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Pure and Mixed Nash Equilibria Domain of a Discrete Game Model with Dichotomous Strategy Space

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Abstract—We present a discrete game theoretical model with homogeneous individuals who make simultaneous decisions. In this model the strategy space of all individuals is a discrete and dichotomous set which consists of two strategies. We fully characterize the coherent, split and mixed strategies that form Nash equilibria and we determine the corresponding Nash domains for all individuals. We find all strategic thresholds in which individuals can change their mind if small perturbations in the parameters of the model occurs.

Keywords—Coherent strategy, split strategy, pure strategy, mixed strategy, Nash Equilibrium, game theory.

I. Introduction

GAME theory becomes a pioneer research of interest for many scholars who develop game theory models in order to study how optimal strategies (decisions) have been selected by individuals among other strategies. Ajzen [1] developed the Planned Behavior and Reasoned Action theories in which the measures of perceived behavioral control should contain items that assess self-efficacy and controllability in order to understand and predict the way individuals turn intentions into behaviors. A psychological game theoretical model was developed by Almeida et al. [2] for Reasoned Action and Planned Behavior Theory and a Bayesian-Nash Equilibria are characterized. Brida et al. [3] introduced a game theory model and studied the crowding type effect of individuals over their decisions. This work inspired by the results of Cownley and Wooders [4].

Mousa et al. [5] presented a resort game model and determine the resort's prices that attract all customers and and leave the other resort to go bankruptcy, such prices depends the characteristics of the tourists present in that resorts. The characteristics of individuals have been studied widely in the *Dichotomous Decision Model* [6] introduced by Mousa *et al.* in 2011. This game model has two types of individuals who can make the decision yes or no and can influence the decisions of others. In addition, individuals make decisions according to their preferences. The preferences have an interesting feature by taking into account not only how much the individuals like or dislike a certain decision but also the decisions made by the other individuals .

Making an appropriate decision helps in choosing the optimal strategies when facing a certain optimization problems. For instance, (i) to find the optimal strategies for the management of an economic agent's consumption from a basket of K goods that may become unavailable for consumption from some random time onwards, see [7]; (ii) or to find the optimal consumption, investment and life-insurance selection and purchase strategies for a wage-earner with an uncertain lifetime, see [8].

Resorting to the dichotomous decision model, Mousa and Pinto [9] constructed all the decision tilings whose axes reflect the personal preferences of the individuals to make decision yes or to make the decision no. They show how these tilings include geometrically all the pure and mixed Nash equilibria. In [10], Mousa et al. introduced a game theory model and studied the impact and repercussion of the individual decisions in a competitive market perspective.

Recently, Mousa et al. [11] presented an envy behavioral game theoretical model with two types of homogeneous players. They studied the influence of the envy behavior parameters on the Cartesian position of the equilibria. A modified version of the dichotomous decision model is presented in [12], where all strategies (decisions) that form pure and mixed Nash equilibria are characterized taking into account the way individuals influence the decisions of others. In this paper, they show how Nash equilibria form degenerated hysteresis with respect to the replicator dynamics, with the property that the pure Nash equilibria are asymptotically stable and the strict mixed equilibria are unstable. This results in the observation of the existence of limit cycles for the dynamics associated to situations where the individuals keep changing their decisions with time, but exhibiting a periodic and attracting repetition in their decisions.

A pure strategy is cohesive if all the individuals of same type make the same decision. Soeiro et al. [13] show that individuals with same type can make different decisions at certain Nash equilibria, mainly when individuals are characterized according to their valuation type, externality type and crowding type. Furthermore, they show how positive externalities lead to type symmetries in the set of Nash equilibria, while negative externalities allow the existence of equilibria that are not type-symmetric.

In this paper, we will introduce a simplified version of the game decision model [12] by considering a discrete game model with homogenous individuals who make simultaneous decisions. The strategy space is visible to all individuals and contains two possibilities. We will determine all pure and mixed Nash equilibria strategies and the find the corresponding

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Nash intervals for all individuals. Finally, we will show how the order of the *strategic thresholds* can preserve certain decision for all individuals and so may keep them united. However, small perturbations in the parameters of the model allow individuals to make different decisions, and so may divide the community.

