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Classical relativistic dynamics of a free electron in an intense laser field

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Abstract. We present a classical analytical investigation of the dynamics of a relativistic electron in a super-intense, ultra-short laser pulse. The set of equations used in this study were obtained from a Hamilton–Jacobi formulation. Motion, in vacuum, of a free electron, in a linearly polarized laser pulse is discussed and the recent related results of Hartemann *et al* are confirmed and extended to other possible initial conditions on the direction of motion of the electron.

1. Introduction

With the advent of table-top pulsed laser systems capable of delivering field intensities far in excess of 10^{18} W cm^{-2}, we have recently begun to witness a renewed interest in a number of important problems pertaining to the interaction of a free electron with intense radiation. In the past few years, several schemes for accelerating electrons by means of powerful laser fields were proposed [1]. Recently, Moore and coworkers [2] and Meyerhofer *et al* [3] have observed scattering by a high-intensity laser pulse of an electron, produced via ionization in its focal region. Körmendi and Farkas [4] have shown that soft x-rays may be generated by multiphoton scattering of a laser beam from fast free electrons. The observation of harmonic generation by the scattering of high-intensity laser light from fast electrons has indeed been recently reported [5].

The present work addresses the dynamics of a fast electron in an ultra-short laser pulse. This topic has recently been the subject of an analytical study by Hartemann *et al* [6]. In the past, related studies were mostly done by computer simulations [7].

In their analytical study of the dynamics of the electron in an electromagnetic field, Hartemann and his coworkers [6] used the relativistic momentum transfer equations, integrated them subject to a suitably chosen set of initial conditions and found expressions for the electron energy, position coordinates and velocity components at all space-time points during the interaction with the field. They reviewed first the motion in vacuum of an electron in the presence of a linearly polarized plane wavefield of electric component $\boldsymbol{E} = \hat{\boldsymbol{\imath}} E_0 \sin\eta$ and the associated magnetic component. This was followed by a similar analysis in the presence of an ultrashort laser pulse modelled by a $\sin^2$ pulse-shape function. The main results presented by Hartemann and coworkers included the electron trajectory, velocity components and ponderomotive scattering angle, all corresponding to initial motion parallel to the direction of laser propagation and for fields produced by laser systems operational at present.

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The present paper aims also to introduce a set of equations, arrived at within a classical relativistic formulation, which are likely to minimize the effort in the analytical study of the dynamics of an electron, or any other charged particle, in the presence of a radiation field, with no restrictions on the initial electron velocity or field intensity and polarization. The equations are derived from a Hamilton–Jacobi construction. Dynamics of a fast free electron will be studied using our equations. We arrive at analytical results identical to those of Hartemann *et al* [6] for similar situations (the same electron and laser field parameters and the same initial geometry). This work, however, is different from that of Hartemann *et al* [6] in several respects. First, the approach is not the same, our equations employ a vector potential as opposed to their separation of the electromagnetic field into electric and magnetic components. Secondly, we employ a different pulse-shape function of infinite range while they use a (finite-range) $\sin^2$ pulse-shape function. Thirdly, the cases of an electron initially moving opposite to the laser field propagation direction and perpendicular to it, not taken up by Hartemann *et al*, will be studied with emphasis on the angle, θ, its velocity vector makes with the direction of propagation of the laser during interaction with the field. With the spatial extension of the focal spot of the pulse properly taken care of, θ becomes the *ponderomotive scattering angle* when the amplitude of transverse oscillation exceeds the diameter of the focus. Finally, in view of the recent high-energy laser-accelerator experiments [5], the same angle for electrons having initial kinetic energies in the GeV range, is considered.

Derivation of our general equations will be given in section 2. In section 3, the equations will be used to study the dynamics of a free electron as it is overtaken by an intense laser pulse of a 10 fs duration. In particular, the net transverse electron excursion away from its initial direction of motion will be derived by employing the phase of the laser field as a parameter. Results, which correspond to laser and electron beam parameters in use in some real experimental situations, will be presented and briefly discussed in section 4. In the same section, the angle θ, giving the direction of motion of the electron, will be derived, as a function of its kinetic energy, for the cases of an electron moving initially opposite the direction of propagation of the laser beam and one in which the electron initially moves perpendicular to the laser propagation direction. The equation corresponding to the case of initial motion of the electron at some arbitrary angle θ_0 relative to the laser beam direction will also be given. Initial electron kinetic energies of up to several tens of GeV will be considered. Some concluding remarks will be given in section 5.

2. The equations

Our approach to the study of the dynamics of the electron takes account of the effects of both the electric and magnetic forces in a unified way through the use of a vector potential. It is also founded on a relativistic Hamilton–Jacobi equation [8, 9]. The last point gives the approach the advantage of being easily generalized as a quantum treatment.

We represent the laser field by a transverse monochromatic wave, with laboratory propagation vector $\boldsymbol{k}$ and frequency ω, whose fields are derived from a vector potential $\boldsymbol{A}(\eta) = f(\eta)\boldsymbol{a}(\eta)$, where $\eta = \omega t - \boldsymbol{k}\cdot\boldsymbol{r}$ and $f(\eta)$ is some suitably chosen pulse-shape function. The equations of motion of the electron, of mass m and charge $-e$, in the presence of such a field may be found from the following Hamilton–Jacobi equation

$$\left(\frac{\partial S}{\partial t}\right)^2 = c^2\left(\boldsymbol{\nabla} S + \frac{e}{c}\boldsymbol{A}\right)^2 + (mc^2)^2 \tag{1}$$

where $S(\boldsymbol{r}, t)$ is the Hamilton principal function and c is the speed of light. One usually looks for a solution of the form [8, 9]

$$S(\boldsymbol{r}, t) = \boldsymbol{s} \cdot \boldsymbol{r} + \xi ct + F(\eta) \tag{2}$$

where $\boldsymbol{s}$ and ξ are constants to be determined from the initial conditions of the problem. Substituting equation (2) into (1) and using the transversality condition $\boldsymbol{k} \cdot \boldsymbol{A} = 0$, leads us to a first-order differential equation for F. A single integration with respect to η then gives

$$F(\eta) = \tfrac{1}{2}(\boldsymbol{s} \cdot \boldsymbol{k} + \xi k)^{-1} \int_{\eta_0}^{\eta} [s^2 - \xi^2 + (mc)^2 + 2(e/c)\boldsymbol{s} \cdot \boldsymbol{A}(\eta') + (e/c)^2 A^2(\eta')]\, \mathrm{d}\eta'. \tag{3}$$

