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On the dynamics of a rational difference equation $x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-k}}{Bx_n+Cx_{n-k}}$

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A B S T R A C T

In this paper, we will investigate the dynamics of a nonlinear rational difference equation of a higher order. Our concentration is on invariant intervals, periodic character, the character of semi-cycles and global asymptotic stability of all positive solutions of

$$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-k}}{Bx_n+Cx_{n-k}},\qquad n=0,1,\ldots$$

It is worth to mention that our results solve partially some of the open problems proposed by Kulenvic and Ladas in their monographs [17], [18].

1. Introduction

In this paper, we will study the nonlinear rational difference equation

$$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-k}}{Bx_n+Cx_{n-k}},\qquad n=0,1,\ldots \tag{1.1}$$

where the parameters α, β, γ, B, C and the initial conditions $x_{-k},\ldots,x_{-1},x_0$ are positive real numbers, $k=\{1,2,3,\ldots\}$.

Our concentration is on invariant intervals, periodic character, the character of semi-cycles and global asymptotic stability of all positive solutions of Eq. (1.1).

It is worth mentioning that the results in [4,5,7,12,20] are special cases of our main results. Some recent work on rational equations can be find in [1–3,8–11,15,17–19,21–25].

The global stability of Eq. (1.1) for $k=1$ has been investigated in [7]. They showed, in respect to variation of the parameters, the positive equilibrium point is globally asymptotically stable or every solution lies eventually in an invariant interval. Kulenovic and Ladas, in addition, considered Eq. (1.1) in their monograph [16].

In [7], the equation

$$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{Bx_n+Cx_{n-1}} \tag{1.2}$$

was studied. Our interest now to study and solve Eq. (1.1) in the general case which is a generalization of those studied in [7].

The change of variables $x_n=\frac{\beta}{B}y_n$, will change the equation

$$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-k}}{Bx_n+Cx_{n-k}},\qquad n=0,1,\ldots$$

into the equation

$$y_{n+1}=\frac{D+y_n+py_{n-k}}{y_n+qy_{n-k}},\qquad n=0,1,\ldots \tag{1.3}$$

where $D=\frac{B\alpha}{\beta^2}$, $p=\frac{\gamma}{\beta}$, $q=\frac{C}{B}$.

Before studying the behavior of solutions of this rational difference equation, we will review some definitions and basic results that will be used throughout this paper.

Lemma 1.1 [13]**.** *Let I be some interval of real numbers and let*

$$f: I^{k+1}\to I$$

be a continuously differentiable function. Then for every initial conditions $x_{-k},\ldots,x_{-1},x_0\in I$, $k=\{1,2,3,\ldots\}$., the difference equation

$$x_{n+1}=f(x_n,x_{n-k}),\quad n=0,1,\ldots \tag{1.4}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1.1. We say that a solution $\{y_n\}_{n=-k}^{\infty}$ of a difference equation $y_{n+1}=f(y_n,y_{n-1},\ldots,y_{n-k})$ is periodic if there exists a positive integer p such that $y_{n+p}=y_n$. The smallest such positive integer p is called the prime period of the solution of the difference equation.

Definition 1.2. The equilibrium point $\bar{y}$ of the equation

$$y_{n+1}=f(y_n,y_{n-1},\ldots,y_{n-k}),\ n=0,1,\ldots \tag{1.5}$$

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is the point that satisfies the condition

$$\bar{y} = f(\bar{y}, \bar{y}, \ldots, \bar{y}).$$

Definition 1.3 [6]**.** Let $\bar{y}$ be an equilibrium point of Eq. (1.5). Then the equilibrium point $\bar{y}$ is called

(1) locally stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $y_{-k}, y_{-k+1}, \ldots, y_0 \in I$ with $|y_{-k} - \bar{y}| + |y_{-k+1} - \bar{y}| + \cdots + |y_0 - \bar{y}| < \delta$, we have $|y_n - \bar{y}| < \epsilon$ for all $n \geq -k$,
(2) locally asymptotically stable if it is locally stable and if there exists $\gamma > 0$ such that for all $y_{-k}, y_{-k+1}, \ldots, y_0 \in I$ with $|y_{-k} - \bar{y}| + |y_{-k+1} - \bar{y}| + \ldots + |y_0 - \bar{y}| < \gamma$, we have $lim_{n\to\infty} y_n = \bar{y}$.
(3) a global attractor if for all $y_{-k}, y_{-k+1}, \ldots, y_0 \in I$, we have $lim_{n\to\infty} y_n = \bar{y}$.
(4) globally asymptotically stable if $\bar{y}$ is locally stable and $\bar{y}$ is a global attractor.

Let

$$p = \frac{\partial f}{\partial u}(\bar{x}, \bar{x}) \quad and \quad q = \frac{\partial f}{\partial v}(\bar{x}, \bar{x})$$

where $f(u, v)$ is the function in Eq. (1.4) and $\bar{x}$ is an equilibrium of the equation. Then the equation

$$y_{n+1} = py_n + qy_{n-k}, \qquad n = 0, 1, \ldots \tag{1.6}$$

is called the linearized equation associated with Eq. (1.4) about the equilibrium point $\bar{x}$. Its characteristic equation is

$$\lambda^{k+1} - p\lambda^k - q = 0 \tag{1.7}$$

Theorem 1.1 [12] Linearized Stability**.** *Consider the difference equation*

$$y_{n+1} = py_n + qy_{n-k}, \qquad n = 0, 1, \ldots$$

(a) *If both roots of the equation have absolute values less than one, then the equilibrium $\bar{y}$ of the equation is locally asymptotically stable.*
(b) *If at least one of the roots of the equation has an absolute value greater than one, then $\bar{y}$ is unstable.*
(c) *Both roots of the equation have absolute values less than one if and only if*

$$|p| < 1 - q < 2$$

in this case, $\bar{y}$ is a locally asymptotically stable.

(d) *Both roots of the equation have absolute values greater than one if and only if*

$$|q| > 1 \quad \text{and} \quad |p| > |1 - q|$$

in this case, $\bar{y}$ is a repeller.