This paper is organized as follow: In Section II, we introduce the model set up. In Section III, we characterize the pure coherent strategies that form Nash equilibria and determine the bifurcated coherent thresholds. In Section IV, we study the pure split strategies that form Nash equilibria. In Section V, we determine the mixed strategies that form Nash equilibria. We conclude in Section VI.

II. The Model Setup

The model has $m \geq 2$ homogenous individuals

$$i \in \mathbf{I} = \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{m}\} \ .$$

Each individual has to make one decision

$$d \in \mathbf{D} = \{d_1, d_2\} \ .$$

Note that we can consider the case where a single individual makes m decisions, or we can also consider a mixed model using these two possibilities. We define the *preference decision vector*

$$(\omega^{d_1} \quad \omega^{d_2})$$

whose *coordinates* $\omega^d \in \mathbb{R}$ indicate how much an individual likes $\omega^d > 0$, or dislikes $\omega^d < 0$, or indifference $\omega^d = 0$ to make decision $d \in \mathbf{D}$. We define the *preference neighbors vector*

$$(\alpha^{d_1} \quad \alpha^{d_2})$$

whose *coordinates* $\alpha^d \in \mathbb{R}$ indicate how much an individual likes $\alpha^d > 0$, or dislikes $\alpha^d < 0$, or indifference $\alpha^d = 0$ to be with other individuals making decision $d \in \mathbf{D}$. We describe the individuals' decision by a *strategy map*

$$S : \mathbf{I} \to \mathbf{D}$$

that associates to each individual $i \in \mathbf{I}$ his decision $S(i) \in \mathbf{D}$. Let $\mathbf{S}$ be the space of all strategies S. Given a strategy $S \in \mathbf{S}$, let $\mathcal{O}_S$ be the *strategic occupation vector*

$$\mathcal{O}_S = (l^{d_1} \quad l^{d_2})$$

whose coordinates $l^d = l^d(S)$ indicate the number of individuals who make decision $d \in \mathbf{D}$ under strategy S. Hence, for a given a strategy $S \in \mathbf{S}$, let

$$l^{d_1}(S) = l \tag{1}$$

be the number of individuals who make the decision d_1. Hence,

$$l^{d_2}(S) = m - l^{d_1}(S) = m - l \tag{2}$$

is the number of individuals who make the decision d_2. Let $\mathbf{O}$ be the *occupation set* defined by

$$\mathbf{O} = \{l : l \in \{0, 1, 2, \ldots, m\}\}.$$

Given a strategy $S \in \mathbf{S}$, the *utility* $U_i(S)$ of an individual $i \in \mathbf{I}$ is then given by

$$U_i(S(i); l^d(S)) \ , \quad d \in \mathbf{D} \ .$$

Let $U_i : \mathbf{D} \times \mathbf{O} \to \mathbb{R}$ be the *pure utility function* for any individual $i \in \mathbf{I}$ given by

$$U_i(S(i); l) = \begin{cases} \omega^{d_1} + \alpha^{d_1}(l^{d_1} - 1), & S(i) = d_1 \ , \\ \omega^{d_2} + \alpha^{d_2}(l^{d_2} - 1), & S(i) = d_2 \ . \end{cases} \tag{3}$$

Definition 1: A strategy $S^* \in \mathbf{S}$ defined by

$$S^* : \mathbf{I} \to \mathbf{D}$$

is a *Nash equilibrium* if and only if for every individual $i \in \mathbf{I}$ and for every strategy $S \in \mathbf{S}$, we have

$$U_i(S^*) \geq U_i(S) \ .$$

III. Coherent Nash Equilibrium Strategies

In this section, we will study the Nash domain intervals for all coherent strategies $S \in \mathbf{S}$ that form Nash equilibria.

Definition 2: A *coherent strategy* is a strategy in which all individuals prefer to make the same decision $d \in \mathbf{D}$.
A *coherent strategy* $C \in \mathbf{S}$ is described by a map

$$C : \mathbf{I} \to \mathbf{D}$$

that indicates for every individual $i \in \mathbf{I}$ his coherent decision $C(i) \in \mathbf{D}$. We observe that there are two distinct *coherent strategies*:

(i) d_1^c *coherent strategy*

$$C_m : \mathbf{I} \to \{\mathbf{d_1}\}$$

in which all individuals make the decision d_1, i.e $l = m$;

(ii) d_2^c *coherent strategy*

$$C_0 : \mathbf{I} \to \{\mathbf{d_2}\}$$

in which all individuals make the decision d_2, i.e $l = 0$.

