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**POWERFUL IDEALS, STRONGLY
PRIMARY IDEALS, ALMOST
PSEUDO-VALUATION DOMAINS, AND
CONDUCTIVE DOMAINS**

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ABSTRACT

Let R be a domain with quotient field K , and let I be an ideal of R . We say that I is powerful (strongly primary) if whenever $x, y \in K$ and $xy \in I$, we have $x \in R$ or $y \in R$ ($x \in I$ or $y^n \in I$ for some $n \geq 1$). We show that an ideal with either of these properties is comparable to every prime ideal of R , that an ideal is strongly primary \Leftrightarrow it is a primary ideal in some valuation overring of R , and that R admits a powerful ideal $\Leftrightarrow R$ admits a strongly primary ideal $\Leftrightarrow R$ is conducive in the sense of Dobbs-Fedder. Finally, we study domains each of whose prime ideals is strongly primary.

INTRODUCTION

Throughout this work R will denote an integral domain with quotient field K . Recall from [10] that a prime ideal P of R is said to be *strongly prime* if, whenever $xy \in P$ for elements $x, y \in K$, we have $x \in P$ or $y \in P$. In this paper, we consider two generalizations of this concept. We define a nonzero ideal I of R to be *powerful* if, whenever $xy \in I$ for elements $x, y \in K$, we have $x \in R$ or $y \in R$. It is easy to see that R itself is powerful $\Leftrightarrow R$ is a valuation domain. In the first section, we show that a powerful prime ideal is strongly prime. We also show that if I is a proper powerful ideal of R , then its radical $\text{Rad}(I)$ is prime in general and strongly prime when R is seminormal [Propositions 1.9 and 1.12]. Moreover, a powerful ideal I is comparable to every nonzero prime of R , from which it follows that the prime ideals contained in $\text{Rad}(I)$ are linearly ordered [Theorem 1.5]. The remainder of the first section is devoted to a study of overrings. We show in Proposition 1.17 that if R admits a powerful ideal and T is an overring of R , then R and T share an ideal which is powerful in both rings. Conversely, in Proposition 1.18, we prove that if T is an overring of R such that R and T share a common ideal J which is powerful in T , then J^3 is a powerful ideal of R .

As another generalization of the notion of “strongly prime,” in Sec. 2 we define an ideal I of R to be *strongly primary* if, whenever $xy \in I$ with $x, y \in K$, we have $x \in I$ or $y^n \in I$ for some $n \geq 1$. Simple examples show that “powerful” and “strongly primary” are different notions. A proper strongly primary ideal of R is clearly primary, and we observe that the converse is true in a valuation domain [Proposition 2.1]. In fact, noticing that the property of being strongly primary is independent of the domain in which I happens to be an ideal, this characterizes whether a given ideal is strongly primary. More precisely, we show in Theorem 2.11 that an ideal I of R is strongly primary $\Leftrightarrow I$ is a primary ideal in some valuation overring of R . We also show that a strongly primary ideal of R is comparable to every radical ideal of R [Theorem 2.8] and that a proper strongly primary ideal I of a seminormal domain is powerful with $\text{Rad}(I)$ strongly prime [Theorem 2.4].

Section 3 is devoted to a study of a generalization of pseudo-valuation domains (PVDs). Recall from [10] that a PVD is a domain in which each nonzero ideal is strongly prime and that PVDs are characterized as quasilocal domains $(R : M)$ with the property that $(M : M)$ is a valuation domain with maximal ideal M . We define an *almost pseudo-valuation domain* (APVD) to be a domain each of whose prime ideals is strongly primary. The main result of this section then characterizes APVDs as quasilocal domains (R, M) such that $(M : M)$ is a valuation domain with M primary to the maximal ideal of $(M : M)$. We also consider overrings of an APVD, and we prove that the integral closure of an APVD is a PVD.

Finally, in Sec. 4, we relate our work to the conducive domains of Dobbs-Fedder. In [7], they define a conducive domain to be a domain R such that for each overring $T \neq K$ of R , the conductor $(R : T)$ is nonzero. We show that a domain R is conducive $\Leftrightarrow R$ admits a powerful ideal $\Leftrightarrow R$ admits a strongly primary ideal. We also use our techniques to recover the Dobbs-Fedder characterization of conducive Prüfer domains and the Barucci-Dobbs-Fontana characterization of conducive Noetherian domains.

1. POWERFUL IDEALS

We begin with a simple but useful restatement of the definition of *powerful*.

Lemma 1.1. *An ideal I of R is powerful $\Leftrightarrow x^{-1}I \subseteq R$ for each $x \in K \setminus R$.*

Proof. Assume that I is powerful, and let $x \in K \setminus R$. Then for $a \in I$ we have $xx^{-1}a = a \in I$, whence $x^{-1}a \in R$. For the converse, let $yz \in I$, $y, z \in K$. Suppose $y \notin R$. Then $z = y^{-1}yz \in y^{-1}I \subseteq R$, as desired. \square

As an easy consequence of the lemma, we obtain the fact that powerfulness is preserved upon passage to homomorphic images.

