Journal of Difference Equations and Applications

ISSN: 1023-6198 (Print) 1563-5120 (Online) Journal homepage: https://www.tandfonline.com/loi/gdea20

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To cite this article: Raghib M. Abu-Saris & Neda'a K. Al-Jubouri (2004) Characterization of Rational Periodic Sequences II, Journal of Difference Equations and Applications, 10:4, 409-418, DOI: 10.1080/10236190410001647799

To link to this article: https://doi.org/10.1080/10236190410001647799

Published online: 16 Aug 2006.

Journal of Difference Equations and Applications,
VOL. 10, No. 4, 10 April 2004, pp. 409–418

Characterization of Rational Periodic Sequences II

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(Received 9 August 2003; In final form 28 October 2003)

In this research, we continue our investigation of the difference equation:

$$x_{n+1} = \frac{f(x_n)}{x_{n-1}},$$

and characterize those functions for which all solutions of the above-mentioned difference equation are periodic of the same fundamental period p. In particular, we address the cases $p = 3, 4, 5, 6$.

Keywords: Rational difference equation; Periodicity and periodic solution; Periodic of prime period; Rational periodic sequences

Mathematics Subject Classification: 39A10

INTRODUCTION

Consider the rational difference equation

$$x_{n+1} = \frac{f(x_n)}{x_{n-1}}, \quad n = 0, 1, 2, \ldots \tag{1}$$

where $f(0,\infty) \subseteq (0,\infty)$ and x_{-1}, x_0 are positive real numbers.

Our objective in this article is to completely characterize those functions with minimal assumptions of continuity and smoothness so that every solution of Eq. (1) is periodic of prime period $p = 3, 4, 5$ and 6. This paper is organized as follows.

In second section, we consider the cases $p = 3$ and $p = 4$. The cases $p = 5$ and $p = 6$ are considered in third and fourth sections, respectively. The main results in the present work are Theorems 2.1, 2.2, 3.1, 3.2, 4.1 and 4.2. These numbers are on section basis.

Our work horse in this research is the following result which was established in Ref. [2]:

THEOREM (NSP) *A solution* $\{x_n\}_{n=-1}^{\infty}$ *of Eq.* (1) *is periodic of period p if and only if*

$$\prod_{j=0}^{p-1} \frac{f(x_j)}{x_j^2} = 1,$$

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Journal of Difference Equations and Applications
ISSN 1023-6198 print/ISSN 1563-5120 online
http://www.tandf.co.uk/journals
DOI: 10.1080/10236190410001647799

and

$$\prod_{j=0}^{p-1}\left(\frac{f(x_j)}{x_j^2}\right)^{p-j} = \left(\frac{x_{-1}}{x_0}\right)^p.$$

THE CASES: $p = 3$ AND $p = 4$

In this section, we characterize completely all functions f for which all solutions of Eq. (1) are periodic of prime periods 3 and 4.

THEOREM 2.1 *Every solution of Eq.* (1) *is periodic of prime period* $p = 3$ *if and only if* $f(x) = c/x$ *for some constant c.*

Proof *Suppose that every solution of Eq.* (1) *is periodic of prime period* $p = 3$. *By Theorem (NSP) we have*

$$\frac{f(x_0)}{x_0^2}\frac{f(x_1)}{x_1^2}\frac{f(x_{-1})}{x_{-1}^2} = 1,$$

and

$$\left(\frac{f(x_0)}{x_0^2}\right)^3\left(\frac{f(x_1)}{x_1^2}\right)^2\left(\frac{f(x_{-1})}{x_{-1}^2}\right) = \left(\frac{x_{-1}}{x_0}\right)^3.$$

This implies that

$$\frac{f(x_0)/x_0^2}{f(x_{-1})/x_{-1}^2} = \left(\frac{x_{-1}}{x_0}\right)^3,$$

and so we have

$$x_0 f(x_0) = x_{-1} f(x_{-1}).$$

Since x_0 *is independent of* x_{-1}, *we must have* $f(x) = c/x$.
The proof of the converse is straight forward and will be omitted. □

Remark 2.1 It is worth mentioning that Theorem 2.1 answers Open Problem 3.4.4 in Ref. [6].