We now define the *difference decision parameter* which plays a major role in classifying the equilibria.

Definition 3: The *difference decision parameter* of the individuals is defined by

$$x = \omega^{d_1} - \omega^{d_2} \ . \tag{4}$$

Independently from the influence of the other individuals: if $x > 0$, then individuals prefer to decide d_1; if $x = 0$, then individuals are indifferent to decide d_1 or d_2; and if $x < 0$, then individuals prefer to decide d_2.

We now introduce one of the main results.

Lemma 1: The *coherent strategy* d_1^c is *Nash Equilibrium* if and only if

$$x \geq -\alpha^{d_1}(m-1)$$

and the *coherent strategy* d_2^c is *Nash Equilibrium* if and only if

$$x \leq \alpha^{d_2}(m-1) \ .$$

Proof: Let $C \in \mathbf{S}$ be a *coherent Nash Equilibrium strategy*. All individuals prefer to be together making decision d_1 if and only if

$$U_i(d_1, m) \geq U_i(d_2, 1) \ , \quad i \in \mathbf{I} \ .$$

Substituting the utility functions of U_i from (3), we obtain

$$\omega^{d_1} + \alpha^{d_1}(m-1) \geq \omega^{d_2} \ .$$

Rearranging the terms using (4), we get

$$x \geq -\alpha^{d_1}(m-1) \ ,$$

which includes the proof of the first part.

To prove the second part, note that all individuals prefer to be together making decision d_2 if and only if

$$U_i(d_2, m) \geq U_i(d_1, 1) \ .$$

Substituting the utility functions of U_i from (3), we obtain

$$\omega^{d_2} + \alpha^{d_2}(m-1) \geq \omega^{d_1} \ .$$

Rearranging the terms using (4), we get

$$x \leq \alpha^{d_2}(m-1) \ .$$

This includes the proof of the second part. ■

We now introduce the following definition for the *coherent threshold*.

Definition 4: The *coherent thresholds* $C(d_1^c)$ and $C(d_2^c)$ for the *strategies* d_1 and d_2 are, respectively, defined by

$$\begin{aligned} C(d_1^c) &= -\alpha^{d_1}(m-1) \quad \text{and} \\ C(d_2^c) &= \alpha^{d_2}(m-1) \ . \end{aligned} \tag{5}$$

We proceed to define the *decision parameter* $A \in \mathbb{R}$ which measures the influence of the individual's decisions over each other.

Definition 5: The *decision parameter* is defined by

$$A = \alpha^{d_1} + \alpha^{d_2} \ . \tag{6}$$

The *decision parameter* A determines the orders for the *coherent thresholds* $C(d_1^c)$ and $C(d_2^c)$. In Fig. 1, we order the *coherent thresholds* $C(d_1^c)$ and $C(d_2^c)$ when the *decision parameter* $A < 0$.

Fig. 1 Ordering the *coherent thresholds* when $A < 0$

Definition 6: The *Nash Equilibria domain* $N(S)$ of a strategy $S \in \mathbf{S}$ is the set of all *difference decision parameter* x for which S is a Nash Equilibrium.

To characterize the Nash domain intervals, we now introduce the following result.

Lemma 2: Assume $C \in \mathbf{S}$ is *coherent Nash Equilibrium* strategy. The *Nash Equilibria domain* of the *coherent strategy* d_1^c is the interval of all *difference decision parameter* x for which

$$x \in I(d_1^c) = [C(d_1^c), \infty)$$

and the *Nash Equilibria domain* of the *coherent strategy* d_2^c is the interval of all *difference decision parameter* x for which

$$x \in I(d_2^c) = (-\infty, C(d_2^c)] \ ,$$

where the *coherent thresholds* $C(d_1^c)$ and $C(d_2^c)$ are as given in (5).

