Dear Author,

Here are the proofs of your article.

- You can submit your corrections **online**, via **e-mail** or by **fax**.
- For **online** submission please insert your corrections in the online correction form. Always indicate the line number to which the correction refers.
- You can also insert your corrections in the proof PDF and **email** the annotated PDF.
- For fax submission, please ensure that your corrections are clearly legible. Use a fine black pen and write the correction in the margin, not too close to the edge of the page.
- Remember to note the **journal title**, **article number**, and **your name** when sending your response via e-mail or fax.
- **Check** the metadata sheet to make sure that the header information, especially author names and the corresponding affiliations are correctly shown.
- **Check** the questions that may have arisen during copy editing and insert your answers/ corrections.
- **Check** that the text is complete and that all figures, tables and their legends are included. Also check the accuracy of special characters, equations, and electronic supplementary material if applicable. If necessary refer to the *Edited manuscript.*
- The publication of inaccurate data such as dosages and units can have serious consequences. Please take particular care that all such details are correct.
- Please **do not** make changes that involve only matters of style. We have generally introduced forms that follow the journal's style.
  Substantial changes in content, e.g., new results, corrected values, title and authorship are not allowed without the approval of the responsible editor. In such a case, please contact the Editorial Office and return his/her consent together with the proof.
- If we do not receive your corrections **within 48 hours**, we will send you a reminder.
- Your article will be published **Online First** approximately one week after receipt of your corrected proofs. This is the **official first publication** citable with the DOI. **Further changes are, therefore, not possible.**
- The **printed version** will follow in a forthcoming issue.

Please note

After online publication, subscribers (personal/institutional) to this journal will have access to the complete article via the DOI using the URL: http://dx.doi.org/[DOI].
If you would like to know when your article has been published online, take advantage of our free alert service. For registration and further information go to: https://www.lhscientificpublishing.com/.

Due to the electronic nature of the procedure, the manuscript and the original figures will only be returned to you on special request. When you return your corrections, please inform us if you would like to have these documents returned.

Journal of Applied Nonlinear Dynamics 10(3) (2021) 561-576

Journal of Applied Nonlinear Dynamics

https://lhscientificpublishing.com/Journals/JAND-Default.aspx

Dynamics and Bifurcation of a Second Order Rational Difference Equation with Quadratic Terms

Mohammad Saleh†, Shahd Herzallah

Department of Mathematics, Birzeit University, West Bank

Submission Info

Communicated by Hongjun Cao
Received 30 October 2019
Accepted 2 November 2020
Available online 1 October 2021

Keywords

Equilibrium points
Stability
Period-doubling
Bifurcation

Abstract

We study some results concerning dynamics and bifurcation of a special case of a second order rational difference equations with quadratic terms. We consider the second order, quadratic rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{A + Bx^2{}_n + Cx_{n-1}},\ n = 0,\ 1,\ 2,\ ...$$

with positive parameters α, β, A, B, C, and non-negative initial conditions.

We investigate local stability, invariant intervals, boundedness of the solutions, periodic solutions of prime period two and global stability of the positive fixed points. And we study the types of bifurcation exist where the change of stability occurs. Then, we give numerical examples with figures to support our results.

1 Introduction

In mathematics, a dynamical system is a system whose behavior at a given time depends on its behavior at one or more previous time. One of the main objectives in the theory of dynamical systems is the study of the behavior of orbits near fixed points.

Dynamical systems are a fundamental part of bifurcation theory which studies the changes in the qualitative or topological structure of systems. The term bifurcation refers to the phenomenon of a system exhibiting new dynamical behavior as the parameter is varied.

In this paper, we consider the second order, quadratic rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{A + Bx^2{}_n + Cx_{n-1}},\ n = 0,\ 1,\ 2,\ ... \tag{1}$$

With positive parameters α, β, A, B, C, and non-negative initial conditions.

We focus on local stability, invariant intervals, boundedness of the solutions, periodic solutions of prime period two and global stability of the positive fixed points. And we study the types of bifurcation

†Corresponding author.
Email address: msaleh@birzeit.edu

ISSN 2164-6457, eISSN 2164-6473/$-see front materials
DOI:10.5890/JAND.2021.09.014

exist where the change of stability occurs. Then, we use Matlab for numerical discussions with figures to support our results.

Equation (1) is special case of equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1} + \gamma x_n + \eta x^2{}_{n-1} + \zeta x_n x_{n-1} + \ell x^2{}_n}{A + D x_{n-1} + B x_n + C x^2{}_{n-1} + E x_n x_{n-1} + F x^2{}_n},\ n = 0,\ 1,\ 2,\ ... \tag{2}$$

Some special cases of (2) have been considered in many papers [1–18]. In [3] and [4] global stability character, the periodic nature, and the boundedness of solutions of special cases of equation

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + B x_n x_{n-1} + C x_{n-1}},\ n = 0,\ 1,\ 2, \tag{3}$$

have been studied, with non-negative parameters and with arbitrary non-negative initial conditions such that the denominator is always positive.

A. M. Amleh, E. Camouzis and G. Ladas [2] considered equations 24 and 25 in [4], they confirmed some conjectures and solved some open problems stated.

In [5] M. GariT-Demirović et al. investigated global behavior of the equation

$$x_{n+1} = \frac{{x_{n-1}}^2}{a{x_n}^2 + b x_n x_{n-1} + C{x_{n-1}}^2},\ n = 0,\ 1,\ 2, \tag{4}$$

where the parameters a, b, and c are positive numbers and the initial conditions x_{-1} and x_0 are arbitrary non-negative numbers such that $x_{-1} + x_0 > 0$.

2 Preliminaries

Before studying the behavior of solutions of this rational difference equation, we will review some definitions and basic results that will be used throughout this paper.

Lemma 1. *[20] Consider the second order difference equation,*

$$x(n+1) = f(x(n), x(n-1)),\ n = 0,\ 1,\ 2,\ ... \tag{5}$$

Where $f : I \times I \to I$ is a continuously differentiable function, and I is an interval of real numbers. Then for every set of initial conditions $x_{-1},\ x_0 \in I$ the difference equation (5) has a unique solution $\{x_n\}_{n=-1}^{\infty}$.

Definition 1. [20] A point $\bar{x} \in I$ is an equilibrium point of equation (5) if $f(\bar{x}, \bar{x}) = \bar{x}$.

Definition 2. [20] Let $\bar{x}$ be an equilibrium point of equation (5).

