Poincaré Types Solutions of Systems of Difference Equations[1]

Raghib Abu-Saris[a], Saber Elaydi[a2] and Sophia Jang[b]

a. Mathematics Department, Birzeit University, P.O. Box 14, Birzeit, Palestine. b. Department of Mathematics and Statistics, Texas Tech University, Lubbock, Texas 79409-1042 U.S.A.

1 Introduction

The study of asymptotics theory of ordinary difference equations originates from the work of Henry Poincaré. In 1885, Poincaré [19] published a seminal paper on the asymptotics of both ordinary difference and differential equations, where he studied the k^{th} order linear nonautonomous difference equation of the form

$$y(n+k)+(a_1+p_1(n))y(n+k-1)+\cdots+(a_k+p_k(n))y(n)=0 \tag{1.1}$$

with $k \in Z_+,\ a_i \in C$ and $p_i(n) : Z_+ \to C$ for $1 \le i \le k$. This equation is said to be of Poincaré type if $\lim_{n\to\infty} p_i(n) = 0$ for $1 \le i \le k$. We assume that Eq.(1.1) is of Poincaré type and associated with Eq.(1.1) its limiting equation

$$x(n+k)+a_1x(n+k-1)+\cdots+a_kx(n)=0 \tag{1.2}$$

with the corresponding characteristic equation

$$\lambda^k + a_1\lambda^{k-1}+\cdots+a_k=0. \tag{1.3}$$

Suppose that $\lambda_1,\ \lambda_2, ...,\ \lambda_k$ are the characteristic roots of Eq.(1.2), i.e., the roots of Eq.(1.3). It is straightforward to see that solutions of Eq.(1.2) are of the form

$$\sum_{i=1}^{r} q_i(n)\lambda_i^n$$

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[2]This research was done while the author was visiting Birzeit University under a Fulbright Grant. His permanent address is: Trinity University, San Antonio, Texas 78212 USA.

where $q_i(n)$ is a polynomial in n of degree less than the multiplicity of λ_i and $\lambda_1, \cdots, \lambda_r$ are all the distinct characteristic roots of Eq.(1.2). The main goal of the asymptotic theory is to relate solutions of Eq.(1.1) with solutions of Eq.(1.2) in an asymptotic fashion. We now state the fundamental result due to Poincaré [19] as mentioned above. **Poincaré Theorem** *Suppose that λ_i's are the characteristic roots of Eq.(1.2) and $|\lambda_i| \neq |\lambda_j|$ for $i \neq j$. Then every solution $y(n)$ of Eq.(1.1) satisfies either $y(n) = 0$ for all large n or*

$$\lim_{n\to\infty} \frac{y(n+1)}{y(n)} = \lambda_i \tag{1.4}$$

for some characteristic root λ_i.

Oscar Perron [16] later improved this fundamental result of Poincaré. He showed that under the condition $a_k \neq 0$, Eq.(1.1) has a fundamental set of solutions $y_i(n)$ which satisfy Eq.(1.4) for $1 \leq i \leq k$. Subsequently, Perron [17] also removed the conditions imposed on the characteristic roots but gave a weaker conclusion than Eq.(1.4) as stated below. **Perron Theorem** *Suppose that $a_k \neq 0$. Then Eq.(1.1) has k linearly independent solutions $y_i(n)$, $1 \leq i \leq k$, such that*

$$\limsup_{n\to\infty} \sqrt[n]{|y_i(n)|} = |\lambda_i| \tag{1.5}$$

where λ_i's are the characteristic roots of Eq.(1.2).

Based on a result of C.W. Coffman [4], M. Pituk [18] proved a standing conjecture which states that *every solution $y(n)$ of Eq.(1.1) satisfies*

$$\limsup_{n\to\infty} \sqrt[n]{|y(n)|} = |\lambda| \tag{1.6}$$

for some characteristic root λ of Eq.(1.2). This conjecture was first introduced in a seminar at Trinity University by U. Krause and S. Elaydi. Motivated by equations (1.4) and (1.5), U. Krause in the above-mentioned seminar introduced several types of solutions of Eq.(1.1) which he called "Poincaré types" of solutions: weak Poincaré (WP), Poincaré (P), and strong Poincaré (SP). The main objective of this paper is to extend these "Poincaré types" solutions of scalar difference equations to the k-dimensional system

$$\vec{y}(n+1) = [A + B(n)]\, \vec{y}(n), \qquad n \geq 0, \tag{1.7}$$

where A is a $k \times k$ nonsingular matrix and $B(n)$ is a $k \times k$ matrix defined on Z_+. A notion closely related to our Poincaré type solutions is the notion of strong ergodicity which is known in the mathematical ecology literature [6, 14]. In population biology, matrix difference equations have been exploited to study the dynamics of structured population models since the pioneering work of Lewis [13] and Leslie [12] in the 1940s. It is often desirable to understand the long term behavior of population growth. One of the most important aspects in this respect is ergodicity. A population is said to be *ergodic* if its eventual behavior is independent of its initial state [3]. For an age-structured population model with unchanging fertility and mortality rates, it is known that the normalized age distribution approaches a stable age distribution regardless of the initial population. Such property is well documented and often referred to as the fundamental theorem of demography or the strong ergodic theorem of demography [3, 5]. For other types of structured population models, for example the size-structured models, a similar asymptotic property can occur if the vital rates under consideration are also assumed to be independent of time and population density [3]. Motivated by this concept we introduce the more general notion of ergodic Poincaré. We show that strong Poincaré implies Poincaré, Poincaré implies weak Poincaré, and ergodic Poincaré implies Poincaré. For the case when the eigenvalue is positive, strong Poincaré implies ergodic Poincaré. Counterexamples are given to illustrate the fact that these implications may not be reversed. Let R be the set of real numbers and $R^k_+ = \{(x_1, x_2, \cdots, x_k) \in R^k : x_i \geq 0 \text{ for } 1 \leq i \leq k\}$ be the positive cone of R^k. A matrix A is called *nonnegative* if each of its entries is nonnegative, in which case we write $A \geq 0$. A is called *positive* if $A \geq 0$ and $A \neq 0$, we write $A > 0$. A is called *strictly positive*, $A >> 0$ in notation, if each of its entries is positive. Similar terminology is also used for vectors. Let $\vec{e}_i \in R^k_+$ denote the column vector for which the ith entry is 1, with all other entries 0. The celebrated Perron-Frobenius theory [20] states that for any irreducible and primitive $k \times k$ matrix $A > 0$, there exists a unique dominant eigenvalue $\lambda_1 > 0$ which is moreover simple. Corresponding to this eigenvalue there exists a right eigenvector $\vec{v}_1 >> 0$. Moreover, there exists an integer $p > 0$ such that $A^p >> 0$ by the primitivity of A. The main results are Theorems 3.3, 3.5 and 4.3. Theorem 3.3 gives a genuine extension of Poincaré Theorem to systems of difference equations. A sufficient condition for which Eq.(1.7) is of ergodic Poincaré is presented in Theorem 3.4. Theorem 4.3 derives sufficient conditions for the strong Poincaré property of

