $1, p/q$.
  - نقطة التعادل ثابتة مطلقا.

إن النتائج التي حصلنا عليها في هذه الرسالة لها أهمية كبرى في حد ذاتها، بالإضافة إلى أنها تشكل الأساس لمعرفة ثبات الحل للمعادلات الكسرية المختلفة من الدرجة النونية.

إن النتائج و الطرق التي استخدمت في هذه الرسالة لها أهمية كبرى في تحليل النماذج الرياضية المستخدمة في كافة مناحي الحياة مثل النماذج المستخدمة في الأحياء و التطبيقات الأخرى.

Abstract

In resent years, dynamical systems has had many applications to science and engineering; these include mechanical vibration, lasers, biological rhythms, super conducting circuits, insect outbreaks, chemical oscillators, genetic control systems, chaotic water wheels, and even a technique for using chaos to send secret messages. Some of which have gone under the related headings of [nonlinear analysis]. Behind these applications there lies a rich mathematical subject; which we will treat one of them in this thesis. [2].

This subject centers on the orbits of iteration of a nonlinear rational difference equation. In particular, we are interested in the analytic analysis (e.g. the local analysis near a fixed point, the character of semicycles and global asymptotic stability theory). Although the subject has analytic analysis, a geometric or topological flavor plays an important role for suggesting the behavior of this rational difference equation.

In this thesis, we will investigate the nonlinear rational difference equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \qquad n = 0, 1, ... \qquad (1)$$

where the parameters β, γ and B, C and the initial conditions $x_{-k}, ..., x_{-1}$ and x_0 are nonnegative real numbers, $k = \{1, 2, 3, ...\}$.

Our concentration is on invariant intervals, periodic character, the character of semicycles and global asymptotic stability of all positive solutions of equation(1).

In order to investigate the global attractivity, boundedness, periodicity, and global stability of solution of this difference equation, we will use MATLAB to see how the behavior of this difference equation look like. MATLAB now capable of finding approximations of solutions of this difference equation and also producing high quality graphics representations of its behavior. Although MATLAB are a wonderful tool for suggesting the behavior of this difference equation; it is the base on which to build the mathematical theory, it do not normally provide proof of its existence in the strict mathematical sense. We will use different techniques to help us solving this difference equation and prove it.

There have been several papers and monographs on the subject of Dynamical Systems. There are several distinctive aspects which together make this thesis unique.

• First of all, the results of this thesis solve the open problem 6.10.17 (equation(6.100)) proposed by Kulenvic and Ladas in their monograph [Dynamics of Second Order Rational Difference Equations: with Open Problems and Conjectures, Chapman & Hall/CRC, Boca Raton, 2002]. [7]

• Second, this thesis treats the subject from a mathematical perspective with the proofs of most of the results included: the only proofs which are omitted either (i) are left to the reader, (ii) are mentioned in other papers. Although it has a mathematical perspective, readers who are more interested in applied or computational aspects of the subject should find the explicit statements of the results helpful even if they do not concern themselves with the details of the proofs.

• Third, this thesis is meant to be a graduate requisite and not just a paper on the subject. This aspect of the thesis is reflected in the way the background materials are carefully reviewed as we use them. The ideas are introduced through numerical examples to learn the meaning of the theorems and master the techniques of the proofs and topic under consideration.

In this thesis we use difference equation in the kth order to introduce basic ideas and results of dynamical systems. In order to investigate this dynamical system we divided this thesis to four chapters:

Chapter 1 gives an introduction to dynamical systems, it gives some basic information to discrete system, linear system and difference equations. Chapters 2 shows in details the solutions of linear and nonlinear difference equations form the first up to $k^{th} order$.Chapter 3 shows in details the behavior of solutions of linear and nonlinear difference equations. Chapter 4 shows our problem in details; starting with the linearization and the equilibrium point, then conditions under which the equilibrium point will be local stable or global stable, and the others under which the solution will have period two solution, and finally we discuss the semicycles and invariant interval. The ideas are introduced through numerical examples to learn the meaning of the theorems and master the techniques of the proofs and topic under consideration.

As might be expected, the two cases $\mathbf{p} > \mathbf{q}$ and $\mathbf{p} < \mathbf{q}$ give rise to different dynamic behaviors.

We believe that the results about equation(1) are of paramount importance in their own right, the results presented also give the basic theory of the global behavior of solutions of nonlinear difference equations of order k. The techniques and results in this thesis are also extremely useful in analyzing the equations in the mathematical models of various biological systems and other applications. [7].

CHAPTER 1
Introduction

Introduction

In this thesis, we will study the nonlinear rational difference equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{B x_n + C x_{n-k}}, \qquad n = 0, 1, ..., \quad (1)$$

where the parameters β, γ and B, C and the initial conditions $x_{-k}, ..., x_{-1}$ and x_0 are nonnegative real numbers, $k = \{1, 2, 3, ...\}$.

Our concentration is on invariant intervals, periodic character, the character of semicycles and global asymptotic stability of all positive solutions of equation(1).

As we mention the thesis solve the open problem 6.10.17 (equation(6.100)) proposed by Kulenvic and Ladas in their monograph [Dynamics of Second Order Rational Difference Equations: with Open Problems and Conjectures, Chapman & Hall/CRC, Boca Raton, 2002]. [7]
Before studying the behavior of solutions of this rational difference equation, we will review some subjects which will be useful to introduce a basic idea in order to understand the above open problem successfully.

The following diagram shows where the subject of my thesis lie? what is the related topics? what is the background material should the reader know before studying the problem? any way my thesis will provide the reader with these basic information as an introductory sections for the dynamical systems.

dynamical system
↙ ↘
continuous ↓ differential equation out of my research
discrete ↓ difference equation behavior of solution
↙ linear: ↙ $first$ $order$, ↙ $second$ $order$, ... ↘ $k-1$ $order$, ↘ $k^{th} order$
↘ nonlinear: ↙ $first$ $order$, ↙ $second$ $order$, ... ↘ $k-1$ $order$, ↘ $k^{th} order$
↓
$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{B x_n + C x_{n-k}}$
↕
my thesis

The following sections are introductory sections for the dynamical systems. It provides some very basic information. These sections are organized in terms of some frequently asked questions, cursory answers. The ideas are introduced through numerical examples to learn the meaning of the theorems and master the techniques of the proofs and topic under consideration.

1.1 Dynamical System

"The dynamic of any situation refers to how the situation changes over the course of time. A dynamical system is a physical setting together with rules for how the setting changes or evolves from one moment of time to the next". [10].

"In simplest terms, a dynamical system is a system that changes over time. Thus, the solar system is a dynamical system; the united states economy is a dynamical system; the weather is a dynamical system; the human heart is a dynamical system". [10].

When we model a system, we usually idealize the system in term of its state variables of the system, which are quantities that represent the system itself. For example, a moving body may be represented by state variable of velocity and position over time. Model of population dynamic, the system state variable may be the number of population that migrate, born and dead and the existing population. [3].

"Mathematically speaking, a dynamical system is a system whose behavior at a given time depends, in some sense, on its behavior at one or more previous times. The words "in some sense" in the preceding sentence should be taken to mean that we may or may not have a clue as to how a current state of a system depends on a past state; but we have reason to believe that it does. Furthermore, it is the task of the mathematical modeler to come up with a mathematical construct, a model that will describe this relationship between current and past states of the system so that predictions about the future course of events for the system may be made with some degree of accuracy". [10].

In other words, dynamical systems is the study of phenomena that evolve in space and / or time by looking at the dynamic behavior or the geometrical and topological properties of the solutions. Whether a particular system comes from biology, physics, chemistry, or even the social sciences, dynamical systems is the subject that provides the mathematical tools for its analysis.

The seminal work by Lorenz in 1963, gave scientists insight to recognize a new type of motion called Chaos. Chaotic systems, which can be very simple, are capable of generating erratic behavior that is different from the one produced by quasi-periodic systems with a large number of frequencies of oscillations. [4,12].

The subject of dynamical systems was founded towards the end of the nineteenth century by the French mathematician Henri Poincare'. The differential equations in which he interested arose from the study of planetary motion. to make progress in the study of these equations, Poincare' invented new topological methods for studying their solutions, in place of the traditional methods involving series. [4, 10, 13].

One basic goal of the mathematical theory of dynamical systems is to determine or characterize the long term behavior of the system. Three kinds of dynamical systems are common, their characteristics are closely related in surprising ways. [10].

- Symbolic dynamical systems.
- Continuous-time systems.
- Discrete-time systems.

My thesis related to Discrete-time systems. The following section will give some basic information in order to deep understanding the topic.

1.2 Discrete-time systems

"Discrete-time system dynamics is a topic of broad interest; the main reason for this interest comes from the variety of the sources of discrete time dynamical models. We may cite:

- Discrete-time models determined by the nature of the described processes: this is particularly true for economics, biology, physiology and discrete-time information processing.
- Discrete-time models induced by the impulses occurring in continuous-time systems.
- Discrete models occurring in controlled systems when the feed-back information used in control generation is composed of output samples obtained through sampling intervals of time.
- Discrete systems occurring during numerical treatment of continuous time systems.

But the interest in discrete-time systems may also be explained by the simplicity of their treatment; it requires minimal computational and graphical resources to obtain the solutions of the associated difference equations and follow this behavior. Since difference equations may be viewed as recurrence relations, their treatment seems much simpler that the one of differential equations ". [3].

We know that a physical setting is reduced to a set of measurement, for example, temperature, pressure, stock market prices, etc. In discrete systems, we give these measurements at a sequence of specific times. We would hope that given the measurements at time n that we have a rule to determine the measurements at time n+1. [10]. If y_n represents the measurements at time n, this rule may take the form

$$y(k+1) = f(y(k))$$

where $f(x)$ is a given function fixed for all time. The evolution of the system is then obtained by iterating the function

$$y_n = \underbrace{f(f(...f(x_0)...)}_{ntimes}$$

Example :

Suppose we have the function

$$f(x) = 2x(1-x)\ , \quad [0,1].$$

If we start with an initial state of $x_0 = 0.1$, it is easy to compute the subsequent states by means of the equation

$$x_{n+1} = f(x_n)$$

The results are shown in the table below .

Iteration of $f(x) = 2x(1-x)$, initial state of $x_0 = 0.1$

x_0	0.1
x_1	0.18
x_2	0.2952
x_3	0.41611392
x_4	0.4859262512
x_5	0.4996038592
x_6	0.4999996862
x_7	0.5000000000

We may easily guess the long term behavior of this system: the limit of x_n is 0.5 as n$\rightarrow \infty$. In fact, x=0.5 has a special property with respect to this dynamical system; it satisfies $f(0.5) = 0.5$,and thus qualifies as a fixed point of $f(x)$,or a point of equilibrium of the dynamical system. A little experimentation will show that any initial state in (0,1) eventually leads to x=0.5 in the limit. Thus, x=0.5 is an example of a stable fixed point . [10]

The following sections will give more details to the equilibrium points; the central in the study of the dynamics of any physical system.

This is the subject of study of stability theory; a topic of great importance to scientists and engineers.

1.3 Linear System

"A system is called a linear system if the system satisfied two conditions:

• System that receive a sum of two inputs will also produce output equal to the sum of the two inputs.
• System that receive a constant multiplication of input will also produce output equal to the constant multiplication of the input.

We can summarize the two conditions of the linear system into a linear combination. Thus, a linear system is a system that produces output equal to the linear combination of the input ". [3].
The following are examples for linear systems

$$y(k+1) = 5y(k) + 7$$

$$y(k) = 3y(k-1) - 2u(k)$$

A system that is not linear is called a non-linear system. The output is not a linear combination of the input. The following are the example of non-linear system

$$y(k+1) = 2(y(k))^2 + 5$$

$$y(k) = 5y(k-1) - (u(k))^2$$

$$y(k) = \frac{4y(k-1)}{u(k)}$$

the next section will introduce to difference equations to have better understanding of the examples above .

1.4 Difference Equations

Dynamical system come with many different names. Our particular interesting dynamical system is for the system whose state depends on the input history. In discrete time system, we call such system difference equation (equivalent to differential equation in continuous time).

Difference equation is an equation involving differences. We can see difference equation from at least three points of views: as sequence of number, discrete dynamical system and iterated function. It is the same thing but we look at different angle. [3,14].

1. Difference equation is a sequence of numbers that generated recursively using a rule to relate each number in the sequence to previous numbers in the sequence.

Example:

· Sequence $\{1, 1, 2, 3, 5, 8, 13, 21, ...\}$ is called Fibonacci sequence, generated with rule

$$y(k+2) = y(k+1) + y(k), k = 0, 1, 2, 3, ...y(0) = y(1) = 1$$

2. Difference equation is an iterated map.

if we see the sequence as an iterated function:

$$y_0, f(y_0), f(f(y_0)), ...$$

Then $f(y_0)$ is the first iterate of y_0 under f. Notation $f^k(y_0)$ is the k-th iterate of under. For example,

$$f^3(y_0) = f(f(f(y_0)))$$

The set of all iterates of y_0 is called the orbit of f.

Example:

Iterated function

$$y(k+1) = f(y(k)) = (y(k))^2$$

for $y_0 = 1$ will produce orbit$\{1, 1, 1, 1, ...\}$. If $y_0 = 2$, the iterated function generate$\{2, 4, 16, 256, ...\}$. When $= 0.5$, the iterated function yield sequence of$\{0.5, 0.25, 0.0625, ...\}$. We see that knowing the rule only is not enough to know the behavior of the sequence. Initial value is also very important. The orbit of y_0= 1 is constant for function

$$y(k+1) = f(y(k)) = (y(k))^2$$

while for y_0= 2, produces unbounded orbit and the orbit is attracted to zero for y_0= 0.5. The figure below show the orbit of $y_0 = 0.5$.

