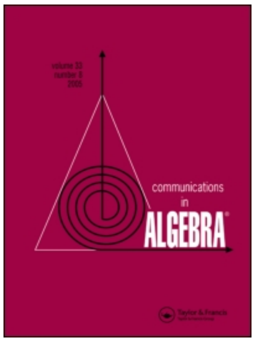


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# DIVISIBILITY CONDITIONS IN COMMUTATIVE RINGS WITH ZERODIVISORS

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## ABSTRACT

Let $R$ be a commutative ring. In this paper, we give several divisibility and ring-theoretic conditions for $R$ or $T(R)$ to be either zero-dimensional or von Neumann regular. We also consider divisibility conditions related to $R$ being completely integrally closed and study several closedness conditions which hold with respect to units of $T(R)$.

## INTRODUCTION

In this paper, we continue our investigation begun in Ref. [1–3, 4] of extending ring-theoretic properties in integral domains to the context of commutative rings with zerodivisors by replacing conditions on elements of the total quotient ring $T(R)$ with internal divisibility conditions on elements



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of the ring $R$. In the first section, we consider divisibility conditions on $R$ which are equivalent to $R$ or $T(R)$ being either zero-dimensional or von Neumann regular. In the second section, we give some additional conditions on $R$ for $T(R)$ to be either zero-dimensional or von Neumann regular and relate this to ideas used in, Refs. [1,2] for subrings of a direct product of integral domains. The two main results (Theorems 2.2 and 2.3) are that $T(R)$ is zero-dimensional (resp., von Neumann regular) if and only if for each $x \in R$, there is a $y \in R$ such that $xy \in \text{nil}(R)$ (resp., $xy = 0$) and $x + y$ is a regular element of $R$. In the third section, we investigate divisibility conditions related to $R$ being completely integrally closed. In the fourth section, we consider several "closedness" properties which hold with respect to units of $T(R)$ and answer some questions raised in Refs. [1,2]. We show in Theorem 4.2 that if $R$ is a Marot ring, then $R$ satisfies these "closedness" properties with respect to $T(R)$ if and only if $R$ satisfies them with respect to units of $T(R)$. In the final section, we briefly consider some conditions related to seminormality from Ref. [1].

Throughout, $R$ is a commutative ring with $1 \neq 0$, group of units $U(R), \text{nil}(R)$ its set of nilpotent elements, $Z(R)$ its set of zerodivisors, $\text{Spec}(R)$ its set of prime ideals, and total quotient ring $T(R) = R_S$, where $S = R - Z(R)$. As usual, an $x \in R - Z(R)$ is called a *regular element* of $R$. When we write $x/y \in T(R)$, we will always mean that $x, y \in R$ with $y$ a regular element of $R$. For any undefined terminology, see Refs. [5,6] or [7]. For an excellent survey of recent work on zero-dimensional commutative rings, see Ref. [8].

## 1 DIVISIBILITY CONDITIONS AND VON NEUMANN REGULAR RINGS

In this section, we give several "divisibility" conditions for a commutative ring $R$ to be either von Neumann regular or a total quotient ring. Recall that a commutative ring $R$ is *von Neumann regular* if for each $x \in R$, there is a $y \in R$ such that $x = x^2y$; equivalently, $R$ is reduced and zero-dimensional.[6, Theorem 3.1] Of course, the most obvious such divisibility condition is that $R$ is von Neumann regular if and only if $x^2 \mid x$ for all $x \in R$. Our first result gives several less obvious divisibility conditions on $R$ which are equivalent to $R$ being von Neumann regular (the equivalence of conditions (4)–(6) is well known).

**Proposition 1.1.** *The following statements are equivalent for a commutative ring $R$.*

(1) *Let $m \geq 2$ be a fixed integer. If $x \mid y^m$ for $x, y \in R$, then $x \mid y$.*

(2) *Let $x, y \in R$. If $x \mid y^n$ for some integer $n \geq 1$, then $x \mid y$.*
(3) *Let $x, y \in R$. If $y^n = xd$ for some integer $n \geq 1$ with $d \in R$ a nonunit, then $x \mid y$.*
(4) *All ideals of $R$ are radical ideals.*
(5) *All principal ideals of $R$ are radical ideals.*
(6) *$R$ is von Neumann regular.*

*Proof*. (1) $\Rightarrow$ (2) Suppose that $x \mid y^n$. If $n \leq m$, then also $x \mid y^m$; so $x \mid y$ by hypothesis. If $n > m$, then also $x \mid y^{km}$ with $k < n$. Thus $x \mid y^k$ by hypothesis, and hence $x \mid y$ by induction on $n$.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (4) Let $I$ be a proper ideal of $R$. Suppose that $x^n = i \in I$ for some $x \in R$ and integer $n \geq 1$. Thus $x^{2n} = i^2$, and hence $i \mid x$ by hypothesis. Thus $x \in I$; so $I$ is a radical ideal of $R$.

(4) $\Rightarrow$ (5) Clear.

(5) $\Rightarrow$ (6) Let $x \in R$. Then $(x^2)$ is a radical ideal of $R$ by hypothesis. Since $x^2 \in (x^2)$, we have $x \in (x^2)$, and hence $x = x^2 y$ for some $y \in R$. Thus $R$ is von Neumann regular.

(6) $\Rightarrow$ (1) Let $x, y \in R$ with $x \mid y^m$; say $y^m = xd$ with $d \in R$. Since $R$ is von Neumann regular, $y = ue$ with $u \in U(R)$ and $e \in R$ idempotent.[6, Corollary 3.3] Thus $y = ue = ue^m = u^{1-m}(ue)^m = u^{1-m}y^m = (u^{1-m}d)x$; so $x \mid y$. □

Proposition 1.1 yields our next result on when $T(R)$ is von Neumann regular (also, see Theorem 2.3). In a similar manner, one may obtain criteria for $T(R)$ to be either strongly root closed or strongly (2,3)-closed (see Sec. 4 for the definitions), zero-dimensional (see Sec. 2), or seminormal (see Sec. 5).