1. $\bar{x}$ is called locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta$, then $|x_n - \bar{x}| < \varepsilon$ for all $n > 0$.
2. $\bar{x}$ is called attracting, if there exists $\gamma > 0$ such that if $|x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma$, then $\lim_{n\to\infty} x_n = \bar{x}$.
3. $\bar{x}$ is called a global attractor if for every $x_{-1},\ x_0 \in I$ we have $\lim_{n\to\infty} x_n = \bar{x}$.
4. $\bar{x}$ is called globally asymptotically stable if it is locally stable and a global attractor.
5. $\bar{x}$ is called unstable if it is not stable.

Definition 3. [20]

1. A solution $\{x_n\}_{n=-1}^{\infty}$ of equation (5) is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -1$.

2. A solution $\{x_n\}_{n=-1}^{\infty}$ of equation (5) is said to be periodic with prime period p, or p-cycle if it is periodic with period p and p is the least positive integer for which $x_{n+p} = x_n$ for all $n \geq -1$.

Definition 4. [20] Consider the difference equation (5). Then the linearized equation associated with this difference equation is

$$y_{n+1} = py_n + qy_{n-1}, \ n = 0, \ 1, \ 2, \ ... \tag{6}$$

Where $a = \frac{\partial f}{\partial u}(\bar{x},\bar{x})$, and $b = \frac{\partial f}{\partial v}(\bar{x},\bar{x})$.

And the characteristic equation of (5) is

$$\lambda^2 - a\lambda - b = 0 \tag{7}$$

Theorem 2. *[21] (Linearized Stability)*
Consider the characteristic equation (7).

1. *If both characteristic roots of (7) lie inside the unit disk in the complex plane, then the equilibrium $\bar{x}$ of (5) is locally asymptotically stable.*

2. *If at least one characteristic root of (7) is outside the unit disk in the complex plane, the equilibrium point $\bar{x}$ is unstable.*

3. *If one characteristic root of (7) is on the unit disk and the other characteristic root is either inside or on the unit disk, then the equilibrium point $\bar{x}$ may be stable, unstable, or asymptotically stable.*

4. *A necessary and sufficient condition for both roots of (7) to lie inside the unit disk in the complex plane, is*

$$|a| < 1 - b < 2. \tag{8}$$

Let $A = Jf(\bar{x})$ is the Jacobian matrix of f at $\bar{x}$, where

$$Jf(\bar{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} |_{\bar{x}} \tag{9}$$

An important way to determine the stability of fixed points is given in the following result.

Theorem 3. *[22] Consider the map $f : H \subset \mathbb{R}^2 \to \mathbb{R}^2$, and let $A = Jf(\bar{x})$, with spectral norm $\rho(A)$. Then $\rho(A) < 1$, if and only if*

$$|tr(A)| - 1 < det(A) < 1 \tag{10}$$

where $tr(A)$ is the trace of A, and $det(A)$ is the determinant of A.

The following theorem will be used to investigate global stability of fixed points.

Theorem 4. *[20] Let $[a,b]$ be an interval of real numbers and assume that $f : [a,b] \times [a,b] \to [a,b]$ is a continuous function satisfying the following properties:*

1. *$f(x,y)$ is non-increasing in $x \in [a,b]$ for each $y \in [a,b]$, $f(x,y)$ is non-decreasing in $y \in [a,b]$ for each $x \in [a,b]$.*

2. *The difference equation (5) has no solutions of prime period two in $[a,b]$.*

   Then (5) has a unique equilibrium $\bar{x} \in [a,b]$ and every solution of (5) converges to $\bar{x}$.

The expression ?bifurcation? is extremely general. We use it to describe the orbit structure near non-hyperbolic fixed points.

Definition 5. Bifurcation is a change of the topological type of the system as its parameters pass through a bifurcation (critical) value.

There are several types of bifurcation, the saddle-node bifurcation, period-doubling bifurcation, Neimark-Sacker bifurcation.

Definition 6. [18] Consider the non-linear difference equation

$$X_{n+1} = AX_n + F(X_n),$$

where A is $k \times k$ matrix, $X_n \in R^k$ for every $n > 0$, $F \in C[R^k, R^k]$.

1. The bifurcation associated with the appearance of an eigenvalue $\mu = 1$ is called fold or (tangent) bifurcation.
   This bifurcation is also referred to as a limit point, saddle-node bifurcation, turning point, among others.

2. The bifurcation associated with the appearance of an eigenvalue $\mu = -1$ is called flip or (period-doubling) bifurcation.

3. The bifurcation corresponding to the presence of two eigenvalues $\lambda_{1,2} = e^{\pm i\theta_0}$, $0 < \theta_0 < \pi$, is called a Neimark-Sacker (or torus) bifurcation.

   The fold and flip bifurcations are possible if $n \geq 1$, but for the Neimark- Sacker bifurcation we need $n \geq 2$.

Theorem 5. *[22] Consider the two-dimensional map*

$$x \mapsto f(x,\mu), \ x \in \mathbb{R}, \ \mu \in \mathbb{R}. \tag{11}$$

Let $(\bar{x},\mu^*)$ be a fixed point of $f(x,\mu)$ and $A = Jf(\bar{x},\mu^*)$. Then the following statements hold:

1. *If $det(A) = tr(A) - 1$, then the eigenvalues of A are $\lambda_1 = det(A)$ and $\lambda_2 = 1$.*
2. *If $det(A) = -tr(A) - 1$, then the eigenvalues of A are $\lambda_1 = -det(A)$ and $\lambda_2 = -1$.*
3. *If $|tr(A)| - 1 < det(A)$ and $det(A) = 1$, then A has a pair of complex conjugate eigenvalues $\lambda_{1,2} = e^{\pm i\theta}$ where $\theta = \cos^{-1}(\frac{tr(A)}{2})$.*

Corollary 6. *Let*

$$x \mapsto f(x,\mu), \ x \in \mathbb{R}^2, \ \mu \in \mathbb{R} \tag{12}$$

be a one-parameter family of two-dimensional maps, with fixed point $(\bar{x},\mu^*)$ and $A = Jf(\bar{x},\mu^*)$. Then the following statements hold:

1. *If $det(A) = tr(A) - 1$, then the system (12) undergoes a saddle-node bifurcation.*
2. *If $det(A) = -tr(A) - 1$, then the system (12) undergoes a period-doubling bifurcation.*
3. *If $|tr(A)| - 1 < det(A)$ and $det(A) = 1$, then the system (12) undergoes a Neimark-sacker bifurcation.*

