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Dynamics of K^{th} Order Rational Difference Equation

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Abstract

In this paper we will investigate the dynamical behavior of the following rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}, \quad n = 0, 1, \ldots \tag{1}$$

where the parameters α, β, γ and A, B, C and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are non-negative real numbers, and the denominator is nonzero.
Our concentration here, is on the global stability, the periodic character, the analysis of semi-cycles and the invariant intervals of the positive solution of the above equation.
It is worth mentioning that our difference equation is the general case of the rational equation which is studied by Kulenovic and Ladas in their monograph (Dynamics of Second Order Rational Difference Equation with Open Problems and Conjectures, 2002).

1 Introduction and preliminaries

Our goal in this paper is to study the dynamics of the higher order nonlinear difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{2}$$

where the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are non-negative real numbers, $k \in \{1, 2, \ldots\}$, and all the parameters α, β, γ,A, B and C are non-negative real numbers, and the denominator is nonzero.

In 2002, Ladas and Kulenovic in [1] studied the special case of our difference equation, when $k = 1$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \qquad n = 0, 1, 2, \ldots$$

where the parameters $\alpha, \beta, \gamma, A, B$ and C are non-negative real numbers, and the initial conditions x_{-1}, x_0 are non-negative real numbers, and the denominator is nonzero.

They investigated the local stability, semi-cycles, periodicity, and the invariant intervals.

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Li and Sun in [2] studied the dynamical characteristics, such as the global asymptotic stability, the invariant interval, the periodic and oscillatory characters of all positive solutions of the equation

$$x_{n+1} = \frac{px_n + x_{n-k}}{q + x_{n-k}}, \qquad n = 0,1,2,\dots$$

where the initial conditions $x_{-k},\dots,x_{-1},x_0$ are non-negative real numbers, $k \in \{1,2,\dots\}$, and the parameters p and q are non-negative real numbers, and the denominator is nonzero.

Devault et al. in [3] investigated the periodic character and the global stability of the solutions of the difference equation

$$x_{n+1} = \frac{p + x_{n-k}}{qx_n + x_{n-k}}, \qquad n = 0,1,2,\dots$$

where the initial conditions $x_{-k},\dots,x_{-1},x_0$ are non-negative real numbers, $k \in \{1,2,\dots\}$, and all the parameters p and q are non-negative real numbers, and the denominator is nonzero.

Dehghan and Sebdani in [4] investigated the global stability, the boundedness of positive solutions and the character of semi-cycles of the difference equation

$$x_{n+1} = \frac{p + qx_n}{1 + x_{n-k}}, \qquad n = 0,1,2,\dots$$

where the initial conditions $x_{-k},\dots,x_{-1},x_0$ are non-negative real numbers, $k \in \{1,2,\dots\}$, and all the parameters p and q are non-negative real numbers, and the denominator is nonzero.

Also, Dehghan and Douraki in [5] investigated the global stability, invariant intervals and the boundedness of positive solutions of the difference equation

$$x_{n+1} = \frac{p + x_n}{x_n + qx_{n-k}}, \qquad n = 0,1,2,\dots$$

where the initial conditions $x_{-k},\dots,x_{-1},x_0$ are non-negative real numbers, $k \in \{1,2,\dots\}$, and all the parameters p and q are non-negative real numbers, and the denominator is nonzero.

[6] studied the local and global stability, invariant intervals, analysis of semi-cycles and the periodic character of solution of the difference equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \qquad n = 0,1,2,\dots$$

where the initial conditions $x_{-k},\dots,x_{-1},x_0$ are non-negative real numbers, $k \in \{1,2,\dots\}$, and all the parameters β,γ,B and C are non-negative real numbers, and the denominator is nonzero.

[7] studied the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \qquad n = 0,1,2,\dots$$

where the initial conditions $x_{-k},\dots,x_{-1},x_0$ are non-negative real numbers, $k \in \{1,2,\dots\}$, and all the parameters α,β,γ,B and C are non-negative real numbers, and the denominator is nonzero.

They studied the periodic character of the positive solution, the invariant intervals, the oscillation and the global stability of all solutions of the above difference equation.

Here, we present the basic definitions and theorems, and some results which will be useful in our investigation of the behavior of solution of Eq. (2), in this part.

Definition 1. [8] The equilibrium point $\bar{y}$ of the equation

$$y_{n+1} = f(y_n, y_{n-1},\dots,y_{n-k}), \qquad n = 0,1,\dots \tag{3}$$

is the point that satisfies the condition $\bar{y} = f(\bar{y},\bar{y},\dots,\bar{y})$.

Definition 2. [4] Let $\bar{y}$ be an equilibrium point of equation Eq. (3), then the equilibrium point $\bar{y}$ is called:

1- Locally stable "or stable" if for every $\varepsilon > 0$ there exist $\delta > 0$ such that for all $y_{-k}, \dots, y_{-1}, y_0 \in I$ with $\sum_{i=-k} \mid y_i - \bar{y} \mid < \delta$ we have $\mid y_n - \bar{y} \mid < \varepsilon$ for all $n \geq -k$.

2- Locally asymptotically stable "asymptotically stable" if it is locally stable and if there exist $\gamma > 0$ such that for all $y_{-k}, \dots, y_{-1}, y_0 \in I$ with $\sum_{i=-k} \mid y_i - \bar{y} \mid < \gamma$, we have

$$\lim_{n \longrightarrow \infty} y_n = \bar{y}.$$

3- Global attractor if for every $y_{-k}, \dots, y_{-1}, y_0 \in I$ we have

$$\lim_{n \longrightarrow \infty} y_n = \bar{y}.$$

4- Globally asymptotically stable if it is locally stable and global attractor.

5- Unstable if it is not stable.

6- A source or a repeller, if there exists $r > 0$ such that for all $y_{-k}, \dots, y_{-1}, y_0 \in I$ with $\sum_{i=-k} \mid y_i - \bar{y} \mid < \gamma$ there exists $N \geq 1$ such that $\mid y_N - \bar{y} \mid \geq r$.

The linearized equation associated with Eq. (3) about the equilibrium point $\bar{y}$ is

$$y_{n+1} = \sum_{i=-k} \frac{\partial f}{\partial u_i}(\bar{y}, \dots, \bar{y}) y_{n-i} \qquad n = 0, 1, \dots \tag{4}$$

and its characteristic equation

$$\lambda^{k+1} = \sum_{i=-k} \frac{\partial f}{\partial u_i}(\bar{y}, \dots, \bar{y}) \lambda^{k-i}. \tag{5}$$

2 Local stability

In this section we investigate the locally asymptotic stability of the unique positive equilibrium point of Eq. (2).

But before investigating the local stability of the positive equilibrium point we utilize the change of variables, let $x_n = \frac{\beta}{B} y_n$ then

$$\frac{\beta}{B} y_{n+1} = \frac{\alpha + \beta \frac{\beta}{B} y_n + \gamma \frac{\beta}{B} y_{n-k}}{A + B \frac{\beta}{B} y_n + C \frac{\beta}{B} y_{n-k}}$$

$\Longrightarrow$

$$\beta y_n = \frac{\alpha B + \beta^2 y_n + \gamma \beta y_{n-k}}{A + \beta y_n + \frac{C\beta}{B} y_{n-k}}$$

then

$$y_{n+1} = \frac{\frac{\alpha B}{\beta} + \beta y_n + \gamma y_{n-k}}{A + \beta y_n + \frac{C\beta}{B} y_{n-k}}$$

hence,

$$y_{n+1} = \frac{\frac{\alpha B}{\beta^2} + y_n + \frac{\gamma}{\beta} y_{n-k}}{\frac{A}{\beta} + y_n + \frac{C}{B} y_{n-k}}.$$

Set $p = \frac{\alpha B}{\beta^2}$, $q = \frac{A}{\beta}$, $L = \frac{\gamma}{\beta}$ and $d = \frac{C}{B}$.

So we get,

$$y_{n+1} = \frac{p + y_n + Ly_{n-k}}{q + y_n + dy_{n-k}}. \tag{6}$$

Let

$$f(x,y) = \frac{p+x+Ly}{q+x+dy}$$

assume that

$$a = \frac{\partial f}{\partial x}(\bar{y},\bar{y}) \qquad and \qquad b = \frac{\partial f}{\partial y}(\bar{y},\bar{y})$$

$$\frac{\partial f}{\partial x} = \frac{(q+x+dy)-(p+x+Ly)}{(q+x+dy)^2} = \frac{(q-p)+y(d-L)}{(q+x+dy)^2}.$$

So,

$$a = \frac{\partial f}{\partial x}(\bar{y},\bar{y}) = \frac{(q-p)+\bar{y}(d-L)}{(q+\bar{y}+d\bar{y})^2}$$

and

$$\frac{\partial f}{\partial y} = \frac{L(q+x+dy)-d(p+x+Ly)}{(q+x+dy)^2} = \frac{(Lq-dp)+x(L-d)}{(q+x+dy)^2}.$$

So,

$$b = \frac{\partial f}{\partial y}(\bar{y},\bar{y}) = \frac{(Lq-dp)+\bar{y}(L-d)}{(q+\bar{y}+d\bar{y})^2}.$$

We notice that the partial derivatives of $f(x,y)$ are evaluated at the equilibrium point $\bar{y}$, so we will find the equilibrium points of Eq. (6).