(e) *One root of the equation has an absolute value greater than one while the other root has an absolute value less than one if and only if*

$$p^2 + 4p > 0 \quad \text{and} \quad |p| > |1 - q|.$$

in this case, $\bar{y}$ is unstable and is called a saddle point.

2. The equilibrium points

Next we investigate the equilibrium points of the nonlinear rational difference equation

$$y_{n+1} = \frac{D + y_n + py_{n-k}}{y_n + qy_{n-k}}, \qquad n = 0, 1, \ldots \tag{2.1}$$

where the parameters p, q, D and the initial conditions $y_{-k}, \ldots, y_{-1}, y_0$ are positive real numbers, $k = \{1, 2, 3, \ldots\}$.

The equilibrium points of Eq. (2.1) are the positive solutions of the equation

$$\bar{y} = \frac{D + \bar{y} + p\bar{y}}{\bar{y} + q\bar{y}} = \frac{D + \bar{y}(p+1)}{\bar{y}(q+1)}$$

By rearranging, we get

$$\bar{y}^2 = \frac{D + \bar{y}(1+p)}{1+q} \tag{2.2}$$

hence the equilibrium point is given by

$$\bar{y} = \frac{1 + p + \sqrt{(1+p)^2 + 4D(1+q)}}{2(q+1)}$$

To find the linearization of our problem, consider

$$f(u, v) = \frac{D + u + pv}{u + qv}$$

so

$$\frac{\partial f}{\partial u} = \frac{v(q-p) - D}{(u+qv)^2}$$

hence

$$\frac{\partial f}{\partial u}(\bar{y}, \bar{y}) = \frac{\bar{y}(q-p) - D}{(q\bar{y} + \bar{y})^2}$$

from

$$\bar{y}^2 = \frac{D + \bar{y}(1+p)}{1+q} \tag{2.3}$$

we get

$$\frac{\partial f}{\partial u}(\bar{y}, \bar{y}) = \frac{(q-p)\bar{y} - D}{(1+q)^2(\frac{D+\bar{y}(1+p)}{1+q})} = \frac{(q-p)\bar{y} - D}{(1+q)(D + (1+p)\bar{y})}$$

also

$$\frac{\partial f}{\partial v} = \frac{u(p-q) - qD}{(u+qv)^2}$$

hence

$$\frac{\partial f}{\partial v}(\bar{y}, \bar{y}) = \frac{\bar{y}(p-q) - qD}{(q\bar{y} + \bar{y})^2}$$

from

$$\bar{y}^2 = \frac{D + \bar{y}(1+p)}{1+q} \tag{2.4}$$

we get

$$\frac{\partial f}{\partial v}(\bar{y}, \bar{y}) = -\frac{(q-D)\bar{y} + Dq}{(\bar{y} + q\bar{y})^2} = -\frac{(q-p)\bar{y} + Dq}{\bar{y}^2(1+q)^2} = -\frac{(q-p)\bar{y} + Dq}{(1+q)^2(\frac{D+\bar{y}(1+p)}{1+q})} = -\frac{(q-p)\bar{y} + Dq}{(1+q)(D + (1+p)\bar{y})}$$

so, the linearized equation is

$$z_{n+1} = \frac{\bar{y}(q-p) - D}{(q\bar{y} + \bar{y})^2} z_n + \frac{\bar{y}(p-q) - qD}{(q\bar{y} + \bar{y})^2} z_{n-k} \tag{2.5}$$

and its characteristic equation is

$$\lambda^{k+1} - \frac{\bar{y}(q-p) - D}{(q\bar{y} + \bar{y})^2}\lambda^k + \frac{\bar{y}(q-p) + qD}{(q\bar{y} + \bar{y})^2} = 0 \tag{2.6}$$

Which can be rewritten as $\bar{y}$ is:

$$z_{n+1} = \frac{(q-p)\bar{y} - D}{(1+q)(D+(1+p)\bar{y})} z_n - \frac{(q-p)\bar{y} + Dq}{(1+q)(D+(1+p)\bar{y})} z_{n-k}$$

i.e

$$z_{n+1} - \frac{(q-p)\bar{y} - D}{(1+q)(D+(1+p)\bar{y})} z_n + \frac{(q-p)\bar{y} + pq}{(1+q)(D+(1+p)\bar{y})} z_{n-k} = 0 \tag{2.7}$$

and its characteristic equation is:

$$\lambda^{k+1} - \frac{(q-p)\bar{y} - D}{(1+q)(D+(1+p)\bar{y})} \lambda^k + \frac{(q-p)\bar{y} + Dq}{(1+q)(D+(1+p)\bar{y})} = 0 \tag{2.8}$$

3. The local stability

The following facts are important to study the local stability

Lemma 3.1 [6]. *Assume that a,b are real numbers and $k \in \{1, 2, \ldots\}$. Then*

$$|a| + |b| < 1 \tag{3.1}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+1} + a y_n + b y_{n-k} = 0, \ n = 0, 1, \ldots. \tag{3.2}$$

Suppose in addition that one of the following two cases holds.

1. *k odd and $b < 0$.*
2. *k even and $ab < 0$.*

Then (3.1) is also a necessary condition for the asymptotic stability of Eq. (3.2).

Lemma 3.2 [6]. *Assume that a,b are real numbers. Then*

$$|a| < b + 1 < 2$$

is a necessary and sufficient condition for the asymptotic stability of the difference equation

$$y_{n+1} + a y_n + b y_{n-k} = 0, \ n = 0, 1, \ldots. \tag{3.3}$$

Lemma 3.3. *Assume that all the roots of the characteristic equation of the above equation lie inside the unit circle, then the positive equilibrium is locally asymptotically stable.*

Theorem 3.1. *Let $\bar{y}$ be an equilibrium point of Eq. (2.1), then $\bar{y}$ is locally asymptotically stable if $q > 1$.*

Proof. The equilibrium point $\bar{y}$ of Eq. (2.1) is

$$\bar{y} = \frac{1 + p + \sqrt{(1+p)^2 + 4D(1+q)}}{2(q+1)}$$

and the linearized equation about it is

$$z_{n+1} = \frac{\bar{y}(q-p) - D}{(q\bar{y} + \bar{y})^2} z_n + \frac{\bar{y}(p-q) - qD}{(q\bar{y} + \bar{y})^2} z_{n-k}$$

