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***Abstract*—This paper presents a new fundamental approach to modal participation analysis of linear time-invariant systems, leading to new insights and new formulas for modal participation factors. Modal participation factors were introduced over a quarter century ago as a way of measuring the relative participation of modes in states, and of states in modes, for linear time-invariant systems. Participation factors have proved their usefulness in the field of electric power systems and in other applications. However, in the current understanding, it is routinely taken for granted that the measure of participation of *modes in states* is identical to that for participation of *states in modes*. Here, a new analysis using averaging over an uncertain set of system initial conditions yields the conclusion that these quantities (participation of modes in states and participation of states in modes) should not be viewed as interchangeable. In fact, it is proposed that a *new definition and calculation* replace the existing ones for *state in mode participation factors*, while the previously existing participation factors definition and formula should be retained but viewed only in the sense of *mode in state participation factors*. Several examples are used to illustrate the issues addressed and the results obtained.**

***Index Terms*—Participation factors, modal participation factors, modal analysis, linear systems, stability, control systems.**

I. Introduction

This paper presents new concepts, results, and formulas in the subject of modal participation analysis of linear time-invariant systems. This topic is an important component of the Selective Modal Analysis (SMA) framework introduced by Perez-Arriaga, Verghese and Schweppe [7], [12] in the early 1980s. A main construct in SMA is the concept of modal participation factors (or simply participation factors). Participation factors are scalars intended to measure the relative contribution of system modes to system states, and of system states to system modes, for linear systems. The work of these authors has had a major impact especially in applications to electric power systems, where participation factors as they were originally introduced have become a routine tool for the practitioner and researcher alike.

Since their introduction, participation factors have been employed widely in electric power systems and other applications. They have been used for stability analysis, order reduction, sensor and actuator placement, and coherency and clustering studies (e.g., [7], [12], [8], [2], [5], [3], [9]). Several researchers have also considered alternate ways of viewing modal participation factors (e.g., [11], [4], [10]).

We study linear time-invariant continuous-time systems

$$\dot{x} \quad = \quad Ax(t) \tag{1}$$

where $x \in R^n$ and A is a real $n \times n$ matrix. We make the blanket assumption that A has a set of n distinct eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. The solution of (1) then takes the form of a sum of modal components:

$$x(t) \quad = \quad \sum_{i=1}^{n} e^{\lambda_i t} c^i \tag{2}$$

where the c^i are constant vectors determined by the initial condition x^0 and by the right and left eigenvectors of A.

In their study of modal participation for the system (1), the authors of [7], [12] selected particular initial conditions and introduced definitions motivated by the calculation of relative state and mode contributions using those initial conditions. In this paper, we take a different approach, building on our previous work [1], in which definitions of modal participation factors are formulated by averaging relative contributions of modes in states and states in modes over an uncertain set of initial conditions. In this approach, we consider initial conditions to be unknown, and we take the view that performing some sort of average over all possible initial conditions should give a more reliable result than focusing attention on one particular possible initial condition. The uncertainty in initial condition can be taken as set-theoretic (unknown but bounded) or probabilistic. We took the same basic approach in our paper [1], but later found a subtle error in the calculation in that paper for the case of state-in-mode participation factors. Upon realizing this subtle error, we embarked on the present research, in which we find a previously unnoticed dichotomy in the two basic types of modal participation factors.

The main contribution of this paper is to reveal this previously unknown dichotomy in modal participation analysis. To wit, although the definitions obtained in [7], [12], and which have been in wide use since their introduction, give identical values for measures of participation of modes in states and for participation of states in modes, these are in fact better viewed as fundamentally different, and should be calculated using two distinct formulas. Summarizing, the main contribution of this paper is as follows: *we propose replacing the existing definition of participation factors with two separate definitions that yield distinct numerical values for participation of modes in states and for participation of*

W. A. Hashlamoun was visiting the Institute for Systems Research, University of Maryland, College Park, MD 20742 USA. He is now at the Department of Electrical and Computer Engineering, Birzeit University, Birzeit, Palestine (email: hwael@birzeit.edu).

M. A. Hassouneh is with the Institute for Systems Research and the Department of Mechanical Engineering, University of Maryland, College Park, MD 20742 USA (email: munther@umd.edu).

E. H. Abed is with the Department of Electrical and Computer Engineering and the Institute for Systems Research, University of Maryland, College Park, MD 20742 USA (email: abed@umd.edu).

states in modes. In this paper, the currently used participation factors measuring participation of ***states in modes*** *are replaced with a new first-principles definition, a particular instance of which is an explicit formula given in Section V. In addition, we show that our formula for participation factors measuring participation of* ***modes in states*** *agrees with the commonly used participation factors formula under reasonable assumptions on the allowed uncertainty in the system initial conditions. Thus, a dichotomy is proposed in the calculation of participation factors.*

The paper proceeds as follows. In Section II, the original definitions of modal participation factors are recalled from [7], [12]. In Section III, basic examples are used to illustrate the need for an approach that yields distinct formulas for measuring the two main types of modal participation: participation of modes in states and participation of states in modes. In Section IV, the approach we introduced to this topic in [1] is recalled and discussed in light of the objectives of the present paper. The discussion makes clear that this approach, based on defining modal participation measures by averaging over an uncertain set of initial conditions, readily yields the original definition for participation of modes in states, but does not easily yield a simple closed-form expression for measuring participation of states in modes. Next, in Section V, a candidate closed-form formula is obtained for modal participation factors that measure the participation of states in modes; this is achieved by careful evaluation of the general averaging formula from Section IV under a simplifying assumption on the initial condition uncertainty. It is important to note that to obtain this simple formula, a specific form is assumed for the uncertainty in the system initial condition; other assumptions on the initial condition uncertainty would not lead to the same formula or any readily useable expression. The derived formula is proposed since it reflects the effect of initial condition uncertainty and can be derived analytically. In Section VI, a mechanical system example is used to illustrate the usefulness of the explicit formula for state in mode participation factors. In Section VII, an additional result is given that relates only to mode in state participation factors. This result expands the initial condition uncertainty assumptions under which the traditional participation factors formula can be shown to accurately measure mode in state participation using the averaging formulation. Concluding remarks and suggestions for future work are collected in Section VIII.

