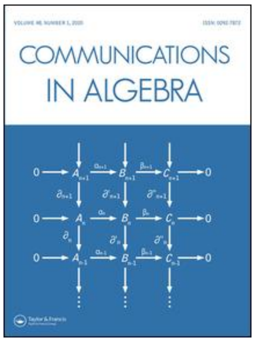

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REMARKS ON PSEUDO-VALUATION RINGS

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ABSTRACT. A prime ideal P of a ring A is said to be a strongly prime ideal if aP and bA are comparable for all $a,b \in A$. We shall say that a ring A is a pseudo-valuation ring (PVR) if each prime ideal of A is a strongly prime ideal. We show that if A is a PVR with maximal ideal M, then every overring of A is a PVR if and only if M is a maximal ideal of every overring of A that does not contain the reciprocal of any element of M. We show that if R is an atomic domain and a PVD, then $\dim(R) \leq 1$. We show that if R is a PVD and a prime ideal of R is finitely generated, then every overring of R is a PVD. We give a characterization of an atomic PVD in terms of the concept of half-factorial domain.

1 INTRODUCTION

Throughout this paper, all rings are commutative with identity and the letter R denotes an integral domain with quotient field K. Hedstrom and Houston

[11] introduced the concept *pseudo-valuation domains (PVD)*. Recall from [11] that an integral domain R, with quotient field K, is called a *pseudo-valuation domain* (PVD) in case each prime ideal P of R is *strongly prime*, in the sense that $xy \in P$, $x \in K$, $y \in K$ implies that either $x \in P$ or $y \in P$. Recently, the author, Anderson, and Dobbs [8] generalized the study of pseudo-valuation domains to the context of arbitrary rings. From [8] a prime ideal P of a ring A is said to be a *strongly prime* ideal if aP and bA are comparable for all $a,b \in A$. If A is an integral domain, this is equivalent to the definition of strongly prime ideal introduced in [11] (see [3, Prop. 3.1], [4, Prop. 4.2], and [7, Prop.3]). We shall say that a ring A is a *pseudo-valuation ring (PVR)* if each prime ideal of A is a strongly prime ideal. For additional characterization of pseudo-valuation rings see [3], [4], [6], [7], and [8].

In this paper, we show that, for a PVR A with maximal ideal M, every overring of A (inside its total quotient ring) is a PVR if and only if M is a maximal ideal of every overring of A that does not contain the reciprocal of any element of M. We show that if R is an atomic domain and a PVD, then $\dim(R) \leq 1$. We show that if R is a PVD and a prime ideal of R is finitely generated, then every overring of R is a PVD. We give a characterization of an atomic PVD in terms of the concept of half-factorial domain. Recall from Zaks [14] an atomic is called a *half-factorial domain (HFD)* if each factorization of a nonzero nonunit element of R into a product of irreducible elements (atoms) in R has the same length. Also, we give an alternative proof of the fact [2, Theorem 6.2] that an atomic PVD is a HFD.

2 RESULTS

We start by recalling some basic facts about a PVR.

FACT 1 [8, Lemma 1]. (a). Let I be an ideal of a ring A and P be a strongly prime ideal of A. Then I and P are comparable.
(b). Any PVR is quasilocal.
Proof. (a). Suppose that I is not contained in P. Then for some $b \in I - P$ and $a = 1$, bA is not contained in $P = aP$, and so $P \subset bA \subset I$.
(b). This follows easily from (a). ■

Fact 2 [8, Theorem 2]. A quasilocal ring A with maximal ideal M is a PVR if and only if M is a strongly prime ideal.

The first part of the following result is taken from [7, Theorem 1] and the second part is a consequence of the above two Facts.

LEMMA 3. (1). If for each a,b in a ring A either $a|b$ or $b|a^2$, then the prime ideals of A are linearly ordered and therefore A is quasilocal.

(2). A ring A is a PVR if and only if it is quasilocal with its maximal ideal strongly prime.

Proof. (1). Suppose that there are two prime ideals P, Q of A that are not comparable. Let $b \in P\backslash Q$ and $a \in Q\backslash P$. Then neither $a|b$ nor $b|a^2$, a contradiction. **(2).** This follows easily from Facts 1 and 2. ■

DEFINITION. Let b be an element of a ring B. Then an element d of B is called a proper divisor of b if $b = dm$ for some nonunit m of B.