THEOREM 2.2 *Every solution of Eq.* (1) *is periodic of prime period* $p = 4$ *if and only if*

$$f\left(\frac{f(x)}{y}\right) = f(y).$$

In particular, if f is continuous, then $f(x) = c$ *for some constant c.*

Proof *Suppose that every solution of Eq.* (1) *is periodic of prime period* $p = 4$. *By Theorem (NSP) we have*

$$\frac{f(x_0)}{x_0^2}\frac{f(x_1)}{x_1^2}\frac{f(x_2)}{x_2^2}\frac{f(x_{-1})}{x_{-1}^2} = 1,$$

and

$$\left(\frac{f(x_0)}{x_0^2}\right)^4\left(\frac{f(x_1)}{x_1^2}\right)^3\left(\frac{f(x_2)}{x_2^2}\right)^2\left(\frac{f(x_{-1})}{x_{-1}^2}\right)=\left(\frac{x_{-1}}{x_0}\right)^4.$$

This implies that

$$\frac{(f(x_0)/x_0^2)^2\,(f(x_1)/x_1^2)}{f(x_{-1})/x_{-1}^2}=\left(\frac{x_{-1}}{x_0}\right)^4,$$

and so, using Eq. (1) *and simplifying, we have*

$$f(f(x_0)/x_{-1})=f(x_{-1}).$$

Since x_0 is independent of x_{-1} and f is continuous, we must have $f(x)=c$.
Again, the proof of the converse is straight forward and will be omitted. □

Remark 2.2 The continuity assumption is essential for the result of Theorem 2.2 to hold. To see this, let $\mathbb{Q}$ denote the set of rational numbers, and consider the following function:

$$f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 2 & \text{if } x\notin\mathbb{Q}\end{cases}$$

It is not difficult to see for this f every solution of Eq. (1) is periodic of prime period 4, yet f is not a constant function.

THE CASE $p = 5$

In this section we investigate the situation when all solutions of Eq. (1) are periodic of prime period 5. Unlike the previous cases, continuity on $(0,\infty)$ or $[0,\infty)$ is not enough to single out a function.

THEOREM 3.1 *Every solution of Eq.* (1) *is periodic of prime period $p=5$ if and only if*

$$\frac{1}{x}f\left(\frac{f(x)}{y}\right)=\frac{1}{y}f\left(\frac{f(y)}{x}\right) \tag{2}$$

for all $x,y\in(0,\infty)$.

Proof *Suppose that condition* (2) *holds. Then, by Eq.* (1)*, for all $x_{-1},x_0\in(0,\infty)$ we have*

$$x_1=\frac{f(x_0)}{x_{-1}},$$

$$x_2=\frac{1}{x_0}f\left(\frac{f(x_0)}{x_{-1}}\right)=\frac{1}{x_{-1}}f\left(\frac{f(x_{-1})}{x_0}\right),$$

$$x_3=\frac{1}{f(x_0)/x_{-1}}f\left(\frac{f(t)}{x_{-1}}\right),\quad t=\frac{f(x_{-1})}{x_0}$$

$$=\frac{t/x_{-1}}{f(x_0)/x_{-1}}f\left(\frac{f(x_{-1})}{t}\right)=\frac{f(x_{-1})}{x_0}=t,$$

$$x_4 = \frac{f(t)}{f(t)/x_{-1}} = x_{-1},$$

$$x_5 = \frac{f(x_{-1})}{t} = x_0.$$

Hence, every solution is periodic of prime period 5.