Proof: Let $C \in \mathbf{S}$ be *coherent Nash Equilibrium* strategy. By Lemma 1 the *coherent strategy* d_1^c is *Nash Equilibrium* if and only if

$$x \geq -\alpha^{d_1}(m-1) \ .$$

Hence, the *Nash Equilibria domain* of the *coherent strategy* d_1^c is the right segment

$$x \in [C(d_1^c), \infty) \ .$$

Similarly, the *coherent strategy* d_2^c is *Nash Equilibrium* if and only if

$$x \leq \alpha^{d_2}(m-1) \ .$$

Hence, the *Nash Equilibria domain* of the *coherent strategy* d_2^c is the left segment

$$x \in (-\infty, C(d_2^c)] \ .$$

■

The representations of the *coherent Nash domains intervals* $I(d_1^c)$ and $I(d_1^c)$ along the horizontal axis determine the coherent decision intervals. The intersections of the *coherent Nash domains intervals* $I(d_1^c)$ and $I(d_1^c)$ are determined by the way the *coherent thresholds* $C(d_1^c)$ and $C(d_2^c)$ are ordered along the horizontal axis. We observe that the order of the *coherent thresholds* are independent of each other.

Lemma 3: Assume that the *decision parameter* $A < 0$. Then there are no *coherent Nash equilibria strategies* for every

$$x \in (C(d_2^c), C(d_1^c)) \ .$$

Proof: Note that the *decision parameter* $A < 0$ if and only if

$$C(d_2^c) < C(d_1^c) \ .$$

Therefore, the *coherent strategy* d_1^c is unique Nash Equilibrium if and only if

$$x \geq C(d_1^c)$$

and the *coherent strategy* d_2^c is unique Nash Equilibrium if and only if

$$x \leq C(d_2^c) \ .$$

Hence, there are no *coherent Nash equilibria strategies* for every

$$x \in (C(d_2^c), C(d_1^c)) \ .$$

■

In the following result, we study the case where the *decision parameter* A takes positive values.

Lemma 4: Assume the decision parameter is such that $A > 0$. Then there exist a *coherent Nash equilibria strategies* for every $x \in \mathbb{R}$.

Proof: Note that the *decision parameter* $A > 0$ if and only if

$$C(d_1^c) < C(d_2^c) \ .$$

The *coherent strategy* d_1^c is unique Nash Equilibrium if and only if

$$x \geq C(d_1^c)$$

and the *coherent strategy* d_2^c is unique Nash Equilibrium if and only if

$$x \leq C(d_2^c) \ .$$

Moreover, the two *coherent strategies* d_1^c and d_2^c are Nash equilibria for every

$$x \in [C(d_1^c), C(d_2^c)] \ .$$

■

In Fig. 2, we order the *coherent thresholds* $C(d_1^c)$ and $C(d_2^c)$ along the horizontal axis when the decision parameter $A > 0$.

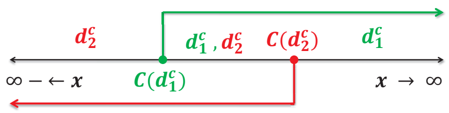

Fig. 2 Overlapping the *coherent thresholds* when $A > 0$

Definition 7: If the *coherent thresholds* $C(d_1^c)$ and $C(d_2^c)$ are coincide, then the *coherent thresholds* are in *bifurcation* position.

Now we study the case where the *decision parameter* A has no influence.

Lemma 5: Assume the *decision parameter* $A = 0$. Then there exist a *unique coherent Nash equilibrium* for every

$$x \in \mathbb{R} \setminus \{C(d_1^c), C(d_2^c)\} \ .$$

Proof: Note that the *decision parameter* $A = 0$ if and only if

$$C(d_1^c) = C(d_2^c) \ .$$

Hence, the *coherent strategies* d_1^c and d_2^c are Nash equilibria only at the *bifurcation* point where

$$x = C(d_1^c) = C(d_2^c) \ .$$

■

As an illustration of Lemma 5, the *coherent thresholds* $C(d_1^c)$ and $C(d_2^c)$ coincide when the decision parameter $A = 0$ (see Fig. 3).

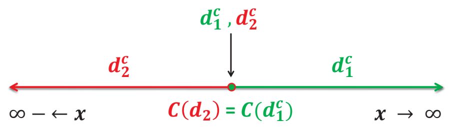

Fig. 3 Bifurcation of the *coherent thresholds* when $A = 0$

IV. Split Nash Equilibrium Strategies

We remark that the *coherent Nash equilibria* requires the group of individuals to be united in their decisions. However, in the this section this will not be the case as the community will be divided in their decisions. Here we will study the *split strategies* or the *no-coherent strategies*.