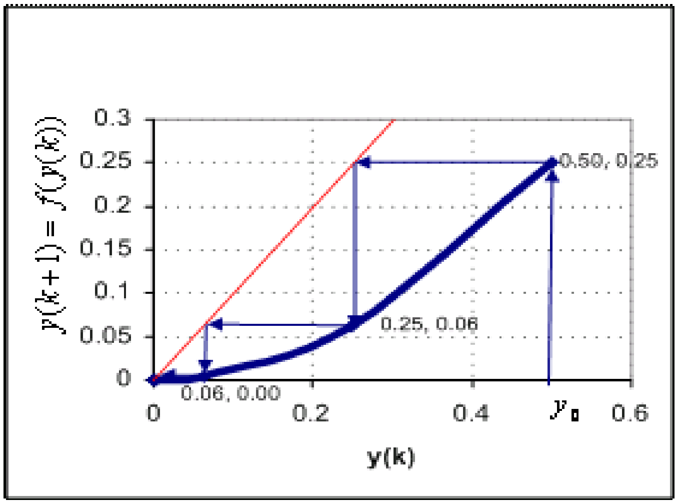

Figure 1: $f(y(k)) = (y(k))^2$

Knowing the initial value and the rule, we can generate the whole sequence recursively. The value of k is an integer (k=...,-2,-1,0,1,2,...) and the rule to generate the sequence is called the difference equation or the dynamical system or iterated function see figure (1). We will discus the meaning of this kind of

figure later when we study the stability theory.

In other words, Difference equations describe the evolution of certain phenomena over the course of time. While a continuous model leading to differential equation is reasonable and attractive for many problems, there are many cases in which a discrete model may be more natural. For example the continuous model of model of compound interest is an approximation to the actual discrete process. Population growth of species whose generations do not overlap and propagate at regular intervals. Then the population y(n+1) at period (n+1) is a function of n and the previous generation y(n). In difference equations we concern about the solution and analytic behavior of the solution if it exist. [3, 9, 10, 14].

1.4.1 Identical Difference Equations

"Two difference equations or dynamcial system are said to be identical if the infinite set of algebraic equation that they represent are identical. They represent the same set of equations". [3].

Example:

$$y(k) - 3y(k-1) = 0 \text{ , for k=1,2,3,...}$$

is identical to

$$y(k-3) = 3y(k-4) \text{ , for k=4,5,6,...}$$

because both dynamical system generate the same sequence for the same initial value. If $y_0 = 1$ they will generate a sequence of $\{1, 3, 6, 18, 24, 54, ...\}$.

1.4.2 Order of Difference Equation

"The order of a dynamcial system or difference equation is the difference between the largest and the smallest arguments k appearing in it". [3].

Example:

$$y(k+1) = a(k)y(k) + b(k) \qquad \text{has order 1}$$

$$y(k+3) + a(k)y(k+1) = b(k)y(k-1) + c(k) \qquad \text{has order 4}$$

CHAPTER 2
Solution to Difference Equations

Solution to Difference Equations

A solution of a difference equation is an expression (or formula) that makes the difference equation true for all values of the integer variable k. The nature of a difference equation allows the solution to be calculated recursively. It is easier to see the solution of the difference equation through algebraic equation. [3].

The following sections will give the details to finding the solution if it's possible and showing the behavior of it .

2.1 Solution to Linear First Order Difference Equations

We have difference equation

$$y(k+1) = ay(k) + b$$

with initial value $y(0) = y_0$.

Then we can calculated the solution recursively :

set

$$y(0) = y_0 \quad \textit{initial value}$$

$$k = 0 : y(1) = ay(0) + b \ = ay_0 + b$$

$$k = 1 : y(2) = ay(1) + b = a(ay_0 + b) + b = a^2y_0 + (a+1)b$$

$$k = 2 : y(3) = ay(2) + b = a(a^2y_0 + (a+1)b) + b = a^3y_0 + (a^2 + a + 1)b$$

$$k = 3 : y(4) = ay(3) + b = a(a^3y_0 + (a^2 + a + 1)b) + b = a^4y_0 + (a^3 + a^2 + a + 1)b$$

$$\vdots$$

$$k = n - 1 : y(n) = ay(n-1) + b == a^ny_0 + (a^{n-1} + ... + a + 1)b$$

However, the series

$$\sum_{i=0}^{n-1} a^i = 1 + a + a^2 + a^3 + ... + a^{n-1}$$

has a closed-form of

$$\sum_{i=0}^{n-1} a^i = \left\{ \begin{array}{cc} n & \text{if } a = 1 \\ \frac{1-a^n}{1-a} & \text{if } a \neq 1 \end{array} \right\}$$

Thus the solution of the difference equation

$$y(k+1) = ay(k) + b$$

with initial value y_0 is

$$y(n) = \left\{ \begin{array}{cc} y_0 + nb & \text{if } a = 1 \\ a^n y_0 + \frac{1-a^n}{1-a} & \text{if a} \neq 1 \end{array} \right\} \qquad (2.1)$$

Example :

find a solution for the equation

$$y(n+1) = 2y(n) + 3^n, \qquad y(1) = 0.5$$

Solution :

from equation(2.1), we have

$$\begin{aligned} y(n) &= \left(\frac{1}{2}\right) 2^{n-1} + \sum_{k=1}^{n-1} 2^{n-k-1} 3^k \\ &= 2^{n-2} + 2^{n-1} \sum_{k=1}^{n-1} \left(\frac{3}{2}\right)^k \\ &= 2^{n-2} + 2^{n-1} \frac{3}{2} \left(\frac{\left(\frac{3}{2}\right)^{n-1} - 1}{\frac{3}{2} - 1} \right) \\ &= 3^n - 2^{n-3}. \end{aligned}$$

2.2 Solution to Linear Difference Equations of Higher Order

The normal form of a kth-order nonhomogeneous linear difference equation given by:

$$y(n+k)+p_1(n)y(n+k-1)+p_2(n)y(n+k-2)+...+p_k(n)y(n) = g(n), \quad (2.2)$$

where p$_i(n)$ and $g(n)$ are real valued functions defined for $n \geq n_0$ and p$_k(n) \neq 0$. If $g(n)$ is identically zero then equation(2.2) is said to be a homogeneous equation. letting n=0 in p$_i(n)$, we obtain

$$y(n+k)+p_1y(n+k-1)+p_2y(n+k-2)+...+p_ky(n) = 0, \quad (2.3)$$

in this section we will give all possible solutions of equation(2.3) , the solutions of equation(2.2) was investigated in [introduction to difference equations, by Saber Elaydi.] in [12].

2.2.1 Solution of a kth-Order Homogeneous Linear Difference Equations with Constant Coefficients

Consider the kth-order homogeneous linear difference equation :

$$y(n+k)+p_1y(n+k-1)+p_2y(n+k-2)+...+p_ky(n) = 0, \quad (2.3)$$

where the $p_{i's}$ are constant and $p_k \neq 0$. Define λ to be the characteristic root of equation(2.3) then λ^n is a solution of equation(2.3) substituting this value into equation(2.3), we obtain

$$\lambda^k+p_1\lambda^{k-1}+...+p_k = 0, \quad (2.4)$$

equation(2.4) is called the characteristic equation of equation(2.3).

The general solution of equation(2.3) has different cases depending on $\lambda' s$.

case (1) : Distinct roots

Suppose that the characteristic roots $\lambda_1, \lambda_2, ..., \lambda_k$ are distinct, i.e.

$$|\lambda_1| \neq |\lambda_2| \neq ... \neq |\lambda_k| .$$

Assume

$$|\lambda_1| > |\lambda_2| > ... > |\lambda_k| , \quad \lambda_i real, 1 \leq i \leq k.$$

So the general solution of equation (2.3) is given by :

$$y(n) = c_1\lambda_1^n + c_2\lambda_2^n + ... + c_k\lambda_k^n.$$

case (2) : Repeated roots
Suppose that the characteristic roots $\lambda_1, \lambda_2, ..., \lambda_k$ are equal, i.e.

$$\lambda_1 = \lambda_2 = ... = \lambda_k, \quad \lambda_i real, 1 \leq i \leq k.$$

So the general solution of equation (2.3) is given by :

$$y(n) = c_1\lambda_1^n + c_2 n\lambda_1^n + ... + c_k n\lambda_1^n.$$

Example :
Find the solution of the following difference equation

$$y(n+2) + 6y(n+1) + 9y(n) = 0$$

Solution :
The characteristic equation of the above difference equation is given by

$$\begin{aligned} \lambda^2 + 6\lambda + 9 &= 0 \\ (\lambda+3)^2 &= 0 \\ \lambda_1 &= \lambda_2 = -3. \end{aligned}$$

So the general solution is

$$y(n) = c_1(-3)^n + c_2 n(-3)^n.$$

case (3) : The absolute value of the roots are equal
Suppose that the absolute value of the characteristic roots are equal, i.e.

$$|\lambda_1| = |\lambda_2| = ... = |\lambda_k|.$$

In this case there are two subcases :

subcase(3.1) :
The characteristic roots $\lambda_1, \lambda_2, ..., \lambda_k$ are equal, i.e.

$$\lambda_1 = \lambda_2 = ... = \lambda_k, \quad \lambda_i real, 1 \leq i \leq k.$$

This subcase is similar to case (2) .

subcase(3.2) :
If
$$\lambda_1 = \lambda_2 = ... = \lambda_m, \lambda_{m+1} = \lambda_{m+2} = ... = \lambda_k = -\lambda_1.$$
Then the general solution of equation (2.3) is given by :
$$y(n) = (c_1 + c_2 n + ... + c_m n^{m-1})\lambda_1^n + (c_{m+1} + c_{m+2} n^{m+1} + ... + c_k n^{k-1})(-1)^n \lambda_1^n.$$

Example :
Find the solution of the following difference equation
$$y(n+2) - 16y(n) = 0.$$

Solution :
The characteristic equation of the above difference equation is given by
$$\begin{aligned} \lambda^2 - 16 &= 0 \\ \lambda^2 &= 16 \\ \lambda &= \pm 4. \end{aligned}$$

so the general solution is
$$\begin{aligned} y(n) &= c_1(4)^n + c_2(-4)^n \\ &= c_1(4)^n + c_2(-1)^n(4)^n. \end{aligned}$$

case (4) : Complex Roots
Suppose that $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, and that $\lambda_3, \lambda_4, ..., \lambda_k$ are all real and distinct such that
$$|\lambda_3| > |\lambda_4| > ... > |\lambda_k|.$$
Where $\lambda_1 = \alpha + i\beta = re^{i\theta} = r(\cos\theta + i\sin\theta), \lambda_2 = \alpha - i\beta = re^{-i\theta} = r(\cos\theta - i\sin\theta)$.
Then the general solution of equation (2.3) is given by :
$$\begin{aligned} y(n) &= c_1 r^n e^{in\theta} + c_2 r^n e^{-in\theta} + c_3\lambda_3^n + ... + c_k\lambda_k^n. \\ &= c_1 r^n(\cos n\theta + i\sin n\theta) + c_2 r^n(\cos n\theta - i\sin n\theta) + c_3\lambda_3^n + ... + c_k\lambda_k^n. \\ &= (c_1 + c_2)r^n \cos n\theta + (c_1 - c_2)r^n\ i\sin n\theta + c_3\lambda_3^n + ... + c_k\lambda_k^n. \\ &= r^n[(c_1 + c_2)\cos n\theta + (c_1 - c_2)\ i\sin n\theta] + c_3\lambda_3^n + ... + c_k\lambda_k^n. \\ &= r^n[a_1\cos n\theta + a_2\sin n\theta] + c_3\lambda_3^n + ... + c_k\lambda_k^n. \end{aligned}$$

where $a_1 = c_1 + c_2$ and $a_2 = (c_1 - c_2)i$.
Let
$$\cos w = \frac{a_1}{\sqrt{a_1^2 + a_2^2}}, \sin w = \frac{a_2}{\sqrt{a_1^2 + a_2^2}}, w = \tan^{-1}(\frac{a_2}{a_1}).$$

Then the solution becomes

$$\begin{aligned} y(n) &= r^n \sqrt{a_1^2 + a_2^2}[\cos w \cos n\theta + \sin w \sin n\theta] + c_3\lambda_3^n + ... + c_k\lambda_k^n. \\ &= r^n \sqrt{a_1^2 + a_2^2} \cos(n\theta - w) \\ &= A\ r^n \cos(n\theta - w). \end{aligned}$$

Where $A = \sqrt{a_1^2 + a_2^2}$, $r = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1}(\frac{\beta}{\alpha})$.

2.3 Solution to Nonlinear Difference Equations

In general, most nonlinear Difference Equations cannot be solved explicitly. However, a few types of nonlinear equations can be solved, usually by transforming them into linear equations . [12].

Example :
Find the solution of the following difference equation

$$y^2(n+1) - 3y(n+1)y(n) + 2y^2(n) = 0.$$

Solution :
By dividing over $y^2(n)$, the above equation becomes

$$\left(\frac{y(n+1)}{y(n)}\right)^2 - 3\left(\frac{y(n+1)}{y(n)}\right) + 2.$$

Letting

$$z(n) = \frac{y(n+1)}{y(n)}$$

reduces the above equation to

$$z^2(n) - 3z(n) + 2 = 0$$

we can factor this down to

$$(z(n) - 2)\,(z(n) - 1) = 0$$

thus , either $z(n) = 2$ or $z(n) = 1$.
this leads to $y(n+1) = 2y(n)$ or $y(n+1) = y(n)$.

CHAPTER 3
Behavior of Solutions for Difference Equations

Behavior of Solutions for Difference Equations

In this chapter we will give the limiting behavior of solutions for difference equations. To simplify our exposition we restrict our discussion to the first and second order difference equations.

3.1 Behavior of Solutions for First Order Linear Discrete Dynamical System with Constant Coefficient

The general form of first order linear difference equation is given by

$$y(k+1) = f(y(k)) \tag{3.1}$$

To deep understanding the behavior of solutions for first order linear discrete dynamical system with constant coefficient, we need to study the subject of Stability Theory; the subject which give the features of a discrete system. The notion of equilibrium points is central in the study of dynamics.

3.1.1 Equilibrium of Difference Equation

Definition 3.1 *[8]. Equilibrium of Difference Equation.*
A point $\bar{x}$ is called an equilibrium point of equation(3.1) if

$$\bar{y} = f(\bar{y}).$$

That is,

$$y_n = \bar{y}, \quad \text{for } n \geq -1,$$

is a solution of equation(3.1) , or equivalently , $\bar{y}$ is a fixed point of f .

3.1.1.1 Equilibrium to Linear First Order Difference Equations

We have difference equation

$$y(k+1) = ay(k) + b$$

with initial value $y_0 = 0$.

