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Neimark-Sacker bifurcation of a fourth order difference equation

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In this article, we study stability and bifurcation of a fourth order rational difference equation. We give condition for local stability, and we show that the equation undergoes a Neimark-Sacker bifurcation. Moreover, we consider the direction of the Neimark-Sacker bifurcation. Finally, we numerically validate our analytical results.

KEYWORDS

equilibrium, Neimark-Sacker bifurcation, rational difference equation, stability

1 | INTRODUCTION

In addition to their importance in their own right, difference equations have important applications in different fields including, but not limited to, economics, biology, and physics. Consequently, this field of research attracts an increasing number of researchers. As closed form of solutions to nonlinear difference, equations are difficult to obtain in most cases, there is a lot of work on the qualitative behavior of solutions of rational difference equations.

Camouzis et al[1] gave an analytical description of the local stability of the positive equilibrium point of the rational difference equation

$$x_{n+1} = \frac{\sigma x_{n-2} + x_{n-3}}{A + x_{n-3}}, \tag{1}$$

with positive parameters σ and A and nonnegative initial conditions. Moreover, the authors investigated the global attractivity of the positive equilibrium point and proved that the positive fixed point is locally stable if $\sigma^3 + \sigma^2 - (2A^2 + 4A + 2)\sigma + A^3 + A^2 - A - 1 < 0$ and unstable if $\sigma^3 + \sigma^2 - (2A^2 + 4A + 2)\sigma + A^3 + A^2 - A - 1 > 0$. Moreover, if $A - 1 < \sigma \leq A + 1$ then every positive solution of Equation 1 converges to the positive equilibrium point.

Zhang and Ding[2] studied the existence and direction of Neimark-Sacker bifurcation of the same equation. The authors proved that if $\sigma > A - 1$ and if σ satisfies $\sigma^3 + \sigma^2 - (2A^2 + 4A + 2)\sigma + A^3 + A^2 - A - 1 = 0$ then Neimark-Sacker bifurcation occurs.

Camouzis[3] studied the global character of solutions of the third order rational difference equation

$$x_{n+1} = \frac{\beta x_n + \delta x_{n-2}}{A + Bx_n + Cx_{n-1}}, \tag{2}$$

where the parameters β, δ, A are nonnegative $\beta + \delta > 0, B, C > 0$ and the initial conditions x_{-2}, x_{-1}, x_0 are nonnegative real numbers. Using an appropriate change of variables, Equation 2 becomes

$$x_{n+1} = \frac{\beta x_n + x_{n-2}}{A + Bx_n + x_{n-1}}, \tag{3}$$

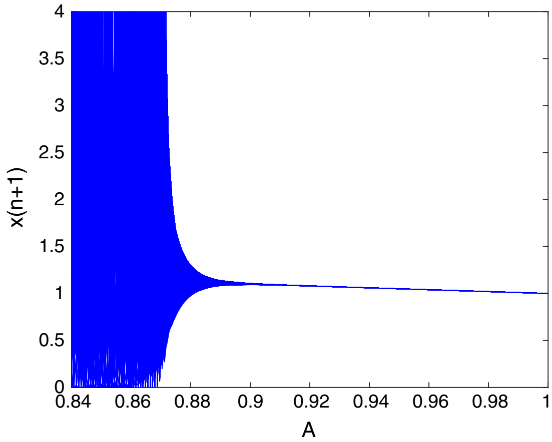

FIGURE 1 Bifurcation diagram of Equation 5 in (A,X) plane [Colour figure can be viewed at wileyonlinelibrary.com]

where $A \geq 0, \beta > 0, B > 0$. The author concentrated on studying the boundedness of solutions of Equation 3. In the same line of research, He and Qiu[4] investigated the existence of Neimark-Sacker bifurcation of the third order difference equation

$$x_{n+1} = \frac{\beta x_n + \alpha x_{n-2}}{1 + x_{n-1}}, \tag{4}$$

with positive parameters α, β and nonnegative initial conditions x_{-2}, x_{-1}, x_0. It has been shown in Camouzis and Ladas[5] that the unique positive equilibrium $x^* = \alpha + \beta - 1, \alpha + \beta > 1$ is locally asymptotically stable when $\beta > \beta^*$ and unstable when $\beta < \beta^*$ where $\beta^* = (\alpha^2 - \alpha)/(\alpha + 1)$. The authors in He and Qiu[4] proved the existence of Neimark-Sacker bifurcation for Equation 4 as β passes through the critical value β^*.

