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On Proximality and Sets of Operators. II. Nonexistence of Best Approximation from the Sets of Finite Rank Operators

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The most important result in this paper is that the set $K_n(I_1, c_0)$ is not proximal in $L(I_1, c_0)$. This gives a negative solution to Problem 5.2.1 and a positive solution to Problem 5.2.4 of Deutsch, Mach, and Saatkamp (*J. Approx. Theory* 33 (1981), 199-213). © 1986 Academic Press, Inc.

INTRODUCTION

If A is a closed subset of the normed linear space X , then A is said to be "proximal" in X if for each $x \in X$, there is $y_0 \in A$ such that

$$\|x - y_0\| = d(x, A) = \inf\{\|x - y\|; y \in A\}.$$

In this case y_0 is called a "best approximation" to x from A . If B is a subset of X then

$$\delta(B, A) = \sup\{d(x, A); x \in B\},$$

is the deviation of B from A , and

$$d_n(B, X) = \inf\{\delta(B, N); N \text{ is an } n\text{-dimensional subspace of } X\},$$

is the Kolmogorov n -width of B in X .

If X and Y are two normed linear spaces, then $L(X, Y)$ denotes the set of all bounded linear operators from X to Y , $K(X, Y)$ the set of all compact operators in $L(X, Y)$, and $K_n(X, Y)$ the set of all operators in $L(X, Y)$ of rank $\leq n$.

* Part of a thesis submitted for the Ph. D. degree at the University of Newcastle Upon Tyne, written under the supervision of Dr. A. L. Brown.

The proximality of $K_n(X, Y)$ in $L(X, Y)$ and $K(X, Y)$ has been studied by several authors. It is known that if Y^* is the dual space of Y then $K_n(X, Y^*)$ is proximal in $L(X, Y^*)$. The more interesting problem turns out to be the proximality of $K_n(X, Y)$ in $K(X, Y)$ and $L(X, Y)$ when $Y = C_0(Q)$, for arbitrary locally compact Hausdorff space Q . Deutsch *et al.* [2] proved that, if X^* is strictly convex then $K_n(X, C_0(Q))$ is proximal in $K(X, C_0(Q))$, and Kamal [4] proved that, if X^* is uniformly convex then $K_n(X, C_0(Q))$ is proximal in $L(X, C_0(Q))$.

This paper contains a further study for the proximality of $K_n(X, C_0(Q))$ in $K(X, C_0(Q))$ and $L(X, C_0(Q))$. In Section 1, it is shown that for each positive integer $n \geq 1$, the set $K_n(l_1, c_0)$ is not proximal in $L(l_1, c_0)$. This gives a negative solution to Problem 5.2.1 of Deutsch *et al.* [2]. Since by Mach and Ward [5], $K(l_1, c_0)$ is proximal in $L(l_1, c_0)$, it follows that the solution of Problem 5.2.4 of Deutch *et al.* [2] is positive, that is there are Banach spaces X and Y such that $K_n(X, Y)$ is not proximal in $L(X, Y)$, whereas $K(X, Y)$ is proximal in $L(X, Y)$.

The Hausdorff space Q will be said to contain Q_0 if it contains an infinite convergent sequence of distinct elements. In Section 2, it is shown that if Q contains Q_0 , then $K_n(l_1, C(Q))$ is not proximal in $K(l_1, C(Q))$. This shows that generally it is not necessary that $K_n(X, Y)$ is proximal even in $K(X, Y)$. Brown [1] proved that if Q satisfies a certain condition then there is a bounded subset A in an $(n + 3)$ -dimensional subspace of $C(Q)$, such that the n -width of A , $d_n(A, C(Q))$ is not attained. Although c_0 is not isometric to any $C(Q)$ for which Q satisfies the given condition, it is shown that for each positive integer $n \geq 1$, there is a bounded subset A of c_0 , such that the n -width of A , $d_n(A, c_0)$ is not attained. Since it is easy to show that for each relatively compact subset K of c_0 , the n -width $d_n(K, c_0)$ is attained, it follows that this result cannot be improved in c_0 .