Definition 8: A *split strategy* $S \in \mathbf{S}$ is a strategy in which individuals prefer to make different decisions.

The *split strategy* is then a strategy in which the individuals split between the two decisions. Recall that $l = l^{d_1}$ *strategy* refers to the number of individuals who make decision d_1 as defined in (1). Therefore, $m - l = l^{d_2}$ refers to the number of individuals who make decision d_2. Note that, a necessary condition for the individuals to split between the two decisions is

$$l \in \{1, 2, \ldots, m-1\} \ .$$

Hence, there are $m - 1$ possibilities for the split strategies. Our aim in this section is to characterize all split strategies that form Nash equilibria by determining the *necessary and sufficient* conditions which guarantee the existence of *split Nash equilibria* strategies.

Definition 9: Let l be as defined in (1) for a given strategy $S \in \mathbf{S}$. The *left split threshold* $S_L(l^{d_1})$ for the strategy d_1 is defined by

$$\begin{aligned} S_L(l) &= S_L(l^{d_1}) \\ &= -\alpha^{d_1}(m-1) + (\alpha^{d_1} + \alpha^{d_2})(m - l^{d_1}) \quad (7) \\ &= -\alpha^{d_1}(m-1) + Al^{d_2} \end{aligned}$$

and the *right split threshold* $S_R(l^{d_2})$ for the strategy d_2 is defined by

$$\begin{aligned} S_R(l^{d_2}) &= S_R(m-l) \\ &= \alpha^{d_2}(m-1) - (\alpha^{d_1} + \alpha^{d_2})(m - l^{d_2}) \\ &= \alpha^{d_2}(m-1) - Al^{d_1} \quad (8) \\ &= S_R(l^{d_1}) \ , \end{aligned}$$

where the preference decision A is as given in (6).

The connection between the *left split threshold* (7) and the *right split threshold* (8) is presented in the following result.

Lemma 6: Given $S \in \mathbf{S}$. For all $l^{d_1} \in \{1, 2, \ldots, m-1\}$ we have

$$S_R(l^{d_1} - 1) = S_L(l^{d_1}) \ .$$

Furthermore,

$$S_R(l^{d_2}) - S_L(l^{d_1}) = -A \ ,$$

where the preference decision A is as given in (6).

Proof: To prove the first part, we start with the *right split threshold* (8) as follows

$$S_R(l^{d_1} - 1) = \alpha^{d_2}(m-1) - A(l^{d_1} - 1) \ .$$

Substituting the identity (2) we obtain

$$S_R(l^{d_1} - 1) = \alpha^{d_2} m - \alpha^{d_2} - A(m - l^{d_2}) + A \ .$$

Using the definition of the decision preference A given in (6) and rearranging terms we get

$$S_R(l^{d_1} - 1) = \alpha^{d_2} m - \alpha^{d_2} - (\alpha^{d_1} + \alpha^{d_2})m + A l^{d_2} + A \ .$$

Rearranging terms we obtain

$$\begin{aligned} S_R(l^{d_1} - 1) &= -\alpha^{d_1} m + \alpha^{d_1} + A l^{d_2} \\ &= S_L(l^{d_1}) \ , \end{aligned}$$

where the *left split threshold* $S_L(l^{d_1})$ is as given in (7). To prove the second part, we remark that the difference between each step (when one individual changes his decision) is the negative of decision parameter, $-A$, i.e

$$\begin{aligned} S_R(l^{d_1}) - S_L(l^{d_2}) &= A(m-1) - Am \\ &= -A \ , \end{aligned}$$

which includes the proof. ∎

As an illustration of Lemma 6, we show in Fig. 4 the process of dividing the group of all individuals between the two decisions d_1 and d_2.

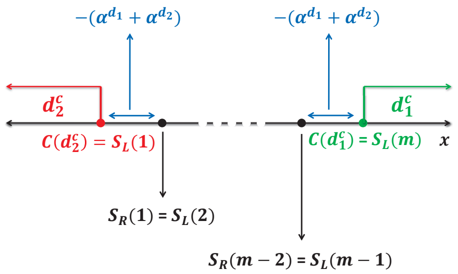

Fig. 4 Split Nash domain when $A < 0$

In the following result we determine the necessary conditions for the split strategy to be Nash equilibrium.