II. Original Definitions of Modal Participation Factors

In this section, the original definitions of modal participation factors are recalled from [7], [12]. Consider the linear system (1), repeated here for convenience:

$$\dot{x} = Ax(t) \tag{3}$$

where $x \in R^n$, and A is a real $n \times n$ matrix. The authors of [7], [12] also make the blanket assumption that A has n distinct eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Let $(r^1, r^2, \ldots, r^n)$ be right eigenvectors of the matrix A associated with the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, respectively. Let $(l^1, l^2, \ldots, l^n)$ denote left (row) eigenvectors of the matrix A associated with the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, respectively. The right and left eigenvectors are taken to satisfy the normalization [6]

$$l^i r^j = \delta_{ij} \tag{4}$$

where δ_{ij} is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

The solution to (3) starting from an initial condition $x(0) = x^0$ is

$$x(t) = e^{At} x^0 \tag{5}$$

Since the eigenvalues of A are distinct, A is similar to a diagonal matrix. Using this, (5) can be rewritten in the form

$$x(t) = \sum_{i=1}^{n} (l^i x^0) e^{\lambda_i t} r^i. \tag{6}$$

From (6), $x_k(t)$ is given by

$$x_k(t) = \sum_{i=1}^{n} (l^i x^0) e^{\lambda_i t} r^i_k. \tag{7}$$

A. Relative participation of the i-th mode in the k-th state

To determine the relative participation of the i-th mode in the k-th state, the authors of [7], [12] select an initial condition $x^0 = e^k$, the unit vector along the k-th coordinate axis. As seen next, this choice is convenient in that it results in a simple formula for mode-in-state participation factors. We note that the derived formula for mode-in-state participation factors agrees with that obtained using an uncertain initial condition under general assumptions, as demonstrated in [1] and in Section IV below. With this choice of x^0, the evolution of the k-th state becomes

$$\begin{aligned} x_k(t) &= \sum_{i=1}^{n} (l^i_k r^i_k) e^{\lambda_i t} \\ &=: \sum_{i=1}^{n} p_{ki} e^{\lambda_i t}. \end{aligned} \tag{8}$$

The quantities

$$p_{ki} := l^i_k r^i_k \tag{9}$$

are found to be unit-independent, and are taken in [7], [12] as measures of the relative participation of the i-th mode in the k-th state; p_{ki} is defined in [7], [12] as the participation factor for the i-th mode in the k-th state.

B. Relative participation of the k-th state in the i-th mode

The relative participation of the k-th state in the i-th mode is studied in [7], [12] by first applying the similarity transformation

$$z := V^{-1} x \tag{10}$$

to system (3), where V is the matrix of right eigenvectors of A:

$$V = [r^1 \ r^2 \ \cdots \ r^n] \tag{11}$$

and V^{-1} is the matrix of left eigenvectors of A:

$$V^{-1} = \begin{bmatrix} l^1 \\ l^2 \\ \vdots \\ l^n \end{bmatrix}. \tag{12}$$

Then z obeys the dynamics

$$\begin{aligned} \dot{z}(t) &= V^{-1}AVz(t) \\ &= \Lambda z(t), \end{aligned} \tag{13}$$

where $\Lambda := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, with initial condition $z^0 := V^{-1}x^0$. This implies that the evolution of the new state vector components z_i, $i = 1, \ldots, n$ is given by

$$\begin{aligned} z_i(t) &= z_i^0 e^{\lambda_i t} \\ &= l^i x^0 e^{\lambda_i t} \\ &= \left[\sum_{k=1}^{n} (l_k^i x_k^0)\right] e^{\lambda_i t}. \end{aligned} \tag{14}$$

For a real eigenvalue λ_i, clearly $z_i(t)$ represents the evolution of the associated mode. If λ_i is not real, then the associated mode is sometimes taken to be $z_i(t)$, but can also be taken as the combination of $z_i(t)$ and its complex conjugate $z_i^*(t)$, which reflects the influence of the eigenvalue λ_i^*. In the latter approach, we view λ_i and λ_i^* as representing the same "complex frequency." In the past, the former convention was used in most publications. In this paper, we allow both interpretations, but we will find it convenient to use the latter point of view when deriving a new state-in-mode participation factors formula for the case of complex eigenvalues.

In order to determine the relative participation of the k-th state in the i-th mode, the authors of [7], [12] select an initial condition $x^0 = r^i$, the right eigenvector associated with λ_i. As seen next, this choice is convenient in that it results in a simple formula for state-in-mode participation factors. We will revisit this later using an uncertain initial condition, and obtain a different result. With this choice of initial condition, the evolution of the i-th mode becomes

$$\begin{aligned} z_i(t) &= l^i r^i e^{\lambda_i t} \\ &= \left[\sum_{k=1}^{n} l_k^i r_k^i\right] e^{\lambda_i t} \\ &= \left[\sum_{k=1}^{n} p_{ki}\right] e^{\lambda_i t}. \end{aligned} \tag{15}$$

Based on (15), the authors of [7], [12] propose the formula

$$p_{ki} = l_k^i r_k^i \tag{16}$$

as a measure of the relative participation of the k-th state in the i-th mode.

Note that (9), (16) provide identical formulas for participation of modes in states and participation of states in modes, respectively. For this reason, the same notation p_{ki} was used for both types of participation factors until now.

III. Motivating Examples Showing Inadequacy of Participation Factors Formula as a Measure of State in Mode Participation

In this section, by way of motivation for the subsequent analysis, two examples are given that show the need for a new definition and a new formula for state in mode participation factors.

Example 1 Consider the two-dimensional system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where a, b and d are constants with $a \neq d$. The eigenvalues of A are $\lambda_1 = a$ and $\lambda_2 = d$. The right eigenvectors associated with λ_1 and λ_2 are

$$r^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } r^2 = \begin{bmatrix} 1 \\ \frac{d-a}{b} \end{bmatrix},$$

respectively. The left eigenvectors associated with λ_1 and λ_2 and satisfying the normalization (4) are

$$l^1 = \begin{bmatrix} 1 & \frac{b}{a-d} \end{bmatrix} \text{ and } l^2 = \begin{bmatrix} 0 & \frac{-b}{a-d} \end{bmatrix},$$

respectively.

Before calculating the participation factors measuring the influence of states x_1 and x_2 in mode 1,[§] we write the evolution of mode 1 explicitly. Using (14), we have

$$\begin{aligned} z_1(t) &= l^1 x^0 e^{\lambda_1 t} \\ &= \begin{bmatrix} 1 & \dfrac{b}{a-d} \end{bmatrix} \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} e^{\lambda_1 t} \\ &= \left(x_1^0 + \frac{b}{a-d} x_2^0\right) e^{\lambda_1 t}. \end{aligned} \tag{17}$$

Note that the evolution of mode 1 is influenced by both x_1^0 and x_2^0, with the relative degree of influence depending on the values of the system parameters a, b and d.

Calculating the participation factors using the original definition as recalled in the foregoing section, we find the participation factor for state x_1 in mode 1 is $p_{11} = l_1^1 r_1^1 = 1$, while the participation factor for state x_2 in mode 1 is $p_{21} = l_2^1 r_2^1 = 0$. Thus, the original definition of participation factors for state in mode participation indicates that state x_2 has much smaller (even zero) influence on mode 1 compared to the influence coming from state x_1, regardless of the values of system parameters a, b and d. This is in stark contradiction to what we observed using the explicit formula (17), and begs for a re-examination of the basic formula for state-in-mode participation factors. ■

[§]For simplicity, we use the terminology 'mode i' in place of 'the mode associated with eigenvalue λ_i.'