The rest of the Introduction will cover some definitions and known results, that will be used frequently in this paper. If Q is a Hausdorff topological space, X is a normed linear space and τ is a topology defined on X , then $C(Q, (X, \tau))$ denotes the set of all bounded functions from Q to X , which are continuous with respect to τ . If $\tau = \|\cdot\|$, then $C_0(Q, X) = \{f \in C(Q, (X, \|\cdot\|)); \forall \varepsilon > 0 \text{ the set } \{q \in Q; \|f(q)\| \geq \varepsilon\} \text{ is compact}\}$. If $X = R$ the set of real numbers, then $C_0(Q, R)$ is denoted by $C_0(Q)$. If X^* is the dual space of X then

$$C_0(Q, (X^*, \omega^*)) = \{f \in C(Q, (X^*, \omega^*)); \hat{x} \circ f \in C_0(Q) \forall x \in X\},$$

where \hat{x} is the image of x under the canonical injection of X in X^{**} .

As a special case if Q is the set of all positive integers, then $C_0(Q, X)$ consists of all bounded sequences in X that converge to zero, and will be denoted by $c_0(X)$.

The importance of introducing the Banach space $C_0(Q, X)$ can be seen in

0.1. LEMMA. Let X be a Banach space, Q a locally compact Hausdorff space, and for each nonnegative integer n , let

$$C_n = \bigcup_N \{C_0(Q, N); N \text{ is an } n\text{-dimensional subspace of } X^*\}.$$

The set $K_n(X, C_0(Q))$ is proximal in $L(X, C_0(Q))$ [resp. $K(X, C_0(Q))$] if and only if, for each $f \in C_0(Q, (X^*, \omega^*))$ [resp. $f \in C_0(Q, X^*)$], there is an n -dimensional subspace N_0 of X^* , and $g \in C_0(Q, N_0)$ such that

$$\|f - g\| = d(f, C_n).$$

The proof of this lemma can be obtained from Deutsch et al. [2], and can be found in Kamal [4].

0.2. DEFINITION. Let X be a Banach space, Q a locally compact Hausdorff space, and let C_n be as in Lemma 0.1.

- (a) For each $f \in C_0(Q, (X^*, \omega^*))$ let $a_n(f)$ denotes $d(f, C_n)$.
- (b) For each $T \in L(X, C_0(Q))$ let $a_n(T)$ denotes $d(T, K_n(X, C_0(Q)))$.

It is obvious from Lemma 0.1 that there is no problem in introducing the same symbol " a_n " in both cases of Definition 0.2, since $a_n(f)$ is attained for each $f \in C_0(Q, (X^*, \omega^*))$ [resp. $f \in C_0(Q, X^*)$], if and only if $a_n(T)$ is attained for each $T \in L(X, C_0(Q))$ [resp. $T \in K(X, C_0(Q))$].

1. $K_n(l_1, c_0)$ IS NOT PROXIMAL IN $L(l_1, c_0)$

In this section it will be shown that for each positive integer $n \geq 1$, the set $K_n(l_1, c_0)$ is not proximal in $L(l_1, c_0)$. After referring to Lemma 0.1, "taking Q to be the set of positive integers," it is enough to construct for each $n \geq 1$, a bounded sequence in l_∞ , such that $\eta_i \rightarrow \omega^* 0$ and $a_n(\{\eta_i\}_{i=1}^\infty)$ is not attained. The main steps in the proof are to construct in $l_{2^n+1}^\infty$ a finite subset satisfying a certain condition (Lemma 1.2), the by injecting this set in a certain way in l_∞ (Lemma 1.3) a bounded sequence $\{\eta_i\}_{i=1}^\infty$ will be constructed in l_∞ with the required properties.