Consider the period-doubling bifurcation case for any n-dimensional map

$$\tilde{x} = Ax + G(x), \ x \in \mathbb{R} \tag{13}$$

where $G(x) = O(\|x\|^2)$ is a smooth function and its Taylor expansion is

$$G(x) = \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + O(\|x^4\|) \tag{14}$$

where

$$B_i(x,y) = \sum_{k,j=1}^{n} \frac{\partial^2 Y_i(\eta)}{\partial \eta_k \partial \eta_j}|_{\eta=0}(x_k y_j) \tag{15}$$

and

$$C_i(x,y,z) = \sum_{l,k,j=1}^{n} \frac{\partial^3 Y_i(\eta)}{\partial \eta_l \partial \eta_k \partial \eta_j}|_{\eta=0}(x_l y_k z_j). \tag{16}$$

And the Jacobian matrix A has the eigenvalue $\lambda = -1$ and the corresponding critical eigenspace T^c is one-dimensional and spanned by an eigenvector $\hat{q} \in \mathbb{R}^n$ such that $A\hat{q} = \lambda \hat{q}$. Let $\hat{p} \in \mathbb{R}^n$ be the adjoint eigenvector, that is, $A^T \hat{p} = \lambda \hat{p}$, where A^T is the transposed matrix. Normalize $\hat{p}$ with respect to $\hat{q}$ such that $\langle \hat{p}, \hat{q} \rangle = 1$. Let T^{su} denote an $(n-1)$-dimensional linear eigenspace of A corresponding to all eigenvalues other than λ. Note that the matrix $(A - \lambda I_n)$ has common invariant spaces with the matrix A, so we conclude that $y \in T^{su}$ if and only if $\langle \hat{p}, y \rangle = 0$.

To predict the direction of period-doubling bifurcation, we use the critical normal form coefficient $c(0)$. $c(0)$ is given by the following invariant formula:

$$c(0) = \frac{1}{6}\langle \hat{p}, C(\hat{q},\hat{q},\hat{q})\rangle - \frac{1}{2}\langle \hat{p}, B(\hat{q},(A - I_n)^{-1}B(\hat{q},\hat{q}))\rangle. \tag{17}$$

If $c(0) > 0$, then a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point. [18]

3 Dynamics of $x_{n+1} = \frac{\alpha + \beta x_{n-1}}{A + Bx^2{}_n + Cx_{n-1}}$

In this section we return to our problem

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{A + Bx^2{}_n + Cx_{n-1}}, \ n = 0, \ 1, \ 2, \ ... \tag{18}$$

with positive parameters α, β, A, B, C, and non-negative initial conditions.

3.1 Change of variables

The change of variables

$$x_n = \frac{\sqrt{A}}{\sqrt{B}} y_n. \tag{19}$$

reduces equation (18) to the difference equation

$$y_{n+1} = \frac{p + q y_{n-1}}{1 + y^2{}_n + r y_{n-1}}, \ n = 0, \ 1, \ 2, \ ... \tag{20}$$

Where $p = \alpha \frac{\sqrt{B}}{\sqrt{A^3}}$, $q = \frac{\beta}{A}$, and $r = \frac{C}{\sqrt{AB}}$.

3.2 Equilibrium points

We will prove the existence of the unique positive equilibrium point of the rational difference equation

$$y_{n+1} = \frac{p+qy_{n-1}}{1+y^2{}_n+ry_{n-1}},\ n=0,\ 1,\ 2,\ ... \tag{21}$$

with positive parameters p, q, r, and non-negative initial conditions. And we use a Matlab code to find it.

To find the equilibrium point, we solve the following equation

$$\bar{y} = \frac{p+q\bar{y}}{1+\bar{y}^2+r\bar{y}} \tag{22}$$

hence

$$\bar{y}^3+r\bar{y}^2+(1-q)\bar{y}-p=0. \tag{23}$$

By Descartes' rule of signs equation (23) has one positive root, which is the unique positive equilibrium point of equation (21).

To find the roots of equation (23) we use a Matlab code.

And then we choose the positive root to be $\bar{y}$.

3.3 Linearized equation

To find the linearized equation of (21) about the equilibrium point $\bar{y}$, let

$$f(x,y) = \frac{p+qy}{1+x^2+ry} \tag{24}$$

We have

$$\frac{\partial f}{\partial x}(x,y) = \frac{-2x(p+qy)}{(1+x^2+ry)^2}. \tag{25}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{q(1+x^2+ry)-r(p+qy)}{(1+x^2+ry)^2}. \tag{26}$$

by substituting $\bar{y} = \frac{p+q\bar{y}}{1+\bar{y}^2+r\bar{y}}$ from equation (22) we get

$$\frac{\partial f}{\partial x}(\bar{y},\bar{y}) = \frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}}. \tag{27}$$

And

$$\frac{\partial f}{\partial y}(\bar{y},\bar{y}) = \frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}}. \tag{28}$$

The linearized equation is

$$y_{n+1} = \frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}}y_n + \frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}}y_{n-1}. \tag{29}$$

And the characteristic equation is

$$\lambda^2 + \frac{2\bar{y}^2}{1+\bar{y}^2+r\bar{y}}\lambda - \frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} = 0. \tag{30}$$

3.4 Local stability

To check when the unique positive equilibrium point $\bar{y}$ of equation (21) is locally asymptotically stable, let

$$a = \frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}},\; b = \frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} \tag{31}$$

Using Theorem 2 (4) a sufficient condition for asymptotic stability of $\bar{y}$ is $|a| < 1-b < 2$. Which is equivalent to

$$-b < 1, \tag{32}$$

$$\text{and } |a| < 1-b. \tag{33}$$

The inequality (32) always holds, since it is equvilent to

$$1+\bar{y}^2+q > 0. \tag{34}$$

Which always holds.

And (33) is equivalent to

$$a > -1+b, \tag{35}$$

$$\text{and } a < 1-b. \tag{36}$$

(35) holds when

$$q < 1-\bar{y}^2+2r\bar{y}. \tag{37}$$

And (36) holds when

$$q < 1+3\bar{y}^2+2r\bar{y}. \tag{38}$$

Hence a sufficient conditions for asymptotic stability of $\bar{y}$ is

$$q < 1-\bar{y}^2+2r\bar{y}. \tag{39}$$

$$\text{and } q < 1+3\bar{y}^2+2r\bar{y}. \tag{40}$$

Note that if (39) holds, then (40) holds, thus $q < 1-\bar{y}^2+2r\bar{y}$ is a sufficient condition for asymptotic stability of $\bar{y}$.