Example 2 Consider the two-dimensional system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ -d & -d \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $d \neq 1$ is a constant. The eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 1 - d$. The right eigenvectors associated with λ_1 and λ_2 are

$$r^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } r^2 = \begin{bmatrix} 1 \\ -d \end{bmatrix},$$

respectively. The left eigenvectors associated with λ_1 and λ_2 and satisfying the normalization (4) are

$$l^1 = \begin{bmatrix} \frac{-d}{1-d} & \frac{-1}{1-d} \end{bmatrix} \text{ and } l^2 = \begin{bmatrix} \frac{1}{1-d} & \frac{1}{1-d} \end{bmatrix},$$

respectively. Denote by V the matrix of right eigenvectors of A:

$$V = [r^1 \;\; r^2] = \begin{bmatrix} 1 & 1 \\ -1 & -d \end{bmatrix}.$$

From the normalization condition (4), we can immediately write

$$V^{-1} = \begin{bmatrix} l^1 \\ l^2 \end{bmatrix} = \begin{bmatrix} \frac{-d}{1-d} & \frac{-1}{1-d} \\ \frac{1}{1-d} & \frac{1}{1-d} \end{bmatrix}.$$

The evolution of the modes can be obtained using the diagonalizing transformation $z := V^{-1}x$ as was done in (10)-(13). The system modes are found to be

$$\begin{aligned} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} &= \begin{bmatrix} l^1 x^0 e^{\lambda_1 t} \\ l^2 x^0 e^{\lambda_2 t} \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{-d}{1-d} x_1^0 - \frac{1}{1-d} x_2^0\right) e^{\lambda_1 t} \\ \frac{1}{1-d}\left(x_1^0 + x_2^0\right) e^{\lambda_2 t} \end{bmatrix}. \end{aligned} \tag{18}$$

Based on the original definition of participation factors, the participation factor for state x_1 in mode 2 is $p_{12} = r_1^2 l_1^2 = \frac{1}{1-d}$, and the participation factor for state x_2 in mode 2 is $p_{22} = r_2^2 l_2^2 = \frac{-d}{1-d}$. Clearly, in general $p_{12} \neq p_{22}$. However, from (18) we have the equation

$$z_2(t) \;=\; \frac{1}{1-d}\left(x_1^0 + x_2^0\right) e^{\lambda_2 t} \tag{19}$$

for the second mode $z_2(t)$, from which we observe that state x_1 and state x_2 *participate equally* in mode 2 since $z_2(t)$ depends on the initial condition x^0 through the sum $x_1^0 + x_2^0$. Again, we find that the state-in-mode participation factors as commonly calculated yield conclusions that are very much at odds with what one might consider reasonable based on explicit calculation of the evolution of system modes as they depend on initial conditions of the state variables. ■

The inadequacy of the original state-in-mode participation factors formula has been demonstrated in the two examples above. This motivates the need for a new formula that better assesses the influence of system states on system modes.

IV. Initial Condition Uncertainty Approach: Application to Derivation of Mode-in-State Participation Factors

For systems operating near equilibrium, it is often reasonable to view the system initial condition as being an uncertain vector in the vicinity of the system equilibrium point. In this paper, and in the authors' previous work [1], we approach the problem of measuring modal participation by averaging relative contributions over an uncertain set of initial conditions. In this section, we summarize this approach as it applies to the definition and calculation of mode-in-state participation factors. We carry the approach through to its conclusion for this problem, obtaining an explicit formula for mode-in-state participation factors. As mentioned above, the final formula we obtain using this approach agrees in this case with the previously existing expression p_{ki}. However, in the next section, such a happy coincidence will not occur for the more delicate situation of defining and calculating state-in-mode participation factors.

Next, we recall from our previous work [1] a basic definition of relative participation of a mode in a state. This definition involves taking an average over system initial conditions of a measure of the relative influence of a particular system mode on a system state. The initial condition uncertainty can be taken as set-theoretic or probabilistic. In the set-theoretic formulation, the participation factor measuring relative influence of the mode associated with λ_i on state x_k can be defined as

$$p_{ki} \;:=\; \underset{x^0 \in \mathcal{S}}{\text{avg}} \; \frac{(l^i x^0) r_k^i}{x_k^0} \tag{20}$$

whenever this quantity exists (however, see Remark 1 below for another possible definition for the case of complex λ_i). Here, $x_k^0 = \sum_{i=1}^n (l^i x^0) r_k^i$ is the value of $x_k(t)$ at $t = 0$, and "$\text{avg}_{x^0 \in \mathcal{S}}$" is an operator that computes the average of a function over a set $\mathcal{S} \subset R^n$ (representing the set of possible values of the initial condition x^0). We assume that the initial condition uncertainty set $\mathcal{S}$ is symmetric with respect to each of the hyperplanes $\{x_k = 0\}$, $k = 1, \ldots, n$.

In the definition in [1] that starts with a probabilistic description of the uncertainty in the initial condition x^0, the average in (20) is replaced by a mathematical expectation. The general formula for the participation factor p_{ki} measuring participation of mode i in state x_k becomes

$$p_{ki} \;:=\; E\left\{ \frac{(l^i x^0) r_k^i}{x_k^0} \right\} \tag{21}$$

where the expectation is evaluated using some assumed joint probability density function $f(x^0)$ for the initial condition uncertainty (of course, this definition applies only when the expectation exists).

Expanding the inner product term in (21), we find

$$\begin{aligned} p_{ki} &= E\left\{\sum_{j=1}^{n}\frac{(l_j^i x_j^0) r_k^i}{x_k^0}\right\} \\ &= E\left\{\frac{(l_k^i x_k^0) r_k^i}{x_k^0}\right\} + E\left\{\sum_{j=1,\ j\neq k}^{n}\frac{(l_j^i x_j^0) r_k^i}{x_k^0}\right\} \\ &= l_k^i r_k^i + \sum_{j=1,\ j\neq k}^{n} l_j^i r_k^i E\left\{\frac{x_j^0}{x_k^0}\right\}. \end{aligned} \tag{22}$$

The second term in (22) vanishes when the components of the initial condition vector $x_1^0, x_2^0, \ldots, x_n^0$ are independent with zero mean [1]. Therefore, under the assumption that the initial condition components $x_1^0, x_2^0, \ldots, x_n^0$ are independent with zero mean, the participation of the i-th mode in the k-th state is given by the same expression originally introduced by Perez-Arriaga, Verghese and Schweppe [7], [12]:

$$p_{ki} = l_k^i r_k^i. \tag{23}$$

This result can also be obtained using the set-theoretic averaging formula (20) [1].