Motivated by the above work, we consider the fourth order rational difference equation

$$X_{n+1} = \frac{\beta X_n + X_{n-3}}{A + X_{n-1}}, \tag{5}$$

with positive parameters β and A and nonnegative initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0$. In Camouzis and Ladas,[5] the authors conjectured that the difference equation

$$x_{n+1} = \frac{\beta x_n + \varepsilon x_{n-3}}{A + C x_{n-1}}$$

has unbounded solutions in some range of its parameters and for some initial conditions. We will not prove this conjecture in this paper. Equation 5 is a special case of the previous equation when $\epsilon = C = 1$. Numerical simulations showed that for the range of parameter $A < A^*$ in Figure 1, the solution of Equation 5 is unbounded. This resulted in the absence of parabolic shape near the bifurcation value. Notice that, in Equation 1, x_{n+1} depends on x_{n-2} and x_{n-3} only, whereas in our Equation 5, X_{n+1} depends on X_n, X_{n-1} and X_{n-3}. Although the calculation involves the same steps to prove the existence of Neimark-Sacker bifurcation, including the proof of existence of a complex conjugate pair of eigenvalues of modulus one and studying the direction of the bifurcation, the calculations are, however, different. The details of the calculations are interesting and can be used to study similar equations. Equation 5 has a unique positive equilibrium $X^* = \beta - A + 1$ when $\beta + 1 > A$. In the next section, we study local stability of this equilibrium. Then, by considering A as a parameter, we show that this equation undergoes a Neimark-Sacker bifurcation as A crosses a certain critical value A^*. Then, we study the direction of the bifurcation. Finally, we present some numerical simulation that supports our theoretical findings.

2 | LOCAL STABILITY AND SOME PRELIMINARY RESULTS

Consider Equation 5 with positive parameters β and A and nonnegative initial conditions x_{-3}, x_{-2}, x_{-1} and x_0. This equation has one positive equilibrium point $X^* = \beta - A + 1$. We assume that $\beta + 1 > A$. Let $u_n = X_n, v_n = X_{n-1}, w_n = X_{n-2}$ and $z_n = X_{n-3}$. Then, Equation 5 is equivalent to the following first order system

$$\begin{aligned} u_{n+1} &= \frac{\beta u_n + z_n}{A + v_n} \\ v_{n+1} &= u_n \\ w_{n+1} &= v_n \\ z_{n+1} &= w_n. \end{aligned} \tag{6}$$

The Jacobian matrix of (6) at the positive equilibrium point is

$$J = \begin{pmatrix} \frac{\beta}{\beta+1} & \frac{-(\beta-A+1)}{\beta+1} & 0 & \frac{1}{\beta+1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of the Jacobian matrix is

$$P(\lambda) = \lambda^4 - \frac{\beta}{\beta+1}\lambda^3 + \frac{\beta - A + 1}{\beta+1}\lambda^2 - \frac{1}{\beta+1}. \tag{7}$$

To investigate the local stability of the positive equilibrium point, we need the following theorem:

Theorem 2.1. *The polynomial $F(\lambda) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ has roots in the unit circle if the following conditions are satisfied*[5]*:*

$$|a_1 + a_3| < 1 + a_0 + a_2, \quad |a_1 - a_3| < 2(1 - a_0), a_2 - 3a_0 < 3,$$

$$a_0 + a_2 + a_0^2 + a_1^2 + a_0^2 a_2 + a_0 a_3^2 < 1 + 2a_0 a_2 + a_1 a_3 + a_0 a_1 a_3 + a_0^3.$$

The following couple of theorems will be used:

Theorem 2.2. (Descartes theorem[6])
The number of positive roots (counted considering their multiplicity) of a polynomial $P_n(x)$ with real coefficients is either equal to the number of sign alterations between consecutive nonzero coefficients or is less than it by a multiple of 2.

Applying Descartes theorem to $P_n(-x)$, we obtain a similar theorem for the negative roots of the polynomial $P_n(x)$. So the number of negative roots of a polynomial $P_n(x)$ is equal to the number of positive roots of the polynomial $P_n(-x)$.