Eq.(1.7) when Eq.(1.7) is regarded as a perturbation of the corresponding linear system. The final section provides conditions under which the nonlinear system is strong ergodic. We refer the reader to the treatise [1] and [9] for basic material on asymptotic theory of difference equations.

2 Classification of Solutions

In this section we define several types of solutions of the following linear nonautonomous system of difference equations

$$\vec{y}(n+1) = [A+B(n)]\vec{y}(n), n = 0, 1, 2, \cdots, \tag{2.1}$$

where A is a $k \times k$ nonsingular matrix and $B(n)$ is a $k \times k$ matrix defined on Z_+. We then discuss relationships between these types of solutions. Counterexamples will be given to demonstrate irreversible of the relationship. **Definition 2.1** *Let $\vec{y}(n)$ be a solution of Eq.(2.1). Then $\vec{y}(n)$ is said to be of*

(1) weak Poincaré type (WP) *if*

$$\lim_{n\to\infty} \sqrt[n]{\|\vec{y}(n)\|} = |\lambda|$$

for some eigenvalue λ of A.

(2) Poincaré type (P) *if*

$$\lim_{n\to\infty} \frac{\|\vec{y}(n+1)\|}{\|\vec{y}(n)\|} = |\lambda|$$

for some eigenvalue λ of A.

(3) strong Poincaré type (SP) *if*

$$\lim_{n\to\infty} \frac{\vec{y}(n)}{\lambda^n} = \vec{c}$$

for some eigenvalue λ of A and some vector $\vec{c} \neq \vec{0}$.

(4) ergodic Poincaré (EP) *if*

$$\lim_{n\to\infty} \frac{\vec{y}(n)}{\|\vec{y}(n)\|} = \vec{\xi}$$

for some eigenvector $\vec{\xi}$ *of* A.

We say that Eq.(2.1) has one of the above-mentioned properties if each one of its nontrivial solutions has the property. Eq.(2.1) possesses strong ergodic *property if there exists an eigenvector* $\vec{\xi} >> 0$ *of* $A > 0$ *such that every solution* $\vec{y}(n)$ *of Eq.(2.1) with* $\vec{y}(0) > 0$ *is of ergodic Poincaré with the same* $\vec{\xi}$ *as its limit.*

Before investigating the interrelations between the four types of solutions introduced above, we establish the following lemma. **Lemma 2.1** *Let* $\vec{y}(n)$ *be a solution of Eq.(2.1). Then*

(a) $\vec{y}(n)$ *is of WP if* $\vec{y}(n)$ *is of P.*

(b) $\vec{y}(n)$ *is SP if and only if*

$$\vec{y}(n) = [\vec{\xi} + o(1)]\lambda^n$$

for some eigenvector $\vec{\xi}$ *of* A *belonging to the eigenvalue* λ*, given that* $\lim_{n\to\infty} B(n) = 0$.

Proof. (a) Suppose that $\vec{y}(n)$ is of P, i.e., there exists an eigenvalue λ of A such that

$$\lim_{n\to\infty} \frac{\|\vec{y}(n+1)\|}{\|\vec{y}(n)\|} = |\lambda|.$$

Then

$$\|\vec{y}(n)\| = \left(\prod_{j=0}^{n-1} [\ |\lambda| + \gamma(j)]\right) \|\vec{y}(0)\|$$

for some null sequence $\gamma(n)$. Hence

$$\lim_{n\to\infty} \sqrt[n]{\|\vec{y}(n)\|} \quad = \quad |\lambda| \lim_{n\to\infty} \sqrt[n]{\prod_{j=0}^{n-1}\left[1 + \frac{\gamma(j)}{|\lambda|}\right]} \lim_{n\to\infty} \sqrt[n]{\|\vec{y}(0)\|}.$$

Note that $\|\vec{y}(0)\| \neq 0$ by our assumption. As a result, $\lim_{n\to\infty} \sqrt[n]{\|\vec{y}(n)\|} = |\lambda|$ and $\vec{y}(n)$ is of WP. (b) If $\vec{y}(n)$ is SP, then there exists an eigenvalue λ of A and a vector $\vec{c} \neq \vec{0}$ such that $\lim_{n\to\infty} \vec{y}(n)/\lambda^n = \vec{c}$, and so $\vec{y}(n) = [\vec{c}+o(1)]\lambda^n$. Consequently, from Eq.(2.1) we have

$$\lambda \frac{\vec{y}(n+1)}{\lambda^{n+1}} = [A + B(n)]\frac{\vec{y}(n)}{\lambda^n} \Rightarrow \lambda\vec{c} = A\vec{c},$$

i.e., $\vec{c}$ is an eigenvector of A belonging to λ. This proves sufficiency. The necessity is straightforward. ■ Now, we summarize some implications between the four types of solutions introduced in Definition 2.1. **Theorem 2.2** *Let $\vec{y}(n)$ be a solution of Eq.(2.1). Then*

(a) $\vec{y}(n)$ is of SP $\Rightarrow \vec{y}(n)$ is of P $\Rightarrow \vec{y}(n)$ is of WP.