Proposition 1.2. *Let I be a powerful ideal of R , and let Q be a prime ideal of R which is properly contained in I . Then I/Q is powerful in R/Q .*

Proof. Let $\phi : R \rightarrow R/Q$ denote the canonical homomorphism. Suppose that $x = \phi(y)/\phi(z)$ is an element of the quotient field of R/Q with $x \notin R/Q$. Then $y/z \notin R$. Hence if $a \in I$, we have $(z/y)a \in R$, and it follows that $(\phi(z)/\phi(y))\phi(a) \in R/Q$. Thus $x^{-1}(I/Q) \subseteq R/Q$, as desired. \square

Proposition 1.3. *A prime ideal of R is strongly prime \Leftrightarrow it is powerful.*

Proof. Suppose that P is a powerful prime ideal of R . Let $xy \in P$ for some $x, y \in K$. Then $x^2y^2 \in P$. We may assume $x \notin R$ and $y \in R$. If $x^2 \in R$, then, since $x \notin R$, $x^2 \notin P$, and the fact that $x^2y^2 \in P$ then implies that $y^2 \in P$, whence $y \in P$. If $x^2 \notin R$, then, since $(y^2/xy)x^2 \in P$, we have $y^2/xy \in R$. Hence $y^2 = (y^2/xy)xy \in P$, and again we have $y \in P$. The converse is trivial. \square

It is known that if $Q \subseteq P$ are prime ideals of R such that P is strongly prime, then Q is strongly prime [1, Proposition 4.8]. By Proposition 1.3, the following result, whose proof is trivial, generalizes this fact.

Proposition 1.4. *If $J \subseteq I$ are ideals of R with I powerful, then J is also powerful.*

Theorem 1.5. *Let I be a powerful ideal of R .*

- (1) *If J is an ideal of R , then either $J \subseteq I$ or $I^2 \subseteq J$.*
- (2) *If J is a prime ideal of R , then I and J are comparable.*
- (3) *The prime ideals of R contained in $\text{Rad}(I)$ are linearly ordered.*

Proof. To prove (1), suppose that J is an ideal of R with $J \not\subseteq I$. Choose $a \in J \setminus I$, and let $b, c \in I$. Then $(bc/a)(a/b) \in I$, and since I is powerful with $a/b \notin R$, we have $bc/a \in R$. Hence $bc \in aR \subseteq J$, as desired. Statement (2) is immediate from (1). For (3), let P, Q be prime ideals properly contained in $\text{Rad}(I)$. Then P and Q are contained in I and are therefore powerful by Proposition 1.4. Hence they are comparable by (2). \square

The following result is an easy consequence of Proposition 1.3 and Theorem 1.5.

Corollary 1.6. *A domain R is a PVD \Leftrightarrow some maximal ideal of R is powerful.*

Proposition 1.7. *If R contains a powerful ideal, then R contains a unique largest powerful ideal.*

Proof. It suffices to show that the sum of powerful ideals is again powerful. Thus let $\{I_\alpha\}$ denote a family of powerful ideals of R . If $x \in K \setminus R$, then $x^{-1}I_\alpha \subseteq R$ for each α by Lemma 1.1. Hence $x^{-1}\sum_\alpha I_\alpha \subseteq R$, and we have that $\sum_\alpha I_\alpha$ is powerful, again by Lemma 1.1. \square

Our next result generalizes [10, Proposition 2.4].

Proposition 1.8. *If I is a proper powerful ideal of R , then $P = \bigcap_{k=0}^{\infty} I^k$ is a strongly prime ideal.*

Proof. It suffices by Propositions 1.3 and 1.4 to show that P is prime. Let $xy \in P$ with $x \in R \setminus P$. Then $x \notin I^n$ for some $n > 0$, whence by Theorem 1.5, $I^{2n} \subseteq xR$. Hence for each $k > 0$, we have $xy \in P \subseteq I^{2n+k} \subseteq xI^k$. Thus $y \in I^k$ for each $k > 0$. It follows that $y \in P$. \square

Proposition 1.9. *Let I be a powerful ideal of R . If $x, y \in K$ and $xy \in \text{Rad}(I)$, then there is a positive integer m such that either $x^m \in I$ or $y^m \in I$. In particular, if I is a proper powerful ideal, then $\text{Rad}(I)$ is prime.*

Proof. We have $(xy)^n \in I$ for some $n > 0$. Hence $(x^{3n}/x^n y^n) (y^{3n}/x^n y^n) = x^n y^n \in I$. Since I is powerful, either $x^{3n}/x^n y^n \in R$ or $y^{3n}/x^n y^n \in R$, whence either $x^{3n} \in I$ or $y^{3n} \in I$. \square

In spite of Proposition 1.9, the radical of a powerful ideal need not be powerful, as the following example shows.