Remark 1: (*Alternate Definition of Mode-in-State Participation Factor for a Complex Mode*) For a complex eigenvalue λ_i, the associated "mode" is taken above as the term containing $e^{\lambda_i t}$ in the system response (2). However, we can alternately view this mode as consisting of the combined contributions from λ_i and its complex conjugate eigenvalue λ_i^*. This viewpoint is easily seen to lead, under the same symmetry hypotheses as above, to the following alternate expression for the participation factor of the mode associated with λ_i and λ_i^* in state x_k:

$$\tilde{p}_{ki} = 2\mathrm{Re}\,\{l_k^i r_k^i\}. \tag{24}$$

V. New Definition of Participation Factors Measuring Participation of States in Modes

In this section, a new definition and calculation are given for participation factors measuring contribution of states in modes. The probabilistic approach presented in the previous section is used, where the initial condition is assumed to satisfy a joint probability density function. In order to obtain an explicit formula from the new general definition of state-in-mode participation factors, we find that it is necessary to make an assumption on the probability distribution of the initial condition which is more constraining than what was needed in the analysis above for mode-in-state participation factors. Thus, the explicit formula derived in this section should be viewed in the pragmatic sense that it provides an easy to use expression that reflects initial condition uncertainty. Other assumed forms of uncertainty may not lead to explicit formulas, although a formula requiring numerical evaluation of integrals can always be obtained from the definition. The explicit formula obtained here differs from the single formula (16) that is currently used to measure both state-in-mode participation and mode-in-state participation, while the currently used formula (16) is retained here as a measure of mode-in-state participation (noting that the alternate formula (24) can also be used for the case of a complex mode). This dichotomy represents a significant departure from current practice. We will also use the new formula to revisit the examples of Section III.

Consider the general linear time-invariant continuous-time system given in (3), repeated here for convenience:

$$\dot{x} = Ax(t) \tag{25}$$

The evolution of $z_i(t)$, $i = 1, \ldots, n$, was obtained in Section II as

$$\begin{aligned} z_i(t) &= e^{\lambda_i t} l^i x^0 \\ &= e^{\lambda_i t}\sum_{j=1}^{n}(l_j^i x_j^0). \end{aligned} \tag{26}$$

This equation shows the contribution of each component x_j^0, $j = 1, \ldots, n$ of the initial state x^0 to $z_i(t)$. Recall also that for the case of a real eigenvalue λ_i, $z_i(t)$ is identically the i-th mode, while, for a complex eigenvalue λ_i, the associated mode can be taken as $z_i(t)$ or as the combination of $z_i(t)$ and its conjugate: $z_i(t) + z_i^*(t) = 2\mathrm{Re}\,\{z_i(t)\}$. The following general definition of state-in-mode participation factors is obtained by averaging the relative contribution of x_k^0 in the i-th mode and evaluating the result at $t = 0$. In this definition, we take the mode associated with a complex eigenvalue as $2\mathrm{Re}\,\{z_i(t)\}$, i.e., the combination of modal components due to the eigenvalue and its conjugate. Had we decided to view the mode associated with a complex eigenvalue λ_i as $z_i(t)$ alone, we would use the first expression in the definition below for both the case of a real and a complex eigenvalue. However, the derivation following the basic definition below of a simple final formula would become unwieldy for the complex eigenvalue case.

Definition 1: For a linear time-invariant continuous-time system (25), the participation factor for the k-th state in the i-th mode is

$$\pi_{ki} := \begin{cases} E\left\{\frac{l_k^i x_k^0}{z_i^0}\right\} & \text{if } \lambda_i \text{ is real} \\ E\left\{\frac{(l_k^i + l_k^{i*}) x_k^0}{z_i^0 + z_i^{0*}}\right\} & \text{if } \lambda_i \text{ is complex} \end{cases} \tag{27}$$

whenever the expectation exists.

Note that in (27), the notation z_i^0 means $z_i(t=0) = l^i x^0$ and the asterisk denotes complex conjugation. Also, analogous to the approach in Section IV and the original work [7], [12], the quantities being evaluated represent the contribution of state x_k to a mode divided by the total mode evaluated at time $t = 0$ (however, see the Conclusions section about the possibility of measuring modal participation effects over time). Note also that Definition 1 always yields real-valued participation factors.

Unfortunately, even under an assumption such as symmetry of the initial condition uncertainty, there is no single closed-form expression for the state in mode participation factors π_{ki}. To obtain a simple closed-form expression for the state in mode participation factors π_{ki} using (27), we need to find an assumption on the probability density function $f(x^0)$ governing the uncertainty in the initial condition x^0 that allows us to explicitly evaluate the integrals inherent in the definition.

In the remainder of this section, we assume that the units of the state variables have been scaled to ensure that the probability density function $f(x^0)$ is such that the components $x_1^0, x_2^0, \ldots, x_n^0$ are jointly uniformly distributed over the unit sphere in R^n centered at the origin:

$$f(x^0) = \begin{cases} k & ||x^0|| \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

(This is the same as assuming a uniform distribution in an ellipsoid that is centered at the origin and symmetric with respect to the coordinate hyperplanes in the original state variable units, a physically palatable assumption and independent of units by construction.) The constant k is chosen to ensure the normalization

$$\int_{||x^0|| \leq 1} f(x^0) dx^0 = 1. \tag{29}$$

The value of the constant k can be determined by evaluating the integral in (29) using $f(x^0)$ given in (28):

$$\int_{||x^0|| \leq 1} f(x^0) dx^0 = \int_{||x^0|| \leq 1} k dx_1^0 dx_2^0 \ldots dx_n^0 = kV_n = 1 \tag{30}$$

where V_n is the volume of the unit sphere in R^n. The constant k is then given by

$$k = \frac{1}{V_n}. \tag{31}$$

The following Lemma will be used below.

Lemma 1: For vectors $a \in C^n$, $b \in R^n$ with $b \neq 0$ we have

$$\int_{||x|| \leq 1} \frac{a^T x}{b^T x} d_n x \quad = \quad \frac{a^T b}{b^T b} V_n \tag{32}$$

where $d_n x$ denotes the differential volume element $dx_1 dx_2 \cdots dx_n$, and V_n is the volume of a unit sphere in R^n which is given by

$$V_n = \begin{cases} 2, & n = 1 \\ \pi, & n = 2 \\ \frac{2\pi}{n} V_{n-2}, & n \geq 3 \end{cases} \tag{33}$$

Proof: The proof below is for the case $a, b \in R^n$, $b \neq 0$. The case $a \in C^n$ and $b \in R^n$ follows by linearity. Consider the transformation $x = Qy$ where the matrix Q is chosen to be an orthogonal matrix (i.e., $Q^{-1} = Q^T$) with first column $b^1 := \frac{b}{||b||}$ (i.e., $Q = [b^1 \;\; Q_1]$). The remaining columns of Q_1 are chosen to be orthogonal to b^1, i.e.,

$$(b^1)^T Q_1 = 0. \tag{34}$$

With this transformation, since $Q^{-1} = Q^T$, we have that $||x||^2 = x^T x = (Qy)^T Qy = y^T Q^T Qy = y^T y = ||y||^2$. Also, since Q is orthogonal, $\det(Q) = 1$ and $d_n x = d_n y$.