Lemma 7: Let S be Nash Equilibrium strategy. If $A > 0$ then the individuals can not be splitted between the strategies d_1 and d_2.

Proof: By contradiction. Assume that S is *split Nash equilibrium strategy*. Let $l^{d_1} = l^{d_1}(S)$ and $l^{d_2} = l^{d_2}(S)$. We note that

$$\begin{aligned} U(d_1; l^{d_1} + 1) &= \alpha^{d_1} + U(d_1; l^{d_1}) \quad \text{and} \\ U(d_2, l^{d_2} + 1) &= \alpha^{d_2} + U(d_2, l^{d_2}) \ . \end{aligned} \tag{9}$$

Since the strategy S is *split Nash equilibrium*, we have that

$$\begin{aligned} U(d_2, l^{d_2} + 1) &\leq U(d_1; l^{d_1}) \quad \text{and} \\ U(d_1; l^{d_1} + 1) &\leq U(d_2, l^{d_2}) \ . \end{aligned} \tag{10}$$

By substituting equalities from (9) in the inequalities of (10), we obtain that

$$\begin{aligned} \alpha^{d_1} + \alpha^{d_2} + U(d_2, l^{d_2}) &\leq U(d_1; l^{d_1} + 1) \ , \\ \alpha^{d_1} + \alpha^{d_2} + U(d_2, l^{d_2}) &\leq U(d_2, l^{d_2}). \end{aligned}$$

Using (10) and rearranging the terms we get

$$\alpha^{d_1} + \alpha^{d_2} \leq 0 \ .$$

This gives $A \leq 0$ which contradicts our previous assumption $A > 0$. ∎

Now we will determine the values of the *difference decision parameter* x in which the *split strategy* $S \in \mathbf{S}$ form a Nash Equilibrium.

Lemma 8: Assume $S \in \mathbf{S}$ is a *split strategy*. Then, S is Nash equilibrium strategy if and only if the *difference decision parameter* x is such that

$$x \in [S_L(l^{d_1}), S_R(l^{d_2})] \ .$$

Proof: Let $S \in \mathbf{S}$ be *split Nash equilibrium strategy*. If the individual $i \in \mathbf{I}$ decides d_1, then the individual i does not like to change his decision to d_2, i.e.

$$U_i(d_1, l) \geq U(d_2, l^{d_2} + 1) \ .$$

Similarly, if the individual $i \in \mathbf{I}$ decides d_2, then the individual i does not like to change his decision to d_1, i.e.

$$U_i(d_1, l + 1) \leq U(d_2, l^{d_2}) \ .$$

Substituting the utility functions (3) in the last inequality and rearranging the terms, we obtain

$$\begin{aligned} \omega^{d_1} + \alpha^{d_1}(l^{d_1} - 1) &\geq \omega^{d_2} + \alpha^{d_2} l^{d_2} \ , \\ \omega^{d_1} + \alpha^{d_1} l^{d_1} &\leq \omega^{d_2} + \alpha^{d_2}(l^{d_2} - 1) \ . \end{aligned} \tag{11}$$

Hence, the strategy S is Nash Equilibrium if and only if the two inequalities in (11) are satisfied. Substituting the identities (1) and (2) above, we obtain

$$\omega^{d_1} + \alpha^{d_1}(l-1) \geq \omega^{d_2} + \alpha^{d_2}(m-l)$$

$$\omega^{d_1} + \alpha^{d_1} l \leq \omega^{d_2} + \alpha^{d_2}(m-l-1).$$

Rearranging the last two inequalities using the *difference decision parameter* from (4), we get

$$\begin{aligned} x &\geq -\alpha^{d_1}(m-1) + (\alpha^{d_1} + \alpha^{d_2})(m - l^{d_1}) \ , \\ x &\leq \alpha^{d_2}(m-1) - (\alpha^{d_1} + \alpha^{d_2})(m - l^{d_2}) \ . \end{aligned}$$

Resorting to the two identities defined in (7) and (8), the last two inequalities become, respectively

$$\begin{aligned} x &\geq S_L(l^{d_1}) \ , \\ x &\leq S_R(l^{d_2}) \ . \end{aligned}$$