Example 1.10. Let $V = k + M$ be a rank one discrete valuation domain, where k is a field and $M = tV$ is the maximal ideal of V , and let $R = k + M^2$. Claim: M^3 is a powerful ideal of R . To see this, let $xy \in M^3$, with $x, y \in K$ (the common quotient field of R and V). We may write $x = ut^n, y = vt^m$, where u, v are units of V and n, m are integers. Since $xy \in M^3$, we must have $n + m \geq 3$. Hence either $n \geq 2$ or $m \geq 2$, say $n \geq 2$. Then $x = ut^n \in M^2 \subseteq R$. This proves the claim. However, (in R) $\text{Rad}(M^3) = M^2$ is not powerful since $t^2 \in M^2$ but $t \notin R$.

We prove below that in a seminormal domain the radical of a powerful ideal is powerful. First, we need a lemma.

Lemma 1.11. *Let I be a powerful ideal of R . If $x \in K$ and $x^n \in I$ for some $n > 0$, then $x^{n+k} \in R$ for each $k \geq 0$.*

Proof. Let $e = \min\{m \geq 1 \mid x^m \in R\}$. Let k be a positive integer, and write $k = qe + r$ with $0 \leq r < e$. If $r = 0$, then it is easy to see that $x^{n+k} \in R$. Suppose that $r > 0$. We have $x^{e-r} x^{qe+n+r} = x^n x^{(q+1)e} \in I$. Since $x^{e-r} \notin R$, we have $x^{n+k} = x^{qe+n+r} \in R$, as desired. \square

Now recall from [2] that a radical ideal J of R is said to be *strongly radical* if $x \in K$ and $x^n \in J$ for some $n > 0$ implies that $x \in J$. We also recall that R is seminormal if $x \in R$ whenever $x^n \in R$ for all sufficiently large n .

Proposition 1.12. *Let I be a proper powerful ideal of R . Then $\text{Rad}(I)$ is powerful (and therefore strongly prime) $\Leftrightarrow \text{Rad}(I)$ is strongly radical. In particular, if R is seminormal, then, $\text{Rad}(I)$ is strongly prime.*

Proof. It is easy to see that a powerful radical ideal must be strongly radical. Suppose that $\text{Rad}(I)$ is strongly radical, and let $xy \in \text{Rad}(I)$ with $x, y \in K$. Then by Proposition 1.9, we have $x^m \in I$ or $y^m \in I$ for some $m > 0$. We may suppose that $x^m \in I$. Then $x^m \in \text{Rad}(I)$, whence $x \in R$, as desired. The “in particular” statement now follows from Lemma 1.11. \square

Proposition 1.13. *Let I be a powerful ideal of R , and let T be an overring of R . Then IT is a powerful ideal of T . In particular, if $IT = T$, then T is a valuation domain.*

Proof. Let $x \in K \setminus T$. Then $x \notin R$, whence $x^{-1}I \subseteq R$ by Lemma 1.1. It follows that $x^{-1}IT \subseteq T$. Hence IT is powerful in T , again by Lemma 1.1. It is easy to see that a domain T is a valuation domain $\Leftrightarrow T$ is powerful as an ideal of T . \square

Proposition 1.14. *Let I be a powerful ideal of R , and suppose that $P \subseteq I$ is a nonzero finitely generated prime ideal of R . Then R is a PVD with maximal ideal P .*

Proof. If P is not maximal, then R contains a nonunit x with $x \notin P$. Since P is strongly prime and $xx^{-1}P \subseteq P$ with $x \notin P$, we have $x^{-1}P \subseteq P$. Hence x^{-1} is integral over R , which is impossible. Thus P is maximal, and it follows that R is a PVD. \square

We use R' to denote the integral closure of a domain R .

Theorem 1.15. *Suppose that R admits a powerful ideal I and that $M = \text{Rad}(I)$ is a maximal ideal of R . Then:*

- (1) R is quasilocal with maximal ideal M .
- (2) $IR' \subseteq M$, and therefore IR' is an ideal of R .
- (3) R' is a PVD with maximal ideal $N = \text{Rad}(IR')$, and hence $(N : N) = \{x \in K \mid xN \subseteq N\}$ is a valuation overring of R with maximal ideal N .