We will use Lemma 3.2 to show that $\bar{y}$ is asymptotically stable if $q > 1$. From our linearized equation we have

$$a = -\frac{\bar{y}(q-p) - D}{(q\bar{y} + \bar{y})^2}$$

$$b = \frac{\bar{y}(q-p) + qD}{(q\bar{y} + \bar{y})^2}$$

But as

$$(\bar{y} + q\bar{y})^2 = (1+q)(D + (1+p)\bar{y})$$

Then

$$a = -\frac{\bar{y}(q-p) - D}{(1+q)(D+(1+p)\bar{y})}$$

$$b = \frac{\bar{y}(q-p) + qD}{(1+q)(D+(1+p)\bar{y})}$$

It is easy to show that

$$|a| < b < b + 1$$

and as

$$b < 1$$

Then we proved that

$$|a| < b + 1 < 2$$

Then by Lemma 3.2 we can say that the equilibrium point $\bar{y}$ is asymptotically stable if $q > 1$.

This completes the proof. □

Theorem 3.2. *The unique equilibrium point $\bar{y}$ of Eq. (2.1) is locally asymptotically stable in the following cases:*

1. *$q > p$ there are two cases*
   - *$p > 1$*
   - *$q < 1$.*
2. *$q = p$*
3. *$q < p$*
   - *if $\bar{y}(p-q) < Dq$ i.e. $\bar{y} < \frac{Dq}{p-q}$*
   - *if $\bar{y}(p-q) > Dq$ and $\bar{y} < \frac{2Dq}{p-3q-1-qp}$ i.e. $\bar{y} > \frac{Dq}{p-q}$ and $\bar{y} < \frac{2Dq}{p-3q-1-qp}$*

Proof.

1. When $q > p$ there are two cases:
   - $(q-p)\bar{y} > D$ i.e. $\bar{y} > \frac{D}{q-p}$ Assume

     $$|a| + |b| < 1,$$

     by substituting values of a and b, we have

     $$\frac{(q-p)\bar{y} - D}{(1+q)(D+(1+p)\bar{y})} + \frac{(q-p)\bar{y} + Dq}{(1+q)(D+(1+p)\bar{y})} < 1$$

     then

     $$\frac{(q-p)\bar{y} - D + (q-p)\bar{y} + Dq}{(1+q)(D+(1+p)\bar{y})} < 1$$

     by multiplying, we have

     $$(q-p)\bar{y} - D + (q-p)\bar{y} + Dq < (1+q)(D+(1+p)\bar{y})$$

     then

     $$2(q-p)\bar{y} - D + Dq < D + (1+p)\bar{y} + Dq + q(1+p)\bar{y}$$

     so

     $$2(q-p)\bar{y} - 2D < \bar{y}(1+p)(1+q)$$

     hence

     $$-2D < \bar{y}[(1+p)(1+q) - 2(q-l)]$$

     then

     $$-2D < \bar{y}[1 + p + q + pq - 2q + 2p]$$

     so

     $$-2D < \bar{y}[1 - q + 3p + pq]$$

     note that when $p > 1$, $\bar{y}[1 - q + 3p + pq]$ is strictly greater than zero. And when $q < 1$, $\bar{y}[1 - q + 3p + pq]$ is strictly greater than zero

- $(q-p)\bar{y} < D$, i.e $\bar{y} < \frac{D}{q-p}$, we have the following inequality

$$\frac{D-(q-p)\bar{y}}{(1+q)(D+(1+p)\bar{y})} + \frac{(q-p)\bar{y}+Dq}{(1+q)(D+(1+p)\bar{y})} < 1$$

then

$$\frac{D-(q-p)\bar{y}+(q-p)\bar{y}+Dq}{(1+q)(D+(1+p)\bar{y})} < 1$$

hence

$$D+Dq < D+(1+p)\bar{y}+Dq+q(1+p)\bar{y}$$

cancel the common terms in both sides, we get

$$0 < (1+p)\bar{y}+q(1+p)\bar{y}$$

which is true for all values of $p, q, \bar{y}$

(2) When $q=p$, we have

$$\frac{D}{(1+q)[D+(1+p)\bar{y}]} + \frac{Dq}{(1+q)[D+(1+p)\bar{y}]} < 1$$

then

$$D+Dq < (1+q)[D+(1+p)\bar{y}]$$

$$D(1+q) < (1+q)[D+(1+p)\bar{y}]$$

cancel the term $1+q$ from both sides, we get

$$D < D+(1+p)\bar{y}$$

also cancel D from both sides to get

$$0 < (1+p)\bar{y}$$

which is true for all values of $\bar{y}, p$ under the condition $q=p$

(3) When $q<p$
there are two cases:

- when $\bar{y} < \frac{Dq}{p-q}$
$|a| = \frac{(p-q)\bar{y}+D}{(1+q)[D+(1+p)\bar{y}]}$ and $|b| = \frac{Dq-(p-q)\bar{y}}{(1+q)[D+(1+p)\bar{y}]}$ implies

$$\frac{(p-q)\bar{y}+D}{(1+q)[D+(1+p)\bar{y}]} + \frac{Dq-(p-q)\bar{y}}{(1+q)[D+(1+p)\bar{y}]} < 1$$

which reduces to

$$\frac{(p-q)\bar{y}+D+Dq-(p-q)\bar{y}}{(1+q)[D+(1+p)\bar{y}]} < 1$$

and thus

$$\frac{D+Dq}{(1+q)[D+(1+p)\bar{y}]} < 1$$

so

$$D+Dq < (1+q)[D+(1+p)\bar{y}]$$

hence

$$D(1+q) < (1+q)[D+(1+p)\bar{y}]$$

this is reduced into

$$D < [D+(1+p)\bar{y}]$$

and consequently we get

$$0 < (1+p)\bar{y}$$

- when $\bar{y} > \frac{Dq}{p-q}$
$|a| = \frac{(p-q)\bar{y}+D}{(1+q)[D+(1+p)\bar{y}]}$ and $|b| = \frac{(p-q)\bar{y}-Dq}{(1+q)[D+(1+p\bar{y}]}$