1.1. LEMMA. Let n be a fixed positive integer, $m = 2^n$, A the set of all $\sigma = (\sigma_1, \dots, \sigma_n) \in l_n^\infty$ with $|\sigma_i| = 1$ for $i = 1, 2, \dots, n$ and let $A = (a_{ij})_{(i,j)=(1,1)}^{(n,m)}$ be the matrix in which the rows are the element of A . Let b_1, \dots, b_n be the columns of A , $z'_i = 2b_i$ for $i = 1, 2, \dots, n$ and $\gamma = (1, 1, \dots, 1) \in l_m^\infty$. Then

- (1) $d_{n-1}(\{z'_1, \dots, z'_n\}, l_m^\infty) = 2$.
- (2) For any number a , with $0 \leq a \leq 2$

$$d_n(\{z'_1, \dots, z'_n, a\gamma\}, l_m^\infty) = a.$$

Proof. (2) Let A be the balanced convex hull of $\{z'_1, \dots, z'_n, a\gamma\}$, and let $F_r(A)$ be the boundary of A . By Brown [1] $d_n(A, l_m^\infty) = \inf\{\|x\|; x \in F_r(A)\}$, thus $d_n(A, l_m^\infty) \leq a$. Second, let $x = \sum_{i=1}^n \alpha_i z'_i + \alpha_{n+1} a\gamma \in F_r(A)$, by the construction of the z'_i, s , and γ

$$\begin{aligned} \|x\| &= \left\| \sum_{i=1}^n \alpha_i z'_i + \alpha_{n+1} a\gamma \right\| = 2 \sum_{i=1}^n |\alpha_i| + a |\alpha_{n+1}| \\ &= 2 \sum_{i=1}^{n+1} |\alpha_i| + (a-2) |\alpha_{n+1}| \\ &= 2 + (a-2) |\alpha_{n+1}| \\ &= 2(1 - |\alpha_{n+1}|) + a |\alpha_{n+1}| \\ &\geq \min\{2, a\} = a. \quad \blacksquare \end{aligned}$$

In Lemma 1.2, for x and y in l_m^∞ and a, b in R , let $(ax, by) = (ax_1, ax_2, \dots, ax_m, by_1, by_2, \dots, by_m) \in l_m^\infty \times l_m^\infty = l_{2m}^\infty$. Conversely if $z = (z_1, \dots, z_{2m}) \in l_{2m}^\infty$ let;

$$\begin{aligned} P_1: l_{2m}^\infty &\rightarrow l_m^\infty, \\ P_1(z) &= (z_1, \dots, z_m) \end{aligned}$$

and

$$\begin{aligned} P_2: l_{2m}^\infty &\rightarrow l_m^\infty, \\ P_2(z) &= (z_{m+1}, \dots, z_{2m}). \end{aligned}$$

Clearly if F is an n -dimensional subspace of l_{2m}^∞ then $P_1(F)$ and $P_2(F)$ are subspaces of l_m^∞ , each of dimension less than or equal to n .

1.2. LEMMA. Let $z'_1, \dots, z'_n, \gamma$, n and m be as in Lemma 1.1. Let $Z_i = (z'_i, z'_i) \in l_{2m}^\infty$, $\phi = (\theta_1\gamma, \theta_2\gamma)$ and $\Psi = (\psi_1\gamma, \psi_2\gamma)$, where $\{\psi_1, \psi_2, \theta_1, \theta_2\}$ satisfies the following conditions.

- (1) $\theta_1 = 2$ and $\psi_1 > 1$.
- (2) $\theta_2\psi_2 < 0$, $|\psi_2| > |\theta_2|$ and $|\psi_2| + |\theta_2| \geq 2$.

Let F be an $(n+1)$ dimensional subspace of l_{2m}^∞ such that

$$\delta(\{Z_1, Z_2, \dots, Z_n, \Phi, \Psi\}, F) \leq 1.$$

If $\beta \in F$ and $\|\Psi - \beta\| \leq 1$ then $\|\beta\| \geq 1$.