Proof. Statement (1) follows from Theorem 1.5. Now let $x \in R' \setminus R$. Note that since R' is integral over R , we must have $x^{-1} \notin R$. Hence $xI \subseteq R$ by Lemma 1.1. In fact, $xI \subseteq M$ (otherwise, $xI = R$, and $x^{-1} \in R$). It follows that $IR' \subseteq M$, proving (2). For (3), note that IR' is powerful in R' by Proposition 1.13. Hence $N = \text{Rad}(IR')$ is strongly prime in R' by Proposition 1.12. By (1), R' is quasilocal with maximal ideal N . Hence R' is a PVD by [10, Theorem 1.4], and $(N : N)$ is a valuation overring with maximal ideal N by [10, Theorem 2.10]. \square

Corollary 1.16. *Let I be powerful in R , and let $P = \text{Rad}(I)$. Then $(R_P)'$ is a PVD with maximal ideal $N = \text{Rad}(I(R_P)')$. It follows that $(N : N)$ is a valuation overring of R with maximal ideal N .*

Proof. Note that P is prime by Proposition 1.19. By Proposition 1.13, IR_P is a powerful ideal of R_P , and the result now follows from Theorem 1.15. \square

Proposition 1.17. *Let I be a powerful ideal of R , and let $T \neq K$ be an overring of R . Then R and T share an ideal which is powerful in both R and T . In fact:*

- (1) If $IT = T$, then $P = N \cap R$, where N is the maximal ideal of T , is a common ideal which is powerful in both rings.
- (2) If $IT \neq T$, then I^2T is a common ideal, and I^3T is powerful in both rings.

Proof. For (1), recall that T is a valuation domain by Proposition 1.13. By Theorem 1.5, I is comparable to P . The fact that $IT = T$ then implies that $P \subseteq I$, whence P is powerful, and therefore strongly prime, in R . Note that PT is powerful in T by Proposition 1.13. We claim that $PT = P$. To verify this, let $x \in T \setminus R$. Clearly, $x^{-1} \notin P$. Hence, since $x^{-1}xP \subseteq P$ and P is strongly prime, we have $xP \subseteq P$, as claimed. For (2), let $x \in T \setminus R$. If $x^{-1} \notin R$, then $xI^2 \subseteq xI \subseteq R$ by Lemma 1.1. If $x^{-1} \in R$, then, by hypothesis, $x^{-1} \notin I$, whence $I^2 \subseteq x^{-1}R$ by Theorem 1.5. Hence, again, $xI^2 \subseteq R$. Thus I^2T is an ideal of R . Since $I^3T \subseteq I$, I^3T is powerful in R by Proposition 1.4, and I^3T is powerful in T by Proposition 1.13. □

Proposition 1.18. *Suppose that T is an overring of R and that R and T share the nonzero ideal J . If J is powerful in T , then J^3 is a powerful ideal of R .*

Proof. Let $x \in K \setminus R$. If $x \notin T$, then $x^{-1}J \subseteq T$ by Lemma 1.1. In this case, we have $x^{-1}J^3 \subseteq J^2T \subseteq R$. Now assume $x \in T$. Since $x \notin J$, we have $J^2 \subseteq xT$ by Theorem 1.5. Hence $x^{-1}J^3 \subseteq JT = J \subseteq R$, and the proof is complete. □

It is not difficult to show that, with the notation of Proposition 1.18, if T is a valuation domain, then J^2 is powerful in R . However, for general T , the third power is best possible, as the following example shows.

Example 1.19. Let k denote the field $\mathbb{Q}(\sqrt{2})$, and let $V = k[[X]] = k + M$, $M = Xk[[X]]$. Then let $T = \mathbb{Q} + M$, $J = XT$, and $R = \mathbb{Q} + J$. Then R and T share the ideal J , and, since T is a PVD, J is powerful in T . However, J^2 is not powerful in R , since $\sqrt{2}X \cdot \sqrt{2}X = 2X^2 \in J^2$, but $\sqrt{2}X \notin R$.

2. STRONGLY PRIMARY IDEALS

Proposition 2.1. *A primary ideal of a valuation domain is strongly primary.*

Proof. Let V be a valuation domain with quotient field K , let I be a primary ideal of V , let $x, y \in K$ with $xy \in I$, and suppose that $x \notin I$. If $x \notin V$, then $x^{-1} \in V$, and we have $y = x^{-1}xy \in I$. Hence we may as well assume that $x \in V$. Since $x = y^{-1}xy \notin I$, it follows that $y \in V$. Now, since $x, y \in V$ with I primary, we have $y^n \in I$ for some $n \geq 1$, as desired. □

Observe that if V is a valuation domain which is not rank one, then there are ideals which are not primary [9, Exercise 2, p. 292]. Since every

ideal of a valuation domain is powerful, this shows that powerful ideals need not be (strongly) primary. Conversely, strongly primary ideals need not be powerful: In Example 1.10, M^2 is strongly primary but not powerful in R .

Notation 2.2. For a subset S of R , we define $E(S)$ by

$$E(S) = \{x \in K \mid x^n \notin S \text{ for each } n \geq 1\}.$$

The following lemma provides a useful restatement of the definition of strongly primary.