In [8] ([7]) we proved that a ring A (R) is a PVR (PVD) if and only if for every $a,b \in A$ (R) either $a|b$ or $b|ac$ for each nonunit c of A (R). An analog of this result is the following proposition.

PROPOSITION 4. A ring B is a PVR if and only if for every $a,b \in B$, either $a|b$ or $d|a$ for every proper divisor d of b.

Proof. Suppose that B is a PVR with the maximal ideal M. Let $a,b \in B$ and suppose that a does not divide b in B. Let d be a proper divisor of b. Then $b = dm$ for some nonunit m of B. If d does not divide a in B, then $dM \subset aB$ since M is strongly prime. Hence, $a|dm = b$, a contradiction. Thus, $d|a$ for every proper divisor d of b.

Conversely, suppose that for every $a,b \in B$ either $a|b$ or $d|a$ for every proper divisor d of b. Let $a,b \in B$ such that a does not divide b

in B. Then $b|a^2$, for otherwise by hypothesis $a|b$ which is a contradiction. Thus, by Lemma 3 (1) B is quasilocal with maximal ideal M. Now, we need show that $aM \subset bB$. Deny. Then there is a nonunit c of B such that b does not divide ac. Since a is a proper divisor of ac and b does not divide ac, by hypothesis $a|b$ which contradicts the assumption that a does not divide b in B. Hence, our denial is invalid. ■

Anderson and Mott [2,Theorem 6.2] proved that an atomic PVD R is a HFD. Now, we give a proof of this result that relies only on the definitions of a PVD and a HFD.

THEOREM 5 [2, Theorem 6.2]. An atomic PVD R is a HFD.

Proof. Deny. Let M be the maximal ideal of R. Then for some nonunit nonzero element x of R, $x = x_1x_2...x_n = y_1y_2...y_m$ where the x_i's and the y_j's are atoms of R and $m > n$. Hence, $(x_1/y_1)...(x_n/y_n) = y_{n+1}...y_m \in M$. Hence, for some i, $1 \le i \le n$, $x_i/y_i \in M$. Thus, $x_i = y_i m$ for some $m \in M$. A contradiction, since x_i is an atom of R and neither y_i nor m is a unit of R. Hence, our denial is invalid and R is indeed a HFD. ■

Definition. Let R be a HFD and x be a non-zero element of R. Then we define $L(x) = n$ if $x = x_1x_2...x_n$ for some atoms x_i, $1 \le i \le n$, of R. If x is a unit of R, then $L(x) = 0$.

In the following theorem, we give a characterization of an atomic PVD in terms of the concept of HFD.

THEOREM 6. Let R be an atomic domain. The following statements are equivalent :

(1) R is a PVD.

(2) R is a HFD and for ever $x,y \in R$, if $L(x) < L(y)$, then $x|y$ in R.

Proof. **(1) ⇒ (2).** By theorem 1 R is a HFD. Let $x,y \in R$ such that $L(x) < L(y)$. Suppose that x does not divide y in R. Then $y|xt$ for some atom t of R by [8, Prop. 3]. Hence, $xt=ym$ for some nonunit m of R (observe that if m is a unit of R, then $x|y$). But $L(xt) < L(ym)$, a contradiction, since R is a HFD. Thus, $x|y$. **(2)⟹(1).** Let $a,b \in R$ and suppose that a does not divide b in R. Then $L(b) \le L(a)$ by the hypothesis. Hence, $L(b) < L(ac)$ for every nonunit c of R. Thus, $b|ac$ for every nonunit c of R. Therefore, R is a PVD by [8, Prop.3]. ∎

COROLLARY 7 . Let R be an atomic PVD, c is an atom of R, and $x \in R$. If $L(x) = n \ge 2$, then $x = c^{(n-1)}b$ for some atom b of R.