Then we can determine the Equilibrium as following:
Write the difference equation in the form

$$y(k+1) = f(y(k))$$

gives

$$f(k+1) = ay(k) + b.$$

Equate

$$f(y(k)) = \bar{y}$$

produces

$$\bar{y} = a\bar{y} + b$$

or

$$\bar{y} = \frac{b}{1-a} \quad , \quad a \neq 1$$

Example :
Determine the equilibrium point for

$$y(k+1) = 2y(k) - 5.$$

Solution :
Set

$$\begin{aligned} \bar{y} &= 2\bar{y} - 5 \\ \bar{y} &= 5 \end{aligned}$$

so the equilibrium point is 5.

We can be determined graphically the equilibrium value of a difference equation, if it is exist, by plotting the value of $y(k)$ as horizontal axis and $y(k+1)$ as the vertical axis. The point of intersection of the graph of the difference equation with the line $y(k+1) = y(k)$ is the equilibrium values.

Start from initial value y_0, we take vertical line to the graph, and then take horizontal line to the line $y(k+1) = y(k)$. From here we again take vertical line to the graph. Repeating this task will eventually lead us to the equilibrium point if the point exists. [3].

Example: the iterated function $f(y(k)) = (y(k))^2$ for $y_0 = 0.5$ lead to zero equilibrium point. (See figure 1)

Suppose at some point the solution of a difference equation deviates form the equilibrium value. Will the solution return to the equilibrium value? This problem is called stability problem of the difference equation.

3.1.2 Stability of Difference Equations

Definition 3.2 *[8,15]. Let I be some interval of real numbers such that*

$$f : I \rightarrow I$$

Let $\bar{Y}$ be an equilibrium point of equation(3.1) .

(a) The equilibrium $\bar{Y}$ of equation(3.1) is called locally stable(or stable) if for every $\varepsilon > 0$,there exist $\delta > 0$ such that if $\left|y_0 - \bar{y}\right| < \delta$,then $\left|y_n - \bar{y}\right| < \varepsilon$, for all $n \geq 1$.

(b) The equilibrium $\bar{y}$ of equation(3.1) is called locally asymptotically stable(or asymptotically stable) if it is stable and if there exist $\gamma > 0$ such that if $\left|y_0 - \bar{y}\right| < \gamma$,then

$$\lim_{n\to\infty} y_n = \bar{y}.$$

(c) The equilibrium $\bar{y}$ of equation(3.1) is called a global attractor if for every $y_0 \in I$, we have

$$\lim_{n\to\infty} y_n=\bar{y}.$$

(d) The equilibrium $\bar{y}$ of equation(3.1) is called a globally asymptotically stable (or globally stable)if it is stable and is a global attractor.

(e) The equilibrium $\bar{y}$ of equation(3.1) is called unstable if it is not stable.

(f) The equilibrium $\bar{y}$ of equation(3.1) is called a repeller (or a source) if there exist $r >$ 0 such that $y_0 \in I$ and $\left|y_0 - \bar{y}\right| < r$,then there exists $N \geq 1$ such that

$$\left|y_N - \bar{y}\right| \geq r.$$

Clearly , a repeller is an unstable equilibrium point .

3.1.2.1 Stability to Linear First Order Difference Equations

There are ten possible types of solutions of

$$y(k+1) = ay(k) + b.$$

This difference equation is called affine dynamical system. It is one of the simplest forms of difference equation. It has characteristic of first order linear discrete dynamical system with constant coefficient . [3].

type 1:constant (see figure 2).
cases :
1. $a > 1$, $y_0 = \frac{b}{1-a}$.
2. $a = 1$, $b = 0$.
3. $0 < a < 1$, $y_0 = \frac{b}{1-a}$.
4. $-1 < a < 0$, $y_0 = \frac{b}{1-a}$.
5. $a = -1$, $y_0 = \frac{b}{2}$.
6. $a < -1$, $y_0 = \frac{b}{1-a}$.
type 2 : Linearly increasing without bound (see figure 3).
cases :
$a = 1$, $b > 0$.
type 3 : Linearly decreasing without bound (see figure 4).
cases :
$a = 1$, $b < 0$.
type 4 : Exponentially increasing without bound (see figure 5).
cases :
$a > 1$, $y_0 > \frac{b}{1-a}$.
type 5 : Exponentially decreasing without bound (see figure 6).
cases :
$a > 1$, $y_0 < \frac{b}{1-a}$.
type 6 : Exponentially increasing to a bound (see figure 7).
cases :
$0 < a < 1$, $y_0 < \frac{b}{1-a}$.
type7 : Exponentially decreasing to a bound (see figure 8).
cases :
$0 < a < 1$, $y_0 > \frac{b}{1-a}$.
type 8 : Oscillating with constant amplitude (see figure 9).
cases :
1. $a = -1$, $y_0 < \frac{b}{2}$.
2. $a = -1$, $y_0 > \frac{b}{2}$.
type 9 : Oscillating with increasing amplitude (see figure 10).
cases :
1. $a < -1$, $y_0 < \frac{b}{1-a}$.
2. $a < -1$, $y_0 > \frac{b}{1-a}$.
type 10 : Oscillating with decreasing amplitude (see figure 11).
cases :
1. $-1 < a < 0$, $y_0 < \frac{b}{1-a}$.
2. $-1 < a < 0$, $y_0 > \frac{b}{1-a}$.

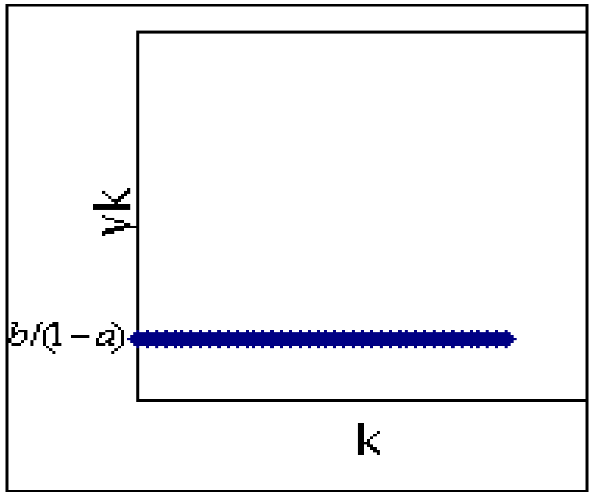

Figure 2: constant type

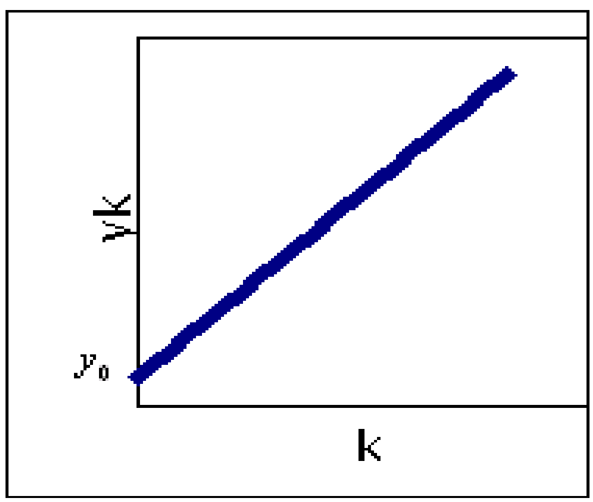

Figure 3: **Linearly increasing without bound** (unstable system)

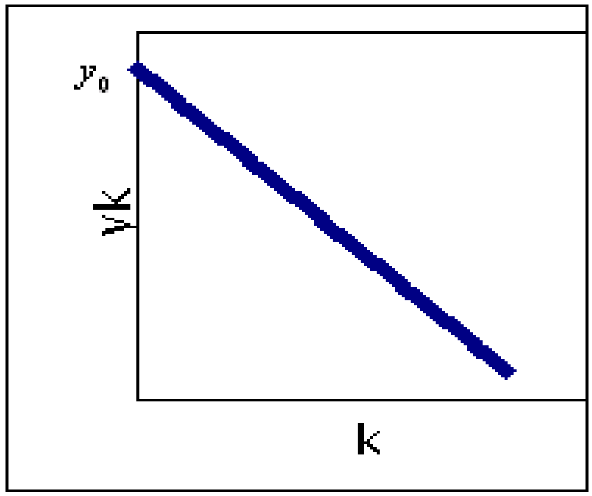

Figure 4: **Linearly decreasing without bound** (unstable system)

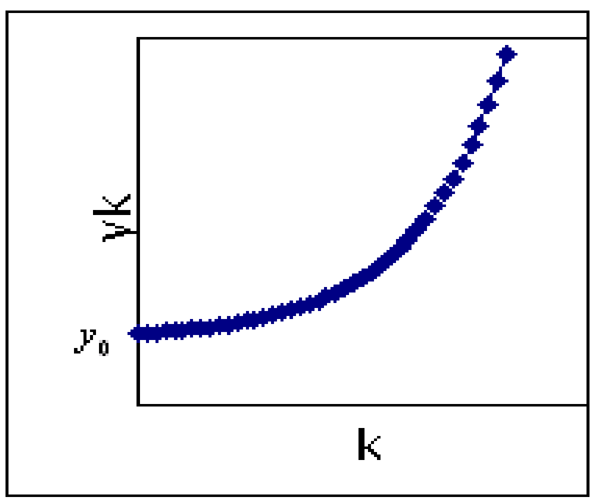

Figure 5: **Exponentially increasing without bound(**unstable system)

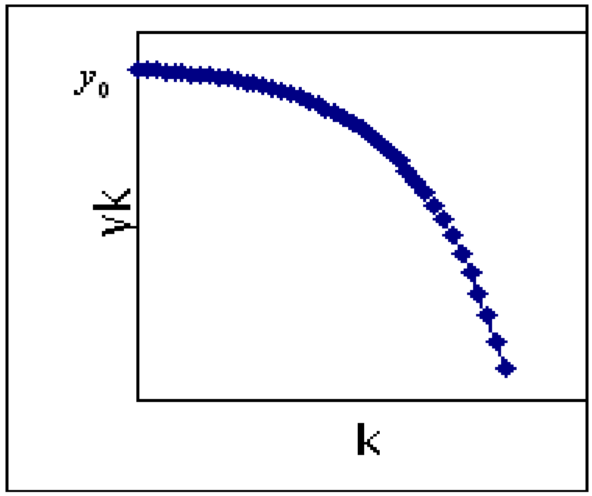

Figure 6: **Exponentially decreasing without bound (**unstable system)

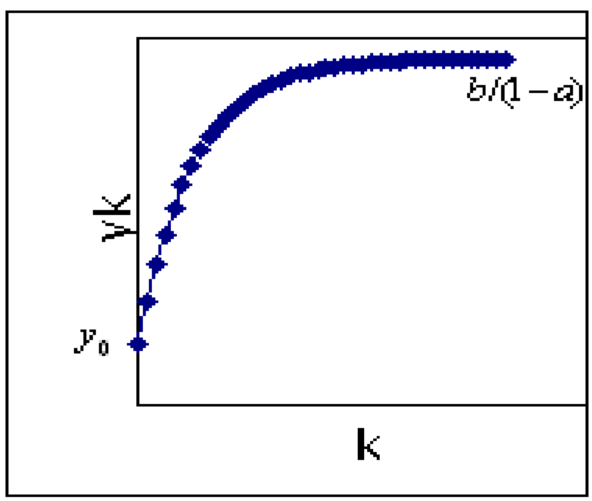

Figure 7: **Exponentially increasing to a bound** (stable system)

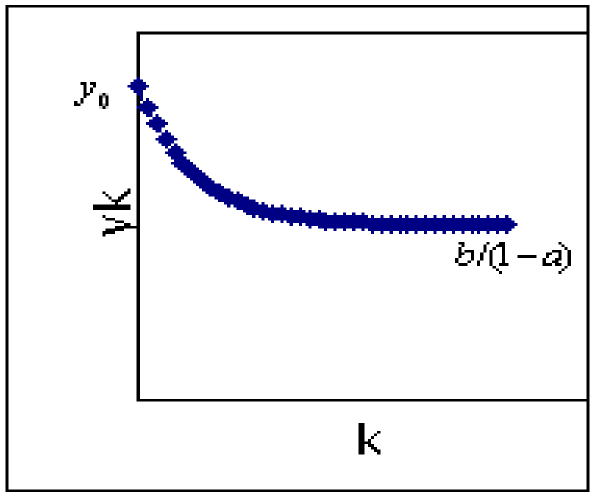

Figure 8: **Exponentially decreasing to a bound** (stable system)

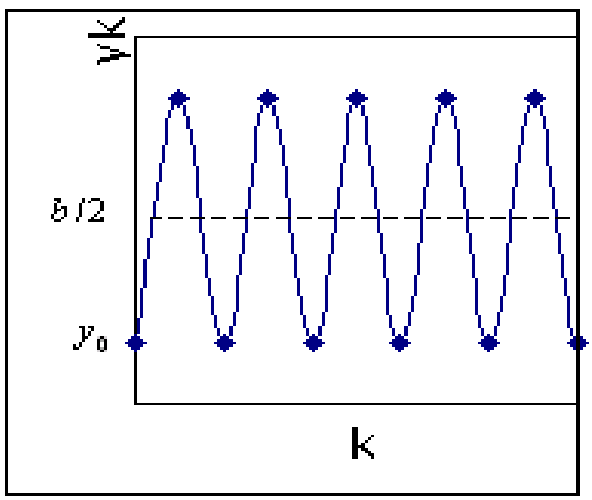

Figure 9: **Oscillating with constant amplitude**

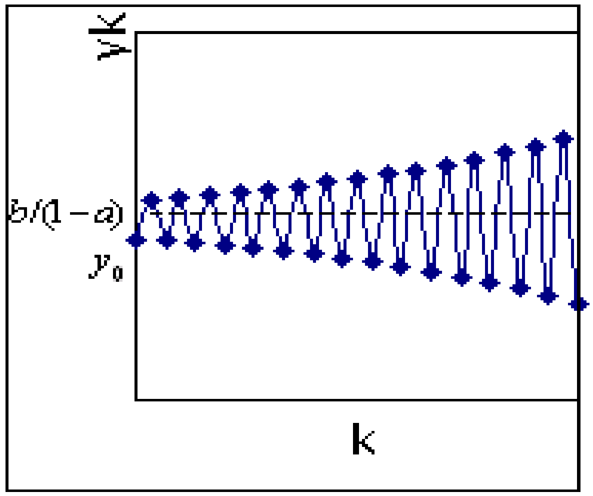

Figure 10: **Oscillating with increasing amplitude (**unstable system)

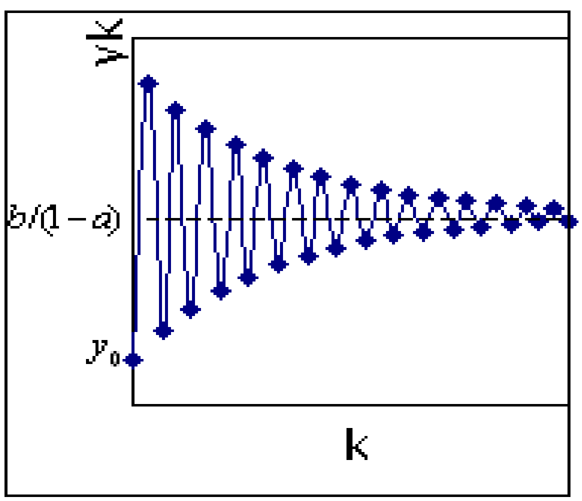

Figure 11: **Oscillating with decreasing amplitude (**stable system)

3.2 behavior of solutions for second order linear discrete dynamical system with constant coefficient

The general form for second order homogenous linear discrete dynamical system with constant coefficient is given by :

$$y(n+2) + py(n+1) + qy(n) = 0, \quad (3.2)$$

where the p, q are constant. define λ to be the characteristic root of equation(3.2) then the characteristic equation to equation(3.2) is given by

$$\lambda^2 + p\lambda + q = 0, \qquad (3.3)$$

.