Let $f(\bar{y},\bar{y}) = \bar{y}$ we get,

$$\bar{y} = \frac{p+\bar{y}+L\bar{y}}{q+\bar{y}+d\bar{y}} \Longrightarrow p+\bar{y}+L\bar{y} = q\bar{y}+\bar{y}^2+d\bar{y}^2$$

$$(1+d)\bar{y}^2 = p-(q-L-1)\bar{y} \tag{7}$$

we solve Eq. (7) and find $\bar{y}$

$$\bar{y} = \frac{(L+1-q) \mp \sqrt{(q-L-1)^2+4p(1+d)}}{2(d+1)}.$$

The only positive equilibrium point is

$$\bar{y} = \frac{(L+1-q) + \sqrt{(q-L-1)^2+4p(1+d)}}{2(d+1)}.$$

For investigation of locally asymptotic stability of the unique positive equilibrium point of Eq. (6) we need the following theorems:

Theorem 1 (Linearized stability). *[9]*

1. *If all the roots of Eq.* (5) *lie in the open unite disk* $|\lambda| < 1$, *then the equilibrium point* $\bar{y}$ *of Eq.* (3) *is locally stable.*

2. *If at least one roots of Eq.* (5) *has absolute value greater than one, then the equilibrium point* $\bar{y}$ *of Eq.* (3) *is unstable.*

An equilibrium point $\bar{y}$ of Eq. (3) is a saddle point if there exists a root of Eq. (5) with absolute value less than one and another root of Eq. (5)with absolute value greater than one.

An equilibrium point $\bar{y}$ of Eq. (3) is called a repeller if all roots of Eq. (5) have absolute value greater than one.

Theorem 2. *[2] Assume that $a,b \in \mathbb{R}$ and $K \in \{1,2,\ldots\}$ then*

$$|\, a \,| + |\, b \,| < 1 \tag{8}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+1} = ay_n + by_{n-k}, \qquad n = 0,1,\ldots \tag{9}$$

Suppose in addition that one of the following two cases hold.

(a) K odd and $b < 0$.

(b) K even and $ab < 0$.

Then Eq. (8) is also a necessary condition for the asymptotically stable of Eq. (9).

Theorem 3. *[9] Assume that $a,b \in \mathbb{R}$. Then $|\, a \,| < b+1 < 2$ is a necessary and sufficient condition for the asymptotically stability of the difference equation*

$$y_{n+1} + ay_n + by_{n-k} = 0, \qquad n = 0,1,\ldots$$

Theorem 4. *[10] The difference equation*

$$y_{n+1} - by_n + by_{n-k} = 0, \qquad n = 0,1,\ldots$$

is asymptotically stable if and only if $0 < |\, b \,| < \frac{1}{2}\cos(\frac{k\pi}{k+2})$.

Note that the linearized equation associated with Eq. (6) about the equilibrium point $\bar{y}$ is

$$z_{n+1} - az_n - bz_{n-k} = 0.$$

Substitute the values of a and b last in the equation to get,

$$z_{n+1} - \frac{(q-p)+\bar{y}(d-L)}{(q+\bar{y}+d\bar{y})^2} z_n - \frac{(Lq-dp)+(L-d)\bar{y}}{(q+\bar{y}+d\bar{y})^2} z_{n-k} = 0. \tag{10}$$

And its characteristic equation is

$$\lambda^{k+1} - \frac{(q-p)+\bar{y}(d-L)}{(q+\bar{y}+d\bar{y})^2}\lambda^k - \frac{(Lq-dp)+(L-d)\bar{y}}{(q+\bar{y}+d\bar{y})^2} = 0.$$

The results presented here and Theorem (2) give the following theorem.

Theorem 5. *The unique equilibrium point $\bar{y}$ of Eq. (6) is locally asymptotically stable in the following cases:*

1. $d > L$, there are two cases:

(a) $(d-L)\bar{y} < (p-q)$

(b) $(d-L)\bar{y} > (p-q)$

69 *2.* $d < L$, *then we have two cases:*

70 *(a)* $(d-L)\bar{y} < (p-q)$

71 *(b)* $(d-L)\bar{y} > (p-q)$

Proof. We use Theorem (2), from the linearized equation we have

$$a = \frac{\bar{y}(d-L)+(q-p)}{(q+\bar{y}+d\bar{y})^2} \quad and \quad b = \frac{\bar{y}(L-d)+(Lq-dp)}{(q+\bar{y}+d\bar{y})^2}$$

72 1. when $d > L$, there are two cases:

73 (a) $(d-L)\bar{y} < (p-q)$ such that $p > q$ so we have,

$$|\, a \,| = \frac{-\bar{y}(d-L)+(p-q)}{(q+\bar{y}+d\bar{y})^2}, \ |\, b \,| = \frac{\bar{y}(d-L)+(dp-Lq)}{(q+\bar{y}+d\bar{y})^2} \tag{11}$$

74 We will prove that, $|\, a \,| + |\, b \,| < 1$.
Substituting the value of a and b we get

$$|a|+|b| = \frac{-\bar{y}(d-L)+(p-q)+\bar{y}(d-L)+(dp-Lq)}{(q+\bar{y}+d\bar{y})^2} = \frac{(p-q)+(dp-Lq)}{(q+\bar{y}+d\bar{y})^2}.$$

But

$$(q+\bar{y}+d\bar{y})^2 = (q+(d+1)\bar{y})^2 = q^2+(d+1)^2\bar{y}^2+2q\bar{y}(1+d)$$

and from Eq. (7) we get,

$$\bar{y}^2 = \frac{p-(q-L-1)\bar{y}}{(1+d)}.$$

So,

$$(q+\bar{y}+d\bar{y})^2 = q^2+(d+1)^2\frac{p-(q-L-1)\bar{y}}{(1+d)}+2q\bar{y}(1+d)$$

$$= q^2+q\bar{y}+qd\bar{y}+p+\bar{y}+L\bar{y}+dp+d\bar{y}+dL\bar{y} > p-q+dp-Lq.$$

75 Proving that $|a|+|b| < 1$.

76 (b) $(d-L)\bar{y} > (p-q)$ and we have two cases:

77 i) $p > q$, $q > 1$, $L \geq 1$

$$|\, a \,| = \frac{\bar{y}(d-L)-(p-q)}{(q+\bar{y}+d\bar{y})^2}, \ |\, b \,| = \frac{\bar{y}(d-L)+(dp-Lq)}{(q+\bar{y}+d\bar{y})^2} \tag{12}$$

$$\frac{\bar{y}(d-L)-(p-q)+\bar{y}(d-L)+(dp-Lq)}{(q+\bar{y}+d\bar{y})^2} < 1.$$

So,

$$q-p+2\bar{y}d-2\bar{y}L+dp-Lq < q^2+q\bar{y}+qd\bar{y}+p+\bar{y}+L\bar{y}+dp+d\bar{y}+dL\bar{y}$$

canceling $d\bar{y}$, dp from both sides, we obtain,

$$\bar{y}d+q-Lq < q^2+q\bar{y}+qd\bar{y}+2p+\bar{y}+3L\bar{y}+dL\bar{y}$$

according to item 2,

$$0 < (q^2-q)+q\bar{y}+qd\bar{y}+2p+\bar{y}+3L\bar{y}+d\bar{y}(L-1)+Lq.$$

78 This is true only if $q > 1$ and $L \geq 1$.

ii) $p<q$. When $p<q$ and $d>L$ then $(dp-Lq)>0$ or $(dp-Lq)<0$ if $(dp-Lq)>0$ then

$$|a|=\frac{\bar{y}(d-L)-(p-q)}{(q+\bar{y}+d\bar{y})^2},\ |b|=\frac{\bar{y}(d-L)+(dp-Lq)}{(q+\bar{y}+d\bar{y})^2}$$

and this case is the same as (12), so $|a|+|b|<1$.
If $(dp-Lq)<0$ and $|(d-L)\bar{y}|<|.dp-Lq|$
then we have

$$|a|=\frac{\bar{y}(d-L)-(p-q)}{(q+\bar{y}+d\bar{y})^2},\ \ |b|=\frac{-\bar{y}(d-L)-(dp-Lq)}{(q+\bar{y}+d\bar{y})^2}. \tag{13}$$

Now we prove that $|a|+|b|<1$
so,

$$-\bar{y}(d-L)-p+q+\bar{y}(d-L)-dp+Lq<(q+\bar{y}+d\bar{y})^2$$

hence,

$$-p+q-dp+Lq<q^2+q\bar{y}+qd\bar{y}+p+\bar{y}+L\bar{y}+dp+d\bar{y}+dL\bar{y}$$

$$0<(q^2-q)+2p+2dp+q\bar{y}+qd\bar{y}+\bar{y}+L\bar{y}+d\bar{y}+dL\bar{y}-Lq$$

then we have,

$$0<(q^2-q)+2p+2dp+q\bar{y}+\bar{y}+L\bar{y}+d\bar{y}+dL\bar{y}+q(d\bar{y}-L).$$

and this is true only if, $d\bar{y}>L$ and $q>1$.