Differentiating the principal function with respect to the (arbitrary) constants and equating the results with the initial space-time coordinates, one gets, in principle, expressions giving the space-time trajectory of the electron. Thus from $\nabla_s S = \boldsymbol{r}_0$, it follows that

$$\boldsymbol{r}(\eta) = \boldsymbol{r}_0 - \int_{\eta_0}^{\eta} \frac{\boldsymbol{s} + (e/c)\boldsymbol{A}(\eta')}{\boldsymbol{s} \cdot \boldsymbol{k} + \xi k}\, \mathrm{d}\eta' + \frac{\boldsymbol{k}}{2} \int_{\eta_0}^{\eta} \frac{s^2 - \xi^2 + (mc)^2 + 2(e/c)\boldsymbol{s} \cdot \boldsymbol{A}(\eta') + (e/c)^2 A^2(\eta')}{(\boldsymbol{s} \cdot \boldsymbol{k} + \xi k)^2}\, \mathrm{d}\eta'. \tag{4}$$

Furthermore, from $\partial S/\partial \xi = ct_0$, one gets

$$ct = ct_0 + \frac{\boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) - (\eta - \eta_0)}{k}. \tag{5}$$

On the other hand, differentiating Hamilton's principal function with respect to the space-time coordinates will give expressions for the energy and momentum of the electron. Equating the gradient of the principal function with the canonical momentum, $\nabla S = \boldsymbol{P}_{\text{can}} = \boldsymbol{p} - \frac{e}{c}\boldsymbol{A}$, gives

$$\boldsymbol{p} = \frac{e}{c}\boldsymbol{A} + \boldsymbol{s} - \frac{\boldsymbol{k}}{2}\left[\frac{s^2 - \xi^2 + (mc)^2 + 2(e/c)\boldsymbol{s} \cdot \boldsymbol{A} + (e/c)^2 A^2}{(\boldsymbol{s} \cdot \boldsymbol{k} + \xi k)}\right] \tag{6}$$

while $E = -\partial S/\partial t$ leads to

$$E = -c\left[\xi + \frac{\boldsymbol{k}}{k} \cdot (\boldsymbol{s} - \boldsymbol{P}_{\text{can}})\right]. \tag{7}$$

Equations (6) and (7) are general in the sense of holding, in any inertial frame, for any initial conditions. The word *initial* is used here to mean *before onset of the particle-field interaction*. In any specific situation the initial conditions take on a specific meaning, as will be shown below. On the other hand, the electron will be assumed to be moving at the (arbitrary in magnitude and direction) velocity $\boldsymbol{v}_0$ before entering the region of interaction with the field, or equivalently, before the field turn-on. Therefore, an appropriate designation of the set of laboratory initial conditions would be to take the vector potential $\boldsymbol{A} = \mathbf{o}$, the canonical momentum of the electron $\boldsymbol{P}_{\text{can}} = \gamma_0 m \boldsymbol{v}_0$, and its energy $E = \gamma_0 mc^2$, where $\gamma_0 = (1 - \beta_0^2)^{-1/2}$ and $\beta_0 = \boldsymbol{v}_0/c$. Using these initial conditions in equations (6) and (7) gives

$$\boldsymbol{s} = \gamma_0 m \boldsymbol{v}_0 + \frac{\boldsymbol{k}}{2k} \frac{s^2 - \xi^2 + (mc)^2}{\boldsymbol{s} \cdot \boldsymbol{k}/k + \xi} \tag{8}$$

and

$$\frac{\boldsymbol{s} \cdot \boldsymbol{k}}{k} + \xi = -\gamma_0 mc\left(1 - \frac{\boldsymbol{k} \cdot \boldsymbol{v}_0}{kc}\right). \tag{9}$$

Cross-multiplying (8) by $\boldsymbol{k}$ and dividing through by k, one gets the component of $\boldsymbol{s}$ normal to the beam propagation direction,

$$\boldsymbol{s}_\perp \equiv \frac{\boldsymbol{k}\cdot\boldsymbol{s}}{k} = \gamma_0 mc\frac{\boldsymbol{k}\cdot\boldsymbol{v}_0}{kc}. \tag{10}$$

An equation involving the component of $\boldsymbol{s}$ parallel to $\boldsymbol{k}$, defined as $s_\parallel \equiv \boldsymbol{k}\cdot\boldsymbol{s}/k$, may be obtained from (8) through scalar multiplying by $\boldsymbol{k}$, dividing the result through by k and adding ξ to both sides. The result of doing so, after some algebra, is

$$\left(\frac{\boldsymbol{k}\cdot\boldsymbol{s}}{k}+\xi\right)^2 = 2\gamma_0 mc\frac{\boldsymbol{k}\cdot\boldsymbol{v}_0}{kc}\left(\frac{\boldsymbol{k}\cdot\boldsymbol{s}}{k}+\xi\right) + s_\perp^2 + (mc)^2. \tag{11}$$