$$\frac{(p-q)\bar{y}+D}{(1+q)[D+(1+p)\bar{y}]} + \frac{(p-q)\bar{y}-Dq}{(1+q)[D+(1+p)\bar{y}]} < 1$$

and thus

$$\frac{(p-q)\bar{y}+D+(p-q)\bar{y}-Dq}{(1+q)[D+(1+p)\bar{y}]} < 1$$

and so

$$2(p-q)\bar{y}+D-Dq < (1+q)[D+(1+p)\bar{y}]$$

hence

$$2(p-q)\bar{y}+D-Dq < D+(1+p)\bar{y}+Dq+q(1+p)\bar{y}$$

so

$$2(p-q)\bar{y}-(1+p)\bar{y}-q(1+p)\bar{y} < 2Dq$$

which is reduced to

$$2p\bar{y}-2q\bar{y}-\bar{y}-p\bar{y}-q\bar{y}-qp\bar{y} < 2Dq$$

so

$$p\bar{y}-3q\bar{y}-\bar{y}-qp\bar{y} < 2Dq$$

and

$$\bar{y}[p-3q-1-qp] < 2Dq$$

consequently

$$\bar{y} < \frac{2Dq}{p-3q-p-qp}$$

□

We have investigated the two cases $q>p$ and $q<p$ in previous theorem. The next theorem about the case $q=p$. When $q=p$, Eq. (2.1) becomes

$$y_{n+1} = \frac{D+y_n+qy_{n-k}}{y_n+qy_{n-k}} \tag{3.4}$$

and the positive equilibrium point $\bar{y} = \frac{1+q+\sqrt{((1+q)^2+4D(1+q))}}{2(1+q)}$. Observe that $\bar{y} > 1$.

Theorem 3.3. *Assume that $q=p$*

(1) *Suppose that k is odd. Then the equilibrium point $\bar{y}$ of Eq. (2.1) is locally asymptotically stable.*
(2) *Suppose that k is even. Then the equilibrium point $\bar{y}$ of Eq. (2.1) is locally asymptotically stable iff $q=p$.*

Proof. Let $f(x,y) = \frac{p+x+qy}{x+qy}$. Assume $a = \frac{\partial f}{\partial x}(\bar{y},\bar{y}) = \frac{-p}{(1+q)[p+(1+q)\bar{y}]}$ and $b = \frac{\partial f}{\partial y}(\bar{y},\bar{y}) = \frac{-pq}{(1+q)[p+(1+q)\bar{y}]}$. Observe that $a<0$ and $ab>0$ and thus $\bar{y}$ is locally stable. □

4. Analysis of semi-cycles

Definition 4.1. We say that a solution $\{y_n\}$ of a difference equation

$$y_{n+1} = f(y_n, y_{n-1}, \ldots, y_{n-k}) \tag{4.1}$$

is *bounded and persists* if there exist positive constants P and Q such that

$$P \le x_n \le Q \quad \text{for} \quad n = -1, 0, \ldots.$$

Definition 4.2. *A positive semi-cycle* of a solution $\{y_n\}$ of Eq. (4.1) consists of a "string" of terms $\{y_l, y_{l+1}, \ldots, y_m\}$, all greater than or equal to the equilibrium $\bar{y}$, with $l \ge -k$ and $m \le \infty$ and such that

either $l=-k$, or $l>-k$ and $y_{l-1} < \bar{y}$

and

either $m=\infty$, or $m<\infty$ and $y_{m+1} < \bar{y}$.

Definition 4.3. *A negative semi-cycle* of a solution $\{y_n\}$ of Eq. (4.1) consists of a "string" of terms $\{y_l, y_{l+1}, \ldots, y_m\}$, all less than the equilibrium $\bar{y}$, with $l \ge -k$ and $m \le \infty$ and such that

either $l=-k$, or $l>-k$ and $y_{l-1} \ge \bar{y}$

and

either $m = \infty$, or $m < \infty$ and $y_{m+1} \geq \bar{y}$.

The first semi-cycle of a solution starts with the term y_{-k} and is positive if $y_{-k} \geq \bar{y}$ and negative if $y_{-k} < \bar{y}$.

Definition 4.4. A solution $\{y_n\}$ of Eq. (4.1) is called *nonoscillatory* if there exists $N \geq -k$ such that $y_n > \bar{y}$ for all $n \geq N$ or $y_n < \bar{y}$ for all $n \geq N$.

And a solution $\{y_n\}$ is called *oscillatory* if it is not nonoscillatory.

Now, we will list some theorems which will be useful in our investigation.

Theorem 4.1 [13]**.** *Assume that $f \in [(0, \infty) \times (0, \infty), (0, \infty)]$ is such that: f(x, y) is increasing in x for each fixed y, and f(x, y) is decreasing in y for each fixed x. Let $\bar{x}$ be a positive equilibrium of equation*

$$x_{n+1} = f(x_n, x_{n-k}) \tag{4.2}$$

Then except possibly for the first semi-cycle, every oscillatory solution of Eq. (4.2) has semi-cycle of length at least k. Furthermore, if we assume that

$$f(u, u) = \bar{x} \tag{4.3}$$

and

$$f(x, y) < x \text{ for every } \bar{x} < y < x \tag{4.4}$$

then $\{x_n\}$ oscillates about the equilibrium $\bar{x}$ with semi-cycles of length $k+1$ or $k+2$, except possibly for the first semi-cycle which may have length k. The extreme in each semi-cycle occurs in the first term if the semi-cycle has two terms and in the second term if the semi-cycle has three terms, and in the $k+1$ term if the semi-cycle has $k+2$ terms.

Theorem 4.2 [13]**.** *Assume that $f \in [(0, \infty) \times (0, \infty), (0, \infty)]$ is such that: f(x, y) is increasing in x for each fixed y, and f(x, y) is increasing in y for each fixed x. Let $\bar{x}$ be a positive equilibrium of Eq. (4.2). Then except possibly for the first semi-cycle, every oscillatory solution of Eq. (4.2) has semi-cycle of length k.*

Now, we give necessary and sufficient condition for Eq. (2.1) to have a prime period-two solution and we exhibit all prime period-two solutions of the equation.