**Proposition 1.2.** *The following statements are equivalent for a commutative ring $R$.*

(1) *$T(R)$ is von Neumann regular.*
(2) *Let $x, y \in R$. If $x \mid y^n$ for some integer $n \geq 1$, then $x \mid sy$ for some regular element $s \in R$.*

*Proof*. (1) $\Rightarrow$ (2) Suppose that $T(R)$ is von Neumann regular and $x \mid y^n$ in $R$. By Proposition 1.1, $x \mid y$ in $T(R)$; so $y = zx$ for some $z = d/s \in T(R)$. Then $x \mid sy$ in $R$ with $s$ a regular element of $R$.

(2) $\Rightarrow$ (1) By Proposition 1.1, we need only show that if $x \mid y^n$ in $T(R)$ for some integer $n \geq 1$ and $x, y \in T(R)$, then $x \mid y$ in $T(R)$. Write $x = a/c$ and $y = b/d$. Then $x \mid y^n$ yields that $d^n ae = cfb^n$ for some $e \in R$ and regular element $f \in R$. Thus $a|(cfb)^n$ in $R$. By hypothesis, $a|s(cfb)$ in $R$ for some regular element $s \in R$. Hence $x \mid y$ in $T(R)$. □

By restricting the divisibility conditions in Proposition 1.1 to regular elements, we obtain several equivalent conditions for $R$ to be a total quotient ring (i.e., $R = T(R)$, equivalently, each regular element of $R$ is a unit of $R$). However, we lose the fact that $R$ is reduced.

**Proposition 1.3.** *The following statements are equivalent for a commutative ring $R$.*

(1) *Let $x, y \in R$ with $x$ a regular element of $R$. If $x \mid y^n$ for some integer $n \geq 1$, then $x \mid y$.*
(2) *Let $x, y \in R$ with $y$ a regular element of $R$. If $x \mid y^n$ for some integer $n \geq 1$, then $x \mid y$.*
(3) *Let $x, y \in R$ with $x, y$ regular elements of $R$. If $x \mid y^n$ for some integer $n \geq 1$, then $x \mid y$.*
(4) $R = T(R)$.

*Proof*. We first show that any of conditions (1)–(3) implies condition (4). Let $x \in R$ be a regular element of $R$. Then $x^2 \mid x^2$ implies that $x^2 \mid x$, and hence $x$ is a unit of $R$. Thus condition (4) holds.

Conversely, suppose that condition (4) holds and that $x \mid y^n$. If either $x$ or $y$ is a regular element of $R$, then $x$ is a unit of $R$ by hypothesis, and hence $x \mid y$. Thus conditions (1)–(3) all hold. □

**Example 1.4.** The ring $\mathbb{Z}/4\mathbb{Z}$ satisfies the equivalent conditions of Proposition 1.3, but not those of Proposition 1.1. Note that $\mathbb{Z}/4\mathbb{Z}$ is a total quotient ring, but it is not reduced.

## 2 ZERO-DIMENSIONAL TOTAL QUOTIENT RINGS

In this section, we give several conditions on $R$ for $T(R)$ to be either zero-dimensional or von Neumann regular. We then show that a condition introduced in Ref. [1], and further used in Ref. [2], on a subring $R$ of a direct product of integral domains is equivalent to $T(R)$ being von Neumann regular.

Recall that a commutative ring $R$ is called *$\pi$-regular* if for each $x \in R$, there is a $y \in R$ and an integer $n \geq 1$ such that $x^{2n}y = x^n$, i.e., $x^{2n} \mid x^n$. Then $R$ is $\pi$-regular if and only if $R$ is zero-dimensional.[6, Theorem 3.1] Thus, in the spirit of Proposition 1.2, one may easily show that $T(R)$ is zero-dimensional if and only if for each $x \in R$, there is an integer $n \geq 1$ and a regular element $s \in R$ such that $x^{2n} \mid sx^n$ in $R$. We give a much more interesting internal characterization of when $T(R)$ is zero-dimensional in Theorem 2.2, but first a lemma.

**Lemma 2.1.** *Let $R$ be a commutative ring and $x, y \in R$.*

(1) *Suppose that $xy \in nil(R)$ and let $n \geq 1$ be an integer. Then $x + y$ is a regular element of $R$ if and only if $x^n + y^n$ is a regular element of $R$.*
(2) *Suppose that $xy = 0$. If $ax + by$ is a regular element of $R$ for some $a, b \in R$, then $x + y$ is also a regular element of $R$. (Thus the ideal $(x, y)$ contains a regular element of $R$ if and only if $x + y$ is a regular element of $R$.)*
(3) *Suppose that $xy = 0$. Then $x + y$ is a regular element of $R$ if and only if $x^m + y^n$ is a regular element of $R$ for some integers $m, n \geq 1$.*

*Proof*. (1) Let $xy \in nil(R)$. By the Binomial Theorem, $(x + y)^n = x^n + y^n + z$ with $z \in nil(R)$. Thus $(x + y)^n$ is a unit in $T(R)$ if and only if $x^n + y^n$ is a unit in $T(R)$. Hence $x + y$ is a regular element of $R$ if and only if $x^n + y^n$ is a regular element of $R$.

(2) Suppose that $xy = 0, ax + by$ is a regular element of $R$, and that $(x + y)d = 0$ for some $0 \neq d \in R$. If $yd = 0$, then also $xd = 0$; and hence $(ax + by)d = 0$, a contradiction. Thus we may assume that $yd \neq 0$. Then $(x + y)yd = 0$ yields $y^2 d = 0$ since $xy = 0$. Hence $(ax + by)yd = 0$, a contradiction. Thus $x + y$ is a regular element of $R$.

(3) This follows easily from part (2) above. □

**Theorem 2.2.** *The following statements are equivalent for a commutative ring $R$.*

(1) *$T(R)$ is zero-dimensional.*
(2) *For each $x \in R$, there is a $y \in R$ and an integer $n \geq 1$ such that $x^n y = 0$ and $x^n + y$ is a regular element of $R$.*
(3) *For each $x \in R$, there is a $y \in R$ such that $xy \in nil(R)$ and $x + y$ is a regular element of $R$.*

*Proof*. (1) $\Rightarrow$ (2) Suppose that $T(R)$ is zero-dimensional, and let $x \in R$. Since $T(R)$ is $\pi$-regular,[6, Theorem 3.1] there is a $z/s \in T(R)$ and an integer $n \geq 1$ such that $x^{2n}(z/s) = x^n$. Thus $x^n(s - x^n z) = 0$. Let $y = s - x^n z$. Then $x^n y = 0$ and $zx^n + y = s$ is a regular element of $R$. Hence $x^n + y$ is a regular element of $R$ by Lemma 2.1(2).