Note that $s_\parallel$ enters in equations (9) and (11) in the combination $(s_\parallel + \xi)$. Without loss of generality, we may take

$$\boldsymbol{s} = \gamma_0 m\boldsymbol{v}_0. \tag{12}$$

With this *choice*, one finds that $s_\parallel = \gamma_0 m\boldsymbol{k}\cdot\boldsymbol{v}_0/k$ and, consequently, equation (9) gives

$$\xi = -\gamma_0 mc. \tag{13}$$

Equations (12) and (13) lead to the numerator of the second term vanishing in (8), resulting thus in a substantial simplification of the algebra to be encountered below. At this point, the scene is set for the derivation of the group of equations of interest to us in this paper. From this point on, a unit vector in the propagation direction of the laser field will be denoted by $\hat{\boldsymbol{k}} \equiv \boldsymbol{k}/k$. Furthermore, the normalized initial velocity $\boldsymbol{\beta}_0 \equiv \boldsymbol{v}_0/c$ will replace $\boldsymbol{v}_0$, for notational convenience.

Using parameters (12) and (13) in (6) and (7) gives the following expressions for the three momenta and total energy of the electron in the presence of the radiation field

$$\boldsymbol{p}(\eta) = \frac{e}{c}\boldsymbol{A}(\eta) + \gamma_0 mc\boldsymbol{\beta}_0 + \hat{\boldsymbol{k}}(\gamma_0 mc)\left[\frac{\frac{1}{2}\left(\frac{e\boldsymbol{A}(\eta)}{\gamma_0 mc^2}\right)^2 + \left(\frac{e\boldsymbol{A}(\eta)}{\gamma_0 mc^2}\right)\cdot\boldsymbol{\beta}_0}{1-\hat{\boldsymbol{k}}\cdot\boldsymbol{\beta}_0}\right] \tag{14}$$

and

$$E(\eta) = \gamma_0 mc^2\left[1 + \frac{\frac{1}{2}\left(\frac{e\boldsymbol{A}(\eta)}{\gamma_0 mc^2}\right)^2 + \left(\frac{e\boldsymbol{A}(\eta)}{\gamma_0 mc^2}\right)\cdot\boldsymbol{\beta}_0}{1-\hat{\boldsymbol{k}}\cdot\boldsymbol{\beta}_0}\right]. \tag{15}$$

Note that at this point, in the absence of the radiation field, equations (14) and (15) reduce to the initial momentum $\gamma_0 m\boldsymbol{v}_0$ and initial energy $\gamma_0 mc^2$, respectively, as they should. In the limit of zero initial velocity, those equations reproduce their counterparts elsewhere [9], as expected. An expression for the electron kinetic energy, $K = (\gamma - 1)mc^2$, follows immediately from (15)

$$K(\eta) = K_0 + \frac{1}{2}\left(\frac{\gamma_0 mc^2}{1-\hat{\boldsymbol{k}}\cdot\boldsymbol{\beta}_0}\right)\left[\left(\frac{e\boldsymbol{A}(\eta)}{\gamma_0 mc^2}+\boldsymbol{\beta}_0\right)^2 - \boldsymbol{\beta}_0^2\right] \tag{16}$$

where $K_0 = (\gamma_0 - 1)mc^2$ is the initial kinetic energy with which the electron is injected into the laser beam.

The following laboratory trajectory equation may now be obtained from equation (4) as

$$\boldsymbol{r}(\eta) = \boldsymbol{r}_0 + \frac{c}{\omega}\int_{\eta_0}^{\eta}\left[\frac{\gamma_0 mc\boldsymbol{\beta}_0 + \frac{e}{c}\boldsymbol{A}(\eta')}{\gamma_0 mc\left(1-\hat{\boldsymbol{k}}\cdot\boldsymbol{\beta}_0\right)}\right]\mathrm{d}\eta'$$
$$+\hat{\boldsymbol{k}}\left(\frac{c}{\omega}\right)\int_{\eta_0}^{\eta}\left[\frac{\frac{1}{2}\left(\frac{e\boldsymbol{A}(\eta')}{\gamma_0 mc^2}\right)^2 + \left(\frac{e\boldsymbol{A}(\eta')}{\gamma_0 mc^2}\right)\cdot\boldsymbol{\beta}_0}{(1-\hat{\boldsymbol{k}}\cdot\boldsymbol{\beta}_0)^2}\right]\mathrm{d}\eta'. \tag{17}$$

Differentiating (17) once with respect to the time variable and rearranging, we get the following expression for the electron normalized velocity, $\boldsymbol{\beta}\equiv\boldsymbol{v}/c$, at any space-time point

$$\boldsymbol{\beta}(\eta) = \frac{\left\{\boldsymbol{\beta}_0 + \frac{e}{\gamma_0 mc^2}\boldsymbol{A}(\eta) + \hat{\boldsymbol{k}}\left[\frac{\frac{1}{2}\left(\frac{e\boldsymbol{A}(\eta)}{\gamma_0 mc^2}\right)^2 + \left(\frac{e\boldsymbol{A}(\eta)}{\gamma_0 mc^2}\right)\cdot\boldsymbol{\beta}_0}{1-\hat{\boldsymbol{k}}\cdot\boldsymbol{\beta}_0}\right]\right\}}{\left\{1 + \left[\frac{\frac{1}{2}\left(\frac{e\boldsymbol{A}(\eta)}{\gamma_0 mc^2}\right)^2 + \left(\frac{e\boldsymbol{A}(\eta)}{\gamma_0 mc^2}\right)\cdot\boldsymbol{\beta}_0}{1-\hat{\boldsymbol{k}}\cdot\boldsymbol{\beta}_0}\right]\right\}}. \tag{18}$$

A more straightforward way of obtaining (18) would be to use (14) and (15) and the relation $\boldsymbol{\beta} = c\boldsymbol{p}/E$. Expressions (17) and (18) look deceptively simple. Dependence, in both expressions, on the space and time coordinates is implicit through η. Fortunately, this causes no inconvenience in most cases of interest to us in this paper, as will be shown via the examples we take up in section 3.