3.5 Invariant intervals

Consider the difference equation (21), and $\{y_n\}_{n=-1}^{\infty}$ as a solution. Then the following are invariant intervals:

1. $[0,q]$ when $r \geq 1,$ *and* $q \geq p$.

2. $[0,\frac{q}{r}]$ when $pr \leq q$.

Proof.

1. Assume that $r \geq 1,$ *and* $q \geq p$, and $y_{N-1},\ y_N \in [0,q]$ for some integer N.

$$\begin{aligned} y_{N+1} &= \frac{p+qy_{N-1}}{1+y^2{}_N+ry_{N-1}} \\ &\leq \frac{p+qy_{N-1}}{1+ry_{N-1}} \\ &\leq \frac{q+qy_{N-1}}{1+y_{N-1}}, \\ &= q. \end{aligned} \tag{41}$$

And working inductively we complete the proof.

2. Assume that $pr \leq q$, and $y_{N-1},\ y_N \in [0, \frac{q}{r}]$ for some integer N.

$$\begin{aligned} y_{N+1} &= \frac{p+qy_{N-1}}{1+y^2{}_N+ry_{N-1}} \\ &= \frac{q(\frac{p}{q}+y_{N-1})}{r(\frac{1}{r}+\frac{1}{r}y^2{}_N+y_{N-1})} \\ &\leq \frac{q(\frac{1}{r}+y_{N-1})}{r(\frac{1}{r}+y_{N-1})} \\ &= \frac{q}{r}. \end{aligned} \tag{42}$$

And working inductively we complete the proof.

3.6 Boundedness

We will show that every solution of the difference equation (21) is bounded. Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of (21). then we have $for\ n = 0,\ 1,\ 2,\ ...$

$$\begin{aligned} 0 < y_{n+1} &= \frac{p+qy_{n-1}}{1+y^2{}_n+ry_{n-1}} \\ &= \frac{p}{1+y^2{}_n+ry_{n-1}} + \frac{qy_{n-1}}{1+y^2{}_n+ry_{n-1}} \\ &\leq \frac{p}{1} + \frac{qy_{n-1}}{ry_{n-1}} \\ &= p + \frac{q}{r}. \end{aligned} \tag{43}$$

Hence the solution is bounded, since it is bounded from below and from above.

3.7 Period two cycles

In general, we say that the solution $\{y_n\}_{n=-1}^{\infty}$ has a prime period two if the solution eventually takes the form:

$$...,\ \phi,\ \psi,\ \phi,\ \psi,\ ... \tag{44}$$

where $\phi\ and\ \psi$ are positive, and $\phi \neq \psi$.

Theorem 7. *Assume that equation (21) has a two periodic cycle $\{\phi,\ \psi\}$, where ϕ and ψ are positive, and $\phi \neq \psi$. Then q must satisfy the following conditions:*

1.

$$q \leq 1 + r(\phi+\psi) \tag{45}$$

2.

$$q > 1 - \phi\psi \tag{46}$$

Proof. Assume $\{\phi,\ \psi\}$ is prime period two solution of equation (21), then $\phi,\ \psi$ satisfy :

$$\phi = \frac{p+q\phi}{1+\psi^2+r\phi} \tag{47}$$

and

$$\psi = \frac{p+q\psi}{1+\phi^2+r\psi}. \tag{48}$$

From Equation (47) we have

$$\phi+\phi\psi^2+r\phi^2=p+q\phi, \tag{49}$$

and from Equation (48) we have

$$\psi+\psi\phi^2+r\psi^2=p+q\psi. \tag{50}$$

Subtracting Equation (50) from (49), we get:

$$(\phi-\psi)-\psi\phi(\phi-\psi)+r(\phi^2-\psi^2)=q(\phi-\psi). \tag{51}$$

Since $\phi\neq\psi$, the last equation can be divided by $(\phi-\psi)$, and we get

$$1-\psi\phi+r(\phi+\psi)=q. \tag{52}$$

So

$$\phi\psi=1+r(\phi+\psi)-q. \tag{53}$$

But $\psi\phi\geq 0$, so

$$1+r(\phi+\psi)-q\geq 0, \tag{54}$$

hence

$$q\leq 1+r(\phi+\psi). \tag{55}$$

Which is the first condition. From (52) we get also:

$$\phi+\psi=\frac{\phi\psi+q-1}{r}. \tag{56}$$

But $\phi+\psi>0$, so

$$\frac{\phi\psi+q-1}{r}>0, \tag{57}$$

since $r>0$ we have

$$\phi\psi+q-1>0, \tag{58}$$

hence

$$q>1-\phi\psi. \tag{59}$$

Which is complete the proof.

3.8 Global stability

Now we will investigate a result about the global stability of the positive equilibrium point of (21) $\bar{y}$.

Theorem 8. *Assume* $pr\leq q\leq\frac{r\sqrt{r^2+4}-r^2}{2}$. *Then the positive equilibrium point* $\bar{y}$ *on the interval* $S=[0,\frac{q}{r}]$ *is globally asymptotically stable.*

Proof. this proof can easily done depending on Theorem (4). Assume $pr\leq q$, and consider the function

$$f(x,y)=\frac{p+qy}{1+x^2+ry}. \tag{60}$$

Note that S is an invariant interval and all non-negative solutions of equation (21) lie in this interval. And $f(x,y)$ on S is non-increasing function in x, and non-decreasing in y.

Now we need to show that the difference equation (21) has no solution of prime period two in S.