(b) $\vec{y}(n)$ is of EP $\Rightarrow \vec{y}(n)$ is of P if $\lim_{n\to\infty} B(n) = 0$.

If $\vec{y}(n)$ is of SP with an associated positive eigenvalue λ and $\lim_{n\to\infty} B(n) = 0$, *then $\vec{y}(n)$ is of EP.*

Proof. (a) If $\vec{y}(n)$ is SP, then

$$\vec{y}(n) = \lambda^n[\vec{c} + \vec{\Gamma}(n)], \quad \text{where} \quad \vec{\Gamma}(n) = o(1) \text{ and } \vec{c} \neq \vec{0}.$$

Consequently,

$$\frac{\vec{y}(n+1) - \lambda\vec{y}(n)}{\|\vec{y}(n)\|}\lambda \left(\frac{\lambda}{|\lambda|}\right)^n \frac{\vec{\Gamma}(n+1) - \vec{\Gamma}(n)}{\|\vec{c} + \vec{\Gamma}(n)\|} \to \vec{0} \quad \text{as} \quad n \to \infty,$$

and $\vec{y}(n+1)$ can be written as

$$\vec{y}(n+1) = \lambda\vec{y}(n) + \vec{\Gamma}_1(n)\|\vec{y}(n)\|, \qquad \text{where} \quad \vec{\Gamma}_1(n) = o(1).$$

By Triangle Inequality,

$$|\lambda| - \|\vec{\Gamma}_1(n)\| \leq \frac{\|\vec{y}(n+1)\|}{\|\vec{y}(n)\|} \leq \lambda| + \|\vec{\Gamma}_1(n)\|.$$

Therefore,

$$\lim_{n\to\infty} \frac{\|\vec{y}(n+1)\|}{\|\vec{y}(n)\|} = |\lambda|.$$

i.e., $\vec{y}(n)$ is P and hence is WP by Lemma 2.1(a).

(b) Suppose now $\vec{y}(n)$ is EP with $\lim_{n\to\infty} \dfrac{\vec{y}(n)}{\|\vec{y}(n)\|} = \vec{\xi}$ for some eigenvector $\vec{\xi}$ of A. Notice that $\|\vec{\xi}\| = 1$ and $A(\lim_{n\to\infty} \dfrac{\vec{y}(n)}{\|\vec{y}(n)\|}) = A\vec{\xi}$. Thus

$$\lim_{n\to\infty} \frac{A\vec{y}(n)}{\|\vec{y}(n)\|} = \lambda\vec{\xi},$$

where λ is the eigenvalue of A with the eigenvector $\vec{\xi}$. Consequently,

$$\lim_{n\to\infty} [A + B(n)] \frac{\vec{y}(n)}{\|\vec{y}(n)\|} = \lambda\vec{\xi}$$

and thus $\lim_{n\to\infty} \dfrac{\|\vec{y}(n+1)\|}{\|\vec{y}(n)\|} = |\lambda|$, i.e., $\vec{y}(n)$ is of P. (c) If $\vec{y}(n)$ is SP, then by Lemma 2.1(b)

$$\vec{y}(n) = \lambda^n \left[\vec{\xi} + o(1)\right],$$

for some eigenvector $\vec{\xi}$. Since $\lambda > 0$,

$$\frac{\vec{y}(n)}{\|\vec{y}(n)\|} = \frac{\vec{\xi} + o(1)}{\|\vec{\xi} + o(1)\|} \to \frac{\vec{\xi}}{\|\vec{\xi}\|}, \quad \text{as} \quad n \to \infty$$

i.e., $\vec{y}(n)$ is EP. ■

The following examples show that the converse of Theorem 2.2 need not be true.

Example 2.1 Consider the following system

$$\vec{y}(n+1) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \vec{y}(n), \qquad n \geq 0.$$

Then

$$\vec{y}(n) = \alpha(-1)^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \beta + \alpha(-1)^n \\ \beta \end{pmatrix},$$

is a solution, where $\alpha, \beta > 0$. By a direct computation one can see that

$$\lim_{n\to\infty} \sqrt[n]{\|\vec{y}(n)\|} = 1,$$

where $\lambda = 1$ is an eigenvalue of A, i.e., $\vec{y}(n)$ is WP. However,

$$\lim_{n\to\infty} \frac{\|\vec{y}(n+1)\|}{\|\vec{y}(n)\|} = \begin{cases} \dfrac{\beta}{\beta+\alpha} & \text{if } n \text{ is even} \\ \dfrac{\beta+\alpha}{\beta} & \text{if } n \text{ is odd.} \end{cases} \tag{2.2}$$

Thus $\lim_{n\to\infty} \dfrac{\|\vec{y}(n+1)\|}{\|\vec{y}(n)\|}$ doesn't exist as $\alpha, \beta > 0$, i.e., $\vec{y}(n)$ is WP but is not P. This demonstrates that the implication of Theorem 2.2(a) can't be reversed. **Example 2.2** Consider the system

$$\begin{aligned} \vec{y}(n+1) &= \begin{pmatrix} -\frac{n+1}{2n} & 0 \\ 0 & 1 \end{pmatrix} \vec{y}(n), \qquad n \geq 1, \\ \vec{y}(1) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

The solution is given by

$$\vec{y}(n) = \frac{(-1)^{n-1}\, n}{2^{n-1}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad n = 1, 2, ...$$