3. Applications

Equations (14)–(18) have a wide range of applications. They are currently being used [10] as the starting point for a relativistic study of harmonic generation by the scattering of intense (plane wave) radiation by fast free electrons. However, we utilize them in this section for an analytical study of the dynamics of a relativistic electron in an intense ultrashort laser pulse. In the past, this has been the subject of theoretical study mostly by computer simulations [7]. Only recently, an analytical study leading to a clear demonstration of, among other things, the phenomenon of *ponderomotive scattering* has been advanced by Hartemann *et al* [6].

3.1. Finite-range pulse

As has been remarked above, our equations take care of both the electric and magnetic effects in a unified way through the use of the vector potential. A single integration of the electric field employed by Hartemann and collaborators, in the plane wave case, gives the vector potential $\boldsymbol{A} = \hat{\boldsymbol{\imath}}(cE_0/\omega)(\cos\eta - 1)$. When this form for the vector potential is used in our equations (15)–(18), all of the results reported in [6] are obtained with a minimum of effort. Moreover, all of their results for the finite-duration pulse follow identically from the following vector potential

$$\boldsymbol{A}(\eta) = \hat{\boldsymbol{\imath}}(cE_0/\omega)\left[\cos\eta + \frac{b^2 - b\sin\eta\sin b\eta - \cos\eta\cos b\eta}{1-b^2}\right] \tag{19}$$

where $b = \pi/(\omega\tau)$ and τ is the pulse duration. This, too, may be arrived at by a single integration of the pulse $\boldsymbol{E} = \hat{\boldsymbol{\imath}}E_0\sin^2(b\eta/2)\sin\eta$, employed by Hartemann and his coworkers.

3.2. Infinite-range pulse

Building upon the success of our equations to reproduce the recent results of Hartemann *et al* [6], we employ the same set of equations, in the remainder of this section, to investigate analytically the dynamics of a free electron as it is overtaken by an ultrashort laser pulse polarized along the x-axis. The pulse will be modelled by the vector potential

$$\boldsymbol{A}(\eta) = \hat{\imath} a e^{-\kappa|\eta|} \cos\eta \tag{20}$$

where $\hat{\imath}$ is a unit vector in the direction of polarization (x-axis), a is the peak amplitude, $\kappa = 1/\omega\tau$, τ is the pulse duration and $\eta = \omega(t - z(\eta)/c)$ is the invariant phase. By making this choice, we have assumed the pulse propagation direction to be along the coordinate z-axis. We will also take the electron to be initially ($\eta \to -\infty$) moving to the right along the same axis with speed $c\beta_0$. The term *initial*, in this context, will be taken to mean $t \to -\infty$. At this point in time the z-coordinate of the electron is meant, in a loose sense, to be finite, large and negative so that the combination $\omega(t - z/c) = \eta$ remains to be $-\infty$. In all our analysis and discussion, however, the word *initial* will refer to the condition $\eta \to -\infty$. Most of the details of the calculation pertaining to the case of initial electron motion parallel to that of the propagation direction of the field will be given. Equations needed to handle the case in which initial motion is antiparallel to the field propagation direction may be obtained from their counterparts of the parallel case by simply changing the sign of the initial speed β_0 of the electron everywhere.

The pulse-shape function employed in our vector potential has been used in the past by Eberly and Sleeper [8] and by Kibble [11] to study the dynamics of an electron initially at rest at the origin. We show the normalized vector potential, A/a, as a function of the phase η, over 30 field cycles, in figure 1.

In the following analysis, the dimensionless intensity parameter defined by $\alpha \equiv ea/mc^2$ will be employed. Direct substitution of equation (20) into (15) yields immediately

$$E(\eta) = \gamma_0 mc^2[1 + \tfrac{1}{2}\alpha^2(1+\beta_0)\cos^2\eta e^{-2\kappa|\eta|}]. \tag{21}$$

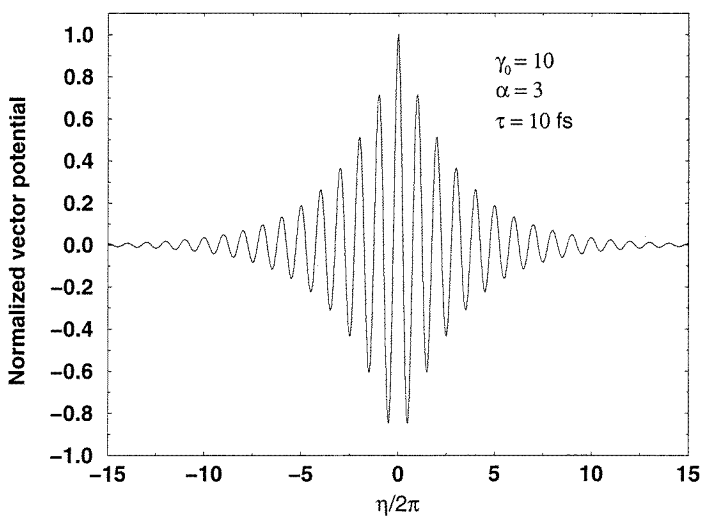

Figure 1. The normalized vector potential, $\cos\eta\exp(-\kappa|\eta|)$, versus the invariant phase $\eta/2\pi$ for $\gamma_0 = 10, \alpha = 3$, $\lambda = 1$ μm, and $\tau = 10$ fs.