2. $d<L$, we have two cases

(a) $(d-L)\bar{y}<(p-q)$ and there are two cases

i) $p<q$

$$|a|=\frac{-\bar{y}(d-L)+(p-q)}{(q+\bar{y}+d\bar{y})^2},\ \ |b|=\frac{-\bar{y}(d-L)-(dp-Lq)}{(q+\bar{y}+d\bar{y})^2}. \tag{14}$$

We will prove that $|a|+|b|<1$. Since,

$$-\bar{y}(d-L)+(p-q)+-\bar{y}(d-L)-(dp-Lq)<(q+\bar{y}+d\bar{y})^2$$

hence,

$$-2\bar{y}d+2L\bar{y}+p-q-dp+Lq<q^2+q\bar{y}+qd\bar{y}+p+\bar{y}+L\bar{y}+dp+d\bar{y}+dL\bar{y}$$

$$\Longrightarrow 0<(q^2+q)+q\bar{y}+qd\bar{y}+\bar{y}-L\bar{y}+2dp+3d\bar{y}+dL\bar{y}-Lq$$

thus

$$0<q(q+1+\bar{y}+d\bar{y}-L)+\bar{y}+2dp+3d\bar{y}+L(d\bar{y}-\bar{y}).$$

So the right hand side strictly greater than zero.

ii) $p>q$ then $(dp-Lq)>0$ or $(dp-Lq)<0$
if $(dp-Lq)>0$ and $|dp-Lq|>|\bar{y}(d-L)|$
then

$$|a|=\frac{-\bar{y}(d-L)+(p-q)}{(q+\bar{y}+d\bar{y})^2},\ \ |b|=\frac{\bar{y}(d-L)+(dp-Lq)}{(q+\bar{y}+d\bar{y})^2} \tag{15}$$

and this case is the same as (11) when $d>L$,
so we have seen that $|a|+|b|<1$

when $(dp-Lq)>0$ such that $\mid (dp-Lq)\mid<\mid \bar{y}(d-L)\mid$ or $(dp-Lq)<0$, then we have

$$\mid a\mid=\frac{-\bar{y}(d-L)+(p-q)}{(q+\bar{y}+d\bar{y})^2},\ \mid b\mid=\frac{-\bar{y}(d-L)-(dp-Lq)}{(q+\bar{y}+d\bar{y})^2}$$

and this case is the same as (14), when $d<L$,
so, $\mid a\mid+\mid b\mid<1$.

(b) $(d-L)\bar{y}>(p-q)$ such that $p<q$ then

$$\mid a\mid=\frac{\bar{y}(d-L)-(p-q)}{(q+\bar{y}+d\bar{y})^2},\ \mid b\mid=\frac{-\bar{y}(d-L)-(dp-Lq)}{(q+\bar{y}+d\bar{y})^2}$$

also, this case is the same as Eq. (13), so we have seen that

$$\mid a\mid+\mid b\mid<1.$$

The proof is complete.

Now we will give the following definition which will be the key concept here.

Definition 3 (Invariant Interval). [5] For the difference equation

$$y_{n+1}=f(y_n,y_{n-1},y_{n-2},\ldots,y_{n-k}),\qquad n=0,1,\ldots \tag{16}$$

is an interval I with the property that if $k+1$ consecutive terms of the solution fall in I, then all subsequent terms of the solution also belong to I. In other words I is an invariant interval for Eq. (16) if $y_{N-k},\ldots,y_{N-1},y_N\in I$ for some $N\geq 0$ then $y_n\in I$, for every $n>N$.

Theorem 6. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (6) then the following are true:*

1. *Suppose that $L<d$, $p<q$ and $dp>Lq$ and assume that for some $N\geq 0$ $y_{N-k},\ldots,y_{N-1},y_N\in[\frac{p+L}{q+d},1]$ then $y_n\in[\frac{p+L}{q+d},1]$, for all $n>N$*

2. *Suppose that $L>d$, $p>q$, $dp>Lq$, and $\mid Lq-dp\mid>\mid x(L-d)\mid$ and assume that for some $N\geq 0$ $y_{N-k},\ldots,y_{N-1},y_N\in[1,\frac{p+L}{q+d}]$ then $y_n\in[1,\frac{p+L}{q+d}]$, for all $n>N$*

Proof. Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (6)

1. Assume that $L<d$, $p<q$ and $dp>Lq$ then we can easily show that $f(x,y)=\frac{p+x+Ly}{q+x+dy}$ is increasing in x and decreasing in y, by using partial derivative [a]

$$\frac{\partial f(x,y)}{\partial x}=\frac{(q+x+dy)-(p+x+Ly)}{(q+x+dy)^2}=\frac{(q-p)+y(d-L)}{(q+x+dy)^2}$$

when $L<d$ and $p<q$ then $\frac{\partial f(x,y)}{\partial x}>0$, so $f(x,y)$ is increasing in x.

Also,

$$\frac{\partial f(x,y)}{\partial y}=\frac{L(q+x+dy)-d(p+x+Ly)}{(q+x+dy)^2}=\frac{(Lq-dp)+x(L-d)}{(q+x+dy)^2}$$

when $L<d$ and $dp>Lq$ then $\frac{\partial f(x,y)}{\partial y}<0$, and so $f(x,y)$ is decreasing in y.

[a] See Theorem 4.2.2 in [11], P.144

Now, for some $N > 0$, and $\frac{p+L}{q+d} \leq y_{N-k}, \ldots, y_{N-1}, y_N \leq 1$, we can say that the following step is true as "$p < q$ and $L < d$"

$$y_{N+1} = \frac{p + y_N + Ly_{N-k}}{q + y_N + dy_{N-k}} \leq \frac{q + y_N + Ly_{N-k}}{q + y_N + dy_{N-k}} \leq \frac{q + y_N + dy_{N-k}}{q + y_N + dy_{N-k}} = 1.$$

So,

$$y_{N+1} \leq 1.$$

And to show that $y_{N+1} \geq \frac{p+L}{q+d}$ we will substitute $y_{N-k} = 1$ and $y_N = \frac{p+L}{q+d}$ in the following function,

$$y_{N+1} = \frac{p + y_N + Ly_{N-k}}{q + y_N + dy_{N-k}}$$

and since y_{N+1} is increasing in y_N and decreasing in y_{N-k}, we get the following,

$$y_{N+1} = \frac{p + y_N + Ly_{N-k}}{q + y_N + dy_{N-k}} \geq \frac{p + \frac{p+L}{q+d} + L(1)}{q + \frac{p+L}{q+d} + d(1)} = \frac{(p+L)[1 + \frac{1}{q+d}]}{(q+d)[1 + \frac{1}{q+d}\frac{p+L}{q+d}]},$$

but $\frac{p+L}{q+d} < 1$.
So,

$$\frac{(p+L)[1 + \frac{1}{q+d}]}{(q+d)[1 + \frac{1}{q+d}\frac{p+L}{q+d}]} > \frac{p+L}{q+d}[\frac{1 + \frac{1}{q+d}}{1 + \frac{1}{q+d}}] = \frac{p+L}{q+d}$$

then

$$y_{N+1} \geq \frac{p+L}{q+d}$$

By Mathematical Induction we can prove that $y_n \in [\frac{p+L}{q+d}, 1]$, for all $n > N$. We proved that $y_{N+1} \in [\frac{p+L}{q+d}, 1]$, so we just will show that if $y_{N+m-1} \in [\frac{p+L}{q+d}, 1]$ then $y_{N+m} \in [\frac{p+L}{q+d}, 1]$.

$$y_{N+m} = \frac{p + y_{N+m-1} + Ly_{N+m-(k+1)}}{q + y_{N+m-1} + dy_{N+m-(k+1)}} \leq \frac{q + y_{N+m-1} + Ly_{N+m-(k+1)}}{q + y_{N+m-1} + dy_{N+m-(k+1)}}$$

also,

$$\frac{q + y_{N+m-1} + Ly_{N+m-(k+1)}}{q + y_{N+m-1} + dy_{N+m-(k+1)}} \leq \frac{q + y_{N+m-1} + dy_{N+m-(k+1)}}{q + y_{N+m-1} + dy_{N+m-(k+1)}} = 1$$

"as $p < q$ and $L < d$".
So,

$$y_{N+m} \leq 1.$$

Now, we will use induction hypothesis and the monotonicity properties of the function y_{N+m}, to show that $y_{N+m} \geq \frac{p+L}{q+d}$.