The general solution of equation(3.2) has different cases depending on $\lambda's$.

case (1) : Distinct roots
Suppose that the characteristic roots λ_1, λ_2 are distinct, i.e. $|\lambda_1| \neq |\lambda_2|$
Assume $|\lambda_1| > |\lambda_2|, \quad \lambda_i$ real,
so the general solution of equation (3.2) is given by :

$$\begin{aligned} y(n) &= c_1\lambda_1^n + c_2\lambda_2^n \\ &= \lambda_1^n \left[c_1 + c_2 \left(\frac{\lambda_2}{\lambda_1} \right)^n \right], \end{aligned}$$

since

$$\left| \frac{\lambda_2}{\lambda_1} \right| < 1, \left(\frac{\lambda_2}{\lambda_1} \right) \to 0 \quad \text{as} \quad n \to \infty$$

consequently,

$$\lim_{n\to\infty} y(n) = \lim_{n\to\infty} c_1\lambda_1^n.$$

There are six different cases here depending on the value of λ_1.[11].

1. $\lambda_1 > 1$: the sequence$\{c_1\lambda_1^n\}$ diverges to ∞ (unstable system).

2. $\lambda_1 = 1$: the sequence$\{c_1\lambda_1^n\}$ is a constant sequence .

3. $0 \leq \lambda_1 < 1$: the sequence$\{c_1\lambda_1^n\}$ is monotonically decreasing (stable system) .

4. $-1 < \lambda_1 < 0$: the sequence$\{c_1\lambda_1^n\}$ is oscillating around zero (alternating in sign) and converging to zero (stable system).

5. $\lambda_1 = -1$: the sequence$\{c_1\lambda_1^n\}$ is oscillating between two values $c_1, -c_1$.

6. $\lambda_1 < -1$: the sequence$\{c_1\lambda_1^n\}$ is oscillating but increasing in amplitude (unstable system)

case (2) : Repeated roots

Suppose that the characteristic roots λ_1, λ_2 are equal, i.e. $\lambda_1 = \lambda_2, \quad \lambda_i$ real so the general solution of equation (3.2) is given by :

$$\begin{aligned} y(n) &= c_1\lambda_1^n + c_2 n\lambda_1^n \\ &= (c_1 + c_2 n)\,\lambda_1^n \end{aligned}$$

if $|\lambda_1| \geq 1$, the solution y(n) diverges either monotonically if $\lambda \geq 1$, or oscillatory if $\lambda \leq -1$.However if $|\lambda| < 1$, then the solution converges to zero .

case (3) : Complex Roots

Suppose that $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, and that $\lambda_3, \lambda_4, ..., \lambda_k$ are all real and distinct such that

$$|\lambda_3| > |\lambda_4| > ... > |\lambda_k|\,.$$

Where $\lambda_1 = \alpha + i\beta = re^{i\theta} = r(\cos\theta + i\sin\theta), \lambda_2 = \alpha - i\beta = re^{-i\theta} = r(\cos\theta - i\sin\theta)$.

As we see in chapter 2 the general solution of equation (2.3) is given by :

$$y(n) = A\ r^n \cos(n\theta - w). \qquad (3.4)$$

Where $r = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1}(\frac{\beta}{\alpha})$.

The solution y(n) oscillates since the cosine function oscillates. However, y(n) oscillates in three different wayes depending on the location of the conjugate characteristic roots:

• $r > 1$: here λ_1 *and* $\lambda_2 = \bar{\lambda_1}$ are outside the unit circle. Hence y(n) oscillating but increasing in magnitude (unstable system).

• $r = 1$: here λ_1 *and* $\lambda_2 = \bar{\lambda_1}$ lie on the unit circle. Hence y(n) oscillating but constant in magnitude.

• $r < 1$: here λ_1 *and* $\lambda_2 = \bar{\lambda_1}$ lie inside the unit circle. Hence y(n) oscillating but converges to zero as $n \to \infty$ (stable system).

We can summarize the above discussion by the following two theorems.

Theorem *[12]. The following statements hold.*

(i) All solutions of equation (3.4) oscillate (about zero) if and only if the equation has no positive real characteristic roots.

(ii) All solutions of equation (3.4) converges to zero (i.e. the zero solution is asymptotically stable) if and only if max $\{|\lambda_1|, |\lambda_2|\} < 1$.

Theorem *[7]. Linearized Stability.*

Consider the difference equation

$$y_{n+1} = f(y_n, y_{n-1}), n = 0, 1, ... \quad (3.5)$$

(a) If both roots of equation(3.3) have absolute values less than one, then the equilibrium $\bar{y}$ of equation(3.5) is locally asymptotically stable.

(b)If at least one of the roots of equation(3.3) has an absolute value greater than one, then $\bar{y}$ is unstable .

(c)Booth roots of equation(3.3) have absolute values less than one if and only if

$$|p| < 1 - q < 2$$

in this case, $\bar{y}$ is a locally asymptotically stable.

(d)Booth roots of equation (3.3) have absolute values greater than one if and only if

$$|q| > 1 \qquad \text{and} \qquad |p| > |1 - q|$$

in this case, $\bar{y}$ is a repeller.

(e)One root of equation (3.3) has an absolute value greater than one while the other root has an absolute value less than one if and only if

$$p^2 + 4p > 0 \qquad \text{and} \qquad |p| > |1 - q| .$$

in this case, $\bar{x}$ is unstable and is called a saddle point .

CHAPTER 4

Dynamics of a k^{th} order Rational Difference Equation Using Theoretical and Computational Approaches

Dynamics of a k$^{\text{th}}$ order Rational Difference Equation Using Theoretical and Computational Approaches

4.1 Introduction and Preliminaries

The dynamical characteristics and the qualitative behavior of positive solutions of some higher order nonlinear differences equations are ivestigated.

Dehghan et al. [6] investigated the global stability, invariant intervals, the character of semicycles, and the boundedness of the equation

$$x_{n+1} = \frac{x_n + p}{x_n + qx_{n-k}}, \quad n = 0, 1, 2, ... \quad (4.1)$$

where the parameters p and q and the initial conditions $x_{-k}, ..., x_{-1}$and x_0 are nonnegative real numbers, $k = \{1, 2, 3, ...\}$.

Li and Sun [15] investigated the periodic character, invariant intervals, oscillation and global stability of all positive solutions of the equation

$$x_{n+1} = \frac{px_n + x_{n-k}}{q + x_{n-k}}, \quad n = 0, 1, 2, ... \quad (4.2)$$

where the parameters p and q and the initial conditions $x_{-k}, ..., x_{-1}$and x_0 are nonnegative real numbers, $k = \{1, 2, 3, ...\}$.

DeVault et al. [11] investigated the global stability and the periodic character of solutions of the equation

$$x_{n+1} = \frac{p + x_{n-k}}{qx_n + x_{n-k}}, \quad n = 0, 1, 2, ... \quad (4.3)$$

where the parameters p and q and the initial conditions $x_{-k}, ..., x_{-1}$and x_0 are nonnegative real numbers, $k = \{1, 2, 3, ...\}$.

All of them showed that the two cases $p < q$ and $p > q$ give rise to different dynamic behaviors.

4.1.1 Statement of the Problem

In this thesis, we will study the nonlinear rational difference equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \quad n = 0, 1, ... \quad (4.4)$$

where the parameters β, γ and B, C and the initial conditions $x_{-k}, ..., x_{-1}$ and x_0 are nonnegative real numbers, $k = \{1, 2, 3, ...\}$.

Our concentration is on invariant intervals, periodic character, the character of semicycles and global asymptotic stability of all positive solutions of equation(4.4).

To analyze equation(4.4) more theoretically, it is a good idea to study the difference equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{Bx_n + Cx_{n-1}}, \quad n = 0, 1, ... \qquad (4.5)$$

where the parameters β, γ and B, C and the initial conditions $x_{-1}$and x_0 are nonnegative real numbers.

The global stability of equation (4.5), the special case of equation (4.4) for $k = 1$, has been investigated. They showed, in respect to variation of the parameters, the positive equilibrium point is globally asymptotically stable or every solution of equation(4.5) lies eventually in an invariant interval. Kulenovic and Ladas, in addition, considered equation(4.5) in their monograph [7]. We interested now to study and solve equation(9.4).

Here, we will list some definitions which will be useful in our investigation.

4.1.2 Equilibrium of kth order Rational Difference Equation

Definition 4.1 [15]. *A point $\bar{x}$ is called an equilibrium point of equation(4.6) if*

$$\bar{x} = f(\bar{x}, \bar{x}).$$

That is,

$$x_n = \bar{x}, \quad for\ n \geq -1$$

is a solution of equation(4.6), or equivalently , $\bar{x}$ is a fixed point of f .

4.1.3 Stability of kth order Rational Difference Equation

Definition 4.2 [6]. *Let $\bar{x}$ be an equilibrium point of equation(4.6) and assume that I is some interval of real numbers .*

(a) The equilibrium $\bar{x}$ of equation(4.6) is called locally stable(or stable) if for every $\varepsilon > 0$, there exist $\delta > 0$ such that if

$$x_{-k}, ...x_{-1}, x_0 \in I$$

and

$$\left|x_{-k} - \bar{x}\right| + ... + \left|x_{-1} - \bar{x}\right| + \left|x_0 - \bar{x}\right| < \delta,$$

$$\left|x_n - \bar{x}\right| < \varepsilon, \qquad \text{for all } n \geq -k.$$

(b) The equilibrium $\bar{x}$ of equation(4.6) is called locally asymptotically stable (or asymptotically stable) if it is stable and if there exist $\gamma > 0$ such that if

$$x_{-k}, ...x_{-1}, x_0 \in I$$

and

$$\left|x_{-k} - \bar{x}\right| + ... + \left|x_{-1} - \bar{x}\right| + \left|x_0 - \bar{x}\right| < \gamma,$$

then

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(c) The equilibrium $\bar{x}$ of equation(4.6) is called a global attractor if for every $x_{-k}, ...\ x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(d) The equilibrium $\bar{x}$ of equation(4.6) is called a globally asymptotically stable (or globally stable) if it is stable and is a global attractor.

(e) The equilibrium $\bar{x}$ of equation(4.6) is called unstable if it is not stable.

(f) The equilibrium $\bar{x}$ of equation(4.6) is called a repeller (or a source) if there exist $r >$ 0 such that if $x_{-k}, ...x_{-1}, x_0 \in I$ and

$$\left|x_{-k} - \bar{x}\right| + ... + \left|x_{-1} - \bar{x}\right| + \left|x_0 - \bar{x}\right| < r,$$

then there exists $N \geq 1$ such that

$$\left|x_N - \bar{x}\right| \geq r.$$

Clearly, a repeller is an unstable equilibrium point .

4.1.4 semicycle of a solution of kth order Rational Difference Equation

Definition 4.3 [5]. *Let* $\bar{x}$ *be a positive equilibrium of equation(4.6). A positive semicycle of a solution* $\{x_n\}_{n=-k}^{\infty}$ *of equation(4.6) consist of a "string" of terms* $\{x_l, x_{l+1}, ..., x_m\}$*,all greater than or equal to the equilibrium* $\bar{x}$ *with*

$$l \geq -1 \quad and \quad m \leq \infty$$

such that

$$either \quad l = -1 \quad \text{or} \quad l > -1 \quad and \quad x_{l-1} < \bar{x}$$

and

$$either \quad m = \infty \quad \text{or} \quad m < \infty \quad and \quad x_{m+1} < \bar{x}.$$

A negative semicycle of a solution $\{x_n\}_{n=-1}^{\infty}$ *of equation(4.6) consist of a "string" of terms* $\{x_l, x_{l+1}, ..., x_m\}$*,all less than or equal to the equilibrium* $\bar{x}$ *with*

$$l \geq -1 \quad and \quad m \leq \infty$$

such that

$$either \quad l = -1 \quad \text{or} \quad l > -1 \quad and \quad x_{l-1} \geq \bar{x}$$

and

$$either \quad m = \infty \quad \text{or} \quad m < \infty \quad and \quad x_{m+1} \geq \bar{x}.$$

4.1.5 Linearization of kth order Rational Difference Equation

Definition 4.4 [15]. *Let*

$$a = \frac{\partial f}{\partial u}\left(\bar{x}, \bar{x}\right) \quad and \quad b = \frac{\partial f}{\partial v}(\bar{x}, \bar{x})$$

where $f(u, v)$ *is the function in equation(4.6) and* $\bar{x}$ *is an equilibrium of the equation . Then the equation*

$$y_{n+1} = ay_n + by_{n-k}, \qquad n = 0, 1, ... \quad (4.7)$$

is called the linearized equation associated with equation(4.6) about the equilibrium point $\bar{x}$*. Its characteristic equation is*

$$\lambda^{k+1} - a\lambda^k - b = 0 \qquad (4.8)$$

Now, we will list and prove some theorems which will be useful in our investigation.