(2) $\Rightarrow$ (3) Suppose that $R$ satisfies condition (2), and let $x \in R$. Then there is a $y \in R$ and an integer $n \geq 1$ such that $x^n y = 0$ and $x^n + y$ is a regular element of $R$. Thus $xy \in nil(R)$. Since $x^n y = 0$ and $x^n + y$ is a regular element of $R$, also $(x^n)^n + y^n = (x^{n^2 - n})x^n + y^n$ is a regular element of $R$ by Lemma 2.1(1). Hence $x^n + y^n$ is a regular element of $R$ by Lemma 2.1(2), and thus $x + y$ is a regular element of $R$ by Lemma 2.1(1) again.

(3) $\Rightarrow$ (1) Suppose that $R$ satisfies condition (3). To show that $T(R)$ is zero-dimensional, it is sufficient to show that each non-minimal prime ideal

$Q$ of $R$ contains a regular element of $R$. Let $P \subset Q$ be distinct prime ideals of $R$, and choose an $x \in Q - P$. By hypothesis, there is a $y \in R$ such that $xy \in nil(R)$ and $x + y$ is a regular element of $R$. Thus $y \in P \subset Q$, and hence $x + y \in Q$. Thus $Q$ contains a regular element of $R$. □

In the next result, a slight modification of the conditions in Theorem 2.2 forces $R$ to be reduced, and hence von Neumann regular.

**Theorem 2.3.** *The following statements are equivalent for a commutative ring $R$.*

(1) *$T(R)$ is von Neumann regular.*
(2) *For each $x \in R$, there is a $y \in R$ such that $xy = 0$ and $x + y$ is a regular element of $R$.*

*Proof*. (1) ⇒ (2) Suppose that $T(R)$ is von Neumann regular. Then $T(R)$, and hence $R$, is reduced. Thus (2) follows from Theorem 2.2 since $nil(R) = \{0\}$.

(2) ⇒ (1) Suppose that $R$ satisfies condition (2). We first show that $R$ is reduced. Let $x \in nil(R)$; say $x^n = 0$ for some integer $n \geq 1$. Then by hypothesis, there is a $y \in R$ such that $xy = 0$ and $x + y$ is a regular element of $R$. Thus $y^n = x^n + y^n$ is a regular element of $R$ by Lemma 2.1(1). Hence $y$ is also a regular element of $R$, so $x = 0$. Thus $R$, and hence $T(R)$, is reduced. It follows from Theorem 2.2 that $T(R)$ is zero-dimensional, and hence von Neumann regular.[6, Theorem 3.1] □

**Remark 2.4.**

(1) Theorems 2.2 and 2.3 also follow from Ref. [6, Theorem 3.2 and Corollary 3.3] via Lemma 2.1 and (for Theorem 2.3) the observation in the proof of (2) ⇒ (1) of Theorem 2.3 that $R$ is reduced. The above two results in Ref. [6] are from Ref. [9].
(2) There are many other characterizations in terms of $R$ of when $T(R)$ is von Neumann regular. For example, see Ref. [6, Theorem 4.5] for conditions concerning when $\mathrm{Min}(R)$, the set of minimal prime ideals of $R$, is compact.

We end this section by showing that condition (2) in Theorem 2.3 is equivalent to a concept introduced in Ref. [1], and further used in Ref. [2], for studying subrings of a direct product of integral domains.

Let $R$ be a subring of the direct product $\Pi R_\alpha$ of a family $\{R_\alpha\}$ of integral domains. As in Ref. [1], for $x = (x_\alpha), y = (y_\alpha) \in R \subset \Pi R_\alpha$, we say that *$y$ extends $x$*, written $yEx$, if $y_\alpha = x_\alpha$ whenever $x_\alpha \neq 0$. We say that $x$ *extends to a regular element of* $R$ if there is a regular element $y \in R$ such that

$yEx$. Note that for $x, y \in R$, we have $xy = 0$ if and only if $x + y$ extends $x$. Thus $y$ extends $x$ if and only if $x(y - x) = 0$.

For any commutative ring $S$, define an ordering on $S$ by $a \leq b$ if either $a = 0$ or $a = b$. For a family $\{R_\alpha\}$ of commutative rings, the induced product order on $\Pi R_\alpha$ is then $(x_\alpha) \leq (y_\alpha) \Leftrightarrow$ either $x_\alpha = 0$ or $x_\alpha = y_\alpha$ for each $\alpha$, i.e., $(y_\alpha)$ extends $(x_\alpha)$ in $\Pi R_\alpha$. This ordering restricts to the ordering $E$ defined above on any subring $R$ of $\Pi R_\alpha$.

Recall that a commutative ring $R$ is reduced if and only if $R$ is a subring of the direct product $\Pi R_\alpha$ of some family $\{R_\alpha\}$ of integral domains, and that $R$ is a subring of the product of a finite number of integral domains if and only if $R$ is reduced with only a finite number of minimal prime ideals. Thus much of our earlier work in Refs. [ 1,2] for reduced rings was set in the context of subrings of a direct product of integral domains, and the concept of "extending to a regular element" played a key role. Our next result shows that the concept of "extending to a regular element" is independent of the embedding of the reduced ring $R$ in a direct product of integral domains. Corollary 2.6 then relates this concept to $T(R)$ being von Neumann regular.

**Proposition 2.5.** *Let $R$ be a subring of a direct product of integral domains. Then $x \in R$ extends to a regular element of $R$ if and only if there is a $y \in R$ such that $xy = 0$ and $x + y$ is a regular element of $R$.*

*Proof.* ($\Rightarrow$) Let $x \in R$ and $zEx$ with $z \in R$ regular. Let $y = z - x$. Then $xy = 0$ and $x + y = z$ is a regular element of $R$.