So we will substitute $y_{N+m-(k+1)} = 1$ and $y_{N+m-1} = \frac{p+L}{q+d}$ in the following function,

$$y_{N+m} = \frac{p + y_{N+m-1} + Ly_{N+m-(k+1)}}{q + y_{N+m-1} + dy_{N+m-(k+1)}}$$

since y_{N+m} is increasing in y_{N+m-1} and decreasing in $y_{N+m-(k+1)}$, we get the following,

$$y_{N+m} = \frac{p + y_{N+m-1} + Ly_{N+m-(k+1)}}{q + y_{N+m-1} + dy_{N+m-(k+1)}} \geq \frac{p + \frac{p+L}{q+d} + L(1)}{q + \frac{p+L}{q+d} + d(1)} = \frac{(p+L)[1 + \frac{1}{q+d}]}{(q+d)[1 + \frac{1}{q+d}\frac{p+L}{q+d}]}$$

but $\frac{p+L}{q+d} < 1$.
So,

$$\frac{(p+L)[1+\frac{1}{q+d}]}{(q+d)[1+\frac{1}{q+d}\frac{p+L}{q+d}]} > \frac{p+L}{q+d}[\frac{1+\frac{1}{q+d}}{1+\frac{1}{q+d}}] = \frac{p+L}{q+d}$$

then

$$y_{N+m} \geq \frac{p+L}{q+d}$$

So,

$$y_{N+m} \in [\frac{p+L}{q+d}, 1]$$

2. Assume that $L > d$, $p > q$, $dp > Lq$ and $\mid Lq - dp \mid > \mid x(L-d) \mid$ then by using partial derivative we can show that $f(x,y)$ is decreasing in both arguments.

Now, for some $N > 0$, and $1 \leq y_{N-k}, \ldots, y_{N-1}, y_N \leq \frac{p+L}{q+d}$
we have the following result as $p > q$ and $L > d$

$$y_{N+1} = \frac{p+y_N+Ly_{N-k}}{q+y_N+dy_{N-k}} \geq \frac{q+y_N+Ly_{N-k}}{q+y_N+dy_{N-k}} \geq \frac{q+y_N+dy_{N-k}}{q+y_N+dy_{N-k}} = 1.$$

So,

$$y_{N+1} \geq 1.$$

Also,

$$y_{N+1} = \frac{p+y_N+Ly_{N-k}}{q+y_N+dy_{N-k}}$$

since y_{N+1} is decreasing in y_{N-k} for each fixed y_N, then by substituting $y_{N-k} = 1$, in the previous function we get the following,

$$y_{N+1} = \frac{p+y_N+Ly_{N-k}}{q+y_N+dy_{N-k}} \leq \frac{p+y_N+L(1)}{q+y_N+d(1)} = \frac{(p+L)[1+\frac{y_N}{p+L}]}{(q+d)[1+\frac{y_N}{q+d}]}$$

$$\text{but} \qquad \frac{1}{p+L} < \frac{1}{q+d}$$

So,

$$\frac{(p+L)[1+\frac{y_N}{p+L}]}{(q+d)[1+\frac{y_N}{q+d}]} < \frac{p+L}{q+d}[\frac{1+\frac{y_N}{q+d}}{1+\frac{y_N}{q+d}}] = \frac{p+L}{q+d}$$

then

$$y_{N+1} \leq \frac{p+L}{q+d}.$$

By Mathematical Induction we can see that $y_n \in [1, \frac{p+L}{q+d}]$, for all $n > N$. We proved that $y_{N+1} \in [1, \frac{p+L}{q+d}]$, so we just will show that if $y_{N+m-1} \in [1, \frac{p+L}{q+d}]$ then $y_{N+m} \in [1, \frac{p+L}{q+d}]$.

$$y_{N+m} = \frac{p+y_{N+m-1}+Ly_{N+m-(k+1)}}{q+y_{N+m-1}+dy_{N+m-(k+1)}} \geq \frac{q+y_{N+m-1}+Ly_{N+m-(k+1)}}{q+y_{N+m-1}+dy_{N+m-(k+1)}}$$

also,

$$\frac{q+y_{N+m-1}+Ly_{N+m-(k+1)}}{q+y_{N+m-1}+dy_{N+m-(k+1)}} \geq \frac{q+y_{N+m-1}+dy_{N+m-(k+1)}}{q+y_{N+m-1}+dy_{N+m-(k+1)}} = 1$$

"since $p > q$ and $L > d$".

So,

$$y_{N+m} \geq 1.$$

Also, we will use induction hypothesis and the monotonicity properties of the function y_{N+m}, to show that $y_{N+m} \leq \frac{p+L}{q+d}$.

Since y_{N+m} is decreasing in $y_{N+m-(k+1)}$ for each fixed y_{N+m-1}, then by substituting $y_{N+m-(k+1)} = 1$, in the previous function we get the following,

$$y_{N+m} = \frac{p + y_{N+m-1} + Ly_{N+m-(k+1)}}{q + y_{N+m-1} + dy_{N+m-(k+1)}} \leq \frac{p + y_{N+m-1} + L(1)}{q + y_{N+m-1} + d(1)} = \frac{(p+L)[1 + \frac{y_{N+m-1}}{p+L}]}{(q+d)[1 + \frac{y_{N+m-1}}{q+d}]}$$

$$\text{but} \qquad \frac{1}{p+L} < \frac{1}{q+d}$$

So,

$$\frac{(p+L)[1 + \frac{y_{N+m-1}}{p+L}]}{(q+d)[1 + \frac{y_{N+m-1}}{q+d}]} < \frac{p+L}{q+d}[\frac{1 + \frac{y_{N+m-1}}{q+d}}{1 + \frac{y_{N+m-1}}{q+d}}] = \frac{p+L}{q+d}$$

then

$$y_{N+m} \leq \frac{p+L}{q+d}.$$

The proof is complete.

3 Analysis of semi-cycles

Our aim in this section is to study the semi-cycles behavior of solutions of Eq. (6) relative to the equilibrium point $\bar{y}$ and relative to the end points of the invariant interval of Eq. (6).

Now we give the definitions for the positive and negative semi-cycle of the solution of Eq. (6), relative to an equilibrium point $\bar{y}$.

Definition 4. [8] A positive semi-cycle of the solution $\{y_n\}$ of Eq. (6) consists of a" string " of terms $\{y_l, y_{l+1}, \ldots, y_m\}$, all greater than or equal to the equilibrium point $\bar{y}$, with $l \geq -k$ and $m \leq \infty$ and such that,

$$\text{either } l = -k \text{ or } l > -k \text{ and } y_{l-1} < \bar{y}$$

and

$$\text{either } m = \infty \text{ or } m < \infty \text{ and } y_{m+1} < \bar{y}.$$

Definition 5. [8] A negative semi-cycle of the solution $\{y_n\}$ of Eq. (6) consists of a" string " of terms $\{y_l, y_{l+1}, \ldots, y_m\}$ all less than or equal to the equilibrium point $\bar{y}$, with $l \geq -k$ and $m \leq \infty$ and such that

$$\text{either } l = -k \text{ or } l > -k \text{ and } y_{l-1} \geq \bar{y}$$

and

$$\text{either } m = \infty \text{ or } m < \infty \text{ and } y_{m+1} \geq \bar{y}.$$

The first semi-cycle of a solution starts with the term y_{-k} it is positive if $y_{-k} \geq \bar{y}$ and negative if $y_{-k} < \bar{y}$

Definition 6. [10] A solution $\{y_n\}$ of Eq. (6) is called non-oscillatory if there exists $N \geq -k$ such that $y_n > \bar{y}$ for all $n \geq N$ or $y_n < \bar{y}$ for all $n \leq N$. and a solution $\{y_n\}$ is called oscillatory if it is not non-oscillatory.

Definition 7. [12]

1. A solution $\{y_n\}_{n=-k}^{\infty}$ of a difference equation is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

2. A solution $\{y_n\}_{n=-k}^{\infty}$ of a difference equation is said to be periodic with prime period p or a p-cycle if it is periodic with period p and p is the least positive integer for which $x_{n+p} = x_n$.

Definition 8. Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (6), we say that the solution has a prime period two if the solution eventually takes the form:

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

where ϕ and ψ are distinct and positive.

Theorem 7. *If k is even, then Eq.* (6) *has no nonnegative prime period two solution.*

Proof. Assume for the sake of contradiction that there exist distinct positive real numbers ϕ and ψ, such that

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

is a prime period two solution of Eq. (6).