Writing the vector y as

$$y = \begin{pmatrix} y_1 \\ y' \end{pmatrix}$$

with $y_1 \in R$ and $y' \in R^{n-1}$, the integral in (32) can be expressed as

$$\begin{aligned} \int_{||x|| \leq 1} \frac{a^T x}{b^T x} d_n x &= \int_{||y|| \leq 1} \frac{a^T Qy}{b^T Qy} d_n y \\ &= \int_{||y|| \leq 1} \frac{a^T [b^1 \;\; Q_1] y}{b^T [b^1 \;\; Q_1] y} d_n y \\ &= \int_{||y|| \leq 1} \frac{a^T b^1 y_1 + a^T Q_1 y'}{b^T b^1 y_1 + \underbrace{b^T Q_1 y'}_{=0}} d_n y. \end{aligned} \tag{35}$$

The expression (35) can be further simplified as follows:

$$\begin{aligned} \int_{||x|| \leq 1} \frac{a^T x}{b^T x} d_n x &= \frac{a^T b^1}{b^T b^1} \int_{||y|| \leq 1} d_n y \\ &+ \frac{1}{b^T b^1} \int_{||y|| \leq 1} \frac{a^T Q_1 y'}{y_1} d_{n-1} y' dy_1. \end{aligned} \tag{36}$$

The first integral on the right of (37) evaluates to V_n, the volume of a unit sphere in R^n, whereas the second integral vanishes (it is improper but the Cauchy principal value is 0). Therefore,

$$\int_{||x|| \leq 1} \frac{a^T x}{b^T x} d_n x \quad = \quad \frac{a^T b}{b^T b} V_n \tag{37}$$

where V_n is given by (33). This completes the proof. ■

Next, the relative participation of the k-th state in the i-th mode is evaluated using Definition 1 under the assumption above on the distribution of the initial condition x^0. Before proceeding, we recall the relationship between x^0 and z^0:

$$\begin{aligned} x^0 &= V z^0 \\ &= \sum_{j=1}^{n} r^j z_j^0. \end{aligned} \tag{38}$$

A. Participation in a mode associated with a real eigenvalue

To determine the participation of the k-th state in a real mode associated with a real eigenvalue λ_i, we substitute $x_k^0 = \sum_{j=1}^n r_k^j z_j^0$ in (27):

$$\begin{aligned} \pi_{ki} &= E\left\{\frac{l_k^i x_k^0}{z_i^0}\right\} \\ &= E\left\{\frac{l_k^i \sum_{j=1}^n r_k^j z_j^0}{z_i^0}\right\} \\ &= E\left\{\frac{l_k^i r_k^i z_i^0}{z_i^0}\right\} + \sum_{j=1,\ j\neq i}^{n} l_k^i r_k^j E\left\{\frac{z_j^0}{z_i^0}\right\} \\ &= l_k^i r_k^i + \sum_{j=1,\ j\neq i}^{n} l_k^i r_k^j E\left\{\frac{z_j^0}{z_i^0}\right\}. \end{aligned} \tag{39}$$

Note that the first term in (39) coincides with p_{ki}, the original participation factors formula. We will find that, in general, the second term in (39) does not vanish. This is true even in case the components $x_1^0, x_2^0, \ldots, x_n^0$ representing the initial conditions of the state are assumed to be independent. This is due to the fact that the second term involves the components of z^0 (i.e., $z_1^0, z_2^0, \ldots, z_n^0$) which need not be independent even under the assumption that the x_k^0 are independent, due to the transformation $z^0 = V^{-1}x^0$. This was overlooked in [1], leading to the incorrect conclusion there that the second term in (39) vanishes.

We now use Lemma 1 to simplify the expression (39) for the participation factor for the k-th state in the i-th (real) mode:

$$\pi_{ki} = l_k^i r_k^i + \sum_{j=1,\ j\neq i}^{n} l_k^i r_k^j E\ \left\{\frac{z_j^0}{z_i^0}\right\}. \quad (40)$$

Substituting $z_i^0 = l^i x^0$ into (40) yields

$$\begin{aligned}\pi_{ki} &= l_k^i r_k^i + \sum_{j=1,\ j\neq i}^{n} l_k^i r_k^j E\ \left\{\frac{l^j x^0}{l^i x^0}\right\} \\ &= l_k^i r_k^i + \sum_{j=1,\ j\neq i}^{n} l_k^i r_k^j E\ \{g(x^0)\} \end{aligned} \quad (41)$$

where $g(x^0)$ takes the form

$$g(x^0) = \frac{l_1^j x_1^0 + l_2^j x_2^0 + \cdots + l_n^j x_n^0}{l_1^i x_1^0 + l_2^i x_2^0 + \cdots + l_n^i x_n^0}. \quad (42)$$

Denote $a := (l_1^j, l_2^j, \ldots, l_n^j)^T$ and $b := (l_1^i, l_2^i, \ldots, l_n^i)^T$. The expected value of $g(x^0)$ is

$$\begin{aligned}E\{g(x^0)\} &= \int_{||x^0||\leq 1} g(x^0) f(x^0)\ dx^0 \\ &= k\int_{||x^0||\leq 1} \frac{a^T x^0}{b^T x^0}\ dx^0 \end{aligned} \quad (43)$$

Using Lemma 1, which applies since b is real, and the normalization $kV_n = 1$ from (31), this integral reduces to

$$\begin{aligned}E\{g(x^0)\} &= \frac{a^T b}{b^T b} k V_n \\ &= \frac{a^T b}{b^T b}. \end{aligned} \quad (44)$$

Substituting (44) into (41) yields a key result of this paper, a new formula for the participation factor for state x_k in a real mode:

$$\pi_{ki} = l_k^i r_k^i + \sum_{j=1,\ j\neq i}^{n} l_k^i r_k^j \frac{l^j (l^i)^T}{l^i (l^i)^T}. \quad (45)$$

Remark 2: Under the initial condition uncertainty assumption based on which (45) was obtained, the participation factor for the i-th mode in the k-th state equals the participation factor for the k-th state in the i-th mode (i.e., $\pi_{ki} = p_{ki}$) if the left eigenvectors of the system matrix A are mutually orthogonal, i.e.,

$$l^j (l^i)^T = 0,\ \text{for}\ j,i = 1,2,\ldots,n,\ i\neq j.$$

This is a very restrictive case (which applies, for instance, when the system matrix, being real, is symmetric).

Next, we derive the following expression equivalent to (45)

$$\begin{aligned}\pi_{ki} &= \frac{(l_k^i)^2}{l^i (l^i)^T} \\ &= \frac{(l_k^i)^2}{\sum_{j=1}^n (l_j^i)^2}. \end{aligned} \quad (46)$$

This expression is easily obtained from Definition 1 and Lemma 1 as follows. Recall Definition 1, the general averaging-based definition for state-in-mode participation factors for the case of a real eigenvalue:

$$\pi_{ki} := E\ \left\{\frac{l_k^i x_k^0}{z_i^0}\right\}. \quad (47)$$

Substituting $z_i^0 = l^i x^0$ in (47) yields

$$\pi_{ki} = E\ \left\{\frac{l_k^i x_k^0}{l^i x^0}\right\} = E\ \left\{\frac{l_k^i e^k x^0}{l^i x^0}\right\}. \quad (48)$$

Denote

$$\begin{aligned}a &:= l_k^i e^k = l_k^i [0\ \ldots\ 0\ \ 1\ \ 0\ldots\ 0]^T, \\ b &:= (l^i)^T. \end{aligned}$$

Using Lemma 1 and the normalization $kV_n = 1$, (48) reduces to

$$\pi_{ki} = \frac{a^T b}{b^T b} = \frac{l_k^i e^k (l^i)^T}{l^i (l^i)^T} = \frac{(l_k^i)^2}{l^i (l^i)^T}, \quad (49)$$

which is exactly (46).