For seek of contradiction assume that the difference equation (21) has a solution of prime period two $\{\phi,\psi\}\in S$. Then q must satisfy

$$q>1-\phi\psi, \tag{61}$$

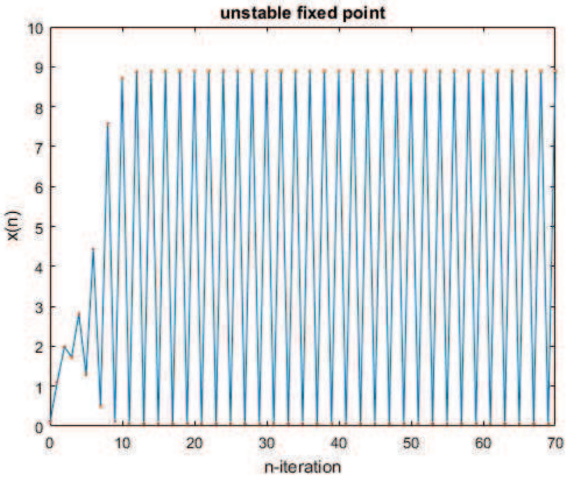

Fig. 1 The positive equilibrium point is unstable.

but since $\{\phi, \psi\} \in S$

$$1 - \phi\psi \geq 1 - \frac{q^2}{r^2}, \tag{62}$$

hence

$$q > 1 - \frac{q^2}{r^2}, \tag{63}$$

which is a contradiction, since $q \leq \frac{r\sqrt{r^2+4}-r^2}{2}$.

So equation (21) has no solution of prime period two in S. Then both conditions of Theorem (4) hold, then (21) has a unique positive equilibrium point $\bar{y} \in S$, and it is globally asymptotically stable.

3.9 Numerical discussion

In this subsectionm we use Matlab to graph an example to support our results.

Example 1. Consider the difference equation (21), take $p = 4,\ q = 5,\ r = 0.5$. Equation (21) becomes

$$y_{n+1} = \frac{4 + 5y_{n-1}}{1 + y^2{}_n + 0.5y_{n-1}}, \ n = 0,\ 1,\ 2,\ ... \tag{64}$$

With initial conditions $y_0 = 0.1,\ y_1 = 1.1$.

The theoretical positive equilibrium point will be $\bar{y} = 2.1786778129$.

Theoretically the positive equilibrium point $\bar{y}$ is unstable since $pr = 2 \leq q$ but $q > \frac{r\sqrt{r^2+4}-r^2}{2} = 0.3903882032$.

Figure (1) shows that the positive equilibrium point is unstable.

4 Bifurcation of $y_{n+1} = \frac{p+qy_{n-1}}{1+y^2{}_n+ry_{n-1}}$

In this section we study the types of bifurcation that occur at $q = q^*$ as q is the bifurcation parameter.

In order to convert equation (21) to a second dimensional system with three parameters p, q, and r, let

$$z_n = y_{n-1}, \tag{65}$$

and

$$v_n = y_n. \tag{66}$$

We get the following system

$$z_{n+1} = v_n \tag{67}$$

$$v_{n+1} = \frac{p+qz_n}{1+v^2{}_n + rz_n},\ n = 0,\ 1,\ 2,\ ... \tag{68}$$

This system has the unique fixed point $(\bar{z},\bar{v})^T = (\bar{y},\bar{y})^T$. Convert this system in to second dimensional map

$$F\begin{pmatrix} z \\ v \end{pmatrix} = \begin{pmatrix} f_1(z,v) \\ f_2(z,v) \end{pmatrix} = \begin{pmatrix} v \\ \frac{p+qz}{1+v^2+rz} \end{pmatrix}. \tag{69}$$

So the Jacobian matrix of $F(z,v)$ at $(\bar{y},\bar{y})$ is

$$JF(z,v)|_{(\bar{y},\bar{y})} = \begin{pmatrix} 0 & 1 \\ \frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} & \frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}} \end{pmatrix} \tag{70}$$

So

$$\det(JF(\bar{y},\bar{y})) = -\frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}}, \tag{71}$$

and

$$tr(JF(\bar{y},\bar{y})) = \frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}}. \tag{72}$$

Theorem 9. *the fixed point $(\bar{y},\bar{y})$ of the system (69) undergoes a saddle-node bifurcation when $q = 2r\bar{y}+3\bar{y}^2+1$.*

Proof. Saddle-node bifurcation happens when

$$\det(J) = tr(J) - 1. \tag{73}$$

So the fixed point $(\bar{y},\bar{y})$ of the system (69) undergoes a saddle-node bifurcation if

$$-\frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} = \frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}} - 1 \tag{74}$$

so

$$q = 2r\bar{y}+3\bar{y}^2+1. \tag{75}$$

So saddle-node bifurcation happens if $q = 2r\bar{y}+3\bar{y}^2+1$.

Theorem 10. *the fixed point $(\bar{y},\bar{y})$ of the system (69) undergoes a period-doubling bifurcation when $q = 2r\bar{y}-\bar{y}^2+1$ if $r > \frac{\bar{y}^2-1}{2\bar{y}}$.*

Proof. Assume $r > \frac{\bar{y}^2-1}{2\bar{y}}$. Period-doubling bifurcation happens when

$$\det(J) = -tr(J) - 1. \tag{76}$$

So the fixed point $(\bar{y},\bar{y})$ of the system (69) undergoes a period-doubling bifurcation if

$$-\frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} = -\frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}} - 1 \tag{77}$$

so

$$q = 2r\bar{y}-\bar{y}^2+1. \tag{78}$$

Which is positive since $r > \frac{\bar{y}^2-1}{2\bar{y}}$. So period-doubling bifurcation happens if $q = 2r\bar{y}-\bar{y}^2+1$.

Note that the system (69) does not undergo Neimark-sacker bifurcation at $(\bar{y},\bar{y})$.

4.1 Direction of the period-doubling (Flip) bifurcation

In this section we will find the direction of Flip bifurcation of system (69) at $q = 2r\bar{y} - \bar{y}^2 + 1$.