As k is even, so $y_n = y_{n-k}$. Now, ϕ and ψ satisfy the systems

$$\phi = \frac{p + \psi + L\psi}{q + \psi + d\psi}$$

and

$$\psi = \frac{p + \phi + L\phi}{q + \phi + d\phi}.$$

So,

$$\phi q + \phi\psi + d\phi\psi = p + \psi + L\psi \quad (17)$$

$$\psi q + \phi\psi + d\phi\psi = p + \phi + L\phi. \quad (18)$$

By subtracting Eq. (18) from Eq. (17), we get

$$q(\phi - \psi) = (\psi - \phi) + L(\psi - \phi)$$

hence

$$(\psi - \phi)\,[q + L + 1] = 0.$$

$$\text{As } q + L + 1 \neq 0, \text{ then } \psi - \phi = 0 \Longrightarrow \psi = \phi$$

which contradicts the hypothesis of $\phi \neq \psi$.

Theorem 8. *If k is odd then we have following results:*

1- The Eq. (6) *has no nonnegative prime period two in these two cases:*

- $L < 1 + q$
- $d > 1$

2- If $L > 1 + q$ and $d < 1$, then Eq. (6) *has a prime period two solution*

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

where the values ψ and ϕ are the solutions of the quadratic equation

$$t^2 - (\phi + \psi)t + \phi\psi = 0$$

Proof.

1- Assume for the sake of contradiction that there exist distinct and positive real numbers ϕ and ψ such that

$$\ldots,\phi,\psi,\phi,\psi,\ldots$$

is a prime period two solution of Eq. (6),

- k is odd then $y_{n-k} = y_{n+1}$ and in this case ϕ and ψ satisfy the following systems

$$\phi = \frac{p+\psi+L\phi}{q+\psi+d\phi}$$

and

$$\psi = \frac{p+\phi+L\psi}{q+\phi+d\psi}.$$

So,

$$q\phi+\phi\psi+d\phi^2 = p+\psi+L\phi \tag{19}$$

$$q\psi+\phi\psi+d\psi^2 = p+\phi+L\psi \tag{20}$$

subtract Eq. (20) from Eq. (19), we have

$$q(\phi-\psi)+d(\phi^2-\psi^2) = (\psi-\phi)+L(\phi-\psi)$$

$$\Longrightarrow (\phi-\psi)[q+d(\phi+\psi)] = (\phi-\psi)[-1+L]$$

so,

$$(\phi+\psi) = \frac{L-1-q}{d} = \frac{L-(1+q)}{d} \tag{21}$$

when $L < 1+q$ then $\phi+\psi < 0$ and this contradicts the assumption that ϕ and ψ are positive distinct real numbers.

- k is odd, from the previous steps we have,

$$\phi q+\phi\psi+d\phi^2 = p+\psi+L\phi \tag{22}$$

$$\psi q+\phi\psi+d\psi^2 = p+\phi+L\psi \tag{23}$$

By adding Eq. (22) and Eq. (23) we get,

$$\begin{aligned} q(\phi+\psi)+2\phi\psi+d(\phi^2+\psi^2) &= 2p+(\phi+\psi)+L(\phi+\psi) \\ q(\phi+\psi)+2\phi\psi+d(\phi^2+2\phi\psi-2\phi\psi+\psi^2) &= 2p+(\phi+\psi)[1+L] \\ q(\phi+\psi)+\phi\psi(2-2d)+d(\phi+\psi)^2 &= 2p+(\phi+\psi)[1+L] \end{aligned}$$

hence

$$\begin{aligned} \phi\psi(2-2d) &= 2p+(\phi+\psi)[1+L]-d(\phi+\psi)^2-q(\phi+\psi) \\ &= 2p+(\phi+\psi)[(1+L)-d(\phi+\psi)-q] \end{aligned}$$

but $\phi+\psi = \frac{L-1-q}{d}$, substitute the value of $(\phi+\psi)$ in the last equation

$$\phi\psi(2-2d) = 2p+(\frac{L-1-q}{d})[(1+L)-d(\frac{L-1-q}{d})-q]$$

then

$$\phi\psi(2-2d) = 2p+2(\frac{L-1-q}{d})$$

so

$$\phi\psi = \frac{[pd + (L-1-q)]}{d(1-d)} \tag{24}$$

when $d > 1$ then $\phi\psi < 0$, this contradicts the assumption that ϕ and ψ are distinct and positive real numbers.

2- If $L > (1+q)$ and $d < 1$, then it is clear from Eq. (24) and Eq. (21) that ϕ and ψ are two distinct real roots of the quadratic equation

$$t^2 - (\frac{L-1-q}{d})t + \frac{[pd + (L-1-q)]}{d(1-d)} = 0$$

which have the following values

$$\psi = \frac{1}{2}(\frac{L-1-q}{d}) - \frac{1}{2}\sqrt{(\frac{L-1-q}{d})^2 - 4(\frac{[pd + (L-1-q)]}{d(1-d)}}),$$

and

$$\phi = \frac{1}{2}(\frac{L-1-q}{d}) + \frac{1}{2}\sqrt{(\frac{L-1-q}{d})^2 - 4(\frac{[pd + (L-1-q)]}{d(1-d)}}).$$

The proof is complete.

Theorem 9. *[5,6] Assume that $f \in \mathscr{C}[(0,\infty)\times(0,\infty),(0,\infty)]$ such that $f(x,y)$ is increasing in x for each fixed y, and decreasing in y for each fixed x. Let $\bar{y}$ be a positive equilibrium of Eq.* (16)*, then every oscillatory solution of Eq.* (16) *has semi-cycles of length at least k.*

Proof. When $k = 1$, the proof is presented as Theorem 1.7.4 in [1]. We just give the proof of the theorem for $k = 2$, the other cases for $k \geq 3$ are similar and we omitted them. Assume that $\{y_n\}$ is an oscillatory solution with three consecutive terms y_{N-1}, y_N, y_{N+1} such that

$$y_{N-1} < \bar{y} \leq y_{N+1}.$$

then by using the increasing character of f, we obtain

$$y_{N+2} = f(y_{N+1}, y_{N-1}) > f(\bar{y},\bar{y}) = \bar{y}$$

which show that the next term y_{N+2} also belongs to the positive semicycle. If $y_N \geq \bar{y}$, then the result follows. Otherwise, if $y_N < \bar{y}$, hence

$$y_{N+3} = f(y_{N+2}, y_N) > f(\bar{y},\bar{y}) = \bar{y}.$$

which shows that it had at least three terms in the positive semicycle. The proof in the case $y_{N-1} \geq \bar{y}$ $> y_{N+1}$ is similar and is omitted.

Theorem 10. *[1,6,12] Assume that $f \in \mathscr{C}[(0,\infty)\times(0,\infty),(0,\infty)]$ and that $f(x,y)$ is increasing in both arguments. Let $\bar{y}$ be a positive equilibrium of Eq.* (16)*. Then except possibly for the first semi-cycle, every oscillatory solution of Eq.* (16) *has semi-cycles of length k.*

Proof. When $k=1$, the proof is presented as theorem 1.7.1 in [13]. We just give the proof of the theorem for $k=2$, the other cases for $k\geq 3$ are similar and we omitted them. Let $\{y_n\}$ be a solution of equation (16) with at least three semicycles, then there exists $N\geq 0$ such that either

$$y_{N-1} < \bar{y} \leq y_{N+1}.$$

or

$$y_{N-1} \geq \bar{y} > y_{N+1}.$$

We will assume that

$$y_{N-1} < \bar{y} \leq y_{N+1}.$$

The other case is similar and will be omitted. By using the monotonic character of $f(x,y)$ we have

$$y_{N+2} = f(y_{N+1}, y_{N-1}) > f(\bar{y}, \bar{y}) = \bar{y}.$$

and

$$y_{N+3} = f(y_{N+2}, y_N) > f(\bar{y}, \bar{y}) = \bar{y}.$$

Thus

$$y_{N+1}, y_{N+2}, y_{N+3}.$$

is a positive semi-cycle with 3 terms. The proof is complete.

The proof of the next two theorems follows by induction in an argument similar to the above two theorems and will be omitted.