B. Participation in a mode associated with a complex conjugate pair of eigenvalues

To determine the participation factor for a state *in a complex mode*, i.e., a mode associated with a complex conjugate pair of nonreal eigenvalues λ_i, λ_i^*, we use the second case of (27):

$$\begin{aligned}\pi_{ki} &= E\left\{\frac{(l_k^i + l_k^{i*}) x_k^0}{z_i^0 + z_i^{0*}}\right\} \\ &= E\left\{\frac{\text{Re}\{l_k^i\} x_k^0}{\text{Re}\{z_i^0\}}\right\}. \end{aligned} \quad (50)$$

Substituting $z_i^0 = l^i x^0$ in (50) yields

$$\pi_{ki} = E\left\{\frac{\text{Re}\{l_k^i\} x_k^0}{\text{Re}\{l^i x^0\}}\right\} = E\left\{\frac{\text{Re}\{l_k^i\} x_k^0}{\text{Re}\{l^i\} x^0}\right\}. \quad (51)$$

Equation (51) can be rewritten as

$$\begin{aligned}\pi_{ki} &= E\left\{\frac{\text{Re}\{l_k^i\} (e^k)^T x^0}{\text{Re}\{l^i\} x^0}\right\} \\ &= \text{Re}\{l_k^i\}\ E\left\{\frac{(e^k)^T x^0}{\text{Re}\{l^i\} x^0}\right\}. \end{aligned} \quad (52)$$

Next, we obtain formulas analogous to (45) and (46) above, but now giving participation factors measuring participation of states in a complex mode.

First, to determine a formula analogous to (45), substitute $z_i^0 = l^i x^0$ and $x_k^0 = \sum_{j=1}^n r_k^j z_j^0 = \sum_{j=1}^n r_k^j l^j x^0$ and apply and Lemma 1 to obtain

$$\begin{aligned}\pi_{ki} &= E\left\{\frac{\text{Re}\{l_k^i\} x_k^0}{\text{Re}\{l^i\} x^0}\right\} \\ &= E\left\{\frac{\text{Re}\{l_k^i\} \sum_{j=1}^n r_k^j l^j x^0}{\text{Re}\{l^i\} x^0}\right\} \\ &= \text{Re}\{l_k^i\}\ E\left\{\frac{\sum_{j=1}^n r_k^j l^j x^0}{\text{Re}\{l^i\} x^0}\right\}. \end{aligned} \quad (53)$$

Let $a = \left(\sum_{j=1}^{n} r_k^j l^j\right)^T$ and $b = (\mathrm{Re}\{l^i\})^T$. Since b is real, we can invoke Lemma 1 and the normalization $kV_n = 1$ to reduce (53) to

$$\pi_{ki} \quad = \quad \mathrm{Re}\{l_k^i\} \, \frac{\left(\sum_{j=1}^{n} r_k^j l^j\right)(\mathrm{Re}\{l^i\})^T}{\mathrm{Re}\{l^i\}(\mathrm{Re}\{l^i\})^T}. \tag{54}$$

This formula can be rewritten as

$$\boxed{\pi_{ki} = \mathrm{Re}\{l_k^i\} \, \frac{\left(r_k^i l^i\right)(\mathrm{Re}\{l^i\})^T}{\mathrm{Re}\{l^i\}(\mathrm{Re}\{l^i\})^T} + \sum_{j=1,\ j\neq i}^{n} \frac{\mathrm{Re}\{l_k^i\} r_k^j l^j (\mathrm{Re}\{l^i\})^T}{\mathrm{Re}\{l^i\}(\mathrm{Re}\{l^i\})^T}.} \tag{55}$$

which is the desired form analogous to (45). We observe that if this formula is applied to a simple real eigenvalue λ_i, implying that the associated eigenvector can also be taken as real, the formula indeed reduces to the formula (45) that was derived for the case of a real mode. *Thus, formula (55) provides a general expression for state-in-mode participation factors for systems without any restriction on the eigenvalues besides that they are distinct.*

Next, we obtain another equivalent formula for the π_{ki} for a complex mode, but this time in a form analogous to the expression (46) derived above for the case of real eigenvalues. We use Lemma 1 to simplify the expression (52). Taking $a = e^k$ and $b = (\mathrm{Re}\{l^i\})^T$ in Lemma 1 and using the normalization $kV_n = 1$, (52) reduces to

$$\begin{aligned} \pi_{ki} &= \mathrm{Re}\{l_k^i\} \frac{(e^k)^T (\mathrm{Re}\{l^i\})^T}{\mathrm{Re}\{l^i\}(\mathrm{Re}\{l^i\})^T} \\ &= \mathrm{Re}\{l_k^i\} \frac{\mathrm{Re}\{l_k^i\}}{\mathrm{Re}\{l^i\}(\mathrm{Re}\{l^i\})^T}. \end{aligned}$$

Finally,

$$\boxed{\pi_{ki} = \frac{(\mathrm{Re}\{l_k^i\})^2}{\mathrm{Re}\{l^i\}(\mathrm{Re}\{l^i\})^T}.} \tag{56}$$

This expression is an alternate form of the newly proposed formula (55) for participation factors measuring participation of a state x_k in a mode. Both expressions apply for a mode associated with a complex conjugate pair of eigenvalues λ_i and λ_i^*, as well as for a simple real eigenvalue λ_i.

Remark 3: Although care was taken with respect to selecting units in deriving the formulas for state-in-mode participation factors, the final formulas are not themselves independent of units. The independence with respect to state variable units that occurs in the definitions of mode-in-state participation factors is a fortunate coincidence for certain choices of initial conditions or under certain initial condition symmetry assumptions. However, no such coincidence occurs in the quantification of state-in-mode participation. One way of viewing this is as follows. Units are important, in the sense that they should be chosen so that a unit variation in the initial condition of any state variable has a similar likelihood as a unit variation in the initial condition of any other state variable of the system. That is the spirit of the assumption made above that the distribution of initial conditions of the state vector can be mapped by changes of units to a uniform distribution on a unit sphere in R^n.

Next, we revisit Examples 1 and 2 using the newly derived formula for state in mode participation factors, and compare the results to the participation factors obtained using the original definitions. Note that all eigenvalues in these examples are real, so the formula (45) applies as a (new) measure of state-in-mode participation factors π_{ki} (as does the equivalent formula (46)).