We need at first to shift the fixed point $(\bar{y}, \bar{y})$ to the origin. Let

$$w_n = z_n - \bar{y}, \tag{79}$$

$$u_n = v_n - \bar{y}. \tag{80}$$

System (69) will be

$$w_{n+1} = u_n \tag{81}$$

$$u_{n+1} = \frac{p + q(w_n + \bar{y})}{1 + (u_n + \bar{y})^2 + r(w_n + \bar{y})}, \; n = 0, \; 1, \; 2, \; ... \tag{82}$$

Or

$$Y_{n+1} = AY_n + G(Y_n), \tag{83}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ \frac{q - r\bar{y}}{1+\bar{y}^2+r\bar{y}} & \frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}} \end{pmatrix}, \; Y_n = \begin{pmatrix} w_n \\ u_n \end{pmatrix}, \tag{84}$$

and

$$G(Y) = \frac{1}{2}B(Y,Y) + \frac{1}{6}C(Y,Y,Y) + O(\|Y\|^4) \tag{85}$$

$$B(Y,Y) = \begin{pmatrix} B_1(Y,Y) \\ B_2(Y,Y) \end{pmatrix} \; and \; C(Y,Y,Y) = \begin{pmatrix} C_1(Y,Y,Y) \\ C_2(Y,Y,Y) \end{pmatrix} \tag{86}$$

where

$$B_i(x,y) = \sum_{k,j=1}^{n} \frac{\partial^2 Y_i(\eta)}{\partial \eta_k \partial \eta_j}|_{\eta=0}(x_k y_j) \tag{87}$$

and

$$C_i(x,y,z) = \sum_{l,k,j=1}^{n} \frac{\partial^3 Y_i(\eta)}{\partial \eta_l \partial \eta_k \partial \eta_j}|_{\eta=0}(x_l y_k z_j). \tag{88}$$

So $B_1(\psi,\phi) = 0$ and $C_1(\psi,\phi,\xi) = 0$,

$$B_2(\psi,\phi) = \frac{-2r(q - r\bar{y})}{(1+\bar{y}^2+r\bar{y})^2}(\psi_1\phi_1) + \frac{2\bar{y}(2r\bar{y} - q)}{(1+\bar{y}^2+r\bar{y})^2}(\psi_1\phi_2 + \psi_2\phi_1) + \frac{8\bar{y}^3 - 2(p + q\bar{y})}{(1+\bar{y}^2+r\bar{y})^2}(\psi_2\phi_2), \tag{89}$$

and

$$\begin{aligned} C_2(\psi,\phi,\xi) =& \frac{6r^2(q - r\bar{y})}{(1+\bar{y}^2+r\bar{y})^3}(\psi_1\phi_1\xi_1) + \frac{4r\bar{y}(2q - 3r\bar{y})}{(1+\bar{y}^2+r\bar{y})^3}(\psi_1\phi_1\xi_2 + \psi_1\phi_2\xi_1 + \psi_2\phi_1\xi_1) \\ &+ \frac{2q(r\bar{y} + 3\bar{y}^2 - 1) + 4r(p - 6\bar{y}^3)}{(1+\bar{y}^2+r\bar{y})^3}(\psi_2\phi_2\xi_1 + \psi_2\phi_1\xi_2 + \psi_1\phi_2\xi_2) \\ &+ \frac{20\bar{y}(p + q\bar{y}) - 48\bar{y}^4}{(1+\bar{y}^2+r\bar{y})^3}(\psi_2\phi_2\xi_2). \end{aligned} \tag{90}$$

Now we find the eigenvectors of A and A^T corresponding to the eigenvalue $\lambda = -1$ at the bifurcation point $q = 2r\bar{y} - \bar{y}^2 + 1$.

Let $\hat{q}$ and p^* be the eigenvectors of A and A^T corresponding to the eigenvalue $\lambda = -1$ respectively. So we have

$$A\hat{q} = -\hat{q}, \; and \; A^T p^* = -p^*. \tag{91}$$

Or

$$(A+I)\hat{q}=0 \tag{92}$$

$$(A^T+I)p^*=0. \tag{93}$$

From equation (92) we get

$$\hat{q}\sim\begin{pmatrix}1\\-1\end{pmatrix}. \tag{94}$$

And from equation (93) we get

$$p^*\sim\begin{pmatrix}\frac{r\bar{y}-q}{1+\bar{y}^2+r\bar{y}}\\1\end{pmatrix}. \tag{95}$$

Now, we normalize p^* and $\hat{q}$, take $\hat{p}=\eta\begin{pmatrix}\frac{r\bar{y}-q}{1+\bar{y}^2+r\bar{y}}\\1\end{pmatrix}$, $\eta=\frac{1}{\frac{r\bar{y}-q}{1+\bar{y}^2+r\bar{y}}-1}=-\frac{1+\bar{y}^2+r\bar{y}}{1+\bar{y}^2+q}$.

The critical eigenspace T^c corresponding to $\lambda=-1$ is one-dimensional and spanned by an eigenvector $\hat{q}$. Let T^{su} denote a one-dimensional linear eigenspace of A corresponding to all eigenvalues other than λ. Note that the matrix $(A-\lambda I_n)$ has common invariant spaces with the matrix A, so we conclude that $y\in T^{su}$ if and only if $\langle\hat{p},y\rangle=0$.

So, to find $c(0)$ which is given by the following invariant formula:

$$c(0)=\frac{1}{6}\langle\hat{p},C(\hat{q},\hat{q},\hat{q})\rangle-\frac{1}{2}\langle\hat{p},B(\hat{q},(A-I_n)^{-1}B(\hat{q},\hat{q}))\rangle. \tag{96}$$

We evaluate

$$B(\hat{q},\hat{q})=\begin{pmatrix}0\\\frac{-2r(q-r\bar{y})-4\bar{y}(2r\bar{y}-q)+8\bar{y}^3-2(p+q\bar{y})}{(1+\bar{y}^2+r\bar{y})^2}\end{pmatrix}. \tag{97}$$

$$C(\hat{q},\hat{q},\hat{q})=\begin{pmatrix}0\\\frac{6r^2(q-r\bar{y})}{(1+\bar{y}^2+r\bar{y})^3}-\frac{12r\bar{y}(2q-3r\bar{y})}{(1+\bar{y}^2+r\bar{y})^3}+3\frac{2q(r\bar{y}+3\bar{y}^2-1)+4r(p-6\bar{y}^3)}{(1+\bar{y}^2+r\bar{y})^3}-\frac{20\bar{y}(p+q\bar{y})-48\bar{y}^4}{(1+\bar{y}^2+r\bar{y})^3}\end{pmatrix}. \tag{98}$$

$$\begin{aligned}\langle\hat{p},C(\hat{q},\hat{q},\hat{q})\rangle=&-\left(\frac{1+\bar{y}^2+r\bar{y}}{1+\bar{y}^2+q}\right)\left[\frac{6r^2(q-r\bar{y})}{(1+\bar{y}^2+r\bar{y})^3}-\frac{12r\bar{y}(2q-3r\bar{y})}{(1+\bar{y}^2+r\bar{y})^3}\right.\\&\left.+3\frac{2q(r\bar{y}+3\bar{y}^2-1)+4r(p-6\bar{y}^3)}{(1+\bar{y}^2+r\bar{y})^3}-\frac{20\bar{y}(p+q\bar{y})-48\bar{y}^4}{(1+\bar{y}^2+r\bar{y})^3}\right].\end{aligned} \tag{99}$$