Theorem 1 *Let I be some interval of real numbers and let*

$$f : I^{k+1} \to I$$

be a continuously differentiable function . Then for every set of initial conditions $x_{-k}, ..., x_{-1}, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-k}), \quad n = 0, 1, ... \quad (4.6)$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$. [6].

Theorem 2 *Linearized Stability. [14]*

(a) If all the roots of equation(4.8) lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\bar{x}$ of equation(4.6) is asymptotically stable.

(b) If at least one of the roots of equation(4.8) has an absolute value greater than one, then $\bar{x}$ is unstable .

Theorem 3 *Assume that a,b $\in R$ and $k \in \{1, 2, ...\}$ then*

$$|a| + |b| < 1 \quad (4.12)$$

is a sufficient condition for the a asymptotic stability of the difference equation(4.6). Suppose in addition that one of the following two cases holds.

(a) k odd and b>0.

(b) k even and ab>0.

Then (4.12) is also a necessary condition for the asymptotic stability of equation(4.6). [10].

Theorem 4 *Assume that $f \in [(0, \infty) \times (0, \infty), (0, \infty)]$ is such that:$f(x, y)$ is increasing in x for each fixed y, and $f(x, y)$ is decreasing in y for each fixed x. Let $\bar{x}$ be a positive equilibrium of equation(4.6). Then except possibly for the first semicycle, every oscillatory solution of equation(4.6) has semicycle of length at least k. Furthermore, if we assume that*

$$f(u, u) = \bar{x} \text{ for every } u \quad (4.13)$$

and

$$f(x,y) < x \text{ for every } \bar{x} < y < x \qquad (4.14)$$

then $\{x_n\}$ *oscillates about the equilibrium* $\bar{y}$ *with semicycles of length* $k+1$ *or* $k+2$*, except possibly for the first semicycle which may have length k. The extreme in each semicycle occurs in the first term if the semicycle has two terms and in the second term if the semicycle has three terms...and in the k+1 term if the semicycle has k+2 terms.*

Proof. *When k=1, the proof is presented as theorem 1.7.4 in[7]. We just give the proof of the theorem for k=2, the other cases for* $k \geq 3$ *are similar and we omitted them. Assume that* $\{x_n\}$ *is an oscillatory solution with three consecutive terms* x_{N-1}, x_N, x_{N+1} *such that*

$$x_{N-1} < \bar{x} \leq x_{N+1}.$$

then by using the increasing character of f *we obtain*

$$x_{N+2} = f(x_{N+1}, x_{N-1}) > f(\bar{x}, \bar{x}) = \bar{x}$$

which show that the next term x_{N+2} *also belongs to the positive semicycle. if* $x_N \geq \bar{x}$ *, then the result follows. otherwise if* $x_N < \bar{x}$*, hence*

$$x_{N+3} = f(x_{N+2}, x_N) > f(\bar{x}, \bar{x}) = \bar{x}.$$

which shows that it had at least three terms in the positive semicycle. The proof in the case $x_{N-1} \geq \bar{x} > x_{N+1}$*is similar and is omitted. Also, assume that* $\{x_n\}$ *is an oscillatory solution with three consecutive terms* x_{N-1}, x_N, x_{N+1} *such that*

$$x_{N-1} > \bar{x} \geq x_{N+1}, \quad \textit{and} \quad x_N < \bar{x}$$

then by using the increasing character of f *we obtain*

$$x_{N+2} = f(x_{N+1}, x_{N-1}) < f(x_{N+1}, x_{N+1}) = \bar{x}$$

which show that the positive semicycle has length three. If

$$x_{N+1} > x_{N-1} \geq \bar{x}$$

then by using the increasing character of f *and condition(4.13) we obtain*

$$x_{N+2} = f(x_{N+1}, x_{N-1}) > f(x_{N+1}, x_{N+1}) = \bar{x}$$

and by using condition(4.14) we find

$$x_{N+2} = f(x_{N+1}, x_{N-1}) < x_{N+1}.$$

The proof is complete. ■

Theorem 5 *Assume that* $f \in [(0,\infty)\times(0,\infty),(0,\infty)]$ *is such that:* $f(x,y)$ *is decreasing in* x *for each fixed* y*, and* $f(x,y)$ *is increasing in* y *for each fixed* x*. Let* $\bar{x}$ *be a positive equilibrium of equation(4.6). Then except possibly for the first semicycle, every oscillatory solution of equation(4.6) has semicycle of length k.*

Proof. *When k=1, the proof is presented as theorem 1.7.1 in[7]. We just give the proof of the theorem for k=2, the other cases for* $k \geq 3$ *are similar and we omitted them. let* $\{x_n\}$ *be a solution of equation(4.6) with at least three semicycles, then there exists* $N \geq 0$ *such that either*

$$x_{N-1} < \bar{x} \leq x_{N+1}.$$

or

$$x_{N-1} \geq \bar{x} > x_{N+1}.$$

we will assume that

$$x_{N-1} < \bar{x} \leq x_{N+1}.$$

the other case is similar and will be omitted,then by using the monotonic character of f(x,y) we have

$$x_{N+2} = f(x_{N+1}, x_{N-1}) < f(\bar{x},\bar{x}) = \bar{x}.$$

and

$$x_{N+3} = f(x_{N+2}, x_N) > f(\bar{x},\bar{x}) = \bar{x}.$$

Thus

$$x_{N+2} < \bar{x} < x_{N+3}.$$

The proof is complete. ■

Theorem 6 *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), n = 0, 1 \quad (4.6)$$

when $k \in \{1,2,...\}$*. let* $I = [a,b]$ *be some interval of real numbers and assume that*

$$f : [a,b] \times [a,b] \to [a,b]\,;$$

is continuous function satisfying the following properties :

(a) f(x,y) is non-increasing in $x \in [a,b]$ *for each* $y \in [a,b]$ *, and f(x,y) is non-decreasing in* $y \in [a,b]$ *for each* $x \in [a,b]$*;*

(b) the difference equation(4.6) has no solutions of prime period two in $[a,b]$*;*

then equation(4.6) has a unique equilibrium $\bar{y} \in [a,b]$ *and every solution of equation(4.6) converges to* $\bar{y}$*.*

Proof. *set* $m_0 = a$ *and* $M_0 = b$ *and for* $i = 1, 2, ...$ *set*

$$M_i = f(m_{i-1}, M_{i-1}) \text{ and } m_i = f(M_{i-1}, m_{i-1})$$

Now, observe that for each $i \geq 0$,

$$m_0 \leq m_1 \leq ... \leq m_i \leq ... \leq M_i \leq ... \leq M_1 \leq M_0$$

and

$$m_i \leq y_n \leq M_i \text{ for } n \geq (i-1)k + i$$

set

$$m = \lim_{i \to \infty} m_i \quad \text{and} \quad M = \lim_{i \to \infty} M_i$$

Then clearly

$$M \geq \lim_{i \to \infty} \sup x_i \quad \geq \lim_{i \to \infty} \inf x_i \geq m$$

and by continuity of f,

$$m = f(M, m) \quad \text{and} \quad M = f(m, M).$$

In view of (b)

$$m = M = \bar{y}$$

The proof is complete. ■

Theorem 7 *Let* $I = [a, b]$ *be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]\,;$$

is continuous function satisfying the following properties :

(a) f(x,y) is non-decreasing in $x \in [a, b]$ *for each* $y \in [a, b]$*, and f(x,y) is non-increasing in* $y \in [a, b]$ *for each* $x \in [a, b]\,$;

(b) if $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$f(m, M) = m \quad \textit{and} \quad f(M, m) = M$$

Then m=M.

Then equation(4.6) has a unique equilibrium $\bar{y} \in [a, b]$ *and every solution of equation(4.6) converges to* $\bar{y}$. *[6]*

4.2 Change of Variables

Return to our problem ,

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \qquad n = 0, 1, ... \qquad (4.4)$$

where the parameters β, γ and B, C and the initial conditions $x_{-k}, ..., x_{-1}$and x_0 are nonnegative real numbers, $k = \{1, 2, 3, ...\}$.

The change of variables

$$x_n = \frac{\gamma}{C} y_n.$$

reduces equation(4.4) to the difference equation

$$y_{n+1} = \frac{py_n + y_{n-k}}{qy_n + y_{n-k}}, \qquad n = 0, 1, ... \qquad (4.9)$$

Where $p = \frac{\beta}{\gamma}$ and $q = \frac{B}{C}$
with

$$p, q \in (0, \infty)$$

$$y_{-k}, ..., \ y_{-1}, y_0 \in (0, \infty).$$

Proof. since

$$\begin{aligned} x_n &= \frac{\gamma}{C} y_n. \\ x_{n+1} &= \frac{\gamma}{C} y_{n+1}. \\ x_{n-k} &= \frac{\gamma}{C} y_{n-k} \end{aligned}$$

substitute in equation(4.4)

$$\frac{\gamma}{C} y_{n+1} = \frac{\beta \frac{\gamma}{C} y_n + \gamma \frac{\gamma}{C} y_{n-k}}{B \frac{\gamma}{C} + C \frac{\gamma}{C} y_{n-k}}$$

so

$$\frac{\gamma}{C} y_{n+1} = \frac{\frac{\gamma}{C}(\beta y_n + \gamma y_{n-k})}{\frac{\gamma}{C}(By_n + Cy_{n-k})}$$

i.e.

$$\frac{\gamma}{C} y_{n+1} = \frac{\gamma(\frac{\beta}{\gamma} y_n + y_{n-k})}{C(\frac{B}{C} y_n + y_{n-k})}$$

hence

$$y_{n+1} = \frac{\frac{\beta}{\gamma} y_n + y_{n-k}}{\frac{B}{C} y_n + y_{n-k}}$$

Let

$$p = \frac{\beta}{\gamma}, q = \frac{B}{C}$$

reduces the above formula to

$$y_{n+1} = \frac{py_n + y_{n-k}}{qy_n + y_{n-k}}, \qquad n = 0, 1, ... \quad (4.9)$$

To avoid a degenerate situation we will also assume that

$$p \neq q.$$

The proof is complete. ■

4.3 Equilibrium Points

Here we investigate the equilibrium points of the nonlinear rational difference equation

$$y_{n+1} = \frac{py_n + y_{n-k}}{qy_n + y_{n-k}}, \qquad n = 0, 1, ... \quad (4.9)$$

where the parameters p, q and the initial conditions $y_{-k}, ...,\ y_{-1}, y_0$ are nonnegative real numbers, $k = \{1, 2, 3, ...\}$.

In view of definition (4.2) and the above restriction on the initial conditions of equation(4.9), the equilibrium points of equation(4.9) are the nonnegative solutions of the equation

$$\begin{aligned} \bar{y} &= \frac{p\bar{y} + \bar{y}}{q\bar{y} + \bar{y}} \\ &= \frac{\bar{y}(p+1)}{\bar{y}(q+1)} \end{aligned}$$

hence

$$\bar{y} = \frac{p+1}{q+1}$$

is the only equilibrium point.

4.4 Linearization

For our problem we have

$$f\,(u,v) = \frac{pu+v}{qu+v}$$

now,

$$\frac{\partial f}{\partial u} = \frac{(qu+v)(p)-(pu+v)(q)}{(qu+v)^2}$$

so

$$\frac{\partial f}{\partial u} = \frac{v(p-q)}{(qu+v)^2}$$

hence

$$\frac{\partial f}{\partial u}(\bar{y},\bar{y}) = \frac{\bar{y}(p-q)}{(q\bar{y}+\bar{y})^2} = \frac{\bar{y}(p-q)}{(\bar{y}(q+1))^2}$$

but

$$\bar{y} = \frac{p+1}{q+1}$$

so

$$\frac{\partial f}{\partial u}(\bar{y},\bar{y}) = \frac{\frac{p+1}{q+1}(p-q)}{(\frac{p+1}{q+1}(q+1))^2} = \frac{(p+1)(p-q)}{(q+1)(p+1)^2}.$$

i.e.

$$\frac{\partial f}{\partial u}(\bar{y},\bar{y}) = \frac{(p-q)}{(q+1)(p+1)}$$

Also,

$$\frac{\partial f}{\partial v} = \frac{(qu+v)-(pu+v)}{(qu+v)^2}$$

so

$$\frac{\partial f}{\partial v} = \frac{u(q-p)}{(qu+v)^2}$$

hence

$$\frac{\partial f}{\partial v}(\bar{y},\bar{y}) = \frac{\bar{y}(q-p)}{(q\bar{y}+\bar{y})^2} = \frac{\bar{y}(q-p)}{(\bar{y}(q+1))^2}$$

but

$$\bar{y} = \frac{p+1}{q+1}$$

so

$$\frac{\partial f}{\partial v}(\bar{y},\bar{y}) = \frac{\frac{p+1}{q+1}(q-p)}{(\frac{p+1}{q+1}(q+1))^2} = \frac{(p+1)(q-p)}{(q+1)(p+1)^2}$$

i.e.
$$\frac{\partial f}{\partial u}(\bar{y}, \bar{y}) = \frac{(q-p)}{(q+1)(p+1)}$$
so, the linearized equation of equation(4.9) is
$$z_{n+1} = \frac{p-q}{(p+1)(q+1)} z_n + \frac{(q-p)}{(q+1)(p+1)} z_{n-k}.$$
Let
$$\frac{p-q}{(p+1)(q+1)} = a,$$
so
$$\frac{(q-p)}{(q+1)(p+1)} = -a.$$
Then
$$z_{n+1} = az_n - az_{n-k}$$
i.e., the linearized equation of equation(4.9) is given by
$$z_{n+1} - az_n + az_{n-k} = 0 \quad (4.10)$$
and its characteristic equation is
$$\lambda^{k+1} - a\lambda^k + a = 0 \qquad (4.11)$$

4.5 Local Stability

As we mentioned in sections (4.3) and (4.4), equation(4.9) has the only positive equilibrium point;

$$\bar{y} = \frac{p+1}{q+1}$$

and the linearized equation is given by

$$z_{n+1} = \frac{p-q}{(p+1)(q+1)} z_n + \frac{(q-p)}{(q+1)(p+1)} z_{n-k}$$

By theorems (2) and (3) we have the following result.