Proof. By Theorem 3, $c^{(n-1)}|x$ since $L(c^{(n-1)}) < L(x)$. Hence, $x = c^{(r-1)}b$ for some $b \in R$. Since R is a PVD, R is a HFD. Hence, b must be an atom of R. ∎

Hedstrom and Houston [11] proved that a Notherian PVD R has a Krull dimension ≤ 1. We strengthen this result in the next theorem. Before stating the following theorem, the following fact is needed :

FACT 8 [7, Corollary 1]. Suppose that the prime ideals of a ring A are linearly ordered and a,b are nonzero elements of A. Let P be the minimum prime ideal of A that contains a and Q be the minimum prime ideal of A that contains b. Then P = Q if and only if there exist $n \geq 1$ and $m \geq 1$ such that $a|b^n$ and $b|a^m$. ∎

THEOREM 9. Let R be an atomic PVD. Then $\mathrm{Dim}(R) \leq 1$.

Proof. Let a,b be nonzero nonunit elements of R. By the above Fact, it suffices to show that $a|b^n$ for some $n \geq 1$. Let $m = L(a)$ and $h = L(b)$. Then for some $n \geq 1$, $m < nh$, that is, $L(a) < L(b^n)$. Hence, by Theorem 6 $a|b^n$. ∎

REMARK : Anderson and Mott [2, Corollary 5.2] proved that R is an atomic PVD with maximal ideal M if and only if $V = M:M = \{ x \in K : xM \subset M \}$ is a discrete valuation domain with maximal ideal M. Since V and R have the same maximal ideal, by [5, Theorem 3.10] the prime ideals of V are the prime ideals of R. Hence, $\mathrm{Dim}(R) \leq 1$ and this is another proof of Theorem 9.

Recall that a domain R is a LT-domain (lowest terms domain) in the sense of [1], if for each nonzero elements $a,b \in R$, there are nonzero elements c,d of R with $a/b = c/d$ and $\gcd(c,d) = 1$.

COROLLARY 10. Let R be an atomic PVD. Then R is a LT-domain.

Proof. Let a,b be nonzero elements of R. We consider three cases. *First case*. Suppose that $L(a) < L(b)$. Then $a|b$. Hence, $a/b = 1/s$ for some $s \in R$ and $\gcd(1,s) = 1$. *Second case*. Suppose that $L(a) > L(b)$. Then $a/b = s/1$ for some $s \in R$ and $\gcd(s,1) = 1$. *Third case*. Suppose that $L(a) = L(b)$. Then $a = vh$ for some atom h of R and $v \in R$. Since $L(v) < L(b)$, $b = vd$ for some $d \in R$. Since $L(b) = L(a) = L(v) + 1$, d is an atom of R. Hence, $a/b = h/d$. Since h,d are atoms of R, $\gcd(h,d) = 1$. ■

In view of the proof of the above Theorem we have

COROLLARY 11. Let R be an atomic PVD and $x = a/b \in K$ where a,b are nonzero elements of R. Then $x = a/b$ must equal to one of the following forms :

(1) $1/s$ for some $s \in R$.

(2) $s/1$ for some $s \in R$.

(3) h/d for some atoms h,d of R.

Definition. For a ring A, let $S = \{ s \in A : s$ is a non-zerodivisor of A, that is, s is regular $\}$. Then $T = R_S$ is the total quotient ring of A. A subring B of T is called an *overring* of A if $A \subset B$.

LEMMA 12. Let P be a strongly prime ideal of a ring A containing the zerodivisors of A and B be an overring of A. If $x = a/b \in B\backslash A$ for some a,b of A, then a is a nonzerodivisor of A and $x^{-1}P \subset P$.

Proof. Suppose that a is a nonzerodivisor of A. Then $b \in P$, for if $b \in A\backslash P$, then $P \subset (b)$ and hence $b|a$ and therefore $x \in A$, a contradiction. Since P is a strongly prime ideal, either $aA \subset bP$ or $bP \subset aA$. If $aA \subset bP$, then $b|a$ and therefore $x \in A$, a contradiction. If $bP \subset aA$, then $a|b^2$ since $b \in P$, a contradiction again, since b is a nonzerodivisor of A and a is a zerodivisor of A. Hence, a is a nonzerodivisor of A. Now, since $x = a/b \in B\backslash A$, $bP \subset aA$. Thus, $x^{-1}P \subset A$. Suppose that for some $p \in P$, $x^{-1}p = q \in A\backslash P$. Then q is a nonzerodivisor of A, since P contains the zerodivisors of A. Since P is a strongly prime ideal and $qA \not\subset P$, $p \in P \subset qA$, and therefore $x = p/q \in A$, a contradiction. Hence, $x^{-1}P \subset P$. ■

The following Theorem is an important tool for the remaining part of this paper.