Since

$$\frac{\|\vec{y}(n+1)\|}{\|\vec{y}(n)\|} \to \frac{1}{2} \text{ as } n \to \infty,$$

where $-1/2$ is an eigenvalue of the corresponding linear system, $\vec{y}(n)$ is P. However,

$$\lim_{n\to\infty} \frac{\vec{y}(n)}{\|\vec{y}(n)\|} = \lim_{n\to\infty} (-1)^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

does not exist, i.e., $\vec{y}(n)$ is not EP. Therefore the converse of Theorem 2.2(b) is not true. **Example 2.3** Consider the difference system

$$\begin{aligned} \vec{y}(n+1) &= \begin{pmatrix} \frac{n+1}{n} & 0 \\ 0 & 1 \end{pmatrix} \vec{y}(n), \qquad n \geq 1, \\ \vec{y}(1) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

The solution is given by

$$\vec{y}(n) = n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad n = 1, 2, ...$$

Since

$$\frac{\vec{y}(n)}{\|\vec{y}(n)\|} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where $(1,0)^T$ is an eigenvector of A belonging to $\lambda = 1$, $\vec{y}(n)$ is EP. However,

$$\frac{\vec{y}(n)}{1^n}$$

diverges, i.e., $\vec{y}(n)$ is not SP. We conclude that the converse of Theorem 2.2 in general is not true. This example also demonstrates that Poincaré doesn't imply strong Poincaré.

3 An Extension and a Generalization of Poincaré Theorem

Observe that Example 2.1 (and there are many other examples as well) shows that the result of Pituk theorem can't be replaced by either P or EP. In this section we strengthen the assumptions on the eigenvalues of A and obtain sufficient conditions for the Poincaré and ergodic Poincaré properties. We begin with a definition and two crucial lemmas. These concept and basic results will enable us to accomplish our goal. **Definition 3.1** *A solution $\vec{y}(n)$ of Eq.(2.1) is said to have the* index for maximum property *(IMP) if there exists an index $l \in \{1, ..., k\}$ such that for sufficiently large n*

$$\|\vec{y}(n)\| = \max_{1 \le i \le k} |y_i(n)| = |y_l(n)|.$$

Clearly, solutions given in Example 2.2 and Example 2.3 have the IMP. The following lemma gives a sufficient condition for which solutions of Eq.(2.1) have the IMP. **Lemma 3.1** *Let $\lim_{n\to\infty} B(n) = 0$. If $A = diag(\lambda_1, ..., \lambda_k)$ such that $0 < |\lambda_1| < ... < |\lambda_k|$, then every solution of Eq.(2.1) has the IMP.*
Proof. Since $\lim_{n\to\infty} B(n) = 0$, for any $\epsilon > 0$, there exists $N_1 > 0$ such that

$\|B(n)\| = \max_{1\le i\le k} \sum_{j=1}^{k} |b_{ij}(n)| < \epsilon$ for $n \ge N_1$. We choose $\epsilon > 0$ such that

$$\frac{|\lambda_i| + \epsilon}{|\lambda_j| - \epsilon} < 1 \text{ for } 1 \le i < j \le k.$$

Let $\vec{y}(n)$ be a nontrivial solution of Eq.(2.1) and l_n be the first index such that

$$\|\vec{y}(n)\| = |y_{l_n}(n)|.$$

We claim that l_n is nondecreasing. To see this suppose that $l_{n+1} < l_n$, then

$$\begin{aligned} y_i(n+1)| &\le |\lambda_i||y_i(n)| + \epsilon|y_{l_n}(n)| \\ y_i(n+1)| &\ge |\lambda_i||y_i(n)| - \epsilon|y_{l_n}(n)| \end{aligned}$$

for all $n \ge N_1$. This implies that

$$\begin{aligned} \frac{|y_{l_{n+1}}(n+1)|}{|y_{l_n}(n+1)|} &\le \frac{|\lambda_{l_{n+1}}|\ |y_{l_{n+1}}(n)| + \epsilon|y_{l_n}(n)|}{|\lambda_{l_n}|\ |y_{l_n}(n)| - \epsilon|y_{l_n}(n)|} \\ &= \frac{|\lambda_{l_{n+1}}|\ |y_{l_{n+1}}(n)|/|y_{l_n}(n)| + \epsilon}{|\lambda_{l_n}| - \epsilon} \\ &\le \frac{|\lambda_{l_{n+1}}| + \epsilon}{|\lambda_{l_n}| - \epsilon} < 1 \end{aligned}$$

which contradicts to the definition of l_{n+1}. Since l_n assumes only finitely many values, the result follows. ■

Lemma 3.2 *Let* $\lim_{n\to\infty} B(n) = 0$. *Suppose that* $A = diag(\lambda_1, ..., \lambda_k)$ *such that* $0 < |\lambda_1| \le ... \le |\lambda_k|$. *Then every nonzero solution* $\vec{y}(n)$ *of Eq.(2.1) that has the IMP with* $\|\vec{y}(n)\| = |y_l(n)|$ *for all large* n *satisfies*

$$\lim_{n\to\infty} \frac{|y_j(n)|}{|y_l(n)|} = 0 \qquad \text{for} \qquad |\lambda_j| \ne |\lambda_l|.$$

Proof. Let $\vec{y}(n)$ be a nonzero solution of Eq.(2.1) that has the IMP. Since $\lim_{n\to\infty} B(n) = 0$, for any $\epsilon > 0$ there exists $N > 0$ such that $\|B(n)\| < \epsilon$ and $\|\vec{y}(n)\| = |y_l(n)|$ for $n \ge N$. We choose $\epsilon > 0$ so that $\dfrac{|\lambda_i|}{|\lambda_j| - \epsilon} < 1$ for