Conversely, suppose that every solution of Eq. (1) *is periodic of prime period 5. Then, by Theorem (NSP), we have*

$$\frac{f(x_0)}{x_0^2}\frac{f(x_1)}{x_1^2}\frac{f(x_2)}{x_2^2}\frac{f(x_3)}{x_3^2}\frac{f(x_{-1})}{x_{-1}^2} = 1,$$

and

$$\left(\frac{f(x_0)}{x_0^2}\right)^5\left(\frac{f(x_1)}{x_1^2}\right)^4\left(\frac{f(x_2)}{x_2^2}\right)^3\left(\frac{f(x_3)}{x_3^2}\right)^2\left(\frac{f(x_{-1})}{x_{-1}^2}\right) = \left(\frac{x_{-1}}{x_0}\right)^5.$$

This implies that

$$\frac{\left(\frac{f(x_0)}{x_0^2}\right)^2\frac{f(x_1)}{x_1^2}}{\frac{f(x_3)}{x_3^2}\left(\frac{f(x_{-1})}{x_{-1}^2}\right)^2} = \left(\frac{x_{-1}}{x_0}\right)^5.$$

But $x_1 = f(x_0)/x_{-1}$ *and* $x_3 = f(x_{-1})/x_0$. *Hence, after some simplifications, one can see that* f *have to satisfy*

$$\frac{1}{x_0} f\left(\frac{f(x_0)}{x_{-1}}\right) = \frac{1}{x_{-1}} f\left(\frac{f(x_{-1})}{x_0}\right),$$

i.e. condition (2) *holds. This completes the proof.* □

Remark 3.1 Notice that Theorem 3.1 states that if $g(x,y) = f(f(x)/y)/x$ is symmetric, i.e. $g(x,y) = g(y,x)$, then every solution of Eq. (1) is periodic of prime period 5. This is an efficient criterion that can be easily applied. For example, it is not difficult to see that this symmetric property is possessed by the functions $1+x$, $\max\{1,x\}$, $\min\{1,x\}$, $x^{-(1+\sqrt{5})/2}$. These functions were studied respectively, in Refs. [5, 4, 3, 2].

THEOREM 3.2 *Suppose that* $f \in C^1[[0,\infty), [0,\infty)]$ *such that* $f(0) = 1$ *and* $f'(0) \neq 0$. *Then every solution of Eq.* (1) *is periodic of prime period 5 if and only if* $f(x) = x+1$. *In this case, Eq.* (1) *reduces to the well known Lyness Equation* [5].

To prove Theorem 3.2, we need several results that will be established below.

An immediate corollary of Theorem 3.1 is the following result:

COROLLARY 3.1 *Let* $b = f(1)$, *and suppose that every solution of Eq.* (1) *is periodic of prime period 5. Then*

(a) $\frac{f(f(x))}{x} = f\left(\frac{b}{x}\right)$, $x \in (0,\infty)$,
(b) $f(x) = \frac{x}{b} f\left(f\left(\frac{b}{x}\right)\right)$, $x \in (0,\infty)$, *and*
(c) $f(f(b)) = b^2$.

Corollary 3.1, together with the continuity assumption, imply that

COROLLARY 3.2 *Let $a = f(0)$, $b = f(1)$, and assume that f is continuous on* [0,∞). *If every solution of Eq.* (1) *is periodic of prime period 5, then*

(a) $\lim_{x\to\infty} \frac{f(x)}{x} = \frac{f(a)}{b}$, *and*
(b) $\lim_{x\to\infty} \frac{f(f(x))}{x} = a$.

Using Theorem 3.1, *Corollary* 3.2 *Part* (a), *and the continuity of f on* [0, ∞) *we obtain:*

LEMMA 3.1 *Suppose that $f \in C[[0,\infty),[0,\infty)]$. If every solution of Eq.* (1) *is periodic of prime period 5, then*

$$\frac{f(a)}{b} f(x) = x f\left(\frac{a}{x}\right) \quad \text{for all } x > 0$$

where $a = f(0)$, and $b = f(1)$.

Proof *By Theorem* 3.1, *we have*

$$x f\left(\frac{f(y)}{x}\right) = f(x) \frac{f(f(x)/y)}{f(x)/y}.$$

Thus, by Corollary 3.2 *Part (a), we obtain for each $x > 0$:*

$$x f\left(\frac{a}{x}\right) = \lim_{y\to 0^+} x\, f\left(\frac{f(y)}{x}\right) = f(x) \quad \lim_{y\to 0^+} \frac{f(f(x)/y)}{f(x)/y} = f(x) \frac{f(a)}{b}$$

which is the required result. □

Now, we turn back to prove Theorem 3.2.