Theorem 4.3. *If k is even, then Eq. (2.1) has no nonnegative prime period-two solution.*

Proof. Let k is even. Assume for the sake of contradiction that there exist distinctive nonnegative real number Φ and Ψ, such that

$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots$

is a prime period two solution of Eq. (2.1), then Φ and Ψ satisfy

$$\Phi = \frac{D + p\Psi + \Psi}{q\Psi + \Psi},$$

and

$$\Psi = \frac{D + p\Phi + \Phi}{q\Phi + \Phi}.$$

so

$$\Phi = \frac{\Psi(p+1) + D}{\Psi(q+1)}$$

and

$$\Psi = \frac{D + \Phi(p+1)}{\Phi(q+1)}$$

By substituting Φ into the equation of Ψ we get easily that$\Phi = \Psi$. This contradicts the hypothesis that Φ and Ψ distinct nonnegative real number.

Thus there exists no prime periodic-two solution for Eq. (2.1). □

Theorem 4.4. *If k is odd, then Eq. (2.1) has no nonnegative prime period-two solution.*

Proof. Let k is odd. Assume for the sake of contradiction that there exist distinctive nonnegative real number Φ and Ψ, such that

$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots$

is a prime period two solution of Eq. (2.1), then Φ and Ψ satisfy

$$\Phi = \frac{D + \Psi + p\Phi}{\Psi + q\Phi}$$

and

$$\Psi = \frac{D + \Phi + p\Psi}{\Phi + q\Psi}$$

We get that

$$\Phi = \Psi$$

and that is contradicts the fact that Ψ and Ψ must be different.

Then Eq. (2.1) has no prime two-periodic solution if k is odd.

This completes the proof. □

Then we can get out this results.

Corollary 4.1. *Eq. (2.1) possess no prime periodic-two solution.*

Semi-cycle analysis of the solution of Eq. (2.1) is a powerful tool for a detailed understanding of the entire character of solutions.

Next, we present some results about the semi-cycle character of solutions of Eq. (2.1).

Theorem 4.5. *Let $\{y_n\}$ be a nontrivial solution of Eq. (2.1), then the following statements are true:*

(1) *Assume $D + p > q$, then $\{y_n\}$ oscillates about the equilibrium $\bar{y}$ with semi-cycles of length $k+1$ or $k+2$, except possibly for the first semi-cycle which may have length k. The extreme in each semi-cycle occurs in the first term if the semi-cycle has two terms and in the second term if the semi-cycle has three terms, and in the $k+1$ term if the semi-cycle has $k+2$ terms.*
(2) *Assume $D + p < q$, then either $\{y_n\}$ oscillates about the equilibrium $\bar{y}$ with semi-cycles of length k after the first semi-cycle, or $\{y_n\}$ converges monotonically to $\bar{y}$.*

Proof.

(1) The proof follows from Theorem 4.1 by observing that the condition $D + p > q$ implies that the function

$$f(x, y) = \frac{D + x + py}{x + qy}$$

is increasing in x and decreasing in y. This function also satisfies conditions 4.3, 4.4.
(2) The proof follows from Theorem 4.2 by observing that the condition $D + p < q$ implies that the function

$$f(x, y) = \frac{D + x + py}{x + qy}$$

is decreasing in x and increasing in y.

The proof is complete. □

We now examine the existence of intervals which attract all solution of Eq. (2.1).

Theorem 4.6. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (2.1). Then the following statements are true :*

(1) *Suppose $D+p<q$ and assume that for some $N \geq 0$.*

$$y_{N-k}, \ldots, y_{N-1}, y_N \in \left[\frac{D+p}{q}, 1\right],$$

then

$$y_n \in \left[\frac{D+p}{q}, 1\right], \text{ for all } n > N.$$

(2) *Suppose $D+p>q$ and assume that for some $N \geq 0$.*

$$y_{N-k}, \ldots, y_{N-1}, y_N \in \left[1, \frac{D+p}{q}\right],$$

then

$$y_n \in \left[1, \frac{D+p}{q}\right], \text{ for all } n > N.$$

Proof.

(1) If for some $N>0$,

$$y_{N-k}, \ldots, y_{N-1}, y_N \in \left[\frac{D+p}{q}, 1\right],$$

then $\frac{D+p}{q} \leq y_{N-k} \leq 1$, then

$$\begin{aligned} y_{N+1} &= \frac{D+y_N+py_{N-k}}{y_N+qy_{N-k}} \\ &\leq \frac{y_N+D+p}{y_N+q\frac{D+p}{q}} \\ &\leq 1. \end{aligned}$$

Now take into consideration that the function

$$f(u,v) = \frac{D+u+pv}{u+qv}$$

is increasing in u and decreasing in v. Then for

$$y_{N+1} = \frac{D+y_N+py_{N-k}}{y_N+qy_{N-k}}$$

y_{N+1} is increasing in y_N for some fixed value for y_{N-k}. We can take this fixed value for y_{N-k} to be 1.and since $\frac{D+p}{q} \leq y_N \leq 1$ then

$$\begin{aligned} y_{N+1} &= \frac{D+y_N+py_{N-k}}{y_N+qy_{N-k}} \\ &\geq \frac{D+\frac{D+p}{q}+p}{\frac{D+p}{q}+q} \\ &\geq \frac{(D+p)(1+\frac{1}{q})}{q(1+\frac{1}{q}\frac{D+p}{q})} \\ &\geq \frac{D+p}{q} \end{aligned}$$

By Mathematical Induction we can prove that

$$\frac{D+p}{q} \leq y_n \leq 1 \ \textit{for all } n \geq N$$

Then

$$y_n \in \left[\frac{D+p}{q}, 1\right], \text{ for all } n > N.$$

(2) This proof is similar to the above one and will be omitted. The proof is complete. □

Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (2.1). Then the following identities are hold :

$$y_{n+1} - 1 = (q-p)\frac{\frac{D}{(q-p)} - y_{n-k}}{y_n+qy_{n-k}} \tag{4.5}$$

$$y_{n+1} - (\frac{D+p}{q}) = \frac{(1-(\frac{D+p}{q}))y_n + D(1-y_{n-k})}{y_n+qy_{n-k}} \tag{4.6}$$

First we will analyze the semi-cycles of the solution $\{y_n\}_{n=-k}^{\infty}$ under the assumption that

$$D+p>q, \ q>p \tag{4.7}$$

The following result is a direct consequences of (4.5)–(4.6).