Lemma 2.3. *A nonzero ideal I of R is strongly primary if and only if $x^{-1}I \subseteq I$ for each $x \in E(I)$.*

Proof. If I is strongly primary and $x \in E(I)$, then the equation $xx^{-1}I = I$ implies that $x^{-1}I \subseteq I$. Conversely, if $yz \in I$ with $y, z \in K$ and $z \in E(I)$, then the hypothesis yields $y = z^{-1}yz \in z^{-1}I \subseteq I$, as desired. \square

Theorem 2.4. *Let R be a seminormal domain. If I is a proper strongly primary ideal of R , then I is powerful, and $\text{Rad}(I)$ is strongly prime. In particular, a prime ideal of R is strongly prime if and only if it is strongly primary.*

Proof. Let $x \in K \setminus R$; we shall show that $x^{-1}I \subseteq I$ (whence $x^{-1}I \subseteq R$). By Lemma 2.3, it suffices to show that $x^n \notin I$ for all $n \geq 1$. Suppose, on the contrary, that $x^r \in I$, with r minimal. It is then easy to see that $x^{-k} \notin I$ for each $k \geq 0$, that is, that $x^{-1} \in E(I)$. By Lemma 2.3, this implies that $x^{r+1} = xx^r \in xI \subseteq I$. By induction, we get $x^t \in I \subseteq R$ for each $t \geq r$. However, the seminormality of R then implies that $x \in R$, a contradiction. \square

Proposition 2.5. *Let I be a proper strongly primary ideal of R , and let T be an overring of R . Then either $IT = T$ or $IT = I$.*

Proof. Assume $IT \neq T$, and pick $x \in T \setminus R$. If $x^{-n} \in I$ for some $n \geq 1$, then, since $IT \neq T$, x^{-n} is a nonunit of T , a contradiction. Hence $x^{-1} \in E(I)$, and we have $xI \subseteq I$ by Lemma 2.3. Thus $IT = I$. \square

Recall that R' denotes the integral closure of the domain R .

Corollary 2.6. *If I is a proper strongly primary ideal of R , then $IR' = I$. Moreover, I^3 is powerful in both R and R' .*

Proof. The first conclusion follows from Proposition 2.5 and the lying over property of integral extensions. Since I is automatically strongly primary in

R' , I is powerful in R' by Theorem 2.4. It follows that I^3 is powerful in R and R' by Proposition 1.18. \square

Corollary 2.7. *If I is a proper strongly primary ideal of R , then $\bigcap_{n=1}^{\infty} I^n$ is a strongly prime ideal of R .*

Proof. This follows from Proposition 1.8 and the fact that I^3 is powerful. \square

Theorem 2.8. *If I is a strongly primary ideal of R , then I is comparable to every radical ideal of R . Moreover, the prime ideals of R which are properly contained in I are strongly prime and linearly ordered.*

Proof. Let J be a radical ideal of R , and suppose that $I \not\subseteq J$. Choose $a \in I \setminus J$, and let $b \in J$. Since $(a^2/b)(b/a) = a \in I$ and $a^2/b \in E(R) \subseteq E(I)$, we have $b/a \in I$. Hence $J \subseteq I$, as desired. If P is a prime ideal which is properly contained in I , then, since I^3 is powerful by Corollary 2.6, and $P \subseteq I^3$, P is also powerful. Then P is strongly prime by Proposition 1.3. The rest follows from Theorem 1.5. \square

The following result is an immediate consequence of Theorem 2.8.

Corollary 2.9. *If P is a prime ideal of R which is strongly primary but not strongly prime, then P is the only prime with this property.*

Theorem 2.10. *Let I be a strongly primary ideal of R , and let $T \neq K$ be an overring of R . Then R and T share a strongly primary ideal. In fact:*

- (1) *if $IT \neq T$, then $IT = I$ is a common strongly primary ideal;*
- (2) *if $IT = T$, then T is strongly primary, and for each maximal ideal N of T , $N \cap R$ is a common strongly prime ideal of R and T .*

Proof. Statement (1) follows from Proposition 2.5. Now assume $IT = T$. If $x \in E(T)$, then $x \in E(I)$, whence $x^{-1}I \subseteq I$. It follows that $x^{-1}T \subseteq T$, and T is strongly primary. Now let N be maximal in T , and let $P = N \cap R$. Then I is comparable to P by Theorem 2.8, and since $IT = T$, we must have $P \subseteq I$. By Theorem 2.8, P is strongly prime. It then follows from (1) that P is a common strongly prime ideal of R and T . \square

Theorem 2.11. *Let I be a proper ideal of a domain R . Then the following statements are equivalent.*

- (1) *I is a strongly primary ideal of R .*
- (2) *I is a primary ideal in some valuation overring of R .*

(3) $V = (I : I)$ is a valuation domain, and I is (an ideal of V which is) primary to the maximal ideal of V .