Proof. Let $\{x_1, \dots, x_n, \alpha, \beta\} \subseteq F$ be so that $\|Z_i - X_i\| \leq 1$ for $i = 1, 2, \dots, n$, $\|\Phi - \alpha\| \leq 1$ and $\|\Psi - \beta\| \leq 1$, and let M be the subspace of F generated by

$\{x_1, \dots, x_n\}$. By Lemma 1.1 $\dim M = n$, so there are two real numbers a_1 and a_2 such that $|a_1| + |a_2| = 1$ and $a_1\alpha + a_2\beta \in M$. By Lemma 1.1(2) for $i = 1, 2$,

$$\begin{aligned} |\theta_i a_1 + \psi_i a_2| &\leq d((\theta_i a_1 + \psi_i a_2)\gamma, P_i(M)) \\ &\leq \|(\theta_i a_1 + \psi_i a_2)\gamma - P_i(a_1\alpha + a_2\beta)\| \\ &\leq |a_1| \|P_i(\Phi) - P_i(\alpha)\| + |a_2| \|P_i(\Psi) - P_i(\beta)\| \\ &\leq |a_1| + |a_2| \\ &= 1. \end{aligned}$$

Since $\theta_1 = 2$ and $\psi_1 > 1$, the case $i = 1$ gives $a_1 a_2 < 0$, and since $\psi_2 \theta_2 < 0$, the case $i = 2$ gives the inequality $|\theta_2| |a_1| + |\psi_2| |a_2| \leq 1$, so $(|\theta_2| - |\psi_2|) |a_1| + |\psi_2| |a_2| \leq 1$, but $|\psi_2| > |\theta_2|$ so

$$|a_1| \geq \frac{|\psi_2| - 1}{|\psi_2| - |\theta_2|} \geq \frac{|\psi_2| - ((|\psi_2| + |\theta_2|)/2)}{|\psi_2| - |\theta_2|} = \frac{1}{2}.$$

Therefore, $|a_1| \geq |a_2|$. Also by Lemma 1.1

$$\begin{aligned} 2 = \theta_1 &\leq d(\theta_1 \gamma, P_1(M)) \leq \left\| \theta_1 \gamma - P_1 \frac{(a_1 \alpha + a_2 \beta)}{a_1} \right\| \\ &\leq \|P_1(\Phi) - P_1(\alpha)\| + \left| \frac{a_2}{a_1} \right| \|P_1(\beta)\| \\ &\leq 1 + \left| \frac{a_2}{a_1} \right| \|\beta\|. \end{aligned}$$

So $\|\beta\| \geq (\theta_1 - 1) |a_1/a_2| \geq 1$. ■

In Lemma 1.3 l_∞ will be considered as $\prod_{i=1}^\infty l_{2^m}^\infty$, where $m = 2^n$, that is, $(y_1, y_2, \dots) \in \prod_{i=1}^\infty l_{2^m}^\infty$ means that $y_i \in l_{2^m}^\infty$.

1.3. LEMMA. Let Z_1, \dots, Z_n, γ , and m be as in Lemma 1.2, and for each positive integer $k \geq 1$ let;

$$\Phi_k = \left(2\gamma, \frac{-(k-1)}{k} \gamma \right) \in l_{2^m}^\infty,$$

and

$$\Psi_k = \left(\frac{k+1}{k} \gamma, \frac{k+1}{k} \gamma \right) \in l_{2^m}^\infty.$$

Define the sequence $\{n_i\}_{i=1}^\infty \prod_{i=1}^\infty l_{2m}^\infty$ as follows:

$$\begin{aligned} n_i &= (Z_i, Z_i, Z_i, \dots) && \text{for } i = 1, 2, \dots, n, \\ \eta_{n+1} &= (\Phi_1, \Phi_2, \Phi_3, \dots), \end{aligned}$$

and

$$\eta_{n+k} = (\underbrace{0, 0, \dots, 0}_{(k-1)\text{times}}, \Psi_k, 0, 0, \dots) \quad \text{for } k = 2, 3, \dots$$

Then;

- (1) $\eta_i \rightarrow^{\omega^*} 0$,
- (2) $a_{n+1}(\{\eta_i\}_{i=1}^\infty) = 1$,
- (3) $a_{n+1}(\{\eta_i\}_{i=1}^\infty)$ is not attained.