$1 \leq i < j \leq k$ and $|\lambda_i| \neq |\lambda_j|$. Observe that for $n \geq N$,

$$\begin{aligned} y_i(n+1)| &\leq |\lambda_i|\ |y_i(n)| + \epsilon |y_l(n)| \\ y_i(n+1)| &\geq |\lambda_i|\ |y_i(n)| - \epsilon |y_l(n)|, \end{aligned}$$

for $1 \leq i \leq k$. Suppose that $|\lambda_j| \neq |\lambda_l|$. We first consider the case when $j > l$. Let

$$s = \sup_n \frac{|y_j(n)|}{|y_l(n)|}.$$

Then there exists a subsequence n_i such that

$$\lim_{n_i \to \infty} \frac{|y_j(n_i)|}{|y_l(n_i)|} = s.$$

Observe that

$$\begin{aligned} \frac{|y_j(n_i+1)|}{|y_l(n_i+1)|} &\geq \frac{|\lambda_j|\ |y_j(n_i)| - \epsilon |y_l(n_i)|}{(|\lambda_l| + \epsilon)|y_l(n_i)|} \\ &= \frac{|\lambda_j|\ |y_j(n_i)|/|y_l(n_i)| - \epsilon}{|\lambda_l| + \epsilon} \end{aligned}$$

for $n_i > N$. Therefore,

$$s \geq \frac{|\lambda_j| s - \epsilon}{|\lambda_l| + \epsilon}$$

and consequently

$$s \leq \frac{\epsilon}{|\lambda_j| - |\lambda_l| - \epsilon}$$

for all sufficiently small ϵ. This implies that $s = 0$ and the assertion is shown.

On the other hand, if $j < l$, then

$$\begin{aligned} \frac{|y_j(n+1)|}{|y_l(n+1)|} &\leq \frac{|\lambda_j|\ |y_j(n)| + \epsilon |y_l(n)|}{(|\lambda_l| - \epsilon)|y_l(n)|} \\ &= \left(\frac{|\lambda_j|}{|\lambda_l| - \epsilon}\right) \frac{|y_j(n)|}{|y_l(n)|} + \frac{\epsilon}{|\lambda_l| - \epsilon} \end{aligned}$$

for $n > N$. Thus,

$$\frac{|y_j(n)|}{|y_l(n)|} \leq \left(\frac{|\lambda_j|}{|\lambda_l| - \epsilon}\right)^{n-N} \frac{|y_j(N)|}{|y_l(N)|} + \left[\frac{1 - \left(\frac{|\lambda_j|}{|\lambda_l| - \epsilon}\right)^{n-N}}{1 - \frac{|\lambda_j|}{|\lambda_l| - \epsilon}}\right] \frac{\epsilon}{|\lambda_l| - \epsilon},$$

and as a result

$$\limsup_{n\to\infty} \frac{|y_j(n)|}{|y_l(n)|} \le \frac{\epsilon}{|\lambda_l| - |\lambda_j| - \epsilon}$$

for all sufficiently small ϵ. This implies that

$$\limsup_{n\to\infty} \frac{|y_j(n)|}{|y_l(n)|} = 0$$

and completes the proof. ■

By using Lemmas 3.1 and 3.2, we present a sufficient condition for which Eq.(2.1) has the Poincaré property. **Theorem 3.3** *Suppose that the eigenvalues of A have distinct moduli and* $\lim_{n\to\infty} B(n) = 0$. *Then Eq.(2.1) possesses the Poincaré property P.* **Proof.** We may assume, without loss of generality, that A is in diagonal form, i.e., $A = \mathrm{diag}(\lambda_1,\ \lambda_2, ..., \lambda_k)$, where $0 < |\lambda_1| < \cdots < |\lambda_k|$. Let $\vec{y}(n)$ be a nontrivial solution of Eq.(2.1). It follows from Lemma 3.1 that

$$\|\vec{y}(n)\| = |y_l(n)|$$

for all large n, for some $1 \le l \le k$. Moreover, Lemma 3.2 implies

$$\lim_{n\to\infty} \frac{|y_i(n)|}{|y_l(n)|} = 0$$

for $1 \le i \le k$ such that $i \ne l$. Therefore, if $i \ne l$, then

$$\lim_{n\to\infty} \frac{y_i(n+1)}{|y_l(n)|} = \lim_{n\to\infty} \left[\lambda_i \ \frac{y_i(n)}{|y_l(n)|} + \sum_{j=1}^{k} b_{ij}(n) \ \frac{y_j(n)}{|y_l(n)|} \right] = 0$$

and if $i = l$ we have

$$\lim_{n\to\infty} \frac{y_l(n+1) - \lambda_l \ y_l(n)}{|y_l(n)|} = \lim_{n\to\infty} \sum_{j=1}^{k} b_{ij}(n) \ \frac{y_j(n)}{|y_l(n)|} = 0.$$

Consequently,

$$\lim_{n\to\infty} \frac{\|\vec{y}(n+1) - \lambda_l \ \vec{y}(n)\|}{\|\vec{y}(n)\|} = 0$$

and the proof of Theorem 2.2(a) can be applied to show that $\vec{y}(n)$ is of P. This completes the proof of the theorem. ■ The following theorem is an immediate consequence of Lemma 3.1 and Theorem 3.3. **Theorem 3.4** *If $A = diag(\lambda_1, ..., \lambda_k)$ such that $0 < \lambda_1 < ... < \lambda_k$ and $\lim_{n\to\infty} B(n) = 0$, then Eq.(2.1) has the ergodic Poincaré property. In fact, any nontrivial solution $\vec{y}(n)$ of Eq.(2.1) satisfies*

$$\lim_{n\to\infty} \frac{\vec{y}(n)}{\|\vec{y}(n)\|} = \pm\vec{e}_j$$

where $\vec{e}_j$ depends on $\vec{y}(n)$.