Theorem 8 *(a) Assume that $p > q$,then the positive equilibrium of equation(4.9) is locally asymptotically stable .*

(b) Assume that $p < q$,and k odd ,then the positive equilibrium of equation(4.9) is locally asymptotically stable when

$$q < pq + 3p + 1.$$

and unstable when

$$q > pq + 3p + 1.$$

(c) Assume that $p < q$,and k even, then the positive equilibrium of equation(4.9) is locally asymptotically stable.

Proof. Applying to theorem(3)

$$|a| + |a| < 1$$

so

$$2\,|a| < 1$$

i.e.

$$|a| < \frac{1}{2}$$

hence

$$\frac{-1}{2} < a < \frac{1}{2}$$

then

$$\frac{-1}{2} < \frac{p-q}{(p+1)(q+1)} < \frac{1}{2}$$

so

$$-(p+1)(q+1) < 2(p-q) < (p+1)(q+1)$$

i.e.

$$\underbrace{-pq - p - q - 1 < 2p}_{(1)} - \underbrace{2q < pq + p + q + 1}_{(2)}$$

from the first inequality we have

$$-pq - 3p - q - 1 < -2q$$

so

$$-pq - 3p - 1 < -q$$

then

$$pq + 3p + 1 < q$$

from the second inequality we have

$$2p < pq + p + 3q + 1$$

so

$$-3q < pq - p + 1$$

i.e.

$$q > \frac{-1}{3}(pq - p + 1)$$

then

$$q > \frac{-1}{3}(p(q-1) + 1)$$

which is always satisfied.
notice that :

- for k odd and $b > 0$
  i.e.
  $$\frac{q-p}{(p+1)(q+1)} > 0$$
  implies that $p < q$.
- for k even and $ab > 0$
  i.e. $a(-a) > 0$, (impossible since $a > 0$). ■

4.6 Existence of a Two Cycles

In this section we give necessary and sufficient condition for equation(4.9) to have a prime period-two solution and we exhibit all prime period-two solutions of the equation.

Theorem 9 *(1) When $p > q$.*

Equation(4.9) has no nonnegative prime period-two solutions .

(2) When $p < q$.

Equation(4.9) has prime period-two solutions iff k odd and $q > pq+3p+1$.

$$..., \Phi, \Psi, \Phi, \Psi, ...$$

Where the values of Φ and Ψ are the (positive and distinct)solutions of the quadratic equation

$$t^2 - (1-p)t + \frac{p(1-p)}{q-1} = 0.$$

Proof. *(1) When $p > q$.*

Assume for the sake of contradiction that there exist distinctive nonnegative real number Φ and Ψ , such that

$$..., \Phi, \Psi, \Phi, \Psi, ...$$

is a prime period-two solution of equation (4.9).there are two cases to be considered .

Case (a) : k is odd.

in this case Φ and Ψ satisfy

$$\Phi = \frac{p\Psi + \Phi}{q\Psi + \Phi},$$

and

$$\Psi = \frac{p\Phi + \Psi}{q\Phi + \Psi}.$$

so

$$\Phi(q\Psi + \Phi) = p\Psi + \Phi \qquad (4.15)$$

and

$$\Psi(q\Phi + \Psi) = p\Phi + \Psi \qquad (4.16)$$

subtracting equation(4.16) from(4.15) ,we have

$$\begin{aligned}
\Phi^2 - \Psi^2 &= p(\Psi - \Phi) + (\Phi - \Psi) \\
(\Phi - \Psi)(\Phi + \Psi) &= -p(\Phi - \Psi) + (\Phi - \Psi) \\
(\Phi - \Psi)(\Phi + \Psi) &= (\Phi - \Psi)(1-p)
\end{aligned}$$

hence,

$$\Phi + \Psi = 1 - p$$

Also,adding equation(4.15) to equation(4.16) then we have

$$\begin{aligned}
2q\Phi\Psi + \Phi^2 + \Psi^2 &= p\Psi + \Phi + p\Phi + \Psi \\
2q\Phi\Psi + \Phi^2 + \Psi^2 + \overbrace{2\Phi\Psi - 2\Phi\Psi}^{adding(zero)} &= p(\Psi + \Phi) + (\Psi + \Phi) \\
2q\Phi\Psi + (\Psi + \Phi)^2 - 2\Phi\Psi &= (\Psi + \Phi)(p + 1)
\end{aligned}$$

but

$$\Phi + \Psi = 1 - p$$

so

$$\begin{aligned}
\Phi\Psi(2q - 2) + (\Psi + \Phi)^2 &= (1 - p)(p + 1) \\
\Phi\Psi(2q - 2) + (1 - p)^2 &= p + 1 - p^2 - p \\
\Phi\Psi &= \frac{1 - p^2 - (1 - p)^2}{2q - 2} \\
&= \frac{1 - p^2 - 1 + 2p - p^2}{2q - 2} \\
&= \frac{-2p^2 + 2p}{2q - 2} \\
&= \frac{2(p - p^2)}{2(q - 1)}
\end{aligned}$$

hence,

$$\Phi\Psi = \frac{p(1 - p)}{q - 1}, q \neq 1$$

since $\Phi\Psi > o$ *;*Φ, Ψ *distinctive nonnegative real number,implies that*

$$p(1 - p) > 0 \quad and \quad q - 1 > 0 \qquad (4.17)$$

or

$$p(1 - p) < 0 \quad and \quad q - 1 < 0 \qquad (4.18)$$

Now , from equation(4.17) we have $q > 1$*,and we have* $p > q$*,so* $p > 1$*,hence* $(1 - p) < 0$*,so* $\Phi\Psi < 0$*,this contradicts the hypothesis that* $\Phi\Psi > 0$*.Also,from equation(4.18) we have* $q < 1$*,and we have* $p > q$*,so* $p > 1$*,hence* $(1 - p) < 0$*,so* $\Phi\Psi < 0$*,this contradicts the hypothesis that* $\Phi\Psi > 0$*.*

Case(b) : k is even.

in this case Φ *and* Ψ *satisfy*

$$\Phi = \frac{p\Psi + \Psi}{q\Psi + \Psi},$$

and

$$\Psi = \frac{p\Phi + \Phi}{q\Phi + \Phi}.$$

so

$$\begin{aligned}\Phi &= \frac{\Psi(p+1)}{\Psi(q+1)}\\ &= \frac{p+1}{q+1}\end{aligned}$$

and

$$\begin{aligned}\Psi &= \frac{\Phi(p+1)}{\Phi(q+1)}\\ &= \frac{p+1}{q+1}\end{aligned}$$

so $\Phi = \Psi$*.this contradicts the hypothesis that* Φ *and* Ψ *distinctive nonnegative real number.*

(2) When $p < q$.

Assume that there exist distinctive nonnegative real number Φ *and* Ψ *, such that*

$$..., \Phi, \Psi, \Phi, \Psi, ...$$

is a prime period-two solution of equation(4.9).there are two cases to be considered.

Case (a) : k is odd.

in this case Φ *and* Ψ *satisfy*

$$\Phi = \frac{p\Psi + \Phi}{q\Psi + \Phi},$$

and

$$\Psi = \frac{p\Phi + \Psi}{q\Phi + \Psi}.$$

then we have

$$\Phi + \Psi = 1 - p$$

and

$$\Phi\Psi = \frac{p(1-p)}{q-1}, q \neq 1$$

Now , construct the quadratic equation

$$t^2 - (1-p)t + \frac{p(1-p)}{q-1} = 0.$$

So,the values of Φ *and* Ψ *are the (positive and distinct)solutions of the above*

quadratic equation.
i.e.

$$t = \frac{(1-p) \pm \sqrt{(1-p)^2 - \frac{4p(1-p)}{q-1}}}{2},$$

Case(b) : k is even
in this case Φ *and* Ψ *satisfy*

$$\Psi = \frac{p\Phi + \Phi}{q\Phi + \Phi}.$$

and

$$\Phi = \frac{p\Psi + \Psi}{q\Psi + \Psi},$$

so

$$\begin{aligned} \Phi &= \frac{\Psi(p+1)}{\Psi(q+1)} \\ &= \frac{p+1}{q+1} \end{aligned}$$

and

$$\begin{aligned} \Psi &= \frac{\Phi(p+1)}{\Phi(q+1)} \\ &= \frac{p+1}{q+1} \end{aligned}$$

so $\Phi = \Psi$*.This contradicts the hypothesis that* Φ *and* Ψ *distinctive nonnegative real number. The proof is complete.* $\blacksquare$

4.7 Analysis of Semicycles and Oscillations

We believe that a semicycle analysis of the solution of equation(4.9) is a powerful tool for a detailed understanding of the entire character of solutions.

In this section , we present some results about the semicycle character of solutions of equation(4.9).

Theorem 10 *Let $\{y_n\}$ be a nontrivial solution of equation(4.9),then the following statements are true:*

(a)Assume $p > q$, then $\{y_n\}$ oscillates about the equilibrium $\bar{y}$ with semicycles of length k+1 or k+2, except possibly for the first semicycle which may have length k. The extreme in each semicycle occurs in the first term if the semicycle has two terms and in the second term if the semicycle has three terms...and in the k+1 term if the semicycle has k+2 terms.

(b)Assume $p < q$, then either $\{y_n\}$ oscillates about the equilibrium $\bar{y}$ with semicycles of length k after the first semicycle, or $\{y_n\}$ converges monotonically to $\bar{y}$.

Proof. *(a)The proof follows from theorem (4) by observing that the condition p>q implies that the function*

$$f(x,y) = \frac{px+y}{qx+y}$$

is increasing in x and decreasing in y. This function also satisfies condition(4.14).

(b)The proof follows from theorem (5) by observing that the condition p<q implies that the function

$$f(x,y) = \frac{px+y}{qx+y}$$

is decreasing in x and increasing in y . The proof is complete. ∎

4.8 Attracting Intervals

We now examine the existence of intervals which attract all solution of equation(4.9).

Lemma 1 *let* $\{y_n\}_{n=-k}^{\infty}$ *be a solution of equation(4.9),then the following statements are true :*

(1) suppose $p < q$ *and assume that for some* $N \geq 0$.

$$y_{N-k+1}, ..., y_{N-1}, y_N \in \left[\frac{p}{q}, 1\right],$$

then

$$y_n \in \left[\frac{p}{q}, 1\right] \text{ , for all } n > N.$$

(2) suppose $p > q$ *and assume that for some* $N \geq 0$.

$$y_{N-k+1}, ..., y_{N-1}, y_N \in \left[1, \frac{p}{q}\right],$$

then

$$y_n \in \left[1, \frac{p}{q}\right] \text{ , for all } n > N.$$

Proof. *We prove (1). The proof of (2) is similar and will be omitted.*

If for some $N > 0$,$1 \leq y_N \leq \frac{p}{q}$,*then*

$$\begin{aligned} y_{n+1} &= \frac{py_n + y_{n-k}}{qy_n + y_{n-k}} \\ &\leq \frac{qy_n + y_{n-k}}{qy_n + y_{n-k}} = 1. \text{ since } p < q. \end{aligned}$$

also,since $1 \leq y_N \leq \frac{p}{q}$, *then*

$$\begin{aligned} y_{n+1} &= \frac{py_n + y_{n-k}}{qy_n + y_{n-k}} \\ &\geq \frac{p(1) + \frac{p}{q}}{q(1) + \frac{p}{q}} \\ &= \frac{p + \frac{p}{q}}{q + \frac{p}{q}} \\ &\geq \frac{p}{q}. \end{aligned}$$

The proof is complete. ■

4.9 Global Stability Analysis

In this section we will prove a global attracter for the positive equilibrium of equation(4.9) .

Theorem 11 *Assume that $p < q$,and k odd, then the positive equilibrium of equation(4.9) is globally asymptotically stable when*

$$q < pq + 3p + 1.$$

Proof. *set*

$$f(x, y) = \frac{px + y}{qx + y}$$

we note that $f(x, y)$ is decreasing in x for each fixed y, and increasing in y for each fixed x, also, clearly

$$\frac{p}{q} \leq f(x, y) \leq 1 \text{ for all x,y >0.}$$

Finally, since

$$q < pq + 3p + 1.$$

equation(4.9) has no prime period-two solution. Now the conclusion of theorem(11) follows as a consequence of theorem(6)

and the fact that $\bar{y}$ is locally asymptotically stable.

The proof is complete. ■

Theorem 12 *Assume that $p > q$, and*

$$p \leq pq + 3q + 1$$

then the positive equilibrium of equation(4.9) is globally asymptotically stable .

Proof. *set*

$$f(x, y) = \frac{px + y}{qx + y}$$

when p>q, the function $f(x, y)$is increasing in x for each fixed y , and decreasing in y for each fixed x, also, clearly

$$1 \leq f(x, y) \leq \frac{p}{q} \text{ for all x,y >0.}$$

Finally, when

$$p \leq pq + 3q + 1.$$

the only solution of the system

$$M = \frac{pM + m}{qM + m}, m = \frac{pm + M}{qm + M}$$

is m=M.
Now the result is consequence of theorem (7).
The proof is complete. ∎

CHAPTER 5
Computational Approaches

5.1 Numerical Examples

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to non-linear difference equations.

In this part, to observe this numerical results clearly, we present tables of solutions that were carried out using MATLAB. We choose different values for the parameters p and q. It should be noted that $y_{-k}, y_{-k+1}, ..., y_{-1}, y_0$ are also different initial points.

To simplify our exposition we restrict our discussion to the following sections:

- **First order difference equation**

Here we will represent different types of solutions of equation (4.9) when $k = 1$.

- **Second order difference equation**

Here we will represent different types of solutions of equation (4.9) when $k = 2$.

- **Third order difference equation**

Here we will represent different types of solutions of equation (4.9) when $k = 3$.

- **Fourth order difference equation**

Here we will represent different types of solutions of equation (4.9) when $k = 4$.