Proof. (1) It is clear that $\eta_i \rightarrow^{\omega^*} 0$.

(2) For each positive integer $k \geq 1$, let $y_k = (\gamma, (1/k)\gamma) \in l_{2m}^\infty$, and define $y = (y_1, y_2, y_3, \dots) \in \prod_{i=1}^\infty l_{2m}^\infty$. Let N_0 be the $(n+1)$ -dimensional subspace of $\prod_{i=1}^\infty l_{2m}^\infty$ generated by $\{\eta_1, \dots, \eta_n, y\}$. It will be shown that $d(\{\eta_i\}_{i=1}^\infty, c_0(N_0)) \leq 1$. Let $\varepsilon > 0$ be given, and let i_0 be a positive integer more than $1/\varepsilon$. Define the sequence $\{\tau_i\}_{i=1}^\infty$ in N_0 as follows:

$$\begin{aligned} \tau_i &= \eta_i && \text{for } i = 1, 2, \dots, n, \\ \tau_i &= y && \text{for } i = n+1, n+2, \dots, i_0 \\ \tau_i &= 0 && \text{for } i > i_0. \end{aligned}$$

Obviously $\tau_i \rightarrow 0$, that is $\{\tau_i\}_{i=1}^\infty \in c_0(N_0)$, and

$$\begin{aligned} \|\{\eta_i\}_{i=1}^\infty - \{\tau_i\}_{i=1}^\infty\| &= \sup_i |\eta_i - \tau_i| = \sup_{i > n} \|\eta_i - \tau_i\| \\ &= \sup\{ \|\eta_i - y\|; i = n+1, \dots, i_0 \} \cup \{ \|\eta_i\|; i > i_0 \} \\ &\leq 1 + \varepsilon. \end{aligned}$$

Since ε is arbitrary then $a_{n+1}(\{\eta_i\}_{i=1}^\infty) \leq d(\{\eta_i\}_{i=1}^\infty, c_0(N_0)) \leq 1$.

(3) Let N be an $(n+1)$ -dimensional subspace of $\prod_{i=1}^\infty l_{2m}^\infty$, and assume that $d(\{\eta_i\}_{i=1}^\infty, c_0(N)) \leq 1$. It will be shown that if there is a sequence $\{\tau_i\}_{i=1}^\infty$ in N such that $\|\{\eta_i\}_{i=1}^\infty - \{\tau_i\}_{i=1}^\infty\| \leq 1$, the $\tau_i \not\rightarrow 0$. For

$x = (x_1, x_2, \dots) \in \prod_{i=1}^{\infty} l_{2m}^{\infty}$ let $P_k: \prod_{i=1}^{\infty} l_{2m}^{\infty} \rightarrow l_{2m}^{\infty}$, $P_k(x) = x_k$, $k = 1, 2, \dots$
 Then for $k > 1$

$$P_k(\eta_i) = Z_i \quad \text{for } i = 1, 2, \dots, n,$$

$$P_k(\eta_{n+1}) = \Phi_k,$$

and

$$P_k(\eta_{k+n}) = \Psi_k.$$

So $\delta(\{Z_1, Z_2, \dots, Z_n, \Phi_k, \Psi_k\}, P_k(N)) \leq 1$. Since Φ_k and Ψ_k satisfy the conditions of Lemma 1.2, then $\|\tau_k\| \geq \|P_k(\tau_k)\| \geq 1$. To $\tau_i \rightarrow 0$.