Proof of Theorem 3.2 *First, by Lemma* 3.1, *f(x) satisfies the functional equation*

$$f(x) = x f\left(\frac{1}{x}\right).$$

Differentiating the above relations, we see that f satisfies the differential equation:

$$x f'(x) - f(x) = -f'\left(\frac{1}{x}\right), \quad x > 0. \tag{3}$$

Thus, since $f(0) = 1$ and $f \in C^1[0,\infty)$, taking the limit as $x \to 0^+$, we conclude that

$$\lim_{x\to\infty} f'(x) = 1.$$

Second, multiplying Eq. (2) *by xy, and then differentiating with respect to x, we obtain*

$$f\left(\frac{f(y)}{x}\right) - \frac{f(y)}{x} f'\left(\frac{f(y)}{x}\right) = f'\left(\frac{f(x)}{y}\right) f'(x), \quad \text{for all } x, y > 0. \tag{4}$$

But Eqs. (3) *and* (4) *together lead to the conclusion that*

$$f'\left(\frac{f(x)}{y}\right) f'(x) = f'\left(\frac{x}{f(y)}\right)$$

for all $x, y > 0$.
Finally, taking the limit as $y \to \infty$, and Corollary 3.2 *Part (a), we have*

$$f'(0)\ f'(x) = f'(0) \quad \text{for all } x > 0.$$

Hence,

$$f'(x) = 1, \quad \textit{for all } x > 0$$

Integrating and using the initial condition $f(0) = 1$, *we obtain the desired result. This completes the proof.* □

Remark 3.2 It is important to mention that Theorem 3.2 is a modified version of Conjecture 3.4.2 in Ref. [6]. Without an extra condition such as $f(0) = 1$, Conjecture 3.4.2 is not true. For a counter example, $f(x) = x/(x+1) \in C^1[0, \infty)$ for which every solution of Eq. (1) is periodic of prime period 5. Furthermore, this example shows that there is an error in Theorem 2 in the article [7].[1]

THE CASE $p = 6$

In this section, we proceed further and consider the case when all solutions of Eq. (1) are periodic of prime period 6. However, the analysis becomes more involved.

THEOREM 4.1 *Suppose that there exists no* $c > 0$ *such that* $f(x) = c/x$, $x \in (0, \infty)$. *Then every solution of Eq.* (1) *is periodic of prime period 6 if and only if*

$$f\left(\frac{1}{x} f\left(\frac{1}{y} f(x)\right)\right) = \frac{1}{y^2} f(x) f\left(\frac{1}{x} f(y)\right) \tag{5}$$

for all $x, y \in (0, \infty)$.

Proof *We follow the same line of reasoning used in the proof of Theorem* 3.1. *Therefore, we assume first that condition (5) holds. Then, by Eq. (1), for all* x_{-1}, $x_0 \in (0, \infty)$ *we have*

$$x_1 = \frac{f(x_0)}{x_{-1}},$$

$$x_2 = \frac{1}{x_0} f\left(\frac{f(x_0)}{x_{-1}}\right),$$

$$x_3 = \frac{1}{x_1} f\left(\frac{1}{x_0} f\left(\frac{f(x_0)}{x_{-1}}\right)\right) = \frac{1}{x_{-1}} f\left(\frac{1}{x_0} f(x_{-1})\right)$$

$$x_4 = \frac{f\left(\frac{1}{x_{-1}} f\left(\frac{1}{x_0} f(x_{-1})\right)\right)}{\frac{1}{x_0} f\left(\frac{1}{x_{-1}} f(x_0)\right)} = \frac{\frac{1}{x_0^2} f(x_{-1}) f\left(\frac{1}{x_{-1}} f(x_0)\right)}{\frac{1}{x_0} f\left(\frac{1}{x_{-1}} f(x_0)\right)} = \frac{f(x_{-1})}{x_0}$$

$$x_5 = \frac{f\left(\frac{1}{x_0} f(x_{-1})\right)}{\frac{1}{x_{-1}} f\left(\frac{1}{x_0} f(x_{-1})\right)} = x_{-1}$$

$$x_6 = \frac{f(x_{-1})}{f(x_{-1})/x_0} = x_0.$$

Hence, every solution is periodic of prime period 6.