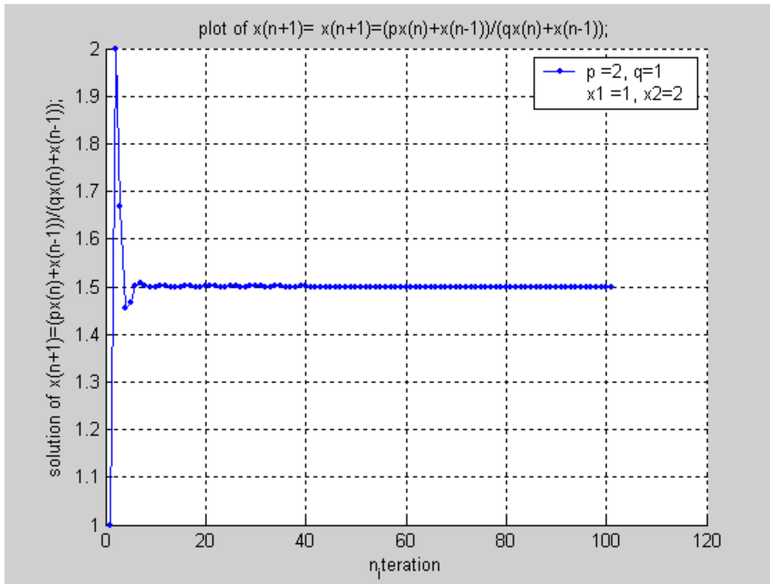

Figure 12: plot of $y_{n+1} = \frac{2y_n + y_{n-1}}{y_n + y_{n-1}}$

5.1.1 First Order Difference Equation

Here we will represent different types of solutions of equation (4.9) when $k = 1$.

case (1) : $p > q$.
Example :

Consider the following difference equation

$$y_{n+1} = \frac{2y_n + y_{n-1}}{y_n + y_{n-1}}, \qquad n = 0, 1, ..$$

with the initial conditions $y_1 = 1, y_2 = 2$.(See figure 12)

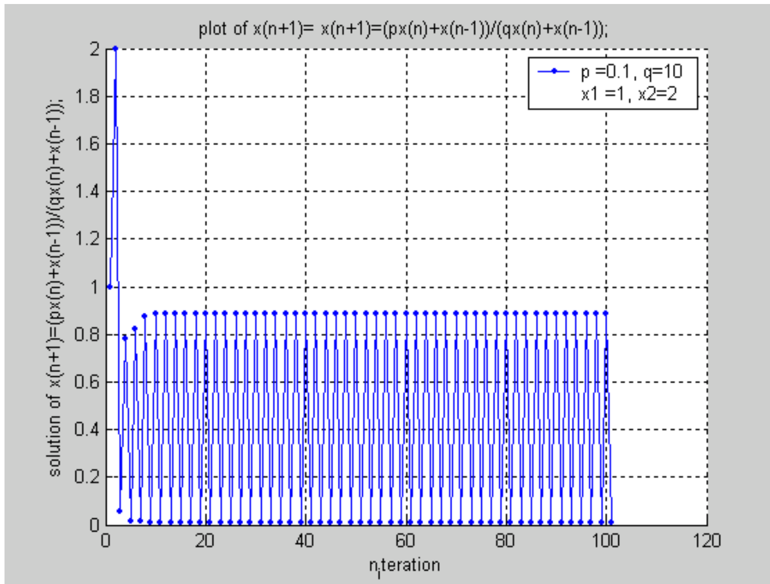

Figure 13: plot of $y_{n+1} = \frac{0.1y_n+y_{n-1}}{10y_n+y_{n-1}}$

case (2) : $p < q$, $q > pq + 3p + 1$.
Example :

Consider the following difference equation

$$y_{n+1} = \frac{0.1y_n + y_{n-1}}{10y_n + y_{n-1}}, \qquad n = 0, 1, ..$$

with the initial conditions $y_1 = 1, y_2 = 2$.(See figure 13)

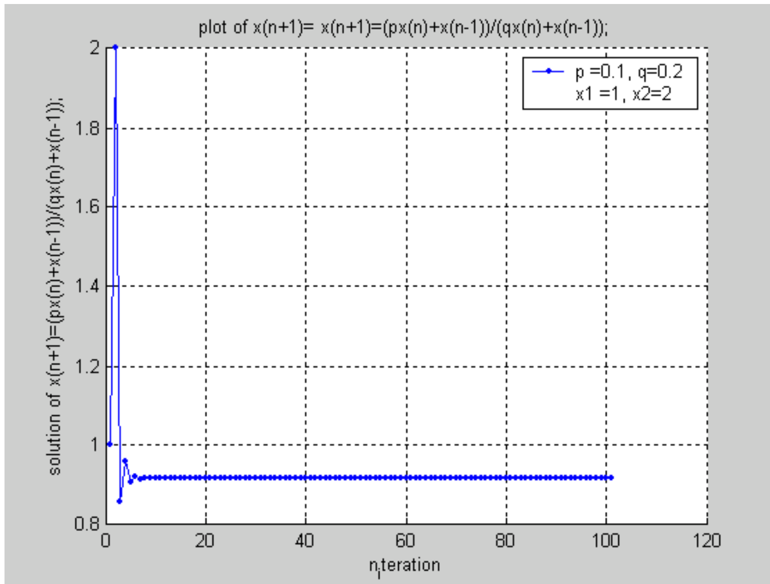

Figure 14: plot of $y_{n+1} = \frac{0.1y_n + y_{n-1}}{0.2y_n + y_{n-1}}$

case (3) : $p < q$, $q < pq + 3p + 1$.
Example :
Consider the following difference equation

$$y_{n+1} = \frac{0.1y_n + y_{n-1}}{0.2y_n + y_{n-1}}, \qquad n = 0, 1, ..$$

with the initial conditions $y_1 = 1, y_2 = 2$.(See figure14).

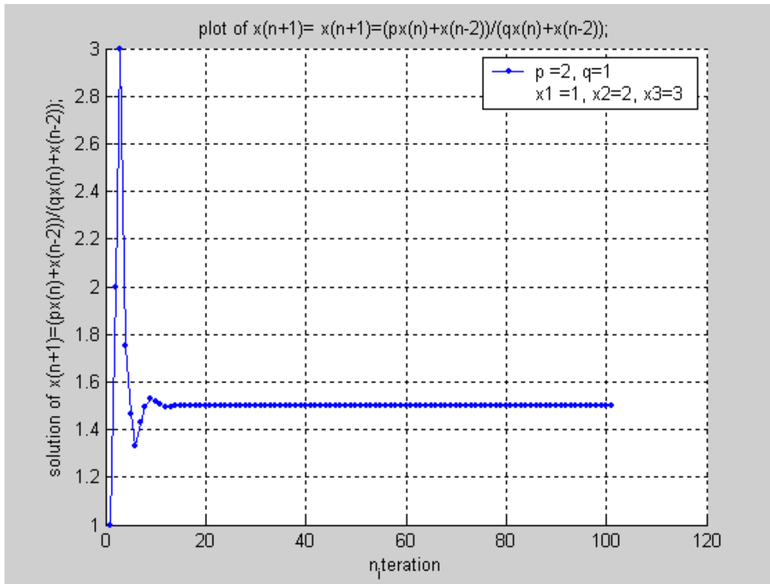

Figure 15: plot of $y_{n+1} = \frac{2y_n + y_{n-2}}{y_n + y_{n-2}}$

5.1.2 Second Order Difference Equation

Here we will represent different types of solutions of equation (4.9) when $k = 2$.

case (1) : $p > q$ *and* $p \leq pq + 3p + 1$.
Example :

Consider the following difference equation

$$y_{n+1} = \frac{2y_n + y_{n-2}}{y_n + y_{n-2}}, \qquad n = 0, 1, ..$$

with the initial conditions $y_1 = 1, y_2 = 2, y_3 = 3$.(See figure 15)

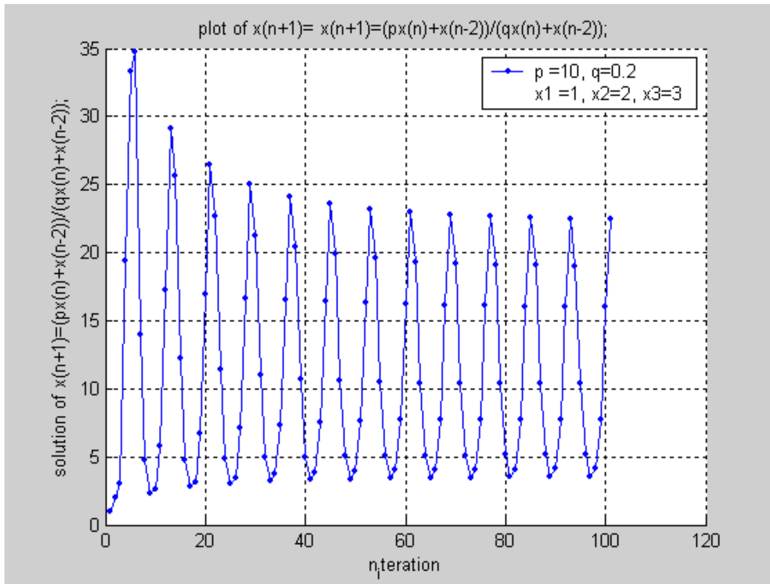

Figure 16: plot of $y_{n+1} = \frac{10y_n + y_{n-2}}{0.2y_n + y_{n-2}}$

case (2) : $p > q$ *and* $p > pq + 3p + 1$.
Example :
Consider the following difference equation

$$y_{n+1} = \frac{10y_n + y_{n-2}}{0.2y_n + y_{n-2}}, \qquad n = 0, 1, ..$$

with the initial conditions $y_1 = 1, y_2 = 2, y_3 = 3$. (See figure 16)

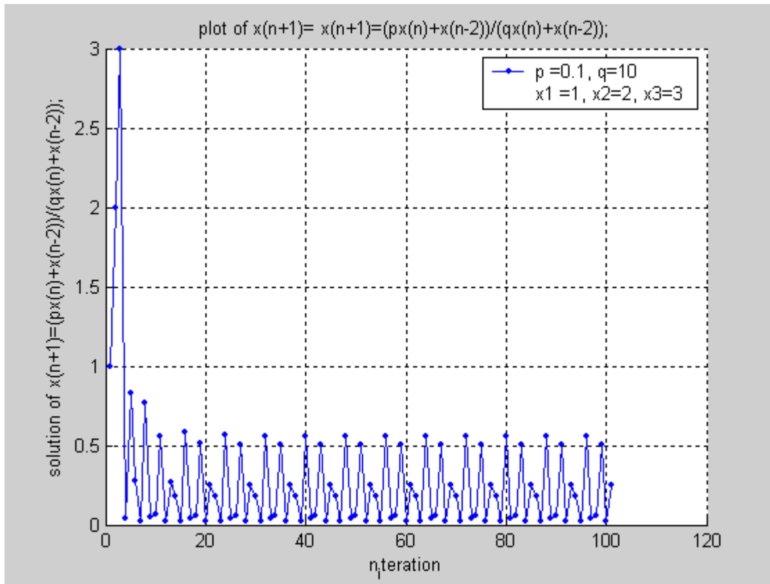

Figure 17: plot of $y_{n+1} = \frac{10y_n + y_{n-2}}{0.2y_n + y_{n-2}}$

case (3) : $p < q$.
Example :
Consider the following difference equation

$$y_{n+1} = \frac{0.1y_n + y_{n-1}}{10y_n + y_{n-1}}, \qquad n = 0, 1, ..$$

with the initial conditions $y_1 = 1, y_2 = 2, y_3 = 3$.(See figure 17).

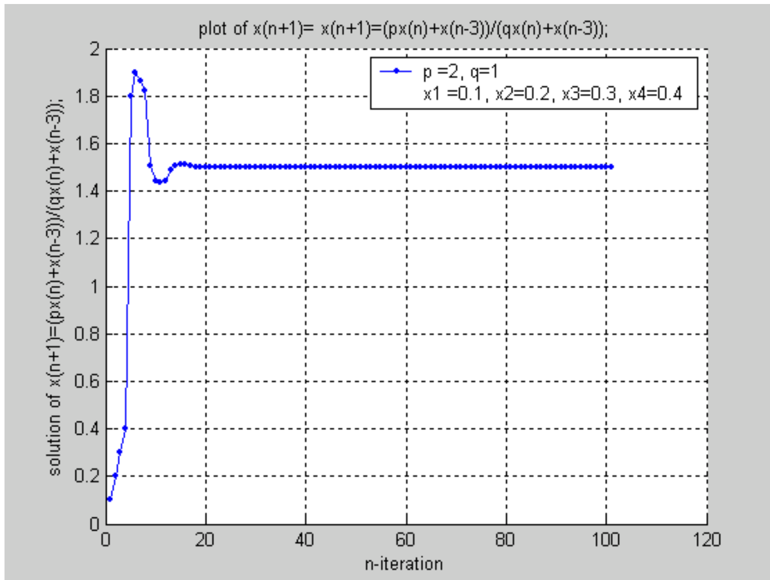

Figure 18: plot of $y_{n+1} = \frac{2y_n + y_{n-3}}{y_n + y_{n-3}}$

5.1.3 Third Order Difference Equation

Here we will represent different types of solutions of equation (4.9) when $k = 3$.

case (1) : $p > q$.
Example :

Consider the following difference equation

$$y_{n+1} = \frac{2y_n + y_{n-3}}{y_n + y_{n-3}}, \qquad n = 0, 1, ..$$

with the initial conditions $y_1 = 0.1, y_2 = 0.2, y_3 = 0.3, y_4 = 0.4$.(See figure 18)

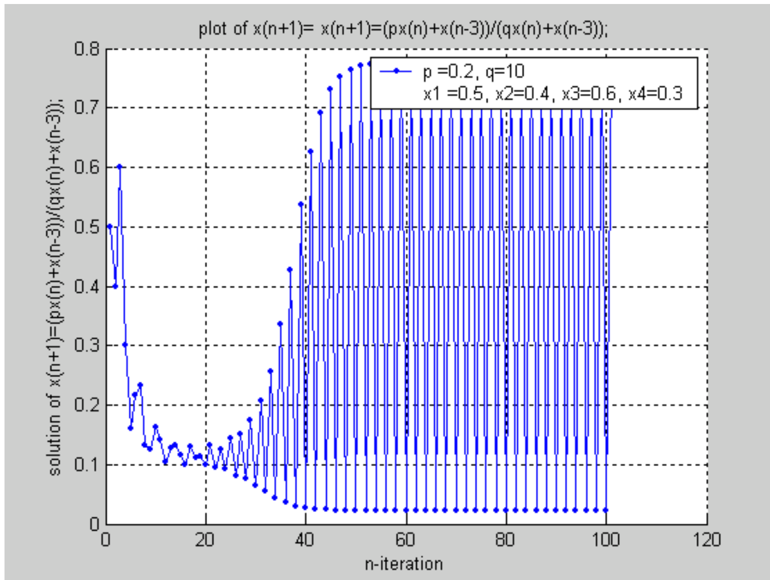

Figure 19: plot of $y_{n+1} = \frac{0.2y_n + y_{n-3}}{10y_n + y_{n-3}}$

case (2) : $p < q$, $q > pq + 3p + 1$.
Example :
Consider the following difference equation

$$y_{n+1} = \frac{0.2y_n + y_{n-3}}{10y_n + y_{n-3}}, \qquad n = 0, 1, ..$$

with the initial conditions $y_1 = 0.5, y_2 = 0.4, y_3 = 0.6, y_4 = 0.3$. (See figure 19).