Joining the above two inequalities, we obtain that the strategy S is Nash Equilibrium if and only if

$$S_L(l^{d_1}) \leq x \leq S_R(l^{d_2}) \ ,$$

which concludes the proof. ■

Definition 10: Assume $A < 0$. The *relative split threshold* is defined by

$$G(x) = (-x + \alpha^{d_2}(m-1))/A \tag{12}$$

and the *matching split threshold* is defined by

$$H(l) = \alpha^{d_2}(m-1) - l^{d_1}A \ . \tag{13}$$

Let us define also $Z(x)$ by

$$Z(x) = x - \alpha^{d_2}m - \alpha^{d_1} \ . \tag{14}$$

We observe that the map $G(x)$ is an increasing affine functions in x with the property that

$$\begin{aligned} G(C(d_2^c) - \alpha^{d_2}) &= 0 \quad \text{and} \\ G(C(d_1^c)) &= m-1 \ . \end{aligned}$$

For a given *split strategy* $S \in \mathbf{S}$, we will show how the *split Nash Equilibrium strategy* $l = l^{d_1}$ can be related to the identities $G(x)$, $H(l)$ and $Z(x)$ defined in (12), (13) and (14), respectively.

Lemma 9: Assume that the *decision parameter* $A < 0$. Given the *split strategy* $S \in \mathbf{S}$. The *split strategy* $l = l^{d_1}$ is a Nash equilibium if and only if

$$x \in [H(l) + A, H(l)] \ .$$

Furthermore, the *split strategy* S is Nash equilibrium if and only if

$$l^{d_1}(S) \in [G(x), G(x)+1] \ .$$

where $G(x)$ is as given in (12) and $H(l)$ is as given in (13).

Proof: Using Lemma 8, the *split strategy* $S \in \mathbf{S}$ is Nash Equilibrium if and only if

$$x \in [S_L(l^{d_1}), S_R(l^{d_2})] \ .$$

We rewrite the last interval in terms of the following two inequalities

$$\begin{aligned} x &\geq S_L(l^{d_1}) \quad \text{and} \\ x &\leq S_R(l^{d_2}) \ . \end{aligned}$$

Substituting the *left split threshold* (7) and the *right split threshold* (8) above we obtain

$$\begin{aligned} x &\geq -\alpha^{d_1}(m-1) + Al^{d_2} \quad \text{and} \\ x &\leq \alpha^{d_2}(m-1) - Al^{d_1} \ . \end{aligned} \tag{15}$$

Rearranging the two inequalities in (15), we get

$$H(l) + A \leq x \leq H(l)$$

and we conclude the first part. To prove the second part, we add the term $-\alpha^{d_2}m - \alpha^{d_1}$ to both sides of the above inequalities, we obtain

$$\begin{aligned} Z(x) + Al^{d_1} &\geq 0 \quad \text{and} \\ Z(x) + A + Al^{d_1} &\leq 0 \ , \end{aligned} \tag{16}$$

where $Z(x)$ is as given in (14). Noting that $l = l^{d_1}(S) \in \{1, 2, \ldots, m-1\}$ and $A < 0$, one can rearrange the inequalities in (16) to get

$$\begin{aligned} 1 \leq \quad l^{d_1}(S) &\leq \frac{-Z(x)}{A} \quad \text{and} \\ m-1 \geq \quad l^{d_1}(S) &\geq \frac{-Z(x) - A}{A} \ . \end{aligned} \tag{17}$$

Hence, the strategy S is Nash Equilibrium if and only if the inequalities in (17) are satisfied. Note that from the definition of $G(x)$ in (12) and $Z(x)$ in (14), we conclude that

$$G(x) = \frac{-Z(x)}{A} - 1 \ .$$

Hence, from the inequalities in (17) we can see that the *split strategy* S is Nash equilibrium if and only if

$$l^{d_1}(S) \in [G(x), G(x)+1]$$

and we conclude the proof of the second part. ■

V. Mixed Strategies

Recall that the *pure strategies* are either *coherent* as we have seen in Section III or *split* as we have seen in Section IV. More generally, individuals may decide with probability, so we describe the mixed decision of the individuals by the *mixed strategy map*

$$S : \mathbf{I} \to [\mathbf{0}, \mathbf{1}]$$

that associates to each individual $i \in \mathbf{I}$ the probability

$$p_i = S(i) \in [0,1]$$

to decide d_1. Hence, each individual $i \in \mathbf{I}$ decides d_2 with probability

$$1 - p_i = 1 - S(i) \ .$$

We assume that the decisions of the individuals are taken independently. Define the following values

$$\begin{aligned} P &= \sum_{i=1}^{m} p_i \\ P_i &= P - p_i \ , \quad i \in \mathbf{I} \ . \end{aligned}$$