Theorem 3.5 *Let $A > 0$ be irreducible and primitive with dominant eigenvalue $\lambda_1 > 0$. Suppose that the eigenvalues of A have distinct moduli, $A + B(n) > 0$ is irreducible for $n = 0, 1, \cdots$, and $\lim_{n\to\infty} B(n) = 0$. Then Eq.(2.1) possesses the strong ergodic property and consequently the ergodic Poincaré property.* **Proof.** Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the eigenvalues of A such that $\lambda_1 > |\lambda_2| > \cdots > |\lambda_k|$. Let $\vec{v}_i$ and $\vec{w}_i$ be the corresponding right and left eigenvectors of A belonging to λ_i respectively. Let $T = (\vec{v}_1, \cdots, \vec{v}_k)$. Observe that $A = TDT^{-1}$, where $D =$diag$(\lambda_1, \cdots, \lambda_k)$. Let $\vec{y}(n)$ be a solution of Eq.(2.1) with $\vec{y}(0) > 0$ and set $\vec{z}(n) = T^{-1}\vec{y}(n)$. Since $A + B(n) > 0$ is irreducible, we have $\vec{y}(n) > 0$ for $n = 1, 2, \cdots$. Eq.(2.1) is transformed into the following system

$$\vec{z}(n+1) = [D + C(n)]\vec{z}(n), \tag{3.1}$$

with $C(n) = T^{-1}B(n)T$ and $\lim_{n\to\infty} C(n) = 0$.

Since Eq.(3.1) has the IMP, we let l be such that $|z_l(n)| = max_{1\leq i\leq k}|z_i(n)|$ for all large n. Note that $\vec{z}(n) = T^{-1}\vec{y}(n) \neq 0$ for all $n \geq 0$, as the rows of T^{-1} are the left eigenvectors $\vec{w}_i$ of A and $\vec{w}_1 >> 0$. It follows from Lemma 3.2 that $\lim_{n\to\infty} \dfrac{|z_j(n)|}{|z_l(n)|} = 0$ for all $j \neq l$. Therefore, $\lim_{n\to\infty} \dfrac{\vec{z}(n)}{\|\vec{z}(n)\|} = \lim\limits_{n\to\infty} \dfrac{\frac{\vec{z}(n)}{|z_l(n)|}}{\|\frac{\vec{z}(n)}{z_l(n)}\|} = \vec{e}_l$ or $-\vec{e}_l$ for some $1 \leq l \leq k$ and hence, $\lim_{n\to\infty} \dfrac{\vec{y}(n)}{\|\vec{y}(n)\|} = \lim\limits_{n\to\infty} \dfrac{T\vec{z}(n)}{\|T\vec{z}(n)\|} = \dfrac{\pm T\vec{e}_l}{\|T\vec{e}_l\|} = \dfrac{\pm\vec{v}_l}{\|\vec{v}_l\|}$. However, since $\vec{y}(n) > 0$ and $\vec{v}_1$ is the only positive eigenvector of A, we conclude that $\lim_{n\to\infty} \dfrac{\vec{y}(n)}{\|\vec{y}(n)\|} = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$. ■

4 A Criterion for Strong Poincaré

In this section we derive a sufficient condition for the existence of strong Poincaré type solutions. The technique used here is the concept of dichotomy. Eq.(2.1) very often can be regarded as the perturbation of the following linear system

$$\vec{x}(n+1) = A\vec{x}(n), n = 0, 1, 2, \cdots. \tag{4.1}$$

As a consequence, Eq.(4.1) can be exploited to study Eq.(2.1). We first recall the definition of dichotomy and a basic result. We refer the reader to [1, 9, 10] for details and proofs of the preliminary. **Definition 4.1** *Let $X(n)$ be a fundamental matrix of Eq.(4.1). Then Eq.(4.1) is said to possess a* dichotomy *if there are constants $M > 0$, $\alpha \in (0, 1]$, and a projection matrix P such that*

$$\begin{aligned} \|X(n)PX^{-1}(m+1)\| &\leq M\alpha^{n-m}, \text{ for } n \geq m \geq 0 \\ \|X(n)(I-P)X^{-1}(m+1)\| &\leq M\alpha^{n-m}, \text{ for } m \geq n \geq 0. \end{aligned}$$

Furthermore, if $\alpha = 1$, then Eq.(4.1) is said to have an ordinary dichotomy, *and if $\alpha \in (0, 1)$, Eq.(4.1) is said to possess an* exponential dichotomy.

Theorem 4.1 *Suppose that Eq.(4.1) possesses an ordinary dichotomy with a projection matrix P and that $B(n) \in \ell^1(Z_+)$. Then there is a homeomorphism between bounded solutions of Eq.(2.1) and Eq.(4.1). If, in addition, $X(n)P \to 0$ as $n \to \infty$, then for each bounded solution $\vec{x}(n)$ of Eq.(4.1) there exists a bounded solution $\vec{y}(n)$ of Eq.(2.1) such that*

$$\vec{y}(n) = \vec{x}(n) + o(1). \tag{4.2}$$

We now prove our first result in this section. **Lemma 4.2** *Suppose that the eigenvalues of A are such that $\lambda_1 > |\lambda_2| \geq ... \geq |\lambda_k| \geq 0$. If $\vec{x}(n)$ is a solution of Eq.(4.1), then either*

$$\lim_{n\to\infty} \frac{\vec{x}(n)}{\lambda_1^n} = \vec{0} \qquad or \qquad \lim_{n\to\infty} \frac{\vec{x}(n)}{\lambda_1^n} = \vec{\xi},$$

where $\vec{\xi}$ is an eigenvector of A associated with λ_1. **Proof.** There exists a nonsingular matrix S such that $A = SJS^{-1}$, where $J = \mathrm{diag}(\lambda_1, J_2, ..., J_r)$ is the Jordan form of A, and $J_i,\ i = 2, ..., r$ are the Jordan blocks corresponding to the eigenvalues $\lambda_2, \cdots, \lambda_r$, respectively. Suppose that $\lim_{n\to\infty} \dfrac{\vec{x}(n)}{\lambda_1^n} \neq \vec{0}$. Setting $\vec{z}(n) = \lambda_1^{-n} S^{-1}\vec{x}(n)$ and $\tilde{J} = \lambda_1^{-1}J = \mathrm{diag}(1, \lambda_1^{-1}J_2, ..., \lambda_1^{-1}J_r)$, Eq.(4.1) reduces to