1.4. *Note.* For the case $n = 1$, there is an easy example. Let $x_1 = (2, 1, 1, 1, \dots) \in l_{\infty}$,

$$x_i = \left(\underset{(i-1)\text{times}}{0}, 0, \dots, 0, \frac{i+1}{i}, \frac{-i}{i+1}, 0, 0, \dots \right) \quad \text{for } i = 2, 3, \dots$$

Then $x_i \rightarrow \omega^* 0$, and if N_0 is the 1-dimensional subspace of l_{∞} generated by the element $y_0 = (1, \frac{1}{2}, \frac{1}{3}, \dots)$, then

$$a_1(\{x_i\}_{i=1}^{\infty}) \leq d(\{x_i\}_{i=1}^{\infty}, c_0(N_0)) \leq 1.$$

Furthermore if $y \in l_{\infty}$ and N is the 1-dimensional subspace generated by y , one can show that if $\|x_i - \alpha_i y\| \leq 1$, $i = 1, 2, \dots$, then $\alpha_i \rightarrow 0$.

1.5. **THEOREM.** *For any positive integer $n \geq 1$, the set $K_n(l_1, c_0)$ is not proximal in $L(l_1, c_0)$.*

Proof. By Lemma 0.1, (taking Q to be the set of positive integer), $K_n(l_1, c_0)$ is proximal in $L(l_1, c_0)$, iff for each bounded sequence $\{x_i\}_{i=1}^{\infty}$ in l_{∞} with $x_i \rightarrow \omega^* 0$, there is an n -dimensional subspace N of l_{∞} , and a sequence $\{\tau_i\}_{i=1}^{\infty}$ in $c_0(N)$, such that $a_n(\{x_i\}_{i=1}^{\infty}) = \|\{x_i\}_{i=1}^{\infty} - \{\tau_i\}_{i=1}^{\infty}\|$. By Lemma 1.3 and note 1.4 this is not true. ■

Theorem 1.5 gives a negative solution for the Problem 5.2.1 in Deutsch *et al.* [2], and since by Mach and Ward [5], the set $K(l_1, c_0)$ is proximal in $L(l_1, c_0)$, it gives a positive solution for the Problem 5.2.4 in the same paper.

2. $K_n(l_1, C(Q))$ IS NOT PROXIMAL IN $K(l_1, C(Q))$ IF Q CONTAINS Q_0

In this section it will be shown that if Q is a compact Hausdorff space Q that contains Q_0 , then for each positive integer $n \geq 1$, the set $K_n(l_1, C(Q))$ is not proximal in $K(l_1, C(Q))$. It will be shown also that for each

positive integer $n \geq 1$, there is a bounded subset A of c_0 , such that the n -width of A , $d_n(A, c_0)$ is not attained.

The proof of the following lemma can be found in Feder [3]:

2.1. LEMMA. *Let $T: l_1 \rightarrow E$ be a bounded linear operator from l_1 into any Banach space E , and let B_{l_1} be the closed unit ball of l_1 . Then*

- (1) $a_n(T) = d_n(T(B_{l_1}), E)$
- (2) $a_n(T)$ is attained iff $d_n(T(B_{l_1}), E)$ is attained.

2.2. COROLLARY. *Let E be a Banach space and let n be any non-negative integer. Then the set $K_n(l_1, E)$ is proximal in $L(l_1, E)$ [resp. $K(l_1, E)$] iff $d_n(A, E)$ is attained for any countable bounded [resp. relatively compact] subset A of E*

Proof. If $A = \{x_1, x_2, \dots\} \subseteq E$ then let $T: l_1 \rightarrow E$ be the linear operator defined by $T(e_i) = x_i, i = 1, 2, \dots$, where $\{e_i\}_{i=1}^\infty$ is the standard basis in l_1 . Clearly T is bounded [resp. compact] if A is bounded [resp. relatively compact], thus the result follows from Lemma 2.1.