For every individual $i \in \mathbf{I}$, we define the d_1*-fitness function*

$$f_{d_1} : [0,1] \times [0,m] \to \mathbb{R}^+$$

by

$$f_{d_1}(p_i; P) = \omega^{d_1} + \alpha^{d_1} P_i \ ; \tag{18}$$

and we define the d_2*-fitness function*

$$f_{d_2} : [0,1] \times [0,m] \to \mathbb{R}^+$$

by

$$f_{d_2}(p_i; P) = \omega^{d_2} + \alpha^{d_2}(m - 1 - P_i) \ . \tag{19}$$

For every individual $i \in \mathbf{I}$, we define the mixed utility function

$$U : [0,1] \times [0,m] \to \mathbb{R}^+$$

by

$$U_1(p_i; P) = p_i \ f_{d_1}(p_i; P,) + (1 - p_i) \ f_{d_2}(p_i; P) \ . \tag{20}$$

Note that for every $i \in \mathbf{I}$, if $p_i \in \{0,1\}$, then every individual decides the pure strategy d_1 with $p_i = 1$ and decides the pure strategy d_2 with $p_i = 0$. In case where $p_i \in \{0,1\}$, the *mixed utility function* defined in (20) coincides with the *pure utility function* defined in (3).

Definition 11: A strategy

$$S^* : \mathbf{I} \to [\mathbf{0},\mathbf{1}]$$

is a *mixed Nash equilibrium* if and only if

$$U_i(S^*) \geq U_i(S)$$

for every individual $i \in \mathbf{I}$ and for every strategy $S \in \mathbf{S}$.

We now introduce the following result.

Lemma 10: Let $S : \mathbf{I} \to [\mathbf{0},\mathbf{1}]$ be a mixed Nash equilibrium. If $0 < p_i < 1$, then the *difference decision parameter* is given by

$$x = -A(P - p_i) + C(d_2^c) \ ,$$

where the *coherent threshold* $C(d_2^c)$ is as given in (5). Hence, if $A \neq 0$, then there is no mixed Nash equilibrium with the property that $0 < p_i \neq p_j < 1$. This is because all individuals are homogeneous.

Proof: Let $S : \mathbf{I} \to [\mathbf{0},\mathbf{1}]$ be a mixed Nash equilibrium. Hence, for every $p \in [0,1]$, we must have

$$U(p_i; P) \geq U(p; P - p_i + p) \ .$$

Since $S \in \mathbf{S}$ is mixed Nash equilibrium, it follows that when $0 < p_i < 1$ we must have

$$f_{d_1}(p_i; P) = f_{d_2}(p_i; P) \ .$$

Substituting the fitness function f_{d_1} from (18) and the fitness function f_{d_2} from (19) we get

$$\omega^{d_1} + \alpha^{d_1} P_i = \omega^{d_2} + \alpha^{d_2}(m - 1 - P_i) \ .$$

Rearranging the last identity we obtain

$$\begin{aligned} \omega^{d_1} - \omega^{d_2} &= \alpha^{d_2}(m-1) - (\alpha^{d_1} + \alpha^{d_2})P_i \\ &= C(d_2^c) - AP_i \ . \end{aligned}$$

Resorting to the *difference decision parameter* (4) we get

$$x = -A(P - p_i) + C(d_2^c) \ ,$$

which concludes the proof. ■

VI. Conclusion

We have studied a simplified game decision model for group of homogenous individuals whose strategy space is the discrete set with two alternatives. We have characterized the coherent strategies that form Nash equilibria and determined the corresponding coherent Nash intervals for all individuals, which explains how individuals preserve together their decisions, and thus may keep the community united. We have characterized the split strategies that form Nash equilibria and determined the corresponding split Nash intervals for all individuals, which explains how the community becomes divided if the *difference decision parameter* of the individuals x passes some split thresholds. Finally, we have characterized the mixed strategies that form Nash equilibria and determined the corresponding mixed Nash intervals for all individuals, which generalize all pure Nash strategies.

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