Example 1 Revisited

For Example 1, the participation factors for states x_1 and x_2 in mode 1 based on the new formula (45) are

$$\pi_{11} \quad = \quad \frac{(a-d)^2}{(a-d)^2+b^2} \quad \text{and} \quad \pi_{21} = \frac{b^2}{(a-d)^2+b^2},$$

respectively. The participation factors for states x_1 and x_2 in mode 1 based on the original formula are $p_{11} = 1$ and $p_{21} = 0$, respectively.

As we observed previously in our discussion of Example 1, the original formula for participation factors erroneously indicates that the participation of state x_2 in mode 1 is zero. The coupling between state x_2 and state x_1 in the system dynamics is not reflected in the original formula for participation factors (the p_{ki}), whereas this coupling between state variables is reflected in the result of applying the new formula (for the π_{ki}).

Example 2 Revisited

For Example 2, the participation factors for states x_1 and x_2 in mode 2 based on the new formula (45) are

$$\pi_{12} = \frac{1}{2} \quad \text{and} \quad \pi_{22} = \frac{1}{2},$$

respectively. The participation factors for states x_1 and x_2 in mode 2 based on the original formula are

$$p_{12} = \frac{1}{1-d} \quad \text{and} \quad p_{22} = \frac{-d}{1-d},$$

respectively.

The results using the new formula more faithfully reflect the relative contributions of the initial conditions of the two state variables to the evolution of mode 2, which is given explicitly by the formula

$$z_2(t) \quad = \quad \frac{1}{1-d}\left(x_1^0 + x_2^0\right) e^{\lambda_2 t}, \tag{57}$$

Here it is clear that $z_2(t)$ is equally influenced by x_1^0 and x_2^0 since it depends on the initial condition x^0 through the sum $x_1^0 + x_2^0$.

VI. A Numerical Example: A Two-Mass Mechanical System

Consider the translational mechanical system depicted in Figure 1, where $y_1(t)$ and $y_2(t)$ denote the displacements of mass 1 and mass 2, respectively, from the static equilibrium [13]. The system parameters are the masses m_1 and m_2, viscous damping coefficients c_1 and c_2, and the spring constants k_1 and k_2.

A state space representation is obtained by defining the system states as

$$\begin{aligned} x_1(t) &= y_1(t) \\ x_2(t) &= \dot{y}_1 = \dot{x}_1 \\ x_3(t) &= y_2(t) \\ x_4(t) &= \dot{y}_2 = \dot{x}_3. \end{aligned}$$

The system dynamics is described by the linear time-invariant differential equation [13]

$$\dot{x} = Ax$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix}.$$

With the system parameters selected as $m_1 = 39$ kg, $m_2 = 17$ kg, $c_1 = 19$ Ns/m, $c_2 = 33$ Ns/m, $k_1 = 374$ N/m, and $k_2 = 196$ N/m, the system state dynamics matrix A becomes

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -14.6154 & -1.3333 & 5.0256 & 0.8462 \\ 0 & 0 & 0 & 1 \\ 11.5294 & 1.9412 & -11.5294 & -1.9412 \end{bmatrix}.$$

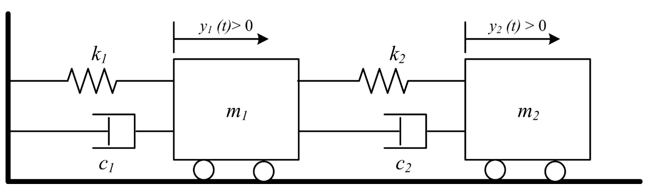

Fig. 1. Mechanical system.

The eigenvalues of A are $\lambda_{1,2} = -0.217 \pm j2.315$ and $\lambda_{3,4} = -1.4203 \pm j4.2935$. We denote the modes associated with the eigenvalues as $z_1(t)$ and $z_2(t)$, respectively. The dominant mode is the one associated with the complex conjugate pair of eigenvalues closest to the imaginary axis, i.e., $\lambda_{1,2}$. Modes associated with eigenvalues close to the imaginary axis are of considerable interest as they can be used as an indication of closeness to system instability. Therefore, our emphasis in this example will be on $z_1(t)$.

The right (column) eigenvectors of the system matrix A associated with λ_1 and λ_3 are

$$r^1 = \begin{bmatrix} 0.0124 - j0.1938 \\ 0.4461 + j0.0708 \\ -0.0321 - j0.3425 \\ 0.7999 \end{bmatrix} \text{ and } \qquad (58)$$

$$r^3 = \begin{bmatrix} 0.0716 + j0.1021 \\ -0.5402 + j0.1624 \\ -0.0553 - j0.1673 \\ 0.7970 \end{bmatrix},$$

respectively. Note that since $\lambda_2 = \lambda_1^*$ and $\lambda_4 = \lambda_3^*$, we have that $r^2 = r^{1*}$ and $r^4 = r^{3*}$, where an asterisk denotes complex conjugation. The left (row) eigenvectors of the system matrix A associated with λ_1 and λ_3 are

$$l^1 = \begin{bmatrix} -0.3122 + j1.2059 \\ 0.4806 + j0.0539 \\ 0.2760 + j0.7884 \\ 0.3730 - j0.0171 \end{bmatrix}^T,$$

$$l^3 = \begin{bmatrix} 0.3134 - j2.4251 \\ -0.4824 - j0.1776 \\ -0.2771 + j1.3357 \\ 0.2530 + j0.1903 \end{bmatrix}^T \qquad (59)$$

respectively, and $l^2 = l^{1*}$; $l^4 = l^{3*}$.

The magnitudes of the original participation factors p_{ki} evaluated using (23) for this example are given in Table I. The state in mode participation factors π_{ki} evaluated using the new formula (56) are given in Table II.

TABLE I

PARTICIPATION FACTORS, p_{ki}, BASED ON ORIGINAL FORMULA.

	mode 1	mode 2
x_1	$\lvert p_{11}\rvert = 0.2420$	$\lvert p_{12}\rvert = 0.3050$
x_2	$\lvert p_{21}\rvert = 0.2184$	$\lvert p_{22}\rvert = 0.2900$
x_3	$\lvert p_{31}\rvert = 0.2874$	$\lvert p_{32}\rvert = 0.2404$
x_4	$\lvert p_{41}\rvert = 0.2987$	$\lvert p_{42}\rvert = 0.2523$

TABLE II

STATE IN MODE PARTICIPATION FACTORS, π_{ki}, BASED ON NEW FORMULA.

	mode 1	mode 2
x_1	$\pi_{11} = 0.1792$	$\pi_{12} = 0.2082$
x_2	$\pi_{21} = 0.4248$	$\pi_{22} = 0.4934$
x_3	$\pi_{31} = 0.1401$	$\pi_{32} = 0.1628$
x_4	$\pi_{41} = 0.2558$	$\pi_{42} = 0.1357$