Theorem 11. *[1, 6] Assume that $f\in\mathscr{C}[(0,\infty)\times(0,\infty),(0,\infty)]$ is such that $f(x,y)$is decreasing in x for each fixed y, and increasing in y for each fixed x. Let $\bar{y}$ be a positive equilibrium of Eq.* (16)*, then except possibly for the first semi-cycle every oscillatory solution of Eq.* (16) *has semi-cycles of length k.*

Theorem 12. *[6, 12] Assume that $f\in\mathscr{C}[(0,\infty)\times(0,\infty),(0,\infty)]$ and that $f(x,y)$ is decreasing in both arguments.*
Let $\bar{y}$ be a positive equilibrium of Eq. (16)*, then every oscillatory solution of Eq.* (16) *has semi-cycles of length at most k.*

Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (6) then the following are true:

$$y_{n+1} - 1 = (d-L)[\frac{(\frac{p-q}{d-L}) - y_{n-k}}{q + y_n + dy_{n-k}}] \tag{25}$$

Notice that $\frac{p-q}{d-L} < 0$, thus $\frac{p-q}{d-L} < \frac{p+L}{d+q}$ so we have the following equation:

$$y_{n+1} - 1 = (d-L)[\frac{(\frac{p-q}{d-L}) - y_{n-k}}{q + y_n + dy_{n-k}}] < (d-L)[\frac{(\frac{p+L}{d+q}) - y_{n-k}}{q + y_n + dy_{n-k}}]. \tag{26}$$

Also,

$$(y_{n+1} - \frac{p+L}{q+d}) = \frac{(1-\frac{p+L}{q+d})y_n + \frac{pd-qL}{q+d}[1-y_{n-k}]}{(q+d)(q+y_n+dy_{n-k})} \tag{27}$$

<u>Case I:</u>

We will analyze the semi-cycles of the solution $\{y_n\}_{n=-k}^{\infty}$ under the assumption that

$$p < q,\ L < d \text{ and } dp > Lq. \tag{28}$$

By using Eqs. (25), (26) and (27) we get the following results:

Lemma 13. *Assume that (28) holds, and let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (6), then the following statements are true:*

1. *For some $N \geq 0$, if $y_{N-k} \geq \frac{p+L}{q+d}$, then $y_{N+1} \leq 1$.*
2. *For some $N \geq 0$, if $y_{N-k} < \frac{p+L}{q+d}$, then $y_{N+1} > 1$.*
3. *For some $N \geq 0$, if $y_{N-k} \leq 1$, then $y_{N+1} \geq \frac{p+L}{q+d}$.*
4. *For some $N \geq 0$, if $\frac{p+L}{q+d} \leq y_{N-k} \leq 1$, then $\frac{p+L}{q+d} \leq y_{N+1} \leq 1$.*
5. *For some $N \geq 0$,if $\frac{p+L}{q+d} \leq y_{N-k},\ldots,y_{N-1},y_N \leq 1$, then $y_n \in [\frac{p+L}{q+d},1]$ for $n \geq N$, where $[\frac{p+L}{q+d},1]$ is an invariant interval of Eq. (3).*
6. *$\frac{p+L}{q+d} < \bar{y} < 1$.*

Proof. Assume that Eq. (28) holds, then

1. for some $N \geq 0$ if $y_{N-k} \geq \frac{p+L}{q+d}$, then we can conclude that $y_{N+1} - 1 \leq 0$ by using Eq. (26). So $y_{N+1} \leq 1$.
2. for some $N \geq 0$ and $y_{N-k} < \frac{p+L}{q+d}$, when $y_{N-k} < \frac{p-q}{d-L}$ then $y_{N+1} - 1 > 0$ but $y_{N-k} < \frac{p-q}{d-L} < \frac{p+L}{d+q}$ then by using Eq. (26) we can conclude that $y_{N+1} - 1 > 0$ and so $y_{N+1} > 1$.
3. for some $N \geq 0$ if $y_{N-k} \leq 1$ then from Eq. (27) we can conclude that $y_{N+1} - \frac{p+L}{q+d} \geq 0$, so $y_{N+1} \geq \frac{p+L}{q+d}$.
4. for some $N \geq 0$, $\frac{p+L}{q+d} \leq y_{N-k} \leq 1$, we see from (1) that if $y_{N-k} \geq \frac{p+L}{q+d}$ then $y_{N+1} \leq 1$, also we see that if $y_{N-k} \leq 1$ then $y_{N+1} \geq \frac{p+L}{q+d}$ so we conclude that $\frac{p+L}{q+d} \leq y_{N+1} \leq 1$.
5. if for some $N \geq 0$, then we see from (4) that if $\frac{p+L}{q+d} \leq y_{N-k} \leq 1$ then $\frac{p+L}{q+d} \leq y_{N+1} \leq 1$. Also we can see that if $\frac{p+L}{q+d} \leq y_{N-k},\ldots,y_{N-1},y_N \leq 1$,
   then $y_n \in [\frac{p+L}{q+d},1]$ for $n \geq N$ by using Eqs. (26) and (27), so $[\frac{p+L}{q+d},1]$ is an invariant interval for Eq. (3).
6. By using (5), as $[\frac{P+L}{q+d},1]$ is an invariant interval, then $\frac{p+L}{q+d} < \bar{y} < 1$.

Theorem 14. *Assume that Eq. (28) holds. Then every non trivial and oscillatory solution of Eq. (6) which lies in the interval $[\frac{p+L}{q+d},1]$ oscillates about $\bar{y}$ with semi-cycles of length at least k.*

Proof. Assume that Eq. (28) holds then Eq. (6) is increasing in x and decreasing in y, $\forall x,y \in [\frac{p+L}{q+d},1]$ so we see by using Theorem (9) that every non trivial and oscillatory solution of Eq. (6) has semi-cycle of length at least k.

Case II:

Now, we will analyze the semi-cycles of the solution $\{y_n\}_{n=-k}^{\infty}$ under the assumption that

$$p > q,\ L > d \text{ and } dp > Lq. \tag{29}$$

The following results is a direct consequences of Eqs. (25), (26) and (27)

Lemma 15. *Assume that (29) holds, and let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (6), then the following statements are true:*

1. *For some $N \geq 0$, if $y_{N-k} \leq \frac{p+L}{q+d}$, then $y_{N+1} \geq 1$.*
2. *For some $N \geq 0$, if $y_{N-k} \geq \frac{p+L}{q+d}$, then $y_{N+1} \leq 1$.*

3. For some $N \geq 0$, *if* $y_{N-k} \geq 1$, *then* $y_{N+1} \leq \frac{p+L}{q+d}$.

4. For some $N \geq 0$, *if* $1 \leq y_{N-k} \leq \frac{p+L}{q+d}$, *then* $1 \leq y_{N+1} \leq \frac{p+L}{q+d}$.

5. For some $N \geq 0$, *if* $1 \leq y_{N-k}, \ldots, y_{N-1}, y_N \leq \frac{p+L}{q+d}$, *then* $y_n \in [1, \frac{p+L}{q+d}]$ *for* $n \geq N$. *where* $[1, \frac{p+L}{q+d}]$ *is an invariant interval of Eq.* (3).

6. $1 < \bar{y} < \frac{p+L}{q+d}$.

Proof. Assume that Eq. (29) holds, then

1. for some $N \geq 0$, when $y_{N-k} \leq \frac{p-q}{d-L}$ then $y_{N+1} - 1 \geq 0$ but $y_{N-k} \leq \frac{p-q}{d-L} < \frac{p+L}{d+q}$ then by using Eq. (26) we can conclude that $y_{N+1} - 1 \geq 0$ and so $y_{N+1} \geq 1$.

2. for some $N \geq 0$ if $y_{N-k} \geq \frac{p+L}{q+d}$, then we can conclude that $y_{N+1} - 1 \leq 0$ by using Eq. (26). So $y_{N+1} \leq 1$.

3. for some $N \geq 0$ if $y_{N-k} \geq 1$ then from Eq. (27) we can conclude that $y_{N+1} - \frac{p+L}{q+d} \leq 0$, so $y_{N+1} \leq \frac{p+L}{q+d}$.

4. for some $N \geq 0$, $1 \leq y_{N-k} \leq \frac{p+L}{q+d}$, we see from (1) that if $y_{N-k} \leq \frac{p+L}{q+d}$ then $y_{N+1} \geq 1$, also we see that if $y_{N-k} \geq 1$ then $y_{N+1} \leq \frac{p+L}{q+d}$ so we conclude that $1 \leq y_{N+1} \leq \frac{p+L}{q+d}$.

5. if for some $N \geq 0$, then we see from (4) that if $1 \leq y_{N-k} \leq \frac{p+L}{q+d}$ then $1 \leq y_{N+1} \leq \frac{p+L}{q+d}$. Also we can see that if $1 \leq y_{N-k}, \ldots, y_{N-1}, y_N \leq \frac{p+L}{q+d}$, then $y_n \in [1, \frac{p+L}{q+d}]$ for $n \geq N$ by using Eqs. (26) and (27), so $[1, \frac{p+L}{q+d}]$ is an invariant interval for Eq. (3).

6. By using (5), as $[1, \frac{P+L}{q+d}]$ is an invariant interval, then $1 < \bar{y} < \frac{p+L}{q+d}$.

Theorem 16. *Assume that Eq.* (29) *holds. Then every non trivial and oscillatory solution of Eq.* (6) *which lies in the interval* $\left[1, \frac{p+L}{q+d}\right]$ *oscillates about* $\bar{y}$ *with semi-cycles of length at most k.*

Proof. Assume that Eq. (29) holds then Eq. (6) is decreasing in both arguments, $\forall x, y \in \left[1, \frac{p+L}{q+d}\right]$ so we see by using Theorem (12) that every non trivial and oscillatory solution of Eq. (6) has semi-cycles of length at most k.