$$(A-I)^{-1}=\begin{pmatrix}-1 & 1\\\frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} & -1+\frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}}\end{pmatrix}^{-1}=\frac{1+\bar{y}^2+r\bar{y}}{2\bar{y}^2}\begin{pmatrix}-1+\frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}} & -1\\-\frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} & -1\end{pmatrix}. \tag{100}$$

$$(A-I)^{-1}B(\hat{q},\hat{q})=\frac{1+\bar{y}^2+r\bar{y}}{2\bar{y}^2}\begin{pmatrix}\frac{2r(q-r\bar{y})+4\bar{y}(2r\bar{y}-q)-8\bar{y}^3+2(p+q\bar{y})}{(1+\bar{y}^2+r\bar{y})^2}\\\frac{2r(q-r\bar{y})+4\bar{y}(2r\bar{y}-q)-8\bar{y}^3+2(p+q\bar{y})}{(1+\bar{y}^2+r\bar{y})^2}\end{pmatrix}. \tag{101}$$

$$B(\hat{q},(A-I_n)^{-1}B(\hat{q},\hat{q}))=\frac{1+\bar{y}^2+r\bar{y}}{2\bar{y}^2}\begin{pmatrix}0\\m\end{pmatrix}, \tag{102}$$

where

$$m=\left(\frac{2r(q-r\bar{y})+4\bar{y}(2r\bar{y}-q)-8\bar{y}^3+2(p+q\bar{y})}{(1+\bar{y}^2+r\bar{y})^2}\right)\left(\frac{-2r(q-r\bar{y})-8\bar{y}^3+2(p+q\bar{y})}{(1+\bar{y}^2+r\bar{y})^2}\right). \tag{103}$$

$$\langle \hat{p}, B(\hat{q},(A-I_n)^{-1}B(\hat{q},\hat{q}))\rangle = ([\frac{2r(q-r\bar{y})+4\bar{y}(2r\bar{y}-q)-8\bar{y}^3+2(p+q\bar{y})}{2\bar{y}^2(1+\bar{y}^2+q)}] \quad [\frac{-2r(q-r\bar{y})-8\bar{y}^3+2(p+q\bar{y})}{(1+\bar{y}^2+r\bar{y})^2}]). \tag{104}$$

If $c(0)>0$, then a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point $q=2r\bar{y}-\bar{y}^2+1$.

4.2 Numerical Results

In this subsection, we use Matlab to give a graph of an example to support our results.

Example 2. Consider the difference equation (21). Fix p, r, and consider q as bifurcation parameter. Take $p=1,\ r=0.9$, and $0<q\le 10$. Equation (21) becomes

$$y_{n+1} = \frac{1+qy_{n-1}}{1+{y^2}_n+0.9y_{n-1}},\ n=0,\ 1,\ 2,\ ... \tag{105}$$

Which is equivalent to

$$\begin{pmatrix} y_1(n+1) \\ y_2(n+1) \end{pmatrix} = \begin{pmatrix} y_2(n) \\ \frac{1+qy_1(n)}{1+{y_2(n)^2}_n+0.9y_1(n)} \end{pmatrix}. \tag{106}$$

The positive equilibrium point $\bar{y}$ of (105) satisfies

$$\bar{y}^3+0.9\bar{y}^2+(1-q)\bar{y}-1=0. \tag{107}$$

Theorem 10 shows that the fixed point undergoes a period-doubling bifurcation at $q^*=1.8\bar{y}-\bar{y}^2+1$. So equation (107) at q^* becomes

$$2\bar{y}^3-0.9\bar{y}^2-1=0. \tag{108}$$

Thus the theoretical fixed point of (105) is

$$\bar{y}=0.97546665.$$

Note that $r=0.9>\frac{\bar{y}^2-1}{2\bar{y}}=-0.0236$, so the condition of Theorem 10 holds. Substituting the value of $\bar{y}$ in q^* we get

$$q^*=1.8043047.$$

Now to determine the direction of period-doubling bifurcation we find $c(0)$.

$$\hat{q}=\begin{pmatrix} 1 \\ -1 \end{pmatrix} \ and\ \hat{p}=(-0.7533482)\begin{pmatrix} -0.2466518 \\ 1 \end{pmatrix}.$$

$$c(0)=\frac{1}{6}\langle \hat{p}, C(\hat{q},\hat{q},\hat{q})\rangle - \frac{1}{2}\langle \hat{p}, B(\hat{q},(A-I_n)^{-1}B(\hat{q},\hat{q}))\rangle.$$

$$\langle \hat{p}, C(\hat{q},\hat{q},\hat{q})\rangle = 1.94576.$$

$$\langle \hat{p}, B(\hat{q},(A-I_n)^{-1}B(\hat{q},\hat{q}))\rangle = 0.0266652827.$$

So

$$c(0)=0.1857633587>0$$

So this shows that a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point $q^*=1.8043047$. Figure (2) shows the stable period-two cycle.

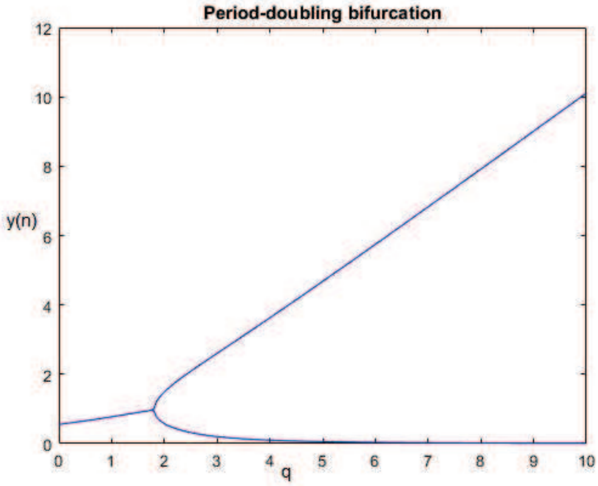

Fig. 2 Period-doubling bifurcation of $y_{n+1} = \frac{1+qy_{n-1}}{1+y^2{}_n+0.9y_{n-1}}$.