Proof. (1) \Rightarrow (3). Let $x \in K \setminus V$. By Corollary 2.6, I is an ideal of R' . Hence $R' \subseteq (I : I) = V$, and, since $x \notin V$, we have $x \notin R'$, whence $x^n \notin R$ for all $n \geq 1$. In particular, $x \in E(I)$, and we have $x^{-1}I \subseteq I$, i.e., $x^{-1} \in V$. Thus V is a valuation domain. Now let $P = \text{Rad}(I)$. Then $IV_P \neq V_P$, so that $IV_P = I$ by Proposition 2.5. Since $(I : I)$ is the largest overring of R in which I is an ideal, we have that $V_P = V$, whence P is the maximal ideal of V .

(3) \Rightarrow (2). Clear.

(2) \Rightarrow (1). Statement (2) implies that I is strongly primary by Proposition 2.1. \square

Corollary 2.12. *If R admits a nonzero proper principal strongly primary ideal, then R is a valuation domain.*

Proof. Let Ra be a nonzero principal strongly primary ideal of R . Then $R = (Ra : Ra)$ is a valuation domain by Theorem 2.11. \square

Proposition 2.13. *Let I be a strongly primary ideal of R . Then*

- (1) $I \subseteq xR$ for every $x \in R \setminus \text{Rad}(I)$, and
- (2) if I is finitely generated, then R is quasilocal with maximal ideal $\text{Rad}(I)$.

Proof. Let $x \in R \setminus \text{Rad}(I)$. Then $x \in E(I)$, and so (by Lemma 2.3) $x^{-1}I \subseteq I$. Hence $I \subseteq xI \subseteq xR$, proving (1). For (2), the relation $x^{-1}I \subseteq I$ shows that x^{-1} is integral over R ; since $x \in R$, this implies that $x^{-1} \in R$. It follows that R is quasilocal with maximal ideal $\text{Rad}(I)$. \square

Proposition 2.14. *Let P be a strongly primary prime ideal of R , and let I be an ideal of R with $\text{Rad}(I) = P$. Then PI is strongly primary. In particular, P^n is strongly primary for $n \geq 1$.*

Proof. Let $x \in E(PI)$. Since $P = \text{Rad}(PI)$, we have $x \in E(P)$. Hence $x^{-1}P \subseteq P$, and we have $x^{-1}PI \subseteq PI$, as desired. \square

3. ALMOST PSEUDO-VALUATION DOMAINS

Definition 3.1. *We say that a domain R is an almost pseudo-valuation domain (APVD) if every prime ideal of R is strongly primary.*

Recall from [6] that a prime ideal of a domain R is said to be *divided* if it is comparable to every ideal of R . If every prime of R is divided, then R is called a *divided domain*.

Proposition 3.2. *Let R be an APVD. Then R is a (quasilocal) divided domain. Moreover, every nonmaximal prime ideal of R is strongly prime.*

Proof. Proposition 2.13(1) shows that a strongly primary prime ideal is divided. The second statement follows from Theorem 2.8. \square

Our next result shows that the requirement that each *primary* ideal of a domain be strongly primary is very restrictive (and hence explains why we defined APVD as we did). Recall that a prime ideal P of a domain R is said to be unbranched if P is the only P -primary ideal of R [9, p. 189]. A prime ideal P of a (pseudo-)valuation domain is unbranched $\Leftrightarrow P$ is the union of the chain of primes properly contained in P .

Proposition 3.3. *The following are equivalent for a domain R .*

- (1) *Each primary ideal of R is strongly primary.*
- (2) *Either R is a valuation domain or R is a PVD with unbranched maximal ideal.*

Proof. Suppose that each primary ideal of R is strongly primary. Then R is an APVD and is therefore quasilocal by Proposition 3.2. Assume that R is not a valuation domain, and let M be the maximal ideal of R . Then M is strongly primary, whence, by Theorem 2.11, $V = (M : M)$ is a valuation overring with M primary to the maximal ideal N of V . Suppose that $N \neq M$. Then by [9, Theorem 17.3], there is a prime ideal P of V with $\text{ht}(N/P) = 1$. We must have $P \subseteq M$, so that P is also a prime ideal of R (possibly, $P = (0)$). Pick $a \in M \setminus P$. Then, since M is maximal in R , Ra is M -primary. Hence Ra is strongly primary. However, by Corollary 2.12, this implies that R is a valuation domain, a contradiction. Hence we must have $N = M$, and so R is a PVD. An argument similar to the one just given shows that M must be unbranched.

For the converse, recall that every primary ideal of a valuation domain is strongly primary by Proposition 2.1. Let (R, M) be a PVD with M unbranched, and let $V = (M : M)$ be the canonical valuation overring (also with maximal ideal M). Let I be a primary ideal of R . Since M is unbranched, $\text{Rad}(I)$ is a prime ideal $P \neq M$. Since I is P -primary, we have $I = IR_P \cap R = IV_P \cap R \supseteq IV \cap R$. Hence $I = IV$. Thus $I = IV_P \cap V$, and I is primary to the prime ideal P in V . Hence I is strongly primary. \square

Theorem 3.4. *The following statements are equivalent for a domain R .*

- (1) R is an APVD.
- (2) Some maximal ideal of R is strongly primary.
- (3) If N is the set of nonunits of R , then $x^{-1}N \subseteq N$ for each element $x \in E(N)$.
- (4) R is a quasilocal domain, and the maximal ideal M of R is such that $(M : M)$ is a valuation domain with M primary to the maximal ideal of $(M : M)$.
- (5) R is a quasilocal domain, and there is a valuation overring in which M is a primary ideal.