($\Leftarrow$) Let $x \in R$. By hypothesis, there is a $y \in R$ such that $xy = 0$ and $x + y$ is a regular element of $R$. Clearly $x + y$ extends $x$ since $xy = 0$. □

**Corollary 2.6.** *Let $R$ be a commutative ring which is a subring of a direct product of integral domains. Then each $x \in R$ extends to a regular element of $R$ if and only if $T(R)$ is von Neumann regular.*

*Proof.* This follows immediately from Theorem 2.3 and Proposition 2.5.

Thus, in several results in Refs. [1,2], the hypothesis "every element of $R$ can be extended to a regular element of $R$" may be replaced by "$T(R)$ is von Neumann regular" (also, see Sec. 4 and 5).

The next corollary generalizes the well-known fact that a reduced commutative ring with only a finite number of minimal prime ideals (in particular, a reduced commutative Noetherian ring) has von Neumann regular total quotient ring.

**Corollary 2.7.** *Let $R$ be a reduced commutative ring such that each nonzero zerodivisor of $R$ is contained in only a finite number of minimal prime ideals of $R$. Then $T(R)$ is von Neumann regular.*

*Proof*. We may view $R$ as a subring of $\Pi(R/P_\alpha)$, where $\{P_\alpha\}$ is the set of minimal prime ideals of $R$. By hypothesis, each $0 \neq (r_\alpha) \in Z(R)$ has only a finite number of zero entries, and hence can be extended to a regular element of $R$ by an argument similar to that in the proof of Ref. [1, Lemma 2.5]. Thus $T(R)$ is von Neumann regular by Corollary 2.6. □

## 3 STRONGLY COMPLETELY INTEGRALLY CLOSED RINGS

Let $R$ be a commutative ring. Recall that $x \in T(R)$ is *almost integral over* $R$ if there is a regular element $s \in R$ such that $sx^n \in R$ for all integers $n \geq 1$. As usual, $R$ is called *completely integrally closed* (CIC) if whenever $x \in T(R)$ is almost integral over $R$, then $x \in R$ (equivalently, if $b^n | sa^n$ for all integers $n \geq 1$ with $a \in R$ and $b, s \in R$ regular elements of $R$, then $b|a$). In the spirit of Ref. [2], we define $R$ to be *strongly completely integrally closed* (SCIC) if whenever $b^n | sa^n$ for all integers $n \geq 1$ with $a, b \in R$ and $s \in R$ a regular element of $R$, then $b|a$. (Here we have replaced $a/b \in R$, where $a \in R, b \in R - Z(R)$, by $b|a$ in $R$, and we allow $b \in Z(R)$.) Clearly, if a ring $R$ is SCIC, then $R$ is also CIC (but not conversely, see Example 3.6).

Our first result gives a trivial case when $R$ is always SCIC.

**Proposition 3.1.** *Let $R$ be a commutative ring. If $R = T(R)$, then $R$ is SCIC. In particular, a zero-dimensional commutative ring is SCIC.*

*Proof*. The first part is clear. For the second part, just recall that $R = T(R)$ when $R$ is zero-dimensional. □

As in Ref. [10], a commutative ring $R$ is called *additively regular* if for each $x \in T(R)$, there is a $y \in R$ such that $x + y$ is a regular element (unit) of $T(R)$; equivalently, $R$ is additively regular if and only if for all $x, y \in R$ with $y$ a regular element of $R$, there is an $a \in R$ such that $x + ay$ is a regular element of $R$. If either $T(R)$ is zero-dimensional or $Z(R)$ is a finite union of prime ideals, then $R$ is additively regular.[6, Theorems 7.4 and 7.2] In particular, von Neumann regular rings, Noetherian rings, and reduced rings with only a finite number of minimal prime ideals are additively regular.

We define $R$ to be *strongly additively regular* if for each $x \in R$, there is a $y \in R$ such that $xy = 0$ and $x + y$ is a regular element of $R$. Of course, by Theorem 2.3, $R$ is strongly additively regular if and only if $T(R)$ is von Neumann regular. Thus a strongly additively regular ring is additively regular (this was also observed in a different context in Ref. [2, Proposition 2.5]). However, the converse is false since any zero-dimensional ring is additively regular; for a reduced example, see Example 3.7.

We next show that a strongly additively regular commutative ring $R$ is SCIC if and only if it is CIC (i.e., when $T(R)$ is von Neumann regular). Example 3.6 shows that it is not enough to just assume that $R$ is additively regular.

**Proposition 3.2.** *Let $R$ be a strongly additively regular commutative ring (equivalently, $T(R)$ is von Neumann regular). Then $R$ is SCIC if and only if $R$ is CIC.*

*Proof*. We need only show that $R$ is SCIC if $R$ is CIC. Suppose that $R$ is CIC, and let $a, b, s \in R$ with $s$ a regular element of $R$ such that $b^n | sa^n$ for all integers $n \geq 1$; say $d_n b^n = sa^n$ with each $d_n \in R$. Since $R$ is strongly additively regular, there is a $z \in R$ such that $bz = 0$ and $b + z$ is a regular element of $R$. Then also $az = 0$ since $s$ is regular and $R$ is reduced by the proof of Theorem 2.3. Thus $(d_n b)(b + z)^n = d_n b^{n+1} = bsa^n = [(b + z)s]a^n$ for all integers $n \geq 1$. Hence $(b + z)^n | [(b + z)s]a^n$ for all integers $n \geq 1$ with $b + z$, $(b + z)s \in R$ regular. Thus $a = (b + z)w$ for some $w \in R$ since $R$ is CIC. Note that $az = bz = 0$ yields $wz^2 = 0$; so $wz = 0$ since $R$ is reduced. Hence $a = bw$, and thus $R$ is SCIC. □

**Corollary 3.3.** *Let $R$ be a reduced commutative ring such that each nonzero zerodivisor of $R$ is contained in only a finite number of minimal prime ideals of $R$. Then $R$ is SCIC if and only if $R$ is CIC. In particular, this holds if $R$ is a reduced commutative Noetherian ring.*

*Proof*. This follows directly from Proposition 3.2 and Corollary 2.7. □

Let $M$ be an $R$-module. Then the *idealization* of $R$ and $M$ is the ring $R(+)M$ with underlying set $R \times M$ under coordinatewise addition, and multiplication given by $(r, x)(s, y) = (rs, ry + sx)$. In the next several results, we will use the facts that $U(R(+)M) = U(R)(+)M$ ([6, Theorem 25.1(6)]) and $Z(R(+)M) = A(+)M$, where $A = Z(R) \cup Z(M)$ ([6, Theorem 25.3]). See Ref. [6, Sec. 25] for more details on idealization. We next determine when certain idealizations are SCIC.