<u>Case III:</u>

We will analyze the semi-cycles of the solution $\{y_n\}_{n=-k}^{\infty}$ under assumption that

$$p = q \text{ and } d = L. \tag{30}$$

The following results follow directly:

Lemma 17. *Let* $\{y_n\}_{n=-k}^{\infty}$ *be a solution of Eq.* (6), *and assume that* (30) *holds, then the following statements are true:*

1. For some $N \geq 0$, $y_{N-k} < 1$, *then* $y_{N+1} > 1$.

2. For some $N \geq 0$, $y_{N-k} = 1$, *then* $y_{N+1} = 1$.

3. For some $N \geq 0$, $y_{N-k} > 1$, *then* $y_{N+1} < 1$.

Proof. Assume that Eq. (30) holds, then

1. for some $N \geq 0$ if $y_{N-k} < 1$ then we conclude that $y_{N+1} - 1 > 0$ and so $y_{N+1} > 1$ by using Eq. (**??**).

2. for some $N \geq 0$ if $y_{N-k} = 1$ then we get $y_{N+1} - 1 = 0$ from Eq. (**??**). So $y_{N+1} = 1$

3. for some $N \geq 0$, if $y_{N-k} > 1$, then $y_{N+1} - 1 < 0$, which implies $y_{N+1} < 1$

Corollary 18. *Assume that Eq.* (30) *holds. Then every non trivial solution of Eq.* (6) *oscillates about the equilibrium point* $\bar{y}$.

Proof. We notice that by using lemma (17) if $y_{N-k} \leq 1$, then $y_{N+1} \geq 1$, also if $y_{N-k} \geq$ then $y_{N+1} \leq 1$, which means that the solution $\{y_n\}_{n=-k}^{\infty}$ oscillates about the equilibrium point $\bar{y}$= 1.

4 Global stability

In this section we consider the global asymptotic stability of Eq. (6). In section (2), we investigated local stability of the positive equilibrium point so it is sufficient to investigate the globally attractive of positive equilibrium point.

Now, we present some theorems which will be used in this section.

Theorem 19. *[14] [1] Let* $I = [a,b]$ *be some interval of real numbers and assume that*

$$f : [a,b] \times [a,b] \longrightarrow [a,b]$$

is a continuous function satisfying the following properties:

(a) $f(x,y)$ *is non decreasing in* x, *and non increasing in* y *where* $x, y \in [a,b]$.

(b) If $(m,\mu) \in [a,b] \times [a,b]$ *is a solution of the system.*

$$m = f(m,\mu) \text{ and } \mu = f(\mu,m),$$

then $m = \mu$.

Then Eq. (16) *has a unique equilibrium point* $\bar{y}$ *and every solution of Eq.* (16) *converges to* $\bar{y}$.

Proof. Set

$$m_0 = a \text{ and } \mu_0 = b$$

and for $i = 1, 2, \ldots$ set

$$\mu_i = f(\mu_{i-1}, m_{i-1}) \text{ and } m_i = f(m_{i-1}, \mu_{i-1}).$$

Now observe that for each $i \geq 0$,

$$m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq \mu_i \leq \cdots \leq \mu_1 \leq \mu_0$$

and

$$m_i \leq y_k \leq \mu_i \text{ for } k \geq 2i+1.$$

Set

$$m = \lim_{i \longrightarrow \infty} m_i \text{ and } \mu = \lim_{i \longrightarrow \infty} \mu_i.$$

Then

$$\mu \geq \limsup_{i \longrightarrow \infty} y_i \geq \liminf_{i \longrightarrow \infty} y_i \geq m$$

and by the continuity of f,

$$m = f(m,\mu) \text{ and } \mu = f(\mu,m).$$

In view of (b),

$$\mu = m$$

from which the result follows.

Theorem 20. *[1] Let $I=[a,b]$ be an interval of real numbers and assume that*

$$f:[a,b]\times[a,b]\longrightarrow[a,b]$$

is a continuous function satisfying the following properties:

(a) $f(x,y)$ is non increasing in each of its arguments.

(b) If $(m,\mu)\in[a,b]\times[a,b]$ is a solution of the system

$$\mu=f(m,m) \text{ and } m=f(\mu,\mu),$$

then $m=\mu$.

Then Eq. (16) *has a unique equilibrium point $\bar{y}$ and every solution of Eq.* (16) *converges to $\bar{y}$.*

Proof. Set

$$m_0=a \text{ and } \mu_0=b$$

and for $i=1,2,\dots$ set

$$\mu_i=f(m_{i-1},m_{i-1}) \text{ and } m_i=f(\mu_{i-1},\mu_{i-1}).$$

Now observe that for each $i\geq 0$,

$$m_0\leq m_1\leq\cdots\leq m_i\leq\cdots\leq\mu_i\leq\dots\mu_1\leq\mu_0$$

and

$$m_i\leq y_k\leq\mu_i \text{ for } k\geq 2i+1.$$

Set

$$m=\lim_{i\longrightarrow\infty} m_i \text{ and } \mu=\lim_{i\longrightarrow\infty}\mu_i.$$

Then clearly

$$\mu\geq\limsup_{i\longrightarrow\infty} y_i\geq\liminf_{i\longrightarrow\infty} y_i\geq m$$

and by the continuity of f,

$$\mu=f(m,m) \text{ and } m=f(\mu,\mu).$$

In view of (b),

$$\mu=m=\bar{y}$$

from which the result follows.

Theorem 21. *[14] [12] Let $I=[a,b]$ be an interval of real numbers and assume that*

$$f:[a,b]\times[a,b]\longrightarrow[a,b]$$

is a continuous function satisfying the following properties:

(a) $f(x,y)$ is non increasing in x for each fixed y and $f(x,y)$ is non decreasing in y for each fixed x, where $x,y\in[a,b]$.

(b) The difference Eq. (16) *has no solutions of prime period two in [a,b]. Then the difference Eq.* (16) *has a unique equilibrium point $\bar{y}\in[a,b]$ and every solution of it converges to $\bar{y}$.*

Proof. Set

$$m_0 = a \text{ and } \mu_0 = b$$

and for $i = 1, 2, \ldots$ set

$$\mu_i = f(m_{i-1}, \mu_{i-1}) \text{ and } m_i = f(\mu_{i-1}, m_{i-1}).$$

Now observe that for each $i \geq 0$,

$$m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq \mu_i \leq \cdots \leq \mu_1 \leq \mu_0,$$

and

$$m_i \leq y_k \leq \mu_i \text{ for } k \geq 2i+1.$$

Set

$$m = \lim_{i \longrightarrow \infty} m_i \text{ and } \mu = \lim_{i \longrightarrow \infty} \mu_i.$$

Then clearly

$$\mu \geq \limsup_{i \longrightarrow \infty} y_i \geq \liminf_{i \longrightarrow \infty} y_i \geq m$$

and by the continuity of f,

$$\mu = f(m, \mu) \text{ and } m = f(\mu, m).$$

In view of (b),

$$\mu = m = \bar{y}$$

from which the result follows.

Theorem 22. *[1] Let $I = [a, b]$ be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \longrightarrow [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(x, y)$ is non decreasing in each of its arguments.

(b) If $(m, \mu) \in [a, b] \times [a, b]$ is a solution of the system

$$\mu = f(\mu, \mu) \text{ and } m = f(m, m)$$

then $m = \mu$.

Then Eq. (16) has a unique positive equilibrium point, and every positive solution of Eq. (16) converges to $\bar{y}$.

Proof. Set

$$m_0 = a \text{ and } \mu_0 = b$$

and for $i = 1, 2, \ldots$ set

$$\mu_i = f(\mu_{i-1}, \mu_{i-1}) \text{ and } m_i = f(m_{i-1}, m_{i-1}).$$

Now observe that for each $i \geq 0$,

$$m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq \mu_i \leq \cdots \leq \mu_1 \leq \mu_0$$

and

$$m_i \leq y_k \leq \mu_i \text{ for } k \geq 2i+1.$$

Set

$$m = \lim_{i \longrightarrow \infty} m_i \text{ and } \mu = \lim_{i \longrightarrow \infty} \mu_i.$$

Then clearly

$$\mu \geq \limsup_{i \longrightarrow \infty} y_i \geq \liminf_{i \longrightarrow \infty} y_i \geq m$$

and by the continuity of f,

$$m = f(m,m) \text{ and } \mu = f(\mu,\mu).$$

In view of (b)

$$\mu = m = \bar{y}$$

from which the result follows.

From the above discussion we have the main result of this section as follows:

Theorem 23. *Assume that $p > q$, $L > d$ and $dp > Lq$, then the equilibrium point of Eq.* (6) *is globally asymptotically stable in the interval* $[1, \frac{p+L}{q+d}]$.

Proof. We use Theorem (20), assume that $p > q$, $L > d$ and $dp > Lq$ and suppose that $[1, \frac{p+L}{q+d}]$ is an invariant interval for the function

$$f(x,y) = \frac{p+x+Ly}{q+x+dy}.$$

We saw that in this interval the function $f(x,y)$ is decreasing in both arguments, so part (a) of Theorem (20) holds.