Proof. The equivalence of (1) and (2) follows from Theorem 2.8, as does (2) \Rightarrow (3). To show that (3) \Rightarrow (2), it suffices to show that R is quasilocal with maximal ideal N . In fact, we claim that the prime ideals of R are linearly ordered. To see this, let P and Q be primes, and suppose that there is an element $b \in Q \setminus P$. Let $a \in P$. Then $b^n/a^n \notin R$ for each $n > 0$. Hence $b/a \in E(N)$, and we have $a/bN \subseteq N$ by assumption. In particular, $a^2/b \in N \subseteq R$, whence $a \in Q$. Thus $P \subseteq Q$, as desired. The equivalence of (1) and (4) follows from Theorems 2.8 and 2.11. Finally, it is clear that (4) \Rightarrow (5), and (5) \Rightarrow (2) by Proposition 2.1. \square

Our next result shows that if R itself is strongly primary, then R is an APVD.

Proposition 3.5. *If R is strongly primary, then R is an APVD (and hence R admits a proper strongly primary ideal).*

Proof. Let M be a maximal ideal of R , and suppose that $xy \in M$ with $x, y \in K$ and $x \in E(M)$. We must show that $y \in M$. We may assume $xy \neq 0$. We have $x^n/xy \in E(R)$ for each $n \geq 1$. We first claim that $y \in R$. If $x \in E(R)$, then $y = x^{-1}xy \in R$, as claimed. On the other hand, if $x^k \in R$ for some k , then $(x^{k+1}/xy)y = x^k \in R$ with $x^{k+1}/xy \in E(R)$ again implies that $y \in R$. Now $(y^3/xy)(x^3/xy) = xy \in M$ with $x^3/xy \in E(R)$ implies that $y^3/xy \in R$, whence $y^3 = (y^3/xy)xy \in M$. Since $y \in R$, this yields $y \in M$, as desired. Thus M is strongly primary, and the result follows from Theorem 3.4. \square

The converse of Proposition 3.5 is easily seen to be false. For example, if R is an integrally closed PVD which is not a valuation domain (e.g., $R = \mathbb{Q} + X\mathbb{Q}(Y)[[X]]$), then R is not strongly primary. (To see this, let V be the canonically associated valuation overring, and pick $u \in V \setminus R$. Then $u \in E(R)$, but $u^{-1} \notin R$.)

Corollary 3.6. *Let R be an APVD with maximal ideal M . If T is an overring of R with $MT = T$, then T is also an APVD.*

Proof. The hypotheses imply that T is strongly primary by Theorem 2.10. Hence T is an APVD by Proposition 3.5. \square

Proposition 3.7. *If R is an APVD with maximal ideal M , then R' is a PVD with maximal ideal $N = \text{Rad}(MR')$.*

Proof. By Corollary 2.6, $MR' = M$. Moreover, by Theorem 2.4, $\text{Rad}(MR')$ is strongly prime. Hence $\text{Rad}(MR')$ is the unique maximal ideal of R' , and R' is a PVD. \square

We close this section by considering domains each of whose overrings is an APVD. For PVD's there is a nice characterization: Every overring of a domain R is a PVD $\Leftrightarrow R'$ is a valuation domain [11, Proposition 2.7]. The situation with APVD's is not so clean.

Proposition 3.8. *If each overring of a domain R is an APVD, then R' is a valuation domain.*

Proof. We have that R' is a PVD by Proposition 3.7. The proof of [11, Proposition 2.7] shows that if R' is not a valuation domain, then there is a non-quasilocal overring of R' . However, such an overring cannot be an APVD by Proposition 3.2. \square

The converse of Proposition 3.8 is false, as the following example shows.

Example 3.9. Let $V = k + N$ be a rank one discrete valuation domain with $N = Va$. Let $S = k + Va^2$, $T = k + Sa^2$, and $R = k + Va^4$. Then R is an APVD and $R' = V$, but T is not an APVD.

In Example 3.9, T is an integral overring of R . The following result shows that this is the only stumbling block.