**Proposition 3.4.** *Let $R$ be a subring of a commutative ring $B$ such that each regular element of $R$ is also a regular element of $B$ (for example, if $B \subset T(R)$). Then $R(+)B$ is SCIC if and only if $R = T(R)$.*

*Proof*. Let $A = R(+)B$. First suppose that $R = T(R)$. Then each element of $R$ is either a unit or a zerodivisor, and hence each element of $A$ is either a unit or a zerodivisor. Thus $A = T(A)$, and hence $A$ is SCIC by Proposition 3.1. Conversely, suppose that $R \neq T(R)$. Let $x \in R$ be a nonunit regular element. Define $s = (x, 0), a = (0, 1)$, and $b = (0, x)$ in $A$. Then $s$ is a regular element of $A$ and $b^n \mid sa^n$ in $A$ for all integers $n \geq 1$. However, $b \nmid a$ in $A$ since $x$ is not a unit of $R$. Thus $A$ is not SCIC. □

**Proposition 3.5.** *Let $R$ be a CIC commutative ring with $R \neq T(R)$. Then $R(+)T(R)$ is CIC, but not SCIC.*

*Proof*. By Proposition 3.4, $A = R(+)T(R)$ is not SCIC. We next show that $A$ is CIC. Suppose that $(b, y)^n \mid (s, z)(a, x)^n$ for all integers $n \geq 1$ and $(a, x), (b, y), (s, z) \in A$ with $(b, y), (s, z)$ regular elements of $A$. Then $b^n \mid sa^n$ in $R$ for all integers $n \geq 1$ with $b, s \in R$ regular. Thus $a = bc$ for some $c \in R$ since $R$ is CIC. Hence $(a, x) = (b, y)(c, w)$ in $A$ with $w = (x - cy)/b \in T(R)$. Thus $A$ is CIC. □

**Example 3.6.** By Proposition 3.5, $R = \mathbb{Z}(+)\mathbb{Q}$ is CIC, but not SCIC. Note that $R$ is additively regular, but not strongly additively regular since $R$ is not reduced.

**Example 3.7.** (A reduced commutative ring $R$ which is additively regular, but not strongly additively regular.) Let $K$ be a field, and let $A = K[X_1, X_2, \ldots] = K[\{X_n\} \mid n \in \mathbb{N}\}]$. Then $I = (\{X_iX_j \mid i, j \in \mathbb{N}, i \neq j\})$ is a radical ideal of $A$ contained in the maximal ideal $N = (\{X_n \mid n \in \mathbb{N}\})$. Thus $R = A_N/I_N$ is a quasilocal reduced ring with maximal ideal $M = N_N/I_N$. Clearly $Z(R) = M$, and thus $T(R) = R$. Hence $R$ is additively regular. However, for any $0 \neq x \in M$, there is no $y \in R$ such that $xy = 0$ and $x + y$ is a regular element of $R$. Thus $R$ is not strongly additively regular.

**Question 3.8.** Is there a reduced commutative ring $R$ which is CIC, but not SCIC?

## 4 CLOSEDNESS WITH RESPECT TO UNITS OF $T(R)$

In this section, we continue our investigation from Refs. [1,2] of when $R$ satisfies certain "closedness" properties with respect to units of $T(R)$, i.e., if $x \in U(T(R))$ (equivalently, $x = a/b$ with $a, b$ regular elements of $R$) satisfies a given closedness property, then $x \in R$ (equivalently, $b \mid a$ in $R$). The "closedness" properties we consider here are completely integrally closed, integrally closed, root closed, and (2,3)-closed. It is clear that these four "closedness" properties with respect to units of $T(R)$ are inherited by direct products, intersections of overrings in $T(R)$, and (except for the CIC case) localizations in $T(R)$. We also briefly discuss rings such that $T(R) - R \subset U(T(R))$.

Recall that a commutative ring $R$ is *root closed* (resp., $(2, 3)$-closed) if whenever $x^n \in R$ for some integer $n \geq 1$ (resp., $x^2, x^3 \in R$) and $x \in T(R)$, then $x \in R$. As in Ref. [2], we say that $R$ is *strongly root closed* (resp., *strongly (2,3)-closed*) if whenever $b^n \mid a^n$ for some integer $n \geq 1$ (resp., $b^2 \mid a^2, b^3 \mid a^3$) and $a, b \in R$, then $b \mid a$ in $R$. (Here we have replaced $a/b \in R$, where $a \in R$ and $b \in R - Z(R)$, by $b \mid a$ in $R$, and we allow $b \in Z(R)$.) Note that $R$ is strongly $(2,3)$-closed (resp., $(2,3)$-closed) if and only if whenever $b^n \mid a^n$ for all sufficiently large integers $n \geq 1$ and $a, b \in R$ (resp., $a \in R$ and $b \in R - Z(R)$), then $b \mid a$ in $R$. Clearly a strongly root closed (resp., strongly $(2,3)$-closed) ring is root closed (resp., $(2,3)$-closed). Examples in Ref. [2] show that the converse is false.

Our first result is the CIC analog of Ref. [2, Propositions 2.2–2.4], i.e., if $R$ is root closed (resp., $(2,3)$-closed, integrally closed) with respect to units of $T(R)$ and $R$ is additively regular (in particular, if $T(R)$ is von Neumann regular), then $R$ is root closed (resp., $(2,3)$-closed, integrally closed). A stronger result is given in Theorem 4.2.