$$\vec{z}(n+1) = \tilde{J}\vec{z}(n), \qquad n \geq 0$$

whose solution can be written as

$$\vec{z}(n) = c_1\vec{e}_1 + o(1),$$

where $c_1 \neq 0$ as $\lim_{n\to\infty} \dfrac{\vec{x}(n)}{\lambda_1^n} \neq \vec{0}$. Since the first column of S is an eigenvector associated with λ_1, we have

$$\frac{\vec{x}(n)}{\lambda_1^n} = S\vec{z}(n) = \vec{\xi} + o(1)$$

where $\vec{\xi}$ is an eigenvector of A belonging to λ_1 and the assertion is shown. ■
By using Lemma 4.2, the following theorem provides a sufficient condition for which solutions of Eq.(2.1) are of strong Poincaré. **Theorem 4.3** *Suppose that the eigenvalues of A are such that $\lambda_1 > |\lambda_2| \geq ... \geq |\lambda_k| \geq 0$, and that $B(n) \in \ell^1(Z_+)$. If $\vec{y}(n)$ is a solution of Eq.(2.1), then either*

$$\lim_{n\to\infty} \frac{\vec{y}(n)}{\lambda_1^n} = \vec{0} \qquad or \qquad \lim_{n\to\infty} \frac{\vec{y}(n)}{\lambda_1^n} = \vec{\xi},$$

where $\vec{\xi}$ is an eigenvector of A associated with λ_1.

Proof. If $\lim_{n\to\infty} \dfrac{\vec{y}(n)}{\lambda_1^n} \neq \vec{0}$, then setting $\vec{w}(n) = \lambda_1^{-n}\vec{y}(n)$ and letting $\tilde{A} = \lambda_1^{-1}A$, Eq.(2.1) reduces to

$$\vec{w}(n+1) = \left[\tilde{A} + \frac{B(n)}{\lambda_1}\right]\vec{w}(n), \qquad n \geq 0.$$

Observe that the eigenvalues of $\tilde{A}$ are such that $1 > |\mu_2| \geq ... \geq |\mu_k| \geq 0$. Therefore, the unperturbed system

$$\vec{z}(n+1) = \tilde{A}\vec{z}(n)$$

possesses an ordinary dichotomy with projection matrix P such that $Z(n)P \to 0$ as $n \to \infty$, where $Z(n)$ is the fundamental matrix.

Since $B(n)/\lambda_1 \in \ell^1(Z_+)$, Theorem 4.1 and Lemma 4.2 imply

$$\vec{w}(n) = \vec{z}(n) + o(1),$$

where $\vec{z}(n)$ is either $\vec{0}+o(1)$ or $\vec{\xi}+o(1)$. However, as $\lim_{n\to\infty} \dfrac{\vec{y}(n)}{\lambda_1^n} \neq \vec{0}$, $\vec{w}(n) = \vec{\xi}(n) + o(1)$, where $\vec{\xi}$ is an eigenvector of $\tilde{A}$ associated with 1. Accordingly, $\vec{y}(n)$ is of SP. ■ Theorem 4.3 has the following immediate consequence.

Corollary 4.4 *Suppose that $B(n) \in \ell^1(Z_+)$, and that the eigenvalues of A are such that $\lambda_1 > |\lambda_2| \geq ... \geq |\lambda_k| \geq 0$. If $A > 0$ is irreducible and primitive, then every solution $\vec{y}(n)$ of Eq.(2.1) with $\vec{y}(0) > 0$ satisfies*

$$\lim_{n\to\infty} \frac{\vec{y}(n)}{||\vec{y}(n)||} = \vec{\xi},$$

where $\vec{\xi} >> 0$ is the eigenvector of A belonging to λ_1 with $\|\vec{\xi}\| = 1$.

5 Strong ergodic theorems for nonlinear systems

In this section we assume that $A > 0$, A is irreducible and primitive. We derive sufficient conditions for which nonlinear systems have the strong ergodic property. Let $\sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ be the spectrum of A with $\lambda_1 > |\lambda_i|$ for $2 \leq i \leq m$ and let $\nu(\lambda_i)$ be the Riesz index of λ_i. In particular, $\nu(\lambda_1) = 1$. Let G be an open subset of the complex plane containing $\sigma(A)$. **Definition 5.1** Let $\mathcal{F}(A)$ denote the family of functions defined and analytic on G. **Definition 5.2** Let $f \in \mathcal{F}(A)$, and let the boundary of G, denoted by ∂G, consist of a finite number of rectifiable Jordan curves. Suppose that $G \cup \partial G$ is contained in the domain of analyticity of f. Then $f(A)$ is defined by the equation

$$f(A) = \frac{1}{2\pi i}\int_{\partial G} f(\lambda)R(\lambda; A)d\lambda, \tag{5.1}$$

where $R(\lambda; A)$ is the resolvent of A at λ.

Note that $f(A)$ depends only on f and not on G for any function $f \in \mathcal{F}(A)$ and thus $f(A)$ is well defined [7]. We restrict ourselves to those $f \in \mathcal{F}(A)$ such that the nonnegative real axis is invariant under f, i.e., we let $\mathcal{F}_+(A) = \{f \in \mathcal{F}(A) : f(z) \geq 0 \text{ for } 0 \leq z \in G\}$. Let $\{f_n\}_{n=0}^{\infty}$ be an arbitrary sequence of functions from $\mathcal{F}_+(A)$ and consider the system of difference equations

$$x(n+1) = f_n(A)x(n), \; n = 0, 1, \cdots \tag{5.2}$$

The following strong ergodic theorem cited from [11, 14] is useful for our study.