Lemma 4.1. *Assume that (4.6) holds and let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (2.1). Then the following statements are true:*

(1) *If for some $N \geq 0$, $y_{N-k} < (D+p)/q$, then $y_{N+1} > 1$;*
(2) *If for some $N \geq 0$, $y_{N-k} = D/(q-p)$, then $y_{N+1} = 1$;*
(3) *If for some $N \geq 0$, $y_{N-k} > D/(q-p)$, then $y_{N+1} < 1$;*
(4) *If for some $N \geq 0$, $y_{N-k} \geq 1$, then $y_{N+1} < (D+p)/q$;*
(5) *If for some $N \geq 0$, $y_{N-k} \leq 1$, then $y_{N+1} \geq 1$;*
(6) *If for some $N \geq 0$, $1 \leq y_{N-k} \leq (D+p)/q$, then $1 \leq y_{N+1} \leq (D+p)/q$;*
(7) *If for some $N \geq 0$, $1 \leq y_{N-k}, y_N \leq (D+p)/q$, then $1 \leq y_n \leq (D+p)/q$, for all $n \geq N$; Thats $[1, (D+p)/q]$ is an invariant interval of Eq. (2.1).*
(8) $1 < \bar{y} < (D+p)/q$,
Indeed: when $D+p>q$

$$Dq + Dp + p^2 > pq + Dq$$

$$Dq > pq + Dq - Dp - p^2$$

$$Dq > (D+p)(q-p)$$

$$\frac{D}{q-p} > \frac{D+p}{q}$$

Theorem 4.7. *Assume that Eq. (2.1) holds. Then every nontrivial and oscillatory solution of Eq. (2.1) which lies in the interval $[1, (D+p)/q]$, oscillates about $\bar{y}$ with semi-cycle of length k or $k+1$.*

Now we will analyze the semi-cycles of the solution $\{y_n\}_{n=-k}^{\infty}$ under the assumption that

$$D+p<q, \ q>p \tag{4.8}$$

The following is a direct consequences of (4.5)–(4.6) and (4.8).

Lemma 4.2. *Assume that (4.8) holds and let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (2.1). Then the following statements are true:*

(1) *If for some $N \geq 0$, $y_{N-k} > (D+p)/q$, then $y_{N+1} < 1$;*
(2) *If for some $N \geq 0$, $y_{N-k} = D/(q-p)$, then $y_{N+1} = 1$;*
(3) *If for some $N \geq 0$, $y_{N-k} < D/(q-p)$, then $y_{N+1} > 1$;*
(4) *If for some $N \geq 0$, $y_{N-k} \leq 1$, then $y_{N+1} > (D+p)/q$;*
(5) *If for some $N \geq 0$, $y_{N-k} \leq (D+p)/q$, then $y_{N+1} > (D+p)/q$;*
(6) *If for some $N \geq 0$, $(D+p)/q \leq y_{N-k} \leq 1$, then $(D+p)/q \leq y_{N+1} \leq 1$;*
(7) *If for some $N \geq 0$, $(D+p)/q \leq y_{N-k}, y_N \leq 1$, then $(D+p)/q \leq y_n \leq 1$, for all $n \geq N$; Thats $[1, (D+p)/q]$ is an invariant interval of Eq. (2.1).*

(8) $(D+p)/q < \bar{y} < 1$,
Indeed: when $D+p<q$

$$Dq + Dp + p^2 < pq + Dq$$

$$Dq < pq + Dq - Dp - p^2$$

$$Dq < (D+p)(q-p)$$

$$\frac{D}{q-p} < \frac{D+p}{q}$$

Theorem 4.8. *Assume that Eq. (2.1) holds. Then every nontrivial and oscillatory solution of Eq. (2.1) which lies in the interval* $[(D+p)/q, 1]$, *oscillates about* $\bar{y}$ *with semi-cycle of length at least* $k+1$.

Next we will analyze the semi-cycles of the solution $\{y_n\}_{n=-k}^{\infty}$ under the assumption that

$$D + p = q \tag{4.9}$$

In this case Eq. (2.1) reduces to

$$y_{n+1} = \frac{D + y_n + p y_{n-k}}{y_n + (D+p) y_{n-k}} \tag{4.10}$$

with the unique equilibrium point $\bar{y} = 1$. Also Eqs. (4.5)–(4.6) reduce to

$$y_{n+1} - 1 = \frac{D(1 - y_{n-k})}{y_n + (D+p) y_{n-k}} \tag{4.11}$$

and so the following results follow immediately.

Lemma 4.3. *Let* $\{y_n\}_{n=-k}^{\infty}$ *be a solution of Eq. (4.10). Then the following statements are true:*

(1) *If for some* $N \geq 0$, $y_{N-k} < 1$, *then* $y_{N+1} > 1$;
(2) *If for some* $N \geq 0$, $y_{N-k} = 1$, *then* $y_{N+1} = 1$;
(3) *If for some* $N \geq 0$, $y_{N-k} > 1$, *then* $y_{N+1} < 1$;

Corollary 4.2. *Let* $\{y_n\}_{n=-k}^{\infty}$ *be a nontrivial solution of Eq. (4.10). Then* $\{y_n\}_{n=-k}^{\infty}$ *oscillates about the equilibrium 1.*

Now assume that $D+p>q$ and let $\{y_n\}_{n=-k}^{\infty}$ be a solution which does not eventually lie in the interval $I = [1, (D+p)/q]$. Then one can see that the solution oscillates relative to the interval $I = [1, (D+p)/q]$, essentially in the following two ways:

(1) k+1 consecutive terms in $((D+p)/q, \infty)$ are followed by k+1 consecutive terms in $((D+p)/q, \infty)$ and so on. The solution never visits the interval $(1, (D+p)/q)$.
(2) There exists exactly k terms in $((D+p)/q, \infty)$, which is followed by exactly k terms in $(1, (D+p)/q)$, which is followed by exactly k terms in (0, 1), which is followed by exactly k terms in $(1, (D+p)/q)$, which is followed by exactly k terms in $((D+p)/q, \infty)$, and so on. The solution visits consecutively the intervals $\ldots, ((D+p)/q, \infty), (1, (D+p)/q), (0, 1), (1, (D+p)/q), ((D+p)/q, \infty), \ldots$ in order with k terms per interval.