**Proposition 4.1.** *The following statements are equivalent for an additively regular commutative ring $R$.*

(1) *$R$ is completely integrally closed.*
(2) *If $x \in U(T(R))$ is almost integral over $R$, then $x \in R$.*

*Proof*. Clearly $(1) \Rightarrow (2)$. Conversely, suppose that condition (2) holds, and let $x \in T(R)$ be almost integral over $R$. Then there is a regular element $s \in R$ such that $sx^n \in R$ for all integers $n \geq 1$. Since $R$ is additively regular, there is a $y \in R$ such that $x + y \in U(T(R))$. Then $s(x+y)^n \in R$ for all integers $n \geq 1$. Thus $x + y \in R$ by hypothesis, and hence $x \in R$. Thus $R$ is completely integrally closed. □

In the spirit of Refs. [1,2], and Sec. 3, we consider the following four conditions related to root closedness on a commutative ring $R$ with $a, b \in R$ and $n \geq 1$ an integer. (Similar conditions can be given for $(2,3)$-closedness, integral closedness, and complete integral closedness; we leave the specific details to the interested reader.)

(1) If $b^n \mid a^n$, then $b \mid a$ (i.e., $R$ is strongly root closed).
(2) If $b^n \mid a^n$ with $b$ regular, then $b \mid a$ (i.e., $R$ is root closed).
(3) If $b^n \mid a^n$ with $a, b$ regular, then $b \mid a$ (i.e., $R$ is root closed with respect to units of $T(R)$).
(4) If $b^n \mid a^n$ with a regular, then $b \mid a$.

Clearly conditions $(1) \Rightarrow (2) \Rightarrow (3)$, and conditions $(3) \Leftrightarrow (4)$. By Ref. [2, Example 3.1], condition (2) does not imply condition (1).

We always have: $R$ is CIC wrt units of $T(R) \Rightarrow R$ is integrally closed wrt units of $T(R) \Rightarrow R$ is root closed wrt units of $T(R) \Rightarrow R$ is $(2,3)$-closed wrt units of $T(R)$. Although a CIC ring is always root closed, an SCIC ring need not be strongly root closed (cf. Proposition 3.1).

In Refs. [2, Question 2.7], we asked if $R$ is integrally closed (resp., $(2,3)$-closed, root closed) with respect to units of $T(R)$ implies that $R$ is integrally closed (resp., $(2,3)$-closed, root closed). In Refs. [2, Propositions 2.2–2.4], we showed that this is true if $R$ is additively regular. We next generalize this to the class of Marot rings. As in Ref. [6], a commutative ring $R$ is called a *Marot ring* if every ideal of $R$ which contains a regular element is generated by regular elements. Examples of Marot rings include integral domains, Noetherian rings, rings such that $Z(R)$ is a finite union of prime ideals, polynomial rings, and rings with zero-dimensional total quotient ring.[6, Theorems 7.2, 7.4, and 7.5] An additively regular ring is a Marot ring,[6, Theorem 7.2] but not conversely (see Refs. [6, Example 12, p. 185] or Ref. [11]).

**Theorem 4.2.** *Let $R$ be a commutative Marot ring. Then $R$ is root closed (resp., $(2,3)$-closed, integrally closed, completely integrally closed) if and only if $R$ is root closed (resp., $(2,3)$-closed, integrally closed, completely integrally closed) with respect to units of $T(R)$.*

*Proof*. Suppose that $R$ is a Marot ring which is root closed with respect to units of $T(R)$. Let $a, b \in R$ with $b$ regular, and suppose that $(a/b)^n \in R$, i.e., $b^n \mid a^n$, for some integer $n \geq 1$. Let $I = (a, b^n)$, and let $x$ be a regular element of $I$. Then $x = ca + db^n$ for some $c, d \in R$. Using the Binomial Theorem, one can easily show that $b^n \mid x^n$ since $b^n \mid a^n$, and hence $b \mid x$ by hypothesis. Since $R$ is a Marot ring, a is a linear combination of regular elements of $I$, and thus $b \mid a$. Hence $a/b \in R$; so $R$ is root closed. The converse is clear. The proofs for the other three closedness properties are similar. For the $(2,3)$-closed case, let $I = (a, b^3)$; for the integrally closed case, let $I = (a, b)$; and for the completely integrally closed case, let $I = (a, b)$. Details are left to the reader. □

We next give an example of a reduced commutative ring $R$ which satisfies the four closedness conditions with respect to units of $T(R)$, but not with respect to $T(R)$. This answers questions raised in Ref. [2, Question 2.7] and Refs. [1, Sec. 3]. We would like to thank Thomas G. Lucas for suggesting this example.

**Example 4.3.** (A reduced commutative ring $R$ which is root closed (resp., $(2,3)$-closed, integrally closed, completely integrally closed) with respect to units of $T(R)$, but $R$ is not root closed (resp., $(2,3)$-closed, integrally closed, completely integrally closed).) We employ the "$A+B$" construction, see Refs. [6, Section 26] and [12] for more details. Let $K$ be a field, $D = K[X^2, X^3, XY, Y]$, and let $\mathcal{P} = \{P \in \mathrm{Spec}(R) \mid htP = 1 \text{ and } Y \notin P\}$. Let $\mathcal{A}$ be an indexing set for $\mathcal{P}$, and let $I = \mathcal{A} \times \mathbb{N}$. For each $i = (\alpha, n) \in I$, let

$K_i = qf(D/P_\alpha)$, and let $B = \oplus_{i\in I} K_i$. Finally, let $R = D + B$. It is easiest to view $R$ as $R = D \oplus B$ with coordinatewise addition and $(d_1, b_1)(d_2, b_2) = (d_1 d_2, d_1 b_2 + d_2 b_1 + b_1 b_2)$. Then $R$ is reduced and one can easily check that $R - Z(R) = \{aY^n \mid 0 \neq a \in K \text{ and } n \geq 1\}$ (here we identify $f \in D$ with $(f, 0)$). One can then easily verify that $R$ satisfies each of the four closedness conditions with respect to units of $T(R)$, but not with respect to $T(R)$. For example, $Y^2 \mid (XY)^2$ in $R$, but $Y \nmid XY$ in $R$. □

Clearly a strongly $(2, 3)$-closed ring is reduced, and $\text{nil}(T(R)) = \text{nil}(R)$ when $R$ is $(2, 3)$-closed. We next show that $\text{nil}(T(R)) = \text{nil}(R)$ when $R$ is $(2, 3)$-closed with respect to units of $T(R)$ (thus also $\text{nil}(T(R)) = \text{nil}(R)$ if $R$ is root closed, integrally closed, or completely integrally closed with respect to units of $T(R)$).