THEOREM 13. Let P be a strongly prime ideal of a ring A containing the zerodivisors of A and b be an overring of A. The following statements are equivalent :

(1) $PB \cap A = P$.

(2) B does not contain the reciprocal of any elements of P.

(3) P is a strongly prime ideal of B.

Proof. (1)⟹(2). Deny. Then there is a nonzerodivisor s of A such that $s \in P$ and $1/s \in B$. Hence, $1 \in PB \cap A = P$ which is a contradiction. **(2)⟹(3).** First, we show that P is an ideal of B. Let $x \in B\setminus A$ and $p \in P$. We consider two cases : *Case 1*. Suppose that x is of the form $1/s$ for some $s \in A$. Then by hypothesis $s \in A\setminus P$. Hence, $p \in (s)$ by Fact 1(a). Thus, $p = sd$ for some $d \in P$. Hence, $xp = d \in P$. *Case 2*. Suppose that x is not of the form of case 1. Then $x = a/b$ for some a,b of A. By Lemma 12, a is a nonzerodivisor of A. Since x is not of the form of case 1, $x^{-1} = b/a \in T\setminus A$. Thus, $xP \subset P$ by Lemma 12. Hence, P is an ideal of B. Now we show that P is a prime ideal of B. Suppose that $xy = p \in P$ for some $x,y \in B$ and $x \in B\setminus P$. If $x \in A$, then x is a nonzerodivisor of A and $y = p/x \in P$ (since $p \in (x)$). If $x \in B\setminus A$, then $y = x^{-1}(xy) = x^{-1}p \in P$ by Lemma 12. Hence, P is a prime ideal of B. Now we show that P is a strongly prime ideal of B. Let $x, y \in B$. Then $x = a/s$ and $y = b/s$ for some $a,b \in B$ and a nonzerodivisor s of A. Since P is a strongly prime ideal of A, either $aA \subset bP$ or $bP \subset aA$. if $bP \subset aA$, then $(b/s)P \subset (a/s)A \subset (a/s)B$. If $aA \subset bP$, then $(a/s)A \subset (b/s)P$ and therefore $xAB = xB \subset yPB = yP$. Thus, P is a strongly prime ideal of B. **(3)⟹(1).** No comments. ■

REMARK. We are unable to construct an example showing the hypothesis that P contains the zerodivisors of A is crucial in the above Theorem.

It was proved in [11] that every overring of a Noetherian PVD R is a PVD. In the following theorem, we see that this result is valid under some weaker conditions.

THEOREM 14. Let R be a PVD with the maximal ideal M. Suppose that a prime ideal P of R is finitely generated. Then R' (the integral closure of R in K) = $M:M = \{ x \in K : xM \subset M \}$. In particular, every overring of R is a PVD.

Proof. Since R is a PVD, it is well-known that M:M is a valuation domain with M as the maximal ideal and $R' \subset M:M$. Let $x \in M:M$. Since P is a prime ideal of M:M by Theorem 13, $xP \subset P$. Since P is finitely generated, $x \in R'$. Thus, $R' = M:M$. Hence, every overring of R is a PVD by [12, Prop. 2.7] or [9, Prop. 4.2], or [8, Theorem 21]. ■

It is well-known that if A is a PVR with the maximal ideal M, then the integral closure A' of A in T is a PVR with the maximal ideal M (see [8, Theorem 19]). Now, If B is an integral overring of a PVR A, then B is a PVR by Theorem 13 and the fact that $A \subset B$ satisfies the INC condition. Recall that $A \subset B$ satisfies the INC condition if any two prime ideals of B with the same contraction in A are incomparable. Hence, we have the following proposition.