The situation is essentially the same relative to the interval $[(D+p)/q, 1]$ when $D+p<q$.

5. The global stability

The next results are about the global stability for the positive equilibrium of Eq. (2.1).

Here are the theorems that we need.

Theorem 5.1. *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), n = 0, 1 \ldots \tag{5.1}$$

where $k \in \{1, 2, \ldots\}$. *Let* $I = [a, b]$ *be some interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \longrightarrow [a, b]$$

is a continuous function satisfying the following properties :

(a) f(x, y) is non-increasing in each of its arguments;

(b) If $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system* $f(M, M) = m$ *and* $f(m, m) = M$ *then* $m = M$.

Then Eq. (5.1) has a unique equilibrium $\bar{y}$ *and every solution of Eq. (5.1) converges to* $\bar{y}$.

Theorem 5.2. *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), n = 0, 1 \tag{5.2}$$

where $k \in \{1, 2, \ldots\}$. *Let* $I = [a, b]$ *be some interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \longrightarrow [a, b]$$

is a continuous function satisfying the following properties :

(a) f(x, y) is non-decreasing in $x \in [a, b]$ *for each* $y \in [a, b]$, *and f(x, y) is non-increasing in* $y \in [a, b]$ *for each* $x \in [a, b]$;

(b) If $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system* $f(m, M) = m$ *and* $f(M, m) = M$ *then* $m = M$.

Then Eq. (5.2) has a unique equilibrium $\bar{y} \in [a, b]$ *and every solution of Eq. (5.2)) converges to* $\bar{y}$.

Now we will apply these theorems on our equation.

Theorem 5.3. *Assume that* $D+p>q$, *then the positive equilibrium of Eq. (2.1)) on the interval* $[1, \frac{D+p}{q}]$ *is globally asymptotically stable.*

Proof. This proof can be done easily depending on Theorem 5.1.

Assume that $D+p>q$ and consider the function

$$f(x, y) = \frac{D + x + py}{x + qy} \tag{5.3}$$

First, note that f(x,y) on the interval $[1, \frac{D+p}{q}]$ is nonincreasing in both of its arguments x, y.

Second, let $(m, M) \in [a, b] \times [a, b]$ is a solution of the system $f(M, M) = m$ and $f(m, m) = M$, then

$$m = \frac{D + M + pM}{M + qM}$$

and

$$M = \frac{D + m + pm}{m + qm}$$

but as we showed that our equation has no periodic-two solutions, then the only solution is $m = M$.

So, the both conditions of Theorem 5.1 hold, so, If $\bar{y}$ is an equilibrium point of Eq. (2.1), then every solution of Eq. (2.1) converges to $\bar{y}$ in the interval $[1, \frac{D+p}{q}]$. As $\bar{y}$ is asymptotically stable, then it is globally asymptotically stable on $[1, \frac{D+p}{q}]$. □

Theorem 5.4. *Assume that* $D+p<q$, *then the positive equilibrium of Eq. (2.1) on the interval* $[\frac{D+p}{q}, 1]$ *is globally asymptotically stable.*

Proof. This proof can be done easily depending on Theorem 5.2.

Assume that $D+p<q$ and consider the function

$$f(x, y) = \frac{D + x + py}{x + qy} \tag{5.4}$$

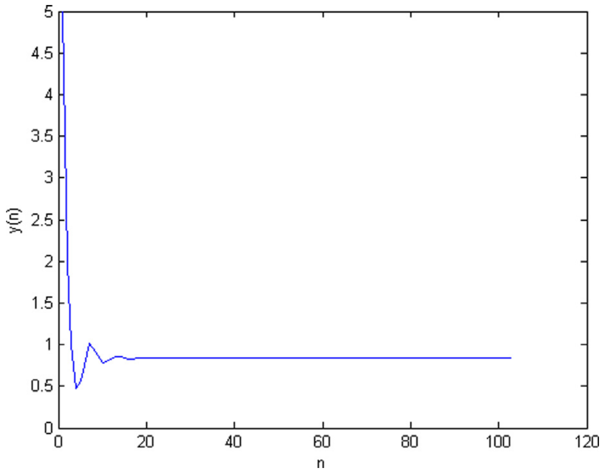

Fig. 1. The Behavior of the Equilibrium point of the Equation $y_{n+1} = \frac{0.5+y_n+0.5y_{n-2}}{y_n+1.5y_{n-2}}$.

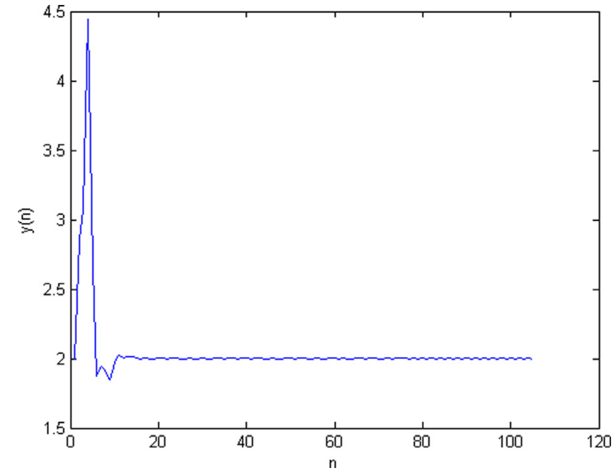

Fig. 2. The Behavior of the Equilibrium point of the Equation $y_{n+1} = \frac{3+y_n+2y_{n-4}}{y_n+1.25y_{n-4}}$.