**Proposition 4.4.** *If a commutative ring $R$ is $(2, 3)$-closed with respect to units of $T(R)$, then $\text{nil}(T(R)) = \text{nil}(R)$.*

*Proof*. Clearly $nil(R) \subset nil(T(R))$. Conversely, suppose that $(x/y)^n = 0$ for some integer $n \geq 1$. Then $(x + y^n)/y = (x/y) + y^{n-1}$ is a unit in $T(R)$ with $[(x + y^n)/y]^m \in R$ for all integers $m \geq n$. Thus $(x + y^n)/y \in R$ by hypothesis, and hence $x/y \in R$. □

**Corollary 4.5.** *Let $R$ be a commutative ring.*

(1) *If $R$ is $(2, 3)$-closed with respect to units of $T(R)$, then* $\text{nil}(R) \subset sR$ *for each regular element $s \in R$.*
(2) *Suppose that $nil(R) = Z(R)$. Then $R$ is $(2, 3)$-closed (resp., root closed, integrally closed, completely integrally closed) if and only if $R$ is $(2, 3)$-closed (resp., root closed, integrally closed, completely integrally closed) with respect to units of $T(R)$.*

*Proof*. (1) This follows directly from Proposition 4.4.

(2) We show that if $R$ is $(2, 3)$-closed with respect to units of $T(R)$ and $\text{nil}(R) = Z(R)$, then $R$ is $(2, 3)$-closed; the proofs of the other cases are left to the reader. Let $x = a/b \in T(R)$ with $x^2, x^3 \in R$. If $a \in R - Z(R)$, then $x \in R$ by hypothesis. If $a \in Z(R) = \text{nil}(R)$, then $x \in \text{nil}(T(R)) = \text{nil}(R) \subset R$. Thus $R$ is $(2, 3)$-closed. □

If $T(R) - R \subset U(T(R))$, then clearly $R$ is $(2, 3)$-closed (resp., root closed, integrally closed, completely integrally closed) if and only if $R$ is $(2, 3)$-closed (resp., root closed, integrally closed, completely integrally closed) with respect to units of $T(R)$. Thus it is of interest to characterize the commutative rings $R$ such that $T(R) - R \subset U(T(R))$. Recall that an ideal of a commutative ring $R$ is said to be *divided* if it is comparable to every other ideal of $R$ (equivalently, to *every* principal ideal of $R$). Note that if $Z(R)$ is a divided ideal (and hence necessarily prime), then $T(R) - R \subset U(T(R))$. We next show that the converse is also true except in trivial cases (note that if $R = T(R)$, then $Z(R) = R - U(R)$; so, in this case, $Z(R)$ is an ideal of $R$ if and only if $R$ is quasilocal with maximal ideal $Z(R)$).

**Proposition 4.6.** *Let $R$ be a commutative ring which contains a nonunit regular element (i.e., $R \neq T(R)$). Then $T(R) - R \subset U(T(R))$ if and only if $Z(R)$ is a divided prime ideal of $R$.*

*Proof*. We have already observed that $T(R) - R \subset U(T(R))$ if $Z(R)$ is a divided (prime) ideal of $R$. Conversely, suppose that $R$ contains a nonunit regular element and $T(R) - R \subset U(T(R))$. Thus $Z(R) \subset sR$ for each regular element $s$ of $R$. We need only show that $Z(R)$ is an ideal of $R$. Let $I = \cap\{xR \mid x \text{ is a regular element of } R\}$. By hypothesis, $I$ is a proper ideal of $R$ containing $Z(R)$. We show that $I = Z(R)$. If not, then there is an $x \in I - Z(R)$. Thus $Z(R) \subset xR \subset I \subset xR$, and hence $I = xR$. Similarly, $I = x^2R$. Thus $x \in U(R)$, a contradiction. □

We next consider several classes of commutative rings $R$ such that $Z(R)$ is a divided prime ideal of $R$. Note that such rings are Marot rings. As in Ref. [4], we say that a commutative ring $R$ is a *pseudo-valuation ring* (PVR) if $aP$ and $bR$ are comparable for all $a, b \in R$ and prime ideals $P$ of $R$. A PVR is necessarily quasilocal (for this and other results about PVRs, see Refs. [3,4]). We say that a commutative ring $R$ is a $\Phi$-pseudo-valuation ring ($\Phi$-PVR) if *nil*$(R)$ is a divided prime ideal of $R$ and for each prime ideal $P \neq \text{nil}(R)$ of $R$, $aP$ and $bR$ are comparable for all $a, b \in R - nil(R)$ (cf.[13, Corollary 7] and [14, Proposition 1.1(6)]); equivalently, nil$(R)$ is a divided prime ideal of $R$ and $R/nil(R)$ is a PVR. [15, Proposition 2.9, 16, Theorem 3.1] Also, see[16] for some other generalizations of PVRs.

**Corollary 4.7.** *Let $R$ be a commutative ring such that $Z(R)$ is a divided prime ideal of $R$. Then $R$ is $(2,3)$-closed (resp., root closed, integrally closed, completely integrally closed) if and only if $R$ is $(2,3)$-closed (resp., root closed, integrally closed, completely integrally closed) with respect to units of $T(R)$. In particular, the above statement holds for a PVR or a $\Phi$-PVR.*

*Proof*. The first statement is clear since in this case $T(R) - R \subset U(T(R))$ by Proposition 4.6. For the "in particular" statement, just note that a PVR or a $\Phi$-PVR clearly satisfies the given hypothesis. □

## 5 SEMINORMAL RINGS

Following Swan,[17] we say that a (necessarily reduced) commutative ring $R$ is *seminormal* if whenever $a^2 = b^3$ for $a, b \in R$, then $a = c^3$ and $b = c^2$

for some $c \in R$. The importance of seminormality is that $\mathrm{Pic}(R[X]) = \mathrm{Pic}(R)$ if and only if $R/\mathrm{nil}(R)$ is seminormal.[17, Theorem 1] In Ref. [1], we studied several variants of seminormality. In this section, we briefly continue that investigation. The following two conditions were conditions (2) and (6) in Ref. [1], respectively.

(5) If $a^2 = b^3$ with $a, b \in R$, then $b \mid a$ in $R$.
(6) If $a^2 = b^3$ with $a, b \in R$ regular, then $b \mid a$ in $R$.