PROPOSITION 15. Let A be a PVR with maximal ideal M. Then every overring B of A such that $B \subset A'$ is a PVR with maximal ideal M. ∎

LEMMA 16. Let A be a PVR with maximal ideal M and B be an overring of A that does not contain the reciprocal of any element of M. Then $B \subset M:M$.
Proof. Let $x \in B \backslash A$. Write $x = a/b$ for some a, b in A. Since a does not divide b in A by hypothesis and M is a strongly prime ideal of A, $aM \subset bM$. Hence, $xM \subset M$. Thus, $x \in M:M$. ∎

THEOREM 17. Let A be a PVR with the maximal ideal M. Then every overring of A is a PVR if and only if M is a maximal ideal of every overring of A that does not contain the reciprocal of any element of M.

Proof. Suppose that every overring of A is a PVR. Then $A' = M:M$ by [8, Theorem 21]. Let C be an overring of A that does not contain the reciprocal of any element of M. Then $C \subset M:M = A'$ by Lemma 16. Hence, C is a PVR with maximal ideal M by Proposition 15.

Conversely, suppose that M is a maximal ideal of every overring of A that does not contain the reciprocal of any element of M. Then every overring of A that does not contain the reciprocal of any element of M is a PVR by Lemma 3(2). Now, suppose that C is an overring of A that contains an element of the form 1/s for some nonzero $s \in M$. Then C is a chained ring (valuation ring) by [8, Lemma 20], and hence a PVR by [8, Corollary 4]. ∎

REMARK. Let A be a PVR with maximal ideal M. Suppose that every overring of A is a PVR. Then $A' = M:M$. Let C be an overring of A that does not contain the reciprocal of any element of M. Then $C \subset M:M = A'$ by Lemma 16. Hence, $A \subset C$ satisfies the INC condition. Conversely, suppose that $A \subset C$ satisfies the INC condition and C does not contain the reciprocal of any element of M. Then M is a strongly prime ideal of C by Theorem 13 and therefore maximal since $A \subset C$ satisfies the INC condition. Hence, C is a PVR.

Combining [8, Theorem 21] with Theorem 17 and Lemma 16 and the above Remark, we arrive to the following corollary :

COROLLARY 18. Let A be a PVR with the maximal ideal M. The following statements are equivalent :

(1) $A' = M:M$.

(2) Every overring of A is a PVR.

(3) Every overring C of A such that $C \subset M:M$ is a PVR.

(4) Every overring C of A such that $C \subset M:M$ is a PVR with maximal ideal M.

(5) Every overring of A that does not contain the reciprocal of any element of M is a PVR.

(6) Every overring of A that does not contain the reciprocal of any element of M is a PVR with maximal ideal M.

(7) M is a maximal ideal of every overring C of A such that $C \subset M:M$.

(8) M is the unique maximal ideal of every overring C of A such that $C \subset M:M$.

(9) $A \subset C$ satisfies the INC condition for every overring C of A such that $C \subset M:M$. ■

3 EXAMPLES

EXAMPLE 1. Choose an infinite dimensional valuation ring (chained ring) V of the form $V = K + M$ where K is a field and M is the maximal ideal of V (see [10, Exercise 12, page 271]). If F is a proper subfield of K and $[K:F]$ is infinite, then $R = F + M$ is a PVD (see [11, Example 2.1]). Observe that R has infinite Krull dimension and therefore is not atomic by Theorem 9.

EXAMPLE 2. Let $R = Z[\sqrt{5}]_{(2,\ 1+\sqrt{5})}$. Then R is a Noetherian PVD and therefore atomic (see [11, Example 3.6]).

EXAMPLE 3. Let k be any field and X, Y be indeterminates. Then $R = k + Xk(Y)[[X]]$ is an atomic PVD that is not Noetherian (see the discussion in [2] following [2, Theorem 5.4]).

EXAMPLE 4. **(a)** Let R be a PVD. If I is an ideal of R, then R/I is a PVR by [8, Corollary 3]. **(b)** Let k be any field and x,y indetrminates. Then $R = k[X,Y]/(X^2,XY,Y^2)$ is a PVR (see [8,Example 10]).

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