2.3. DEFINITION. Let Q be a locally compact Hausdorff space, Q will be said to "contain Q_0 " if it contains a subset that is homeomorphic to the one point compactification of the set of positive integers; that is to say if it contains an infinite convergent sequence of distinct elements. It is obvious that if Q does not contain Q_0 , then every subset Y of Q does not contain Q_0 . If, in the Hausdorff space Q , there is a nonisolated element b_0 that has a countable basis of neighborhoods, then Q contains Q_0 . Thus every first countable nondiscrete Hausdorff space contains Q_0 . Furthermore, there are separable compact infinite Hausdorff spaces, that do not contain Q_0 . Indeed if βN is the Stone-Cech compactification of the set of positive integers, then every subspace of βN does not contain Q_0 .

2.4. PROPOSITION. *Brown [1]; If Q is a compact Hausdorff space such that there is the $C(Q)$ satisfying;*

$$\overline{\{q \in Q; h(q) < 0\}} \cap \overline{\{q \in Q; h(q) > 0\}} \neq \phi,$$

then for any positive integer $n \geq 1$, there is a bounded subset A of $C(Q)$ lying in a $(n + 3)$ -dimensional subspace of $C(Q)$, such that $d_n(A, C(Q))$ is not attained.

2.5. LEMMA. Let Q be a compact Hausdorff space that contains Q_0 . There is a continuous function $h \in C(Q)$ satisfying;

$$\overline{\{q \in Q; h(q) < 0\}} \cap \overline{\{q \in Q; h(q) > 0\}} \neq \emptyset$$

Proof. Let $b_k \rightarrow b_0$ be an infinite convergent sequence in Q , such that $b_i \neq b_j$ when $i \neq j$. Define $f: \{b_k\}_{k=1}^{\infty} \cup \{b_0\} \rightarrow R$ by

$$f(b_k) = \begin{cases} \frac{1}{k} & \text{if } k \text{ is even,} \\ -\frac{1}{k} & \text{if } k \text{ is odd,} \\ 0 & \text{if } k = 0. \end{cases}$$

Then f is continuous. Since Q is compact Hausdorff space and $\{b_k\}_{k=1}^{\infty} \cup \{b_0\}$ is a closed subset of Q , then by Tietze extension Theorem, there is a continuous function $h \in C(Q)$ such that $h|_{\{b_k\}_{k=1}^{\infty} \cup \{b_0\}} = f$. It is obvious that $b_0 \in \overline{\{q \in Q; h(q) < 0\}} \cap \overline{\{q \in Q; h(q) > 0\}}$.

2.6. COROLLARY. If Q is a compact Hausdorff space that contains Q_0 then for each positive integer $n \geq 1$, the set $K_n(l_1, C(Q))$ is not proximal in $K(l_1, C(Q))$.

Proof. By Proposition 2.4 and Lemma 2.5, for each $n \geq 1$, there is a bounded set A of $C(Q)$, such that A lies in an $(n+3)$ -dimensional subspace of $C(Q)$ and $d_n(A, C(Q))$ is not attained. Since A lies in a finite dimensional subspace of $C(Q)$, it follows that A is relatively compact and separable, so the result follows from Corollary 2.2.

2.7. COROLLARY. For any positive integer $n \geq 1$, there is a bounded subset A of c_0 such that $d_n(A, c_0)$ is not attained.

Proof. Follows from Theorem 1.5 and Corollary 2.2.

2.8. COROLLARY. For any relatively compact subset A of c_0 and any non-negative integer $n \geq 0$, the n -width $d_n(A, c_0)$ is attained.

Proof. By Deutsch *et al.* [2], the set $K_n(l_1, c_0)$ is proximal in $K(l_1, c_0)$, thus by Corollary 2.2 for any relatively compact subset A of c_0 and any non-negative integer $n \geq 0$ the n -width $d_n(A, c_0)$ is attained. ■

Corollary 2.6 is the first example in which the set $K_n(X, Y)$ is not

proximal in $K(X, Y)$. Corollary 2.7 cannot be obtained from Proposition 2.4, because c_0 is not isometric to any $C(Q)$, for which Q satisfies the condition of the proposition, indeed Corollary 2.8 shows that the proposition is not true if $C(Q) = c_0$.

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