Theorem 5.1 *Let $A > 0$ be an irreducible and primitive $k \times k$ matrix with spectrum $\sigma(A)\{\lambda_1, \cdots, \lambda_m\}$, where $\lambda_1 > |\lambda_i|$ for $2 \leq i \leq m$. Let $\vec{v}_1 >> 0$ be the corresponding normalized right eigenvector of A belonging to λ_1 and let $\nu(\lambda_i)$ be the Riesz index of λ_i for $1 \leq i \leq m$. Consider a sequence $\{f_n\}_{n=0}^{\infty}$ with $f_n \in \mathcal{F}_+(A)$ for all $n = 0, 1, \cdots$ satisfying*

$$\frac{|f_n(\lambda_i)|}{f_n(\lambda_1)} \leq \delta_i < 1 \text{ for all large } n \tag{5.3}$$

for $2 \leq i \leq m$, for some positive real numbers δ_i, and

$$\frac{|f_n^{(\alpha)}(\lambda_i)|}{f_n(\lambda_1)} = O((n+1)^{k(\alpha,i)}), \; n \to \infty \tag{5.4}$$

for some $k(\alpha, i) \in R$, for all $1 \leq \alpha \leq \nu(\lambda_i) - 1$, if $\nu(\lambda_i) > 1$. If $\vec{x}(n)$ is a solution of Eq.(5.2) with $\vec{x}(0) > 0$, then the normalized solution $\dfrac{\vec{x}(n)}{\|\vec{x}(n)\|}$ satisfies $\lim_{n\to\infty} \dfrac{\vec{x}(n)}{\|\vec{x}(n)\|} = \vec{v}_1$.

Our main result in this section is the following. **Theorem 5.2** *Let*

A, $\sigma(A)$, $\nu(\lambda_i)$ and $\vec{v}_1$ be defined as in Theorem 5.1. Let $B(n) = g_n(A)$, $n = 0, 1, \cdots$ for some $\{g_n(A)\} \subset \mathcal{F}_+(A)$ satisfying (H1) there exists $\delta_i > 0$ such that $\dfrac{|g_n(\lambda_i)|}{g_n(\lambda_1)} \leq \delta_i < 1$ for all large n, for $2 \leq i \leq m$,

(H2) $\dfrac{|g_n^{(\alpha)}(\lambda_i)|}{g_n(\lambda_1)} = O((n+1)^{k(\alpha,i)})$*, $n \to \infty$, for some $k(\alpha, i) \in R$, for all $1 \le \alpha \le \nu(\lambda_i) - 1$ if $\nu(\lambda_i) \ge 2$. Then any solution $\vec{y}(n)$ of Eq.(5.2) with $\vec{y}(0) > 0$ satisfies $\lim_{n\to\infty} \dfrac{\vec{y}(n)}{\|\vec{y}(n)\|} = \vec{v}_1$, i.e., Eq.(5.2) possesses the strong ergodic property.*

Proof. Let $f_n(z) = z + g_n(z)$. Then f_n is analytic in some open set containing $\sigma(A)$ and $f_n(R_+) \subset R_+$, i.e., $\{f_n\}_{n=0}^{\infty} \subset \mathcal{F}_+(A)$. It remains to verify that f_n satisfies Eq.(5.3) and Eq.(5.4). Indeed, let $\hat{\delta}_i$ =max$\{\delta_i, |\lambda_i|/\lambda_1\}$, then $\hat{\delta}_i < 1$ for $2 \le i \le m$ and, for n sufficiently large, $\dfrac{|f_n(\lambda_i)|}{f_n(\lambda_1)} \le \dfrac{|\lambda_i| + |g_n(\lambda_i)|}{\lambda_1 + g_n(\lambda_1)} \le \dfrac{|\lambda_i| + \delta_i g_n(\lambda_1)}{\lambda_1 + g_n(\lambda_1)} \le \hat{\delta}_i < 1$ for $2 \le i \le m$ by (H1). If $\nu(\lambda_i) > 1$, then $f_n'(z) = 1 + g_n'(z)$ and for n sufficiently large $\dfrac{|f_n'(\lambda_i)|}{f_n(\lambda_1)} \le \dfrac{1 + |g_n'(\lambda_i)|}{\lambda_1 + g_n(\lambda_1)} \le \dfrac{1 + M_{1,i}(n+1)^{k(1,i)} g_n(\lambda_1)}{\lambda_1 + g_n(\lambda_1)}$ as $g_n(A)$ satisfies (H2). Thus, let $\hat{M}_{1,i}$ =max$\{1/\lambda_1, M_{1,i}\}$. Then $\dfrac{|f_n'(\lambda_i)|}{f_n(\lambda_1)} \le (n+1)^{|k(1,i)|} \dfrac{1/\lambda_1 + (M_{1,i}/\lambda_1) g_n(\lambda_1)}{1 + g_n(\lambda_1)/\lambda_1} \le \hat{M}_{1,i}(n+1)^{|k(1,i)|}$, i.e., $\dfrac{|f_n'(\lambda_i)|}{f_n(\lambda_1)} = O((n+1)^{|k(1,i)|})$ as $n \to \infty$. Furthermore, for any $j \ge 2$, $f_n^{(j)}(\lambda_i) = g_n^{(j)}(\lambda_i)$. It follows that Eq.(5.4) is satisfied for all $1 \le j \le \nu(\lambda_i) - 1$. Therefore, with $A + B(n) = A + g_n(A) = f_n(A)$, Theorem 5.1 implies that solutions $\vec{y}(n)$ of Eq.(5.2) with $\vec{y}(0) > 0$ satisfy the desired asymptotic behavior. ■ **Remark.** In this paper (sections 2-4) we have assumed that the constant matrix A to be nonsingular. However, using Corollary 1 in [8] one may extend our results to the case when A is noninvertible.

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