Now, let $(m,\mu) \in [a,b] \times [a,b]$ is a solution of the system

$$f(m,m) = \mu \text{ and } f(\mu,\mu) = m$$

then

$$m = \frac{p+\mu+L\mu}{q+\mu+d\mu} \quad \text{and} \quad \mu = \frac{p+m+Lm}{q+m+dm}.$$

But we saw that this equation has no period two solution
" when $y_n = y_{n-k}$, k is even".

So, the only solution is $m = \mu$.

The two conditions of Theorem (20) hold, then every solution of Eq. (6) converge to $\bar{y}$ in the interval $[1, \frac{p+L}{q+d}]$. So the equilibrium point $\bar{y}$ is globally attractive.

Theorem 24. *Assume that $p < q$, $L < d$ and $dp > Lq$, then the equilibrium point of Eq.* (6) *is globally asymptotically stable in the interval* $[\frac{p+L}{q+d}, 1]$.

Proof. We use Theorem (19). Assume that $p < q$, $L < d$ and $dp > Lq$ and suppose that$[\frac{p+L}{q+d}, 1]$ is an invariant interval for the function

$$f(x,y) = \frac{p+x+Ly}{q+x+dy}.$$

We saw that in this interval the function $f(x,y)$ is increasing in x and decreasing in y, so part (a) of Theorem (19) holds.

Now, let $(m,\mu) \in [a,b] \times [a,b]$ be a solution of the system $f(m,\mu) = m$ and $f(\mu,m) = \mu$ then

$$m = \frac{p+m+L\mu}{q+m+d\mu} \qquad \textit{and} \qquad \mu = \frac{p+\mu+Lm}{q+\mu+dm}$$

Then $m = \mu$.

So, the two conditions of Theorem (19) hold. Then by Theorem (19) every solution of Eq. (6) converge to $\bar{y}$ in the interval $[\frac{p+L}{q+d}, 1]$. So the equilibrium point $\bar{y}$ is globally attractive.

Since $\bar{y}$ is asymptotically stable, then by Definition (2), $\bar{y}$ is globally asymptotically stable.

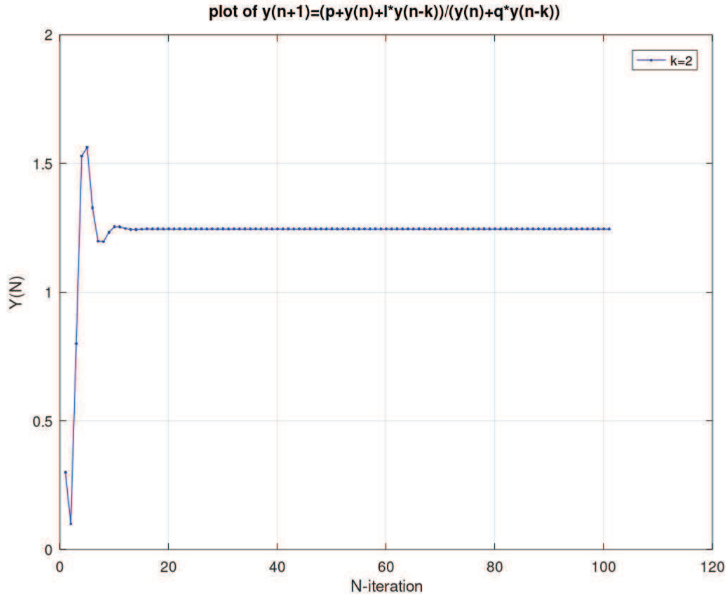

Fig. 1 Plot of $y_{n+1} = \frac{9+y_n+3y_{n-2}}{5+y_n+4y_{n-2}}$.

5 Numerical discussions

To illustrate the results of the previous chapters and to support our theoretical discussion, we consider a few numerical examples in this section.

These examples represent different types of qualitative behavior of solutions to nonlinear difference equations.

Example 1. Consider the third order difference equation when $k = 2$ in Eq. (6):

$$y_{n+1} = \frac{p + y_n + Ly_{n-k}}{q + y_n + dy_{n-k}}.$$

And assume that p=9, q=5, L=3 and d=4. So the equation will be reduced to the following:

$$y_{n+1} = \frac{9 + y_n + 3y_{n-2}}{5 + y_n + 4y_{n-2}}.$$

We assume the initial points $\{y_{-2}, y_{-1}, y_0\}$ are $\{.3, .1, .8\}$.

Then, the results is below.

Example 2. Consider the second order difference equation when k=1 in Eq. (6):

$$y_{n+1} = \frac{p + y_n + Ly_{n-k}}{q + y_n + dy_{n-k}}.$$

And assume that p=2, q=7, L=6 and d=1. So the equation will be reduced to the following:

$$y_{n+1} = \frac{2 + y_n + 6y_{n-1}}{7 + y_n + 1y_{n-1}}.$$

We assume the initial points $\{y_{-1}, y_0\}$ are $\{.9, 2.3\}$.

Then, the results is below.

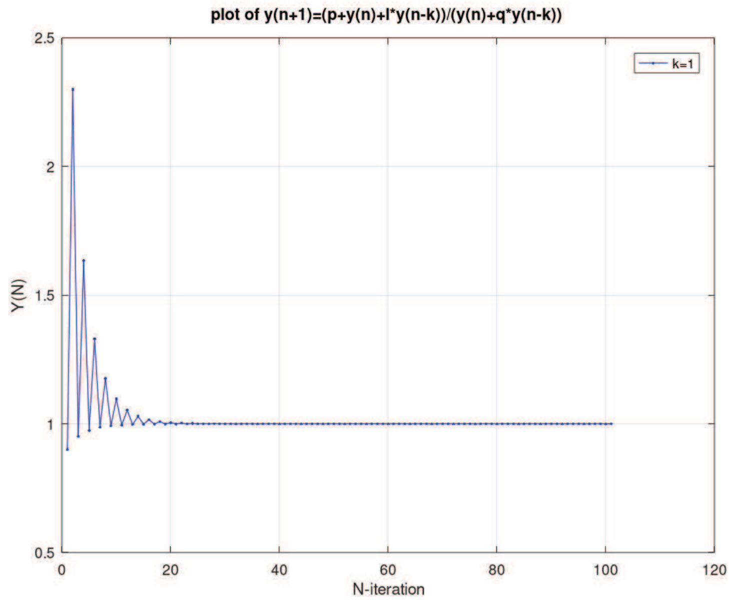

Fig. 2 Plot of $y_{n+1} = \frac{2+y_n+6y_{n-1}}{7+y_n+1y_{n-1}}$.

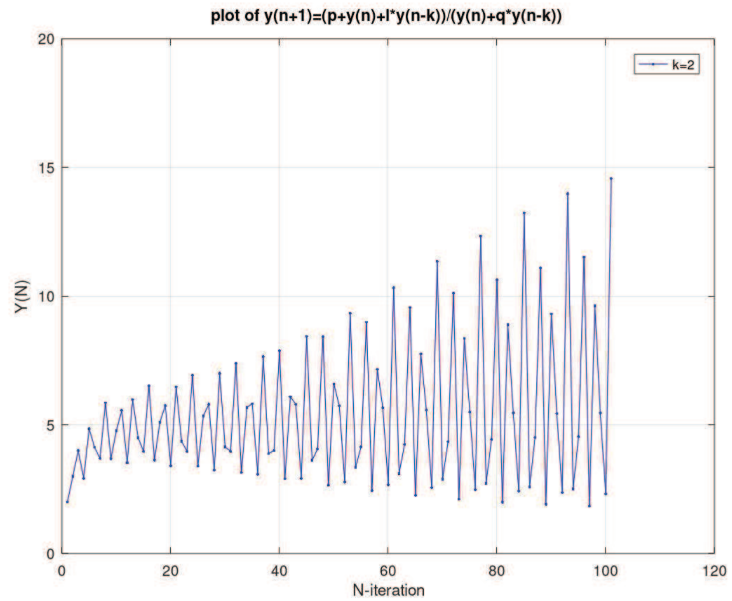

Fig. 3 Plot of $y_{n+1} = \frac{6+y_n+7y_{n-3}}{5+y_n+10y_{n-3}}$.

Example 3. Consider the third order difference equation when $k = 2$ in Eq. (6):

$$y_{n+1} = \frac{p+y_n+Ly_{n-k}}{q+y_n+dy_{n-k}}.$$

And assume that $p = 0$, $q = 0.005$, $L = 4$ and $d = 0.05$, So the equation will be reduced to the following:

$$y_{n+1} = \frac{y_n + 4y_{n-2}}{0.005 + y_n + 0.05y_{n-2}}.$$

We assume the initial points $\{y_{-2}, y_{-1}, y_0\}$ are $\{2,3,4\}$.

Then, the results is below.

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