Theorem 2.3. (Viète theorem[6])
Let $\alpha, \sigma, \gamma, \delta$ be the roots of the polynomial $p(x) = ax^4 + bx^3 + cx^2 + dx + e$ then

$$\begin{aligned} \alpha + \sigma + \gamma + \delta &= \frac{-b}{a} \\ \alpha\sigma + \sigma\gamma + \gamma\delta + \alpha\gamma + \alpha\delta + \sigma\delta &= \frac{c}{a} \\ \alpha\sigma\gamma + \alpha\gamma\delta + \alpha\sigma\delta + \sigma\gamma\delta &= \frac{-d}{a} \\ \alpha\sigma\gamma\delta &= \frac{e}{a}. \end{aligned}$$

Theorem 2.4. *The positive fixed point of Equation 5 is locally asymptotically stable if $A > \frac{4(\beta+1)}{(\beta+2)^2}$ and unstable if $A < \frac{4(\beta+1)}{(\beta+2)^2}$.*

Proof. For the polynomial (7), $a_0 = \frac{-1}{\beta+1}, \quad a_1 = 0, \quad a_2 = \frac{\beta-A+1}{\beta+1}, \quad a_3 = \frac{-\beta}{\beta+1}$. The condition $|a_1 + a_3| < 1 + a_0 + a_2$ is satisfied if and only if $A < \beta+1$, which is equivalent to $\beta - A + 1 > 0$ and the last inequality is satisfied by assumption. The second condition $|a_1 - a_3| < 2(1 - a_0)$ is satisfied if and only if $\beta + 4 > 0$, which is trivially satisfied for every β. The third condition $a_2 - 3a_0 < 3$ is equivalent to $2\beta + A - 1 > 0$. The last inequality holds for every β, since $1 - 2\beta < A < \beta + 1$.

The fourth condition is

$$a_0 + a_2 + a_0^2 + a_1^2 + a_0^2 a_2 + a_0 a_3^2 < 1 + 2a_0 a_2 + a_1 a_3 + a_0 a_1 a_3 + a_0^3.$$

Which is satisfied if and only if

$$\left(\frac{-1}{\beta+1}\right)\left(1+\frac{\beta^2}{(\beta+1)^2}\right)+\frac{1}{(\beta+1)^2}\left(1+\frac{\beta-A+1}{\beta+1}\right)+\frac{\beta-A+1}{\beta+1}$$
$$<1-\frac{2(\beta-A+1)}{(\beta+1)^2}-\frac{1}{(\beta+1)^3}.$$

Multiplying by $(\beta+1)^3$ and simplifying, we find that the last inequality is equivalent to

$$A>4\frac{(\beta+1)}{(\beta+2)^2}:=A^*.$$

Note that $A^*<1$. We conclude that if $A>A^*$ then the eigenvalues of the characteristic equation will lie within the unit circle; hence, the fixed point is stable. □

3 | NEIMARK-SACKER BIFURCATION

In this section, we show that Equation 5 undergoes a Neimark-Sacker bifurcation. Firstly, we show in the next theorem that the Jacobian matrix has a pair of complex conjugate eigenvalues of modulus one.

Theorem 3.1. *If $A=A^*=\frac{4(\beta+1)}{(\beta+2)^2}$ then the characteristic polynomial (7) has 2 complex conjugate roots that lie on the unit circle. Moreover, the Neimark-Sacker bifurcation conditions are satisfied.*

Proof. First, we show that Equation 7 has 2 complex conjugate roots, using Descarte and *Viète* theorem. Applying Descartes theorem to (7), the alteration in sign is $(+-+-)$, so it has one positive root or 3 positive roots. Also applying Descartes theorem to $P(-\lambda)$, the alteration of sign is $(+++-)$, so $P(\lambda)$ has one negative root. Note that

$$P(-1)=\frac{3\beta-A+1}{\beta+1}>0,\quad P(0)=\frac{-1}{\beta+1}<0,\quad P(1)=\frac{\beta-A+1}{\beta+1}>0.$$

Therefore, $P(\lambda)$ has 2 real roots say $\mu_1\in(0,1)$ and $\mu_2\in(-1,0)$. Moreover,

$$P'(\lambda)=\lambda\left(4\lambda^2-\frac{3\beta}{\beta+1}\lambda+\frac{2(\beta-A+1)}{\beta+1}\right).$$

It follows that $P'(\lambda)=0$ if $\lambda=0$ or if $4\lambda^2(\beta+1)-3\beta\lambda+2(\beta-A+1)=0$. The second equality gives

$$\lambda=\frac{3\beta\pm\sqrt{9\beta^2-32(\beta-A+1)(\beta+1)}}{8(\beta+1)}.$$

But the discriminant of the previous quadratic equation is negative since,

$$\Delta=-23\beta^2-64\beta-32+32A(\beta+1)<-23\beta^2-64\beta-32+32(\beta+1)=-23\beta^2-32\beta<0.$$