[1] This was brought to our attention by the referee.

Now, suppose that every solution of Eq. (1) *is periodic of prime period* 6. *Then, by Theorem (NSP), we have*

$$\frac{f(x_0)}{x_0^2}\frac{f(x_1)}{x_1^2}\frac{f(x_2)}{x_2^2}\frac{f(x_3)}{x_3^2}\frac{f(x_4)}{x_4^2}\frac{f(x_{-1})}{x_{-1}^2}=1,$$

and

$$\left(\frac{f(x_0)}{x_0^2}\right)^6\left(\frac{f(x_1)}{x_1^2}\right)^5\left(\frac{f(x_2)}{x_2^2}\right)^4\left(\frac{f(x_3)}{x_3^2}\right)^3\left(\frac{f(x_4)}{x_4^2}\right)^2\left(\frac{f(x_{-1})}{x_{-1}^2}\right)=\left(\frac{x_{-1}}{x_0}\right)^6.$$

This implies that

$$\frac{\left(\frac{f(x_0)}{x_0^2}\right)^3\left(\frac{f(x_1)}{x_1^2}\right)^2\frac{f(x_2)}{x_2^2}}{\frac{f(x_4)}{x_4^2}\left(\frac{f(x_{-1})}{x_{-1}^2}\right)^2}=\left(\frac{x_{-1}}{x_0}\right)^6.$$

But periodicity implies that $x_1=f(x_0)/x_{-1}$ *and* $x_4=f(x_{-1})/x_0$. *Hence, after some simplifications, one can see that f satisfies*

$$x_{-1}^2\, f\left(\frac{1}{x_0}f\left(\frac{1}{x_{-1}}f(x_0)\right)\right)=f(x_0)f\left(\frac{1}{x_0}f(x_{-1})\right)$$

i.e. condition (5) *holds.*

The above argument assures that every solution is periodic of period 6. However, it may not be the minimal one. Since 6 is divisible by the minimal period, the minimal period can be 3.[2] *However, under the given assumption and in view of Theorem 2.1, this cannot happen. This completes the proof.* □

Remark 4.1 Once more, the continuity assumption on $[0,\infty)$ does not single out a unique function with the desired property. For example, it is not difficult to see that the functions x, $\max\{1,x^{\sqrt{2}}\}$, $\min\{1,x^{\sqrt{2}}\}$ satisfy condition (5). These functions are particular cases of more general classes of functions which were studied, respectively, in Refs. [2,1,3].

THEOREM 4.2 *Suppose that* $f\in C^1[0,\infty)$, *with* $f(0)=0$ *and* $f'(0)\neq 0$. *Then every solution of Eq.* (1) *is periodic of prime period 6 if and only if* $f(x)=A\,x$ *where A is an arbitrary positive real number.*
To prove Theorem 4.2, *we need several results that will be established below. The first one follows immediately from Theorem* 4.1.

COROLLARY 4.1 *Let* $b=f(1)$, *and suppose that every solution of Eq.* (1) *is periodic of prime period* 6. *Then*

(a) $f\left(\frac{f(f(x))}{x}\right)=f(x)\ f\left(\frac{b}{x}\right)$ *for all* $x>0$.
(b) $f\left(f\left(\frac{b}{x}\right)\right)=\frac{b}{x^2}f(f(x))$ *for all* $x>0$, *and*
(c) $f(f(b))=bf(b)$.

Corollary 4.1, together with the continuity assumption, imply that

[2] 2 is not a possible minimal period for this type of difference equation.