We observe that the participation factors given in Table I, which are calculated using the original definition of participation factors, differ from the state in mode participation factors in Table II calculated using the new formula. For instance, according to Table I, the state that participates most in mode 1 is x_4, whereas according to Table II, the state that participates most in mode 1 is x_2. To demonstrate that state x_2 participates more than other state variables in mode 1, we calculate the evolution of mode 1 ($z_1(t)$) due to different settings in the initial conditions. Specifically, $z_1(t)$ is calculated for the following set of initial conditions: $x^0 = [0.1, 0, 0, 0]^T$, $x^0 = [0, 0.1, 0, 0]^T$, $x^0 = [0, 0, 0.1, 0]^T$, and $x^0 = [0, 0, 0, 0.1]^T$. We select the initial conditions in this way (0 in all but one of the state variables) to distinguish the influence of each of the state variables on mode 1. The simulation results are depicted in Figure 2, which shows plots of $\mathrm{Re}\{z_1(t)\}$ for the various initial conditions. Figure 2 shows that the initial condition component x_2^0 gives the largest effect on mode 1 at $t = 0$ compared to all other state variables. This agrees with what is predicted using the new formula for state in mode participation factors (see Table II). In other words, mode 1 can be excited by an initial condition on any of the state variables, however, the highest excitation at $t = 0$ comes from x_2^0 and the least excitation comes from x_3^0.

The state in mode participation factors calculated based on the new formula derived in this paper can be used to determine the relative degrees by which system states excite a particular mode in the system. This can be useful in stability monitoring applications. For example, in stressed electric power systems, typically there is one critical mode that needs to be monitored. If the system is operating exactly at an equilibrium point and not influenced by disturbances, this mode cannot be observed in the outputs of the system. In order to monitor the critical mode and detect closeness to instability, a small perturbation signal is applied to the system and the response is measured. The state in mode participation factors can help in selecting a location for applying the perturbation signal in order to achieve the highest excitation in the critical mode to achieve the clearest possible indication on how close the system is to instability.

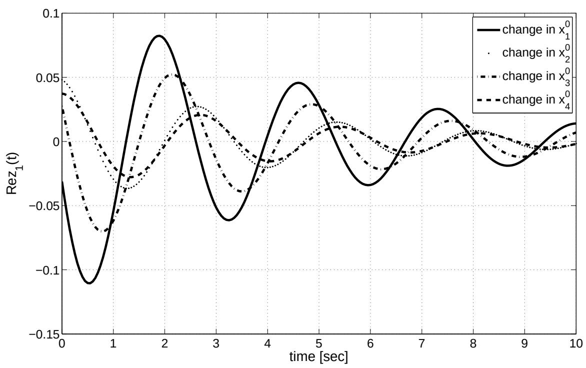

Fig. 2. The effect on mode 1 of a perturbation of 0.1 away from equilibrium in the initial condition component x_1^0 (solid), x_2^0 (dotted), x_3^0 (dash-dot) and x_4^0 (dashed).

VII. A Further Remark on Mode in State Participation Factors

In this section, an additional result is given that relates only to mode in state participation factors. The result is that, in the averaging formulation, an additional possible set of initial condition uncertainty assumption is found to also lead to the traditional participation factors formula for mode in state participation factors.

In Section IV, we showed that when the components of the initial condition vector $x_1^0; x_2^0, \ldots, x_n^0$, are independent random variables with zero mean, the participation of mode i in state x_k is given by

$$p_{ki} = l_k^i r_k^i. \tag{60}$$

In this section, we show that this expression remains valid if the components of the initial condition x^0, x_j^0, $j = 1, 2, \ldots, n$, are assumed to be jointly uniformly distributed over the unit sphere in R^n (see (28) for the expression of the probability density function in this case). Note that under this assumption, the random variables x_j^0 are no longer independent.

Consider the general expression (22) for the mode in state participation factor p_{ki}, repeated here for convenience:

$$p_{ki} = l_k^i r_k^i + \sum_{j=1,\ j\neq k}^{n} l_j^i r_k^i E\left\{\frac{x_j^0}{x_k^0}\right\}. \tag{61}$$

A typical term in the summation on the right side of (61) takes the form

$$\begin{aligned} E\left\{\frac{x_j^0}{x_k^0}\right\}_{j\neq k} &= \int_{\|x^0\|\leq 1} \frac{x_j^0}{x_k^0} f(x^0)\ \ dx^0 \\ &= k\int_{\|x^0\|\leq 1} \frac{x_j^0}{x_k^0}\ \ dx^0 \end{aligned} \tag{62}$$

Denote

$$\begin{aligned} a &:= (0,\ldots,0,\underbrace{1}_{j^{th}},0,\ldots,0)^T, \\ b &:= (0,\ldots,0,\underbrace{1}_{k^{th}},0,\ldots,0)^T. \end{aligned}$$

The integral in (62) can be expressed as

$$E\left\{\frac{x_j^0}{x_k^0}\right\}_{j\neq k} = k\int_{\|x^0\|\leq 1} \frac{a^T x^0}{b^T x^0}\ \ dx^0 \tag{63}$$

Using Lemma 1, this expression reduces to

$$E\left\{\frac{x_j^0}{x_k^0}\right\}_{j\neq k} = \frac{a^T b}{b^T b} = 0. \tag{64}$$

Therefore, the second term in (61) vanishes and, under the new assumptions, the mode in state participation factors are still given by

$$p_{ki} = l_k^i r_k^i \tag{65}$$

We can therefore conclude that (65) is a valid formula for mode in state participation factors under any of the following assumptions on the initial conditions:

1) The initial condition x^0 is taken to lie in an uncertainty set $\mathcal{S}$ which is symmetric with respect to each of the hyperplanes $x_k^0 = 0$, $k = 1, 2, \ldots, n$.
2) The initial condition components are independent random variables with marginal density functions which are symmetric with respect to $x_k^0 = 0$, $k = 1, 2, \ldots, n$
3) The initial condition components, x_j^0, $j = 1, 2, \ldots, n$, are jointly uniformly distributed over a sphere centered at the origin.

VIII. Conclusions

We have presented a new fundamental approach to quantifying the participation of states in modes for linear time-invariant systems. We have proposed that a new definition and formula replace the commonly used participation factors formula for measuring participation of states in modes, while recommending the previously used participation factors formula be retained as a quantification of participation of modes in states. The analysis presented in this paper uses averaging over an uncertain set of system initial conditions. The analysis led to the conclusion that participation of modes in states and participation of states in modes should not be viewed as equivalent or interchangeable. Examples were given to demonstrate that the original formula for participation factors is not convincing as a measure of state in mode participation. Moreover, these examples demonstrated the applicability and usefulness of the new formula for state in mode participation factors.

It is interesting that while the problem addressed in this paper relates to a very simple and well studied class of systems, considerable effort was required to revisit what may seem to be a basic issue, namely modal participation. Indeed, it appears that work is needed on further related matters, such as implications of the results for sensor and actuator placement, for system monitoring to detect impending instability, for order reduction, and possibly for coherency studies of power networks, among other issues. Also, from the results in Figure 2 for the mechanical example, the relative sizes of modal participations at the initial time instant might differ from those over a time interval. For some applications, it may be desirable to have analytical measures of modal participation that quantify modal participation over time.

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