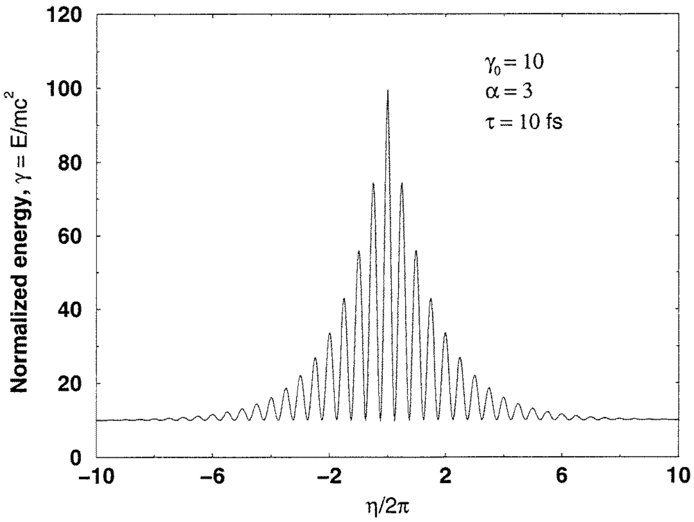

Figure 2. The normalized electron energy, E/mc^2, versus the invariant phase $\eta/2\pi$ using the same set of parameters as in figure 1.

The normalized energy, in units of the rest energy of the electron mc^2 is also shown in figure 2 as a function of $\eta/2\pi$.

The position vector of the electron follows by using (20) in (17). Employing the phase η as a parameter, the following equations give the trajectory of the electron, in presence of the radiation pulse, in parametric form. First, the transverse motion is given by

$$x(\eta) = \frac{c\alpha I(\eta)}{\omega\gamma_0(1-\beta_0)} \tag{22}$$

and

$$y(\eta) = 0. \tag{23}$$

The longitudinal motion, on the other hand, is given by

$$z(\eta) = \frac{c}{\omega(1-\beta_0)}\left[\beta_0\eta + \frac{1}{2}\alpha^2(1+\beta_0)J(\eta)\right]. \tag{24}$$

Hence, motion of the electron is confined to the xz-plane, which contains the polarization and propagation vectors of the radiation field. The integrals I and J in (22) and (24) are evaluated in the appendix. Note that equations (22)–(24) are consistent with the initial conditions $x(-\infty) = y(-\infty) = 0$ and $z(-\infty) = -\infty$.

We show the actual electron trajectory in figure 3. The trajectory is shown over values of $\eta/2\pi$ ranging from −15 to 15, a total of 30 field cycles. Notice the small exit value of the transverse coordinate, about which more will be found in section 4.

Finally, when the vector potential (20) is used in the expression for velocity, equation (18), the following transverse and longitudinal velocity components follow immediately

$$\beta_x(\eta) = \frac{(2\alpha/\gamma_0)\cos\eta e^{-\kappa|\eta|}}{2+\alpha^2(1+\beta_0)\cos^2\eta e^{-2\kappa|\eta|}} \tag{25}$$

and

$$\beta_z(\eta) = 1 - \frac{2(1-\beta_0)}{2+\alpha^2(1+\beta_0)\cos^2\eta e^{-2\kappa|\eta|}}. \tag{26}$$

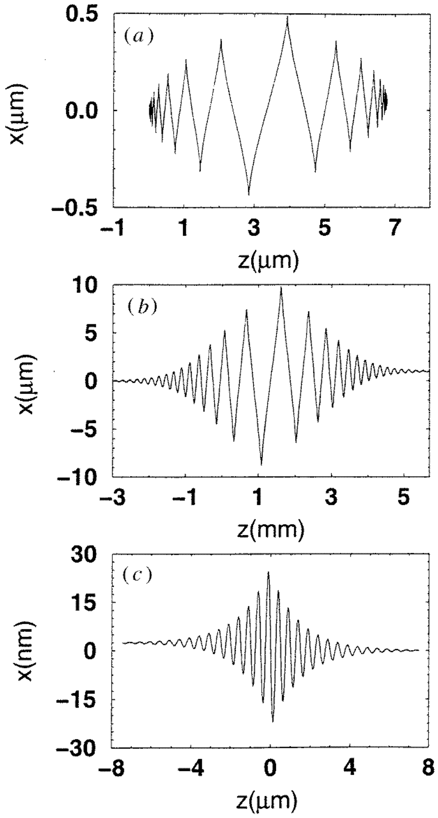

Figure 3. Electron trajectory in the xz-plane in the presence of a laser pulse whose parameters are the same as those in figure 1. (a) The electron initially at rest, (b) the electron initially moving along the laser propagation direction, and (c) the electron initially moving opposite the direction of laser propagation.

Note that (25) and (26) are consistent with the initial conditions on the velocity, namely, $\beta_x(-\infty) = 0$ and $\beta_z(-\infty) = \beta_0$. In figures 4 and 5, we plot the velocity components against the invariant phase.

4. Results and discussion

The set of equations, presented in section 2 and applied in section 3, are of a general nature. They are capable of handling motion of a charged particle in the presence of a laser field of arbitrary intensity and arbitrary polarization, and for any set of physically realizable initial conditions on the motion of the electron. They accomplish that with a minimum of effort because they employ the vector potential and hence involve at most a single integration over the invariant phase. In arriving at our equations, no approximation has been made, other than the neglect of radiation reaction [9].