First, note that f(x,y) on the interval $[\frac{D+p}{q}, 1]$ is nondecreasing in x, and nonincreasing in y.

Second, let $(m, M) \in [a, b] \times [a, b]$ is a solution of the system $f(m, M) = m$ and $f(M, m) = M$, then

$$m = \frac{D + m + pM}{m + qM}$$

and

$$M = \frac{D + M + pm}{M + qm}$$

but as we showed that our equation has no periodic-two solutions, then the only solution is $m = M$.

So, the both conditions of Theorem 5.2 hold, so, If $\bar{y}$ is an equilibrium point of Eq. (2.1), then every solution of Eq. (2.1) converges to $\bar{y}$ in the interval $[\frac{D+p}{q}, 1]$. As $\bar{y}$ is asymptotically stable, then it is globally asymptotically stable on $[\frac{D+p}{q}, 1]$. □

6. Numerical discussion and conclusion

Here in this section, we will study the global stability of our equation numerically based on some data and figures that we can get using MATLAB 6.5.

Example 1. Assume that Eq. (1.1) holds, take $k = 2$, $\alpha = 1$, $\beta = 2$, $\gamma = 1$, $B = 2$, $C = 3$. So the equation will be reduced to the following:

$$x_{n+1} = \frac{1 + 2x_n + x_{n-2}}{2x_n + 3x_{n-2}}, \qquad n = 0, 1, \dots \tag{6.1}$$

We assumed that the initial points $\{x_{-2}, x_{-1}, x_0\}$ *all to be* $\in (0, \infty)$ and are $\{0.2, 0.5, 1\}$.

The change of variable $x_n = \frac{\beta}{B} y_n$ changes the equation into the equation

$$y_{n+1} = \frac{D + y_n + py_{n-2}}{y_n + qy_{n-2}}, \qquad n = 0, 1, \dots \tag{6.2}$$

where $D = \frac{B\alpha}{\beta^2} = 0.5$, $p = \frac{\gamma}{\beta} = 0.5$, $q = \frac{C}{B} = 1.5$. Then the theoretical positive equilibrium point will be $\bar{y} = 0.83851648071345$.

By theory, the equilibrium point $\bar{y}$ is globally asymptotically stable, and it is obvious from Fig. 1 that it is globally asymptotically stable, as we have shown theoretically. Let's take another example now.

Example 2. Assume that Eq. (1.1) holds, take $k = 4$, $\alpha = 3$, $\beta = 1$, $\gamma = 2$, $B = 4$, $C = 5$. So the equation will be reduced to the following:

$$x_{n+1} = \frac{3 + x_n + 2x_{n-4}}{4x_n + 5x_{n-4}}, \qquad n = 0, 1, \dots \tag{6.3}$$

We assumed that the initial points $\{x_{-4}, x_{-3}, \dots, x_0\}$ *all to be* $\in (0, \infty)$ and are $\{2, 1.4, 1.3, 0.9, 1.5\}$.

The change of variable $x_n = \frac{\beta}{B} y_n$ changes the equation into the equation

$$y_{n+1} = \frac{D + y_n + py_{n-4}}{y_n + qy_{n-4}}, \qquad n = 0, 1, \dots \tag{6.4}$$

where $D = \frac{B\alpha}{\beta^2} = 3$, $p = \frac{\gamma}{\beta} = 2$, $q = \frac{C}{B} = 1.25$.then the theoretical positive equilibrium point will be $\bar{y} = 2$.

By theory, the equilibrium point $\bar{y} = 2$ is globally asymptotically stable, and it is obvious from Fig. 2 that it is globally asymptotically stable, as we have shown theoretically Fig. 2.

Here, it is obvious that from Fig. 2 that our equilibrium point is around the point 2.

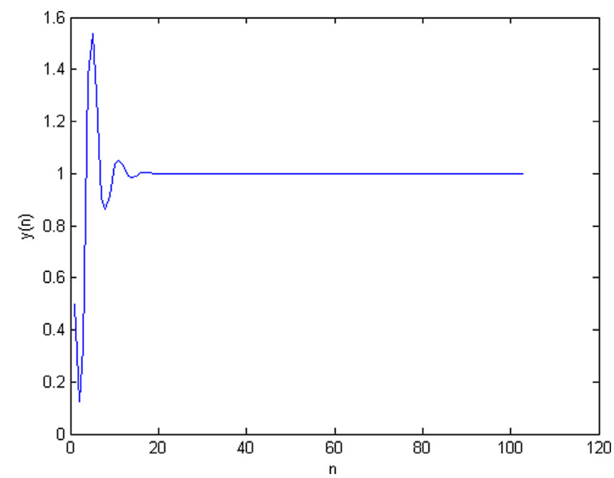

Fig. 3. The Behavior of the Equilibrium point of the Equation $y_{n+1} = \frac{1+y_n+y_{n-2}}{y_n+2y_{n-2}}$.

Example 3. Assume that Eq. (1.1) holds, take $k = 2$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $B = 1$, $C = 2$. So the equation will be reduced to the following:

$$x_{n+1} = \frac{1 + x_n + x_{n-2}}{x_n + 2x_{n-2}}, \qquad n = 0, 1, \dots \tag{6.5}$$

We assumed that the initial points $\{x_{-2}, x_{-1}, x_0\}$ *all to be* $\in (0, \infty)$ and are $\{2, 8, 3\}$.

The change of variable $x_n = \frac{\beta}{B} y_n$ changes the equation into the equation

$$y_{n+1} = \frac{D + y_n + p y_{n-2}}{y_n + q y_{n-2}}, \qquad n = 0, 1, \ldots \tag{6.6}$$

where $D = \frac{B\alpha}{\beta^2} = 1,\ p = \frac{\gamma}{\beta} = 1,\ q = \frac{C}{B} = 2$. Then the theoretical positive equilibrium point will be $\bar{y} = 1$.

Its obvious from Fig. 3 that the equilibrium point $\bar{y} = 1$ is globally asymptotically stable.

We can conclude now that our theoretical discussion was true for this kind of Discrete Nonlinear Equations.

Conclusion

In this manuscript, we have studied higher order rational difference equations a generalization of the second order rational difference equation studied in [7,13,14].

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