COROLLARY 4.2 *Let $a = f(0)$, $b = f(1)$, and assume that f is continuous on $[0,\infty)$. If every solution of Eq.* (1) *is periodic of prime period* 6, *then*

$$\lim_{x\to\infty} \frac{f(f(x))}{x^2} = \frac{f(a)}{b}.$$

Theorem 4.1 together with corollaries 4.1 and 4.2, lead to the following important Lemma:

LEMMA 4.1 *Suppose that $f \in C^1[0,\infty)$ with $f(0) = 0$ and $f'(0) \neq 0$. If every solution of Eq.* (1) *is periodic of prime period 6, then for $y > 0$,*

(a) $\lim_{x\to\infty} \frac{f(f(x))}{x} = \lim_{x\to 0} \frac{f(f(x))}{x} = [f'(0)]^2$.
(b) $\lim_{x\to\infty} f'(x)\, f'\left(\frac{f(x)}{y}\right) = \lim_{x\to 0} f'(x)\, f'\left(\frac{f(x)}{y}\right) = [f'(0)]^2$.
(c) $\lim_{x\to\infty} f(x)\, f\left(\frac{f(y)}{x}\right) = \lim_{x\to 0} f(x)\, f\left(\frac{f(y)}{x}\right) = y^2 f\left(\frac{[f'(0)]^2}{y}\right)$.

Proof

(a) The result follows from Corollary 4.1 Part (b), and L'Hopital's Rule.
(b) First, due to the similarity of the proofs of the two limits, we present the proof of the first one only.

Now, on the one hand, by L'Hopital's Rule, we have

$$\begin{aligned}\lim_{x\to\infty} x\, f\left(\frac{1}{y} f\left(\frac{f(y)}{x}\right)\right) &= \lim_{x\to\infty} \frac{f\left(\frac{1}{y} f\left(\frac{f(y)}{x}\right)\right)}{\frac{1}{x}}\\ &= \lim_{x\to 0^+} \frac{f\left(\frac{1}{y} f(x\, f(y))\right)}{x}\\ &= \lim_{x\to 0^+} \frac{f(y)}{y} f'\left(\frac{1}{y} f(x\, f(y))\right) f'(x\, f(y))\\ &= [f'(0)]^2 \frac{f(y)}{y}.\end{aligned}$$

On the other hand, by Theorem 4.1, we have

$$x\, f\left(\frac{1}{y} f\left(\frac{1}{x} f(y)\right)\right) = \frac{1}{x} f(y)\, f\left(\frac{1}{y} f(x)\right).$$

Therefore,

$$[f'(0)]^2 \frac{f(y)}{y} = \lim_{x\to\infty} \frac{f(y) f\left(\frac{f(x)}{y}\right)}{x}.$$

This means that $f(x) \to \infty$ as $x \to \infty$, and so L'Hopital's Rule is applicable. Thus we have

$$[f'(0)]^2 \frac{1}{y} = \lim_{x\to\infty} f'\left(\frac{f(x)}{y}\right) \frac{f'(x)}{y}.$$

Dividing by (1/y), the result follows.

(c) Once more, by L'Hopital's Rule

$$\lim_{x\to 0^+}\frac{f(f(x)/y)}{x} = \lim_{x\to 0^+} f'\left(\frac{f(x)}{y}\right)\frac{f'(x)}{y} = \frac{[f'(0)]^2}{y}.$$

Applying Theorem 4.1*, the first part follows.*

The proof of the second limit follows immediately from the first limit by substituting $z = f(y)/x$. □

LEMMA 4.2 *Suppose that* $f \in C^1[0,\infty)$ *with* $f(0)=0$ *and* $f'(0)\neq 0$. *If every solution of Eq.* (1) *is periodic of prime period 6, then f satisfies the functional relation*

$$x^2 f\left(\frac{c}{x}\right) = cf(x) \quad \textit{for all } x>0$$

where $c = [f'(0)]^2$.

Proof *By L'Hopital's Rule, we have*

$$\begin{aligned}\lim_{x\to\infty} xf\left(\frac{f(y)}{x}\right) &= \lim_{x\to 0^+}\frac{f(x\ f(y))}{x}\\ &= \lim_{x\to 0^+} f'(x\ f(y))\ f(y)\\ &= f'(0)f(y).\end{aligned}$$

By Lemma 4.1 *Part (b), we have*

$$\lim_{x\to\infty} f'(x) = f'(0),$$

and

$$\lim_{x\to\infty}\frac{f(f(x)/y)}{x} = \lim_{x\to\infty} f'(f(x)/y)f'(x)/y = [f'(0)]^2/y.$$