The equations have been used to treat a specific example in section 3 with the main results displayed in figures 1–6. In all these figures, the laser pulse duration is $\tau = 10$ fs, the wavelength is $\lambda = 1$ μm (hence, $\kappa = \frac{1}{6}\pi$) and the intensity parameter $\alpha = 3$. The initial normalized speed of the electron, β_0, is calculated from an initial injection energy

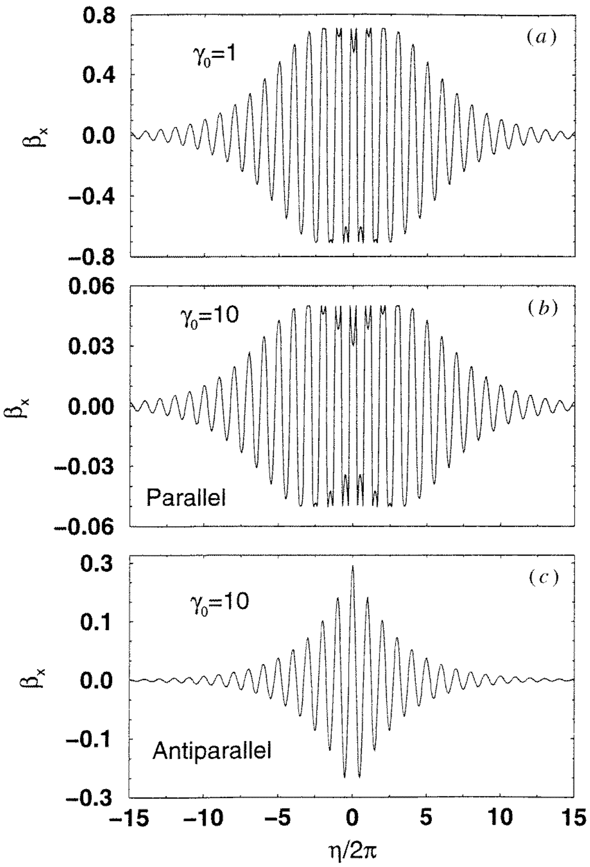

Figure 4. The normalized transverse velocity component β_x versus the invariant phase $\eta/2\pi$. (*a*) The electron initially at rest, $\gamma_0 = 1$, (*b*) the parallel case with $\gamma_0 = 10$, and (*c*) the antiparallel case with $\gamma_0 = 10$. Parameters of the laser pulse used here are also the same as in figure 1.

corresponding to $\gamma_0 = 10$, where applicable. In the following discussions, distinction will be made between four different initial conditions on the velocity of the electron (initial geometries). The case of an electron initially resting at the origin will be referred to as *the case at rest*. The case of an electron initially moving parallel to the direction of laser propagation will be called *the parallel case*. Similarly, we will talk about an *antiparallel case* and a *perpendicular case*, as well.

Of particular interest to us in this paper is the net transverse displacement suffered by the electron as a result of its interaction with the laser pulse. We are talking here about the small finite value the coordinate x has after the left end of the pulse has passed by (see figure 3). This displacement can be analytically calculated from equation (22) as

$$\begin{aligned}\Delta x &= x(\infty) - x(-\infty)\\ &= \frac{2c\alpha\kappa}{\omega(1+\kappa^2)}\sqrt{\frac{1+\beta_0}{1-\beta_0}}.\end{aligned} \qquad (27)$$

For the case at rest, equation (27) yields the value $\Delta x \approx 0.5\ \mu$m for the set of parameters given above (see figure 3(*a*)). For the parallel case, the same equation yields $\Delta x \approx 1\ \mu$m

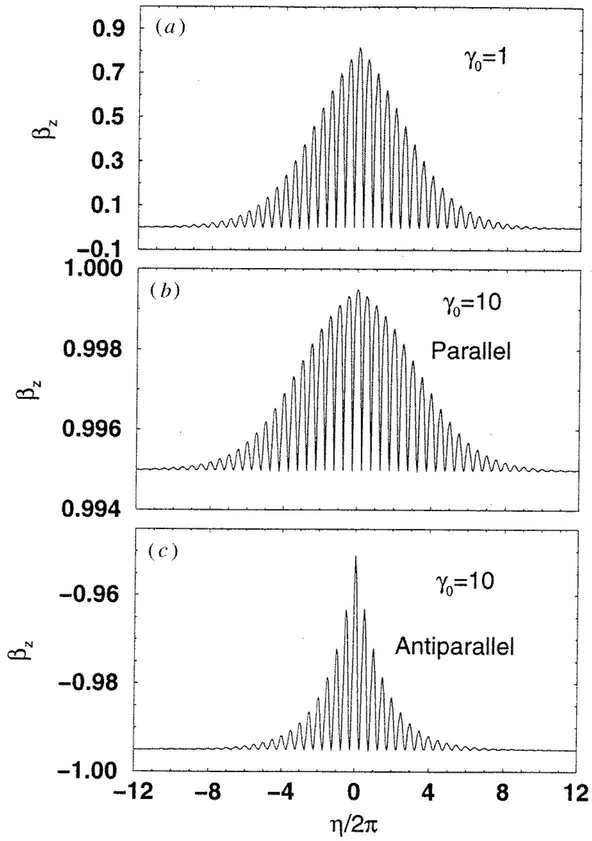

Figure 5. The normalized longitudinal velocity component β_z versus the invariant phase $\eta/2\pi$. (a) The electron initially at rest, $\gamma_0 = 1$, (b) the parallel case with $\gamma_0 = 10$, and (c) the antiparallel case with $\gamma_0 = 10$. Parameters of the laser pulse used here are also the same as in figure 1.

(see also figure 3(b)). This value agrees quite well with the results obtained by Hartemann *et al* [6] who, as has been mentioned above, used a different approach with a different pulse-shape function.

Similarly, an electron initially at rest at the origin suffers a net longitudinal displacement given by

$$\Delta z = z(\infty) - z(-\infty) = \frac{c\alpha^2}{4\omega\kappa}\left[\frac{1+2\kappa^2}{1+\kappa^2}\right]. \tag{28}$$

Using the same set of parameters as before, equation (28) gives $\Delta z \approx 6.8\ \mu$m (see figure 3($a$)).