Proposition 3.10. *Let R be an APVD with R' a valuation domain, and assume that every integral overring of R is an APVD. Then every overring of R is an APVD.*

Proof. Let T be a non-integral overring of R , and pick $x \in T \setminus R'$. If M is the maximal ideal of R , then Proposition 3.7 implies that $x^{-k} \in MR' = M$ for some $k \geq 1$. Hence x^{-k} is a unit of T , and we have $MT = T$. Therefore, T is an APVD by Corollary 3.6. \square

4. CONDUCTIVE DOMAINS

In [5, Theorem 4.5] Bastida and Gilmer prove that a domain R shares an ideal with a valuation overring \Leftrightarrow each overring of R which is different from the quotient field K of R has a nonzero conductor to R . Domains with this property, called *conductive domains*, were explicitly defined and studied by Dobbs and Fedder [7] and further studied by Barucci, Dobbs, and Fontana [4, 8]. Conductive domains, powerful ideals, and strongly primary ideals are intimately connected, as we now show.

Theorem 4.1. *The following conditions are equivalent for a domain R .*

- (1) R is a conductive domain.
- (2) R admits a powerful ideal.
- (3) R admits a strongly primary ideal.
- (4) R shares a nonzero ideal with some conductive overring.

Proof. (1) \Rightarrow (3): Since R is conductive, it follows from [4, Theorem 1] that R shares an ideal I with a valuation overring V such that I is primary in V . By Proposition 2.1, I is strongly primary.

(3) \Rightarrow (2): If R admits a strongly primary ideal, then it admits a proper strongly primary ideal by Proposition 3.5. Hence R admits a powerful ideal by Corollary 2.6.

(2) \Rightarrow (1): Let V be any valuation overring of R . Then by Proposition 1.17, R and V share an ideal. Hence R is conductive.

(1) \Leftrightarrow (4): Assume (4). Let T be a conductive overring of R , and let V be a valuation overring of T . Then R and T share an ideal, and T and V share an ideal, from which it follows that R and V share an ideal. Thus R is conductive. (This argument is similar to that given in the proof of [7, Proposition 3.5], but the result was not stated explicitly there.) The converse is trivial. \square

We close by using our methods to derive the characterizations of conductive Noetherian domains given in [4] and of conductive Prüfer domains given in [7].

Theorem 4.2 (cf. [4, Theorem 6]). *A Noetherian domain R is conductive \Leftrightarrow each of the following conditions holds:*

- (1) R is quasilocal of dimension one,
- (2) R' is a rank one discrete valuation domain, and
- (3) R' is a finitely generated R -module.

Proof. Suppose that R is conductive; then R admits a powerful ideal I . By Theorem 1.5, the primes contained in $P = \text{Rad}(I)$ are linearly ordered, and

since R is Noetherian, this implies that $\text{ht}P = 1$. Moreover, since P is contained in every nonzero prime of R , it follows that R is quasilocal with maximal ideal P . Now R' is a PVD by Theorem 1.15, whence R' is quasilocal and therefore a rank one discrete valuation domain [12, Exercise 14, p. 73]. Also, by Theorem 1.15, the conductor $J = (R : R')$ is nonzero, and since R is Noetherian, $J^{-1} = (R : J)$ is finitely generated. Thus it suffices to show that $J^{-1} = R'$. However, since R' is a rank one discrete valuation ring, we may write $J = xR'$. Hence $J^{-1} = x^{-1}(R : R') = x^{-1}xR' = R'$, as desired. For the converse, note that the fact that R' is a finitely generated R -module implies that R and R' share a common nonzero ideal. Hence the conclusion follows from Proposition 1.18. \square

Let I be a nonzero powerful ideal of R . If $I = R$, then R is a valuation domain. If I is proper, then $\text{Rad}(I)$ is contained in the Jacobson radical of R by Theorem 1.5, and $\text{Rad}(I)$ is prime by Proposition 1.9. In either case, the Jacobson radical of R contains a nonzero prime ideal. For a Prüfer domain, the converse is true.

Theorem 4.3 (cf. [7, Corollary 3.4]). *Let R be a Prüfer domain (which is not a field). Then R is a conducive domain \Leftrightarrow the Jacobson radical of R contains a nonzero prime ideal.*

Proof. Let Q be a nonzero prime ideal contained in the Jacobson radical of R . Let $x \in K \setminus R$; we shall show that $x^{-1}Q \subseteq R_M$ for each maximal ideal M of R . If $x \notin R_M$, this follows from the fact that $x^{-1} \in R_M$ (since R_M is a valuation domain). Suppose $x \in R_M$. If $x \notin QR_M$, then, since QR_M is strongly prime, the fact that $xx^{-1}Q \subseteq QR_M$ implies that $x^{-1}Q \subseteq QR_M \subseteq R_M$. Suppose that $x \in QR_M$, and choose $s \in R \setminus M$ with $sx \in Q$. Now, since $x \notin R$, we must have $x \notin R_N$ for some maximal ideal $N \neq M$. Then $x^{-1} \in R_N$, and hence $x^{-1}Q \subseteq QR_N$. It follows that $s = x^{-1}sx \in QR_N$, whence $s \in Q$. This contradiction completes the proof. \square

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