But, by Theorem 4.1*, we have*

$$xf\left(\frac{f(y)}{x}\right) = y^2\frac{x}{f(x)}f\left(\frac{1}{x}f\left(\frac{f(x)}{y}\right)\right).$$

Therefore,

$$f'(0)\ f(y) = y^2\ f\left(\frac{[f'(0)]^2}{y}\right)\frac{1}{f'(0)}.$$

Rearranging the terms, the result follows. □

We conclude this section by proving Theorem 4.2

Proof of Theorem 4.2 *First, on the one hand, differentiating the equation obtained in Lemma 4.2 and simplifying, we obtain*

$$f'(y) = \frac{2}{y}\ f(y) - f'\left(\frac{c}{y}\right),$$

or equivalently

$$f(y) = y\left[\frac{f'(y) + f'\left(\frac{c}{y}\right)}{2}\right].$$

But this implies that

$$f\left(\frac{f(y)}{x}\right) = \frac{y}{x}\left[\frac{f'(y) + f'\left(\frac{c}{y}\right)}{2}\right]\left[\frac{f'\left(\frac{f(y)}{x}\right) + f'\left(\frac{cx}{f(y)}\right)}{2}\right]. \tag{6}$$

On the other hand, multiplying both sides of Eq. (5) *by* y^2, *differentiation with respect to y, and simplifying, we get*

$$\frac{2}{y}\, f\left(\frac{f(y)}{x}\right) - \frac{1}{x} f'\left(\frac{1}{x}\, f\left(\frac{f(x)}{y}\right)\right) f'\left(\frac{f(x)}{y}\right) = f'\left(\frac{f(y)}{x}\right)\frac{f'(y)}{x},$$

or equivalently

$$f\left(\frac{f(y)}{x}\right) = \frac{y}{x}\left[\frac{f'\left(\frac{1}{x}\, f\left(\frac{f(x)}{y}\right)\right) f'\left(\frac{f(x)}{y}\right) + f'\left(\frac{f(y)}{x}\right) f'(y)}{2}\right]. \tag{7}$$

Now, Eqs. (6) *and* (7) *leads to the conclusion that*

$$\left[\frac{f'(y) + f'\left(\frac{c}{y}\right)}{2}\right] = \left[\frac{f'\left(\frac{1}{x}\, f\left(\frac{f(x)}{y}\right)\right) f'\left(\frac{f(x)}{y}\right) + f'\left(\frac{f(y)}{x}\right) f'(y)}{f'\left(\frac{f(y)}{x}\right) + f'\left(\frac{cx}{f(y)}\right)}\right]$$

for all $x, y > 0$ *which is true only if* $f'(x)$ *is constant.*
Integrating and applying the initial condition the result follows. □

Remark 4.2 Theorem 4.2 furnishes a solution for Conjecture 3.4.1 in Ref. [6]. However, unlike the case of $p = 5$, the additional condition imposed did not single out a unique solution.

Acknowledgements

We would like to extend our thanks to the referees for their insightful suggestions, and for bringing to our attention the existence of the paper by Mestel [7].

References

[1] R. Abu-Saris and F. Allan, Rational recursive sequences involving the maximum function, *Far East J. Math. Scs. (FJMS)*, **1**(3) (1999), 335–342.
[2] R. Abu-Saris, Characterization of rational periodic sequences, *J. Difference Eqs. Appl.*, **6**(2) (2000), 233–242.
[3] R. Abu-Saris, Global behavior of rational sequences involving piecewise power function, *J. Comp. Anal. Appl.*, **2**(1) (2000), 103–109.
[4] E. J. Janowski, V. L. Kocic, G. Ladas and S. W. Schultz, In: S. Elaydi, J. Greaf, G. Ladas and A. Peterson, eds, Global behavior of solutions of $x_{n+1} = \max\{x_n, A\}/x_{n-1}$, *Proc. of the first ICDEA*, pp. 273–282, 1995.
[5] V. L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, 1993.
[6] M. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations, Chapman & Hall/CRC 2002.
[7] B. D. Mestel, On globally periodic solutions of the difference equation $x_{n+1} = f(x_n)/x_{n-1}$, *J. Difference Eqs. Appl.*, **9**(2) (2003), 201–209.