Another interesting result that emerges from equations (25) and (26) is the angle, relative to the laser beam primary direction of propagation, at which the electron moves during interaction with the field. This angle, in the present case, is given by

$$\theta = \tan^{-1}\frac{\beta_x}{\beta_z}$$

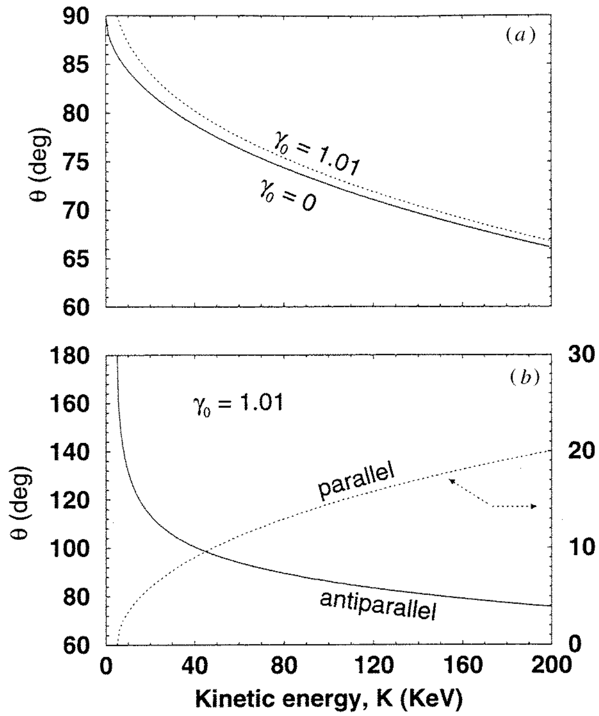

Figure 6. (a) The angle θ, giving the instantaneous direction of electron motion relative to the laser propagation direction, in degrees, versus the electron kinetic energy, $(\gamma - 1)mc^2$, in KeV corresponding to the initial condition of rest at the origin (full curve) and initial electron motion perpendicular to the laser propagation vector and initial kinetic energy $K_0 = 0.01mc^2$ (dotted curve). (b) The same as (a), but for the parallel and antiparallel initial geometries and $K_0 = 0.01mc^2$.

$$= \tan^{-1}\left[\frac{(2\alpha/\gamma_0)\cos\eta e^{-\kappa|\eta|}}{2\beta_0 + \alpha^2(1+\beta_0)\cos^2\eta e^{-2\kappa|\eta|}}\right]. \qquad (29)$$

Note that, with the diameter of the focal spot of the laser pulse properly included in the field amplitude in (20), θ becomes the angle at which the electron is scattered by the ponderomotive force of the laser field, provided the field intensity is high enough to make the amplitude of transverse motion comparable with or greater than the diameter [6]. For the parallel case, an equation giving the same angle as a function of the electron kinetic energy may be obtained from (21) and (29). Writing $E = \gamma mc^2$ in equation (21) gives

$$\gamma = \gamma_0\left(1 + \tfrac{1}{2}\alpha^2\cos^2\eta e^{-2\kappa|\eta|}\right). \qquad (30)$$

Now, when (30) is used in (29), there results

$$\tan\theta = \frac{\sqrt{2(1-\beta_0)(\gamma/\gamma_0 - 1)}}{\beta_0 + (\gamma/\gamma_0 - 1)}. \qquad (31)$$

This equation was obtained by Hartemann *et al* [6] by using a different approach and a different pulse-shape function. Note that the corresponding equation, for the antiparallel case, is obtained from (31) simply by reversing the sign of β_0. Moreover, starting with $\boldsymbol{\beta}_0 = \hat{\imath}\beta_0$, and following a similar procedure, a few lines of algebra yield an equation for the angle θ in the perpendicular case. The result is

$$\tan\theta = \frac{\sqrt{\beta_0^2 + 2(\gamma/\gamma_0 - 1)}}{(\gamma/\gamma_0 - 1)}. \tag{32}$$

In fact, a less straightforward analysis which proceeds without assuming a specific pulse-shape function [12], leads to an equation of which (31) and (32) are special cases. If the electron is assumed to have been injected at the angle θ_0 relative to the direction of laser propagation (in the xz-plane), the following equation results

$$\tan\theta = \frac{\sqrt{\beta_0^2 \sin^2\theta_0 + 2(1 - \beta_0\cos\theta_0)(\gamma/\gamma_0 - 1)}}{\beta_0\cos\theta_0 + (\gamma/\gamma_0 - 1)}. \tag{33}$$

Note at this point that for an electron initially at rest, $\beta_0 = 0$ and $\gamma_0 = 1$, equations (31)–(33) reduce to

$$\tan\theta = \sqrt{\frac{2}{\gamma - 1}}. \tag{34}$$

This equation has been arrived at from conservation of energy considerations by Reiss [13] within the context of a quantum mechanical study of atomic above-threshold ionization. The experimental results of Moore and his coworkers [2] and Meyerhofer *et al* [3] are consistent with this equation.

In figures 6–8, we show the angle θ versus the instantaneous kinetic energy, $K = (\gamma - 1)mc^2$, for a few initial geometries and initial electron injection energies. figure 6(*a*) shows the cases of an electron produced almost at rest near the focus of the pulse [2, 3], and one that is injected there with the initial kinetic energy $K_0 \approx 5.11$ KeV ($\gamma_0 = 1.01$) perpendicular to the laser propagation direction. The parallel and antiparallel cases are shown in figure 6(*b*) for $\gamma_0 = 1.01$. Similar plots are also shown in figures 7 and 8 for injection energies in the MeV and GeV ranges, respectively.