5 Summary

In this paper, we consider the second order, quadratic rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{A + Bx^2{}_n + Cx_{n-1}}, \; n = 0, \; 1, \; 2, \; ...$$

With positive parameters α, β, A, B, C, and non-negative initial conditions.
We focus on local stability, invariant intervals, boundedness of the solutions, periodic solutions of prime period two and global stability of the positive fixed points. And we study the types of bifurcation exist where the change of stability occurs. Then, we give some Matlab codes that use these results and numerical discussions with figures to support our results.

The change of variables

$$x_n = \frac{\sqrt{A}}{\sqrt{B}} y_n. \tag{109}$$

reduces equation (18) to the difference equation

$$y_{n+1} = \frac{p + qy_{n-1}}{1 + y^2{}_n + ry_{n-1}}, \; n = 0, \; 1, \; 2, \; ... \tag{110}$$

Where $p = \alpha \frac{\sqrt{B}}{\sqrt{A^3}}$, $q = \frac{\beta}{A}$, and $r = \frac{C}{\sqrt{AB}}$.

We prove the existence of the unique positive equilibrium point of our difference equation, and then we insert a Matlab code to find it.

Then we find the linearized equation and the characteristic equation. And we check when the unique positive equilibrium point $\bar{y}$ of equation (21) is locally asymptotically stable. We investigate also two invariant intervals. And we show that any solution take its values between 0 and $p + \frac{q}{r}$.

Then we set some conditions on q that must hold when two periodic cycle exist. And we give a case for global stability. And we introduce Matlab code that uses our results for finding the fixed point and its stability and solution behavior, and then we insert an example.

Finally, we study the bifurcation of our difference equation. And we concentrate at the Period-Doubling (Flip) Bifurcation and its direction.

References

[1] Alhalawa, M.A. and Saleh, M. (2017), Dynamics of higher order rational difference equation $x_{n+1} = \frac{\alpha+\beta x_n}{A+Bx_n+x_{n-k}}$, *Int. J. Nonlinear Anal. Appl.*, 363-379.

[2] Anisimova, A. and Bula, I. (2014), Some Problems of Second-Order Rational Difference Equations with Quadratic Terms, *International Journal of Difference Equations*, 11-21.

[3] Amleh, A. M., Camouzis, E., and Ladas, G. (2008), On the Dynamics of a Rational Difference Equation, Part 1, *International Journal of Difference Equations* (IJDE), 1-35.

[4] Amleh, A. M., Camouzis, E., and Ladas, G. (2008), On the Dynamics of a Rational Difference Equation, Part 2, *International Journal of Difference Equations* (IJDE), 195-225.

[5] GariT-Demirović, M., Kulenović, M.R.S., and Nurkanović, M. (2013), Global Dynamics of Certain Homogeneous Second-Order Quadratic Fractional Difference Equation, *The Scientific World Journal*.

[6] Jafar, J. and Saleh, M. (2018), Dynamics of nonlinear difference equation $x_{n+1} = \frac{\beta x_n+\gamma x_{n-k}}{A+Bx_n+Cx_{n-k}}$, *J. Appl. Math. Comput*, 493-524.

[7] Saleh, M. and Aloqeili, M. (2005), On the rational difference equation $y_{n+1} = A + \frac{y_{n-k}}{y_n}$, *Appl. Math. Comput.*, **171**, 862-869.

[8] Saleh, M. and Abu-Baha, S. (2006), Dynamics of higher order Rational Difference Equation, *App. Math. Comp.*, **181**(2006), 84-102.

[9] Saleh, M. and Aloqeili, M. (2005), On the rational difference equation $y_{n+1} = A + \frac{y_{n-k}}{y_n}$, *Appl. Math. Comp.*, **171**(2005), 862-869.

[10] Saleh, M. and Aloqeili, M. (2006), On the rational difference equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$, *Appl. Math. Comput.*, **177**, 189-193.

[11] Saleh, M., Alkoumi, N., and Farhat, A. (2017), *On the Dynamics of a Rational Difference Equation* $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-k}}{Bx_n+Cx_{n-k}}$, *Chaos and Soliton*, 76-84.

[12] Saleh, M. and Farhat, A. (2017), *Global Asymptotic Stability of the Higher Order Equation* $x_{n+1} = \frac{ax_n+bx_{n-k}}{A+Bx_{n-k}}$, *J. Appl. Math. Comput*, DOI 10.1007/s12190-016-1029-4, 135-148.

[13] Saleh, M. and Asad, A. (2021), Dynamics of Kth Order Rational Difference Equation, *J. Applied nonlinear Dynamics*, 125-149, DOI:10.5890/JAND.2021.03.008.

[14] Sebdani, R.M. and Dehghan, M. (2006), Global stability of $y_{n+1} = \frac{p+qy_n+ry_{n-k}}{1+y_n}$, *Appl. Math. Comput.*, **182**, 621-630.

[15] Sebdani, R.M. and Dehghan, M. (2006), The study of a class of rational difference equation, *Appl. Math. Comput.*, **179**, 98-107.

[16] Sebdani, R.M. and Dehghan, M. (2006), Dynamics of a non-linear difference equation, *Appl. Math. Comput.*, **178**, 250-261.

[17] Yan, X., Li, W.T., and Zhao, Z. (2006), Global asymptotic stability for a higher order nonlinear rational difference equations, *Appl. Math. Comput.*, **182**, 1819-1831.

[18] Kuznetsov, A.Y. (2007), *Applied Mathematical Sience, Elements of applied bifurcation theory*, 2nd edition *Journal of Mathematical Analysis and Applications*, **331**, 230-239.

[19] Zayed, E.M.E. and El-Moneam, M.A. (2006), On the Rational Recursive Sequence, Egypt.

[20] Kulenovic, M.R.S. and Ladas, G. (2002), *Dynamics of Second Order Rational Difference Equations With Open Problems and Conjectures,* Chapman. Hall/CRC, Boca Raton.

[21] Elaydi, S. (2000), *An introduction to difference equations,* 3rd edition. Springer.

[22] Elaydi, S. *Discrete Chaos With Applications In Science And Engineering*, 2nd edition. Chapman Hall/CRC.

[23] Kulenovi'c, M.R.S. and Ladas, G. (2002), Dynamics of Second Order Rational Difference Equations With Open Problems and Conjectures.

[24] Li, W.T. and Sun, H.R. (2005), Dynamics of a rational difference equation, *Appl. Math. Comput.*, **163**, 577-591.

[25] Salas Einarhille, S.L. (1982), Calculus One and Several Variables, fourth edition.