Clearly condition (5) implies condition (6) and any seminormal ring satisfies conditions (5) and (6). However, condition (6) is not equivalent to $R$ being seminormal since any total quotient ring satisfies (6). The ring $R$ in Example 4.3 also satisfies (6), but $R$ is not seminormal. In Ref. [1, Example 2.7(a)], an example was given of a reduced commutative ring which satisfies condition (5), but is not seminormal. In fact, $R$ is seminormal if and only if $R$ is $(2, 3)$-closed and $T(R)$ is seminormal.[1, Theorem 3.1(a)] In the spirit of Proposition 1.2, one may easily show that $T(R)$ is seminormal if and only if whenever $a^2 = b^3$ for $a, b \in R$, there are $c, s \in R$ with $s$ regular such that $s^3 a = c^3$ and $s^2 b = c^2$.

We next give several other conditions equivalent to condition (6).

The $(2) \Leftrightarrow (3)$ equivalence of Proposition 5.1 is also in Ref. [1, Theorem 3.1(c)].

**Proposition 5.1.** *The following statements are equivalent for a commutative ring $R$.*

(1) *If $a^2 = b^3$ with $a, b \in R$ regular, then $a = c^3$ and $b = c^2$ for some (regular) $c \in R$ (i.e., $R$ is seminormal with respect to regular elements of $R$).*
(2) *If $a^2 = b^3$ with $a, b \in R$ regular, then $b \mid a$ (i.e., $R$ satisfies condition (6)).*
(3) *If $b^2 \mid a^2$ and $b^3 \mid a^3$ with $a, b \in R$ regular, then $b \mid a$ (i.e., $R$ is $(2, 3)$-closed with respect to units of $T(R)$).*

*Proof*. $(1) \Rightarrow (2)$ Clear.

$(2) \Rightarrow (3)$ Suppose that $b^2 \mid a^2$ and $b^3 \mid a^3$ with $a, b \in R$ regular. Then $\beta = a^2/b^2$ and $\alpha = a^3/b^3$ are regular elements of $R$ which satisfy $\alpha^2 = \beta^3$, and hence $\beta \mid \alpha$ by hypothesis. Thus $b \mid a$.

$(3) \Rightarrow (1)$ Suppose that $a^2 = b^3$ with $a, b \in R$ regular. Then $b^2 \mid a^2$ and $b^3 \mid a^3$, and hence $b \mid a$ by hypothesis. Let $c = a/b \in R$. Then $a = c^3$ and $b = c^2$. □

Our next result is a slight generalization of Ref. [1, Theorem 3.3] via Corollary 2.6. As a special case, if $T(R)$ is von Neumann regular, then $R$ is seminormal if and only if $R$ satisfies any of the three equivalent conditions in Proposition 5.1 (recall that a von Neumann regular ring is seminormal).

**Proposition 5.2.** *Let $R$ be a commutative Marot ring such that $T(R)$ is seminormal. Then $R$ is seminormal if and only if $R$ satisfies condition* (6).

*Proof*. By Theorem 4.2 and Proposition 5.1 $R$ satisfies condition (6) if and only if $R$ is $(2,3)$-closed. As mentioned above, $R$ is seminormal if and only if $T(R)$ is seminormal and $R$ is $(2,3)$-closed.[1, Theorem 3.1(a)] The result follows. □

**Corollary 5.3.** (cf.[1, Theorem 3.3]) *Let $R$ be a strongly regular commutative ring (i.e., $T(R)$ is von Neuman regular). Then $R$ is seminormal if and only if $R$ satisfies condition* (6). □

## REFERENCES

1. Anderson, D.F.; Badawi, A. Conditions Equivalent to Seminormality in Certain Classes of Commutative Rings. *Lecture Notes in Pure and Applied Mathematic*; Marcel Dekker: New York, Basel, 2001; Vol. 220, 49–59.
2. Anderson, D.F.; Badawi, A. On Root Closure in Commutative Rings. Arabian J. Sci. Engrg. **2001**, *26* (*IC*), 17–30.
3. Anderson, D.F; Badawi, A.; Dobbs, D.E. Pseudo-Valuation Rings, II. Boll. Un. Mat. Ital. **B 2000**, *3* (8), 535–545.
4. Badawi, A.; Anderson, D.F.; Dobbs, D.E. Pseudo-Valuation Rings. *Lecture Notes in Pure and Applied Mathematics*; Marcel Dekker: New York, Basel, 1997; Vol. 185, 56–67.
5. Gilmer, R. *Multiplicative Ideal Theory*; Marcel Dekker: New York, Basel, 1972.
6. Huckaba, J.A. *Commutative Rings with Zero Divisors*; Marcel Dekker: New York, Basel, 1988.
7. Kaplansky, I. *Commutative Rings*; Rev. Ed., University of Chicago Press: Chicago, 1974.
8. Gilmer, R. Zero-Dimensional Rings. *Lecture Notes in Pure and Applied Mathematics*; Marcel Dekker: New York, Basel, 1995; Vol. 171, 1–52.
9. Arapovic, M. Characterizations of the 0-Dimensional Rings. Glas. Mat. Ser. III **1983**, *18*, 39–46.
10. Gilmer, R.; Huckaba, J.A. Δ-Rings. J. Algebra **1974**, *28*, 414–432.
11. Matsuda, R. On Marot Rings. Proc. Japan Acad. Ser. A Math. Sci. **1984**, *60*, 134–137.

12. Lucas, T.G. Root Closure and R[X]. Comm. Algebra **1989**, *17*, 2393–2414.
13. Badawi, A. On Φ-Pseudo-Valuation Rings. *Lecture Notes in Pure and Applied Mathematics*; Marcel Dekker: New York, Basel, 1999; Vol. 205, 101–110.
14. Badawi, A. On Φ-Pseudo-Valuation Rings. II, Houston J. Math. **2000**, *26*, 473–480.
15. Badawi, A. On Divided Rings. and Φ-Pseudo-Valuation Rings, Internat. J. Commutative Rings, (to appear).
16. Chang, G.W. Generalizations of Pseudo-Valuation Rings. Internat. J. Commutative Rings, (to appear).
17. Swan, R.G. On Seminormality. J. Algebra **1980**, *67*, 210–229.

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