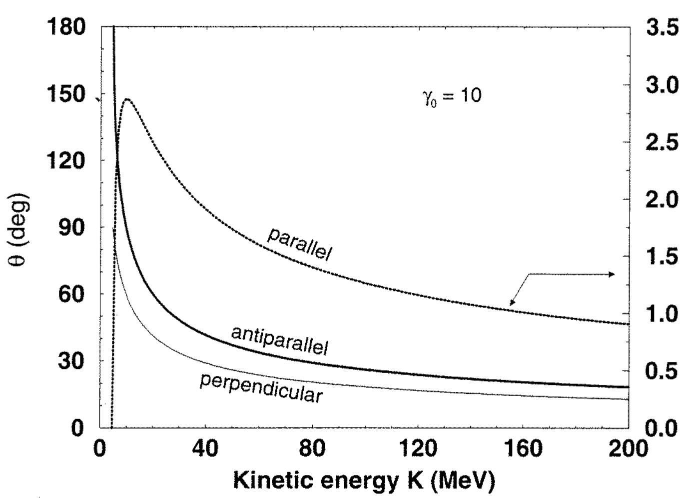

Figure 7. The same as in figure 6, but for the initial conditions of electron motion parallel, antiparallel and perpendicular to the dirction of pulse propagation. For all curves shown here, $\gamma_0 = 10$ and K is in MeV.

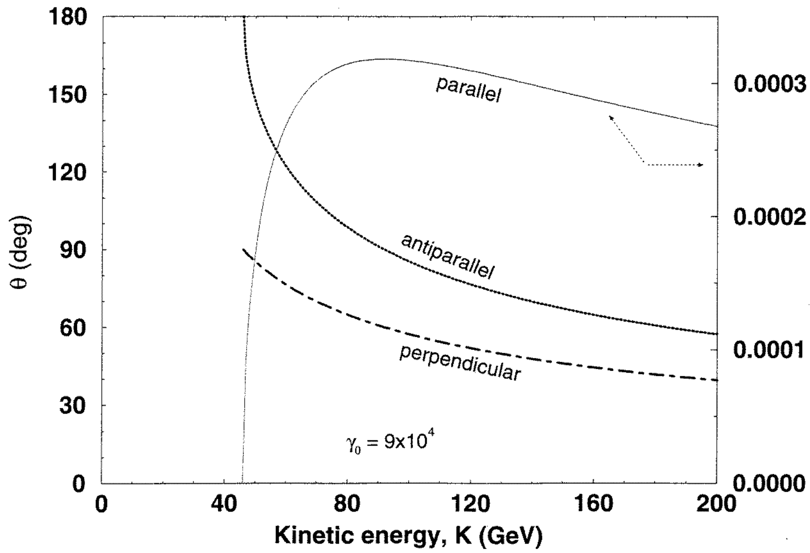

Figure 8. The same as in figure 6, but for initial conditions of electron motion parallel, antiparallel and perpendicular to the dirction of pulse propagation. For all curves shown here, $\gamma_0 = 9 \times 10^4$ and K is in GeV.

5. Concluding remarks

We have presented an explicit analytical derivation for the dynamics of a relativistic electron in the presence of an arbitrary laser field, assuming general conditions on the initial motion of the electron. Our main results have been applied for the investigation of electron motion in a laser pulse. In addition to the case of an electron produced almost at rest near the focal spot, the cases of initial electron motion parallel, antiparallel and perpendicular to the propagation direction of a super-intense, linearly polarized laser pulse of 10 fs duration were considered. Some of the analytical results reported recently by Hartemann *et al* [6] have been confirmed. The expression for the angle θ relative to the laser direction of propagation as a function of the electron kinetic energy, which predicts the recent experimental results of Moore *et al* [2] and Meyerhofer and his coworkers [3], has also been shown to follow from more general ones. Finally, we have presented predictions for the same angle as a function of the kinetic energy, corresponding to injection energies in the MeV and GeV ranges, in anticipation of results from recently proposed laser-accelerator experiments [14].

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Appendix. $I(\eta)$ and $J(\eta)$

The integrals $I(\eta)$ and $J(\eta)$ appearing in the parametric equations (22) and (24) are given below. The starting point for the evaluation of a typical integral is to express the trigonometric function in terms of exponentials.

- For $\eta < 0$

$$I(\eta) = \int_{-\infty}^{\eta} \cos\eta' e^{\kappa\eta'}\, \mathrm{d}\eta' = \frac{\kappa\cos\eta + \sin\eta}{1+\kappa^2} e^{\kappa\eta} \tag{A1}$$

and

$$J(\eta) = \int_{-\infty}^{\eta} \cos^2\eta' e^{2\kappa\eta'}\, \mathrm{d}\eta' = \frac{[1 + 2\kappa^2\cos^2\eta + \kappa\sin 2\eta]}{4\kappa(1+\kappa^2)} e^{2\kappa\eta}. \tag{A2}$$

- For $\eta > 0$

$$I(\eta) = \int_{0}^{\eta} \cos\eta' e^{-\kappa\eta'}\, \mathrm{d}\eta' = \frac{2\kappa + (-\kappa\cos\eta + \sin\eta)e^{-\kappa\eta}}{1+\kappa^2} \tag{A3}$$

and

$$J(\eta) = \int_{0}^{\eta} \cos^2\eta' e^{-2\kappa\eta'}\, \mathrm{d}\eta' = \frac{2(1+2\kappa^2) - (1 + 2\kappa^2\cos^2\eta - \kappa\sin 2\eta)e^{-2\kappa\eta}}{4\kappa(1+\kappa^2)}. \tag{A4}$$

In evaluating I and J for $\eta > 0$, a constant has been added to each, whose value has subsequently been found from matching the respective integrals at $\eta = 0$.

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