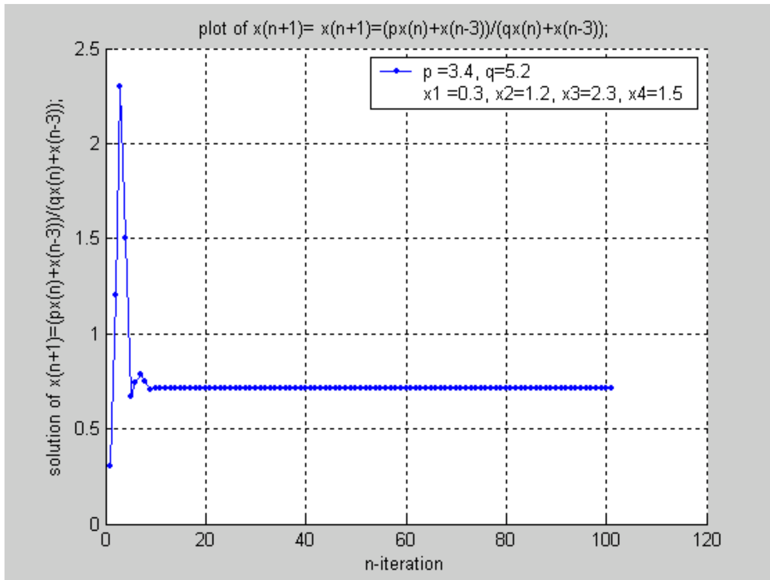

Figure 20: plot of $y_{n+1} = \frac{3.4y_n + y_{n-3}}{5.2y_n + y_{n-3}}$

case (3) : $p < q$, $q < pq + 3p + 1$.
Example :
Consider the following difference equation

$$y_{n+1} = \frac{3.4y_n + y_{n-3}}{5.2y_n + y_{n-3}}, \qquad n = 0, 1, ..$$

with the initial conditions $y_1 = 0.3, y_2 = 1.2, y_3 = 2.3, y_4 = 1.5$. (See figure 20).

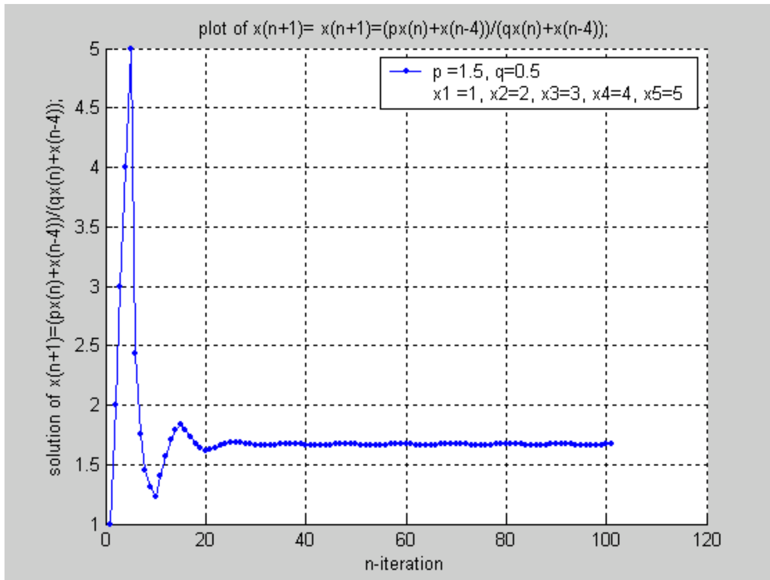

Figure 21: plot of $y_{n+1} = \frac{1.5y_n+y_{n-4}}{0.5y_n+y_{n-4}}$

5.1.4 Fourth Order Difference Equation

Here we will represent different types of solutions of equation (4.9) when $k = 4$.

case (1) : $p > q$ *and* $p \leq pq + 3p + 1$.
Example :

Consider the following difference equation

$$y_{n+1} = \frac{1.5y_n + y_{n-4}}{0.5y_n + y_{n-4}}, \qquad n = 0, 1, ..$$

with the initial conditions $y_1 = 1, y_2 = 2, y_3 = 3, y_4 = 4, y_5 = 5$.
(See figure 21)

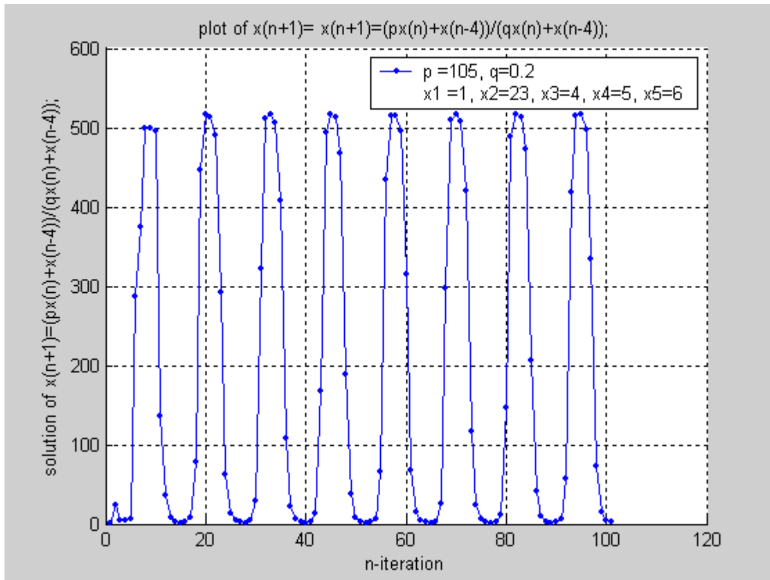

Figure 22: plot of $y_{n+1} = \frac{105y_n + y_{n-4}}{0.2y_n + y_{n-4}}$

case (2) : $p > q$ *and* $p > pq + 3p + 1$.
Example :
Consider the following difference equation

$$y_{n+1} = \frac{105y_n + y_{n-4}}{0.2y_n + y_{n-4}}, \qquad n = 0, 1, ..$$

with the initial conditions $y_1 = 1, y_2 = 23, y_3 = 4, y_4 = 5, y_5 = 6$. (See figure 22).

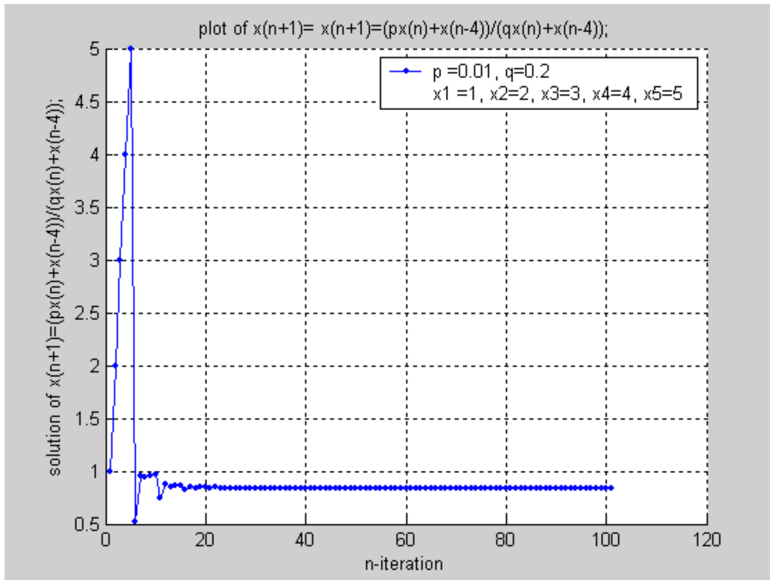

Figure 23: plot of $y_{n+1} = \frac{0.01 y_n + y_{n-4}}{0.2 y_n + y_{n-4}}$

case (3) : $p < q$.
Example :
Consider the following difference equation

$$y_{n+1} = \frac{0.01 y_n + y_{n-4}}{0.2 y_n + y_{n-4}}, \qquad n = 0, 1, ..$$

with the initial conditions $y_1 = 1, y_2 = 2, y_3 = 3, y_4 = 4, y_5 = 5$. (See figure 23).

5.2 MATLAB mfile:

In this mfile solution=difference2(k), we investigate the nonlinear rational difference equation

$$y_{n+1} = \frac{py_n + y_{n-k}}{qy_n + y_{n-k}}, \qquad n = 0, 1, ... \qquad (4.9)$$

where the parameters p, q and the initial conditions $y_{-k}, ..., \ y_{-1}, y_0$ are nonnegative real numbers, $k = \{1, 2, 3, ...\}$

Our concentration is on invariant intervals, periodic character, the character of semicycles and global asymptotic stability of all positive solutions of equation(4.9)

There have been several programs on the subject of Dynamical Systems. There are several distinctive aspects which together make this program unique.

• First of all, this program capable of finding approximations of solutions of this difference equation.

• Second, it can produce high quality graphics representations of solutions behavior.

• Third, give analysis of the result in details.

- The solution x(n) is given in table .
- Calculate the equilibrium point.
- It provide you if the equilibrium is locally asymptotically stable.
- Also tell you if there is period two solution.
- Furthermore show if the solution oscillates about the equilibrium if so it gives the length of the semicycles.
- It show the invariant interval in which the solution takes its values .
- And finally, tell you if the equilibrium point is globally asymptotically stable.

• Fourth, very simple to run; what you should to do is entering the order of the difference equation(k) in difference2(k) on the command window; and then follows the directions which the program asked to do; if you set the value of k to 1,then the program will asked you to enter the values of parameters p, q and the others of the initial conditions y_{-1}, y_0. If you set the value of k to 2,then the program will asked you to enter the values of parameters p, q and the others of the initial conditions y_{-2}, y_{-1}, y_0. And so on.

Conclusion

In this thesis, we investigate the nonlinear rational difference equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{B x_n + C x_{n-k}}, \qquad n = 0, 1, ... \quad (1)$$

where the parameters β, γ and B, C and the initial conditions $x_{-k}, ..., x_{-1}$ and x_0 are nonnegative real numbers, $k = \{1, 2, 3, ...\}$.

Our concentration was on invariant intervals, periodic character, the character of semicycles and global asymptotic stability of all positive solutions of equation(1).

The change of variables

$$x_n = \frac{\gamma}{C} y_n.$$

reduces equation(1) to the difference equation

$$y_{n+1} = \frac{p y_n + y_{n-k}}{q + y_{n-k}}, \qquad n = 0, 1, ... \qquad (2)$$

where

$$p = \frac{\beta}{\gamma}, q = \frac{B}{C}.$$

The parameters p, q and the initial conditions $y_{-k}, ...,$ y_{-1}, y_0 are nonnegative real numbers, $k = \{1, 2, 3, ...\}$.

In order to investigate the global attractivity, boundedness, periodicity, and global stability of solution of this difference equation, we use MATLAB to see how the behavior of this difference equation look like. The solutions that were carried out using MATLAB agree with the theoretical results of this thesis. As might be expected, the two cases $\mathbf{p} > \mathbf{q}$ and $\mathbf{p} < \mathbf{q}$ give rise to different dynamic behaviors.

$\mathbf{Case(1) : p > q}$
- The equilibrium point is locally asymptotically stable.
- There is no period two solution.
- The solution oscillates about the equilibrium $(p+1)/(q+1)$ with semicycle of length $k+1$or $k+2$ except possibly for the first semicycle which may have length k.
- The solution takes its values between 1 and p/q.
- The equilibrium point is globally asymptotically stable if $p \leq pq + 3q + 1$.

$\mathbf{Case(2) : p < q}$
$Subcase(1) : k \ is \ even$
- The equilibrium point is locally asymptotically stable.

• The solution oscillates about the equilibrium $(p+1)/(q+1)$ with semicycle of length k after the first semicycle or it converges monotonically to the equilibrium.

• The solution takes its values between p/q and 1 .

$Subcase(2) : k\ is\ odd$

$Subsubcase(1) : q > pq + 3p + 1.$

• The equilibrium point is unstable.

• There is period two solution.

• The solution oscillates about the equilibrium $(p+1)/(q+1)$ with semicycle of length k after the first semicycle or it converges monotonically to the equilibrium.

• The solution takes its values between p/q and1.

$Subsubcase(2) : q < pq + 3p + 1.$

• The equilibrium point is locally asymptotically stable.

• The solution oscillates about the equilibrium $(p+1)/(q+1)$ with semicycle of length k after the first semicycle or it converges monotonically to the equilibrium.

• The solution takes its values between p/q and 1 .

• The equilibrium point is globally asymptotically stable.

As we mention, this thesis solve the open problem 6.10.17 (equation(6.100)) proposed by Kulenvic and Ladas in their monograph [Dynamics of Second Order Rational Difference Equations: with Open Problems and Conjectures, Chapman & Hall/CRC, Boca Raton, 2002]. [7]

But what about the global behavior of all solutions of the same problem with negative parameters and negative initial conditions.

we proposed this as an open problem and we hope to see the dynamics of it later on.

Open problem:

Invistigate the global behavior of all solutions of

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{B x_n + C x_{n-k}}, \qquad n = 0, 1, ...$$

where the parameters β, γ and B, C and the initial conditions $x_{-k}, ..., x_{-1}$ and x_0 are negative real numbers and $k \in \{1, 2, 3, ...\}$.

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