So $P'(\lambda)$ has one real root, hence, $P(\lambda)$ changes its direction only once. To show that the positive real root is simple, by the way of contradiction suppose it has multiplicity equal 3, then by *Viète* theorem,

$$3\mu_1+\mu_2=\frac{\beta}{\beta+1} \tag{8}$$

$$3\mu_1\mu_2+3\mu_1^2=\frac{\beta-A+1}{\beta+1} \tag{9}$$

$$3\mu_2\mu_1^2+\mu_1^3=0 \tag{10}$$

$$\mu_1^3\mu_2=\frac{-1}{\beta+1}. \tag{11}$$

From Equation $10\mu_1 = -3\mu_2$, substitute in (11), we get

$$\mu_2 = \frac{1}{\sqrt[4]{27(\beta+1)}} > 0.$$

A contradiction. So Equation 7 has 2 real roots and 2 conjugate complex roots. The next step is to show that $|\lambda_{1,2}| = 1$ by applying *Viète* theorem. Let $\mu_{1,2}$ be the real roots of (7), and $\lambda_2 = \bar{\lambda}_1$

$$\mu_1 + \mu_2 + \lambda_1 + \lambda_2 = \frac{\beta}{\beta+1} \tag{12}$$

$$\mu_1\mu_2 + \mu_2\lambda_1 + \lambda_1\lambda_2 + \mu_1\lambda_1 + \mu_1\lambda_2 + \mu_2\lambda_2 = \frac{\beta - A + 1}{\beta+1} \tag{13}$$

$$\mu_1\mu_2\lambda_1 + \mu_1\lambda_1\lambda_2 + \mu_1\mu_2\lambda_2 + \mu_2\lambda_1\lambda_2 = 0 \tag{14}$$

$$\mu_1\mu_2\lambda_1\lambda_2 = \frac{-1}{\beta+1}. \tag{15}$$

Equation 15 gives

$$\mu_1\mu_2\lambda_1\lambda_2 = \mu_1\mu_2 = \frac{-1}{\beta+1}. \tag{16}$$

Substitute for $\mu_1\mu_2$ into Equation 14, we get

$$\frac{1}{\beta+1}(\lambda_1 + \lambda_2) = \mu_1 + \mu_2. \tag{17}$$

Plugging this value of $\mu_1 + \mu_2$ into (12), we find that

$$\lambda_1 + \lambda_2 = \frac{\beta}{\beta+2}. \tag{18}$$

Equations 13, 16, and 17 imply that

$$(\lambda_1 + \lambda_2)(\mu_1 + \mu_2) = \frac{1-A}{\beta+1}.$$

Using (17), the last equation gives

$$(\lambda_1 + \lambda_2)^2 = 1 - A.$$

It follows from Equation 18 that

$$\left(\frac{\beta}{\beta+2}\right)^2 = 1 - A.$$

Hence,

$$A = 1 - \left(\frac{\beta}{\beta+2}\right)^2 = \frac{4(\beta+1)}{(\beta+2)^2} = A^*.$$

Since the roots are uniquely determined, the above argument implies the existence of conjugate pair of complex roots on the unit circle. Let $\lambda = e^{i\theta}$ then

$$P(e^{i\theta}) = e^{4i\theta} - \frac{\beta}{\beta+1}e^{3i\theta} + \frac{\beta - A + 1}{\beta+1}e^{2i\theta} - \frac{1}{\beta+1} = 0.$$

Separate the real and imaginary parts

$$\cos 4\theta - \frac{\beta}{\beta+1}\cos 3\theta + \frac{\beta - A + 1}{\beta+1}\cos 2\theta - \frac{1}{\beta+1} = 0$$

$$\sin 4\theta - \frac{\beta}{\beta+1}\sin 3\theta + \frac{\beta - A + 1}{\beta+1}\sin 2\theta = 0.$$

Rewrite these equations in the form

$$\cos 4\theta - \frac{\beta}{\beta+1}\cos 3\theta = -\frac{\beta-A+1}{\beta+1}\cos 2\theta + \frac{1}{\beta+1}$$

$$\sin 4\theta - \frac{\beta}{\beta+1}\sin 3\theta = -\frac{\beta-A+1}{\beta+1}\sin 2\theta.$$

Squaring, adding up, and simplifying, we finally get

$$\cos^2\theta - \frac{2\beta(\beta+1)}{4(\beta-A+1)}\cos\theta + \frac{2\beta(\beta+1)}{4(\beta-A+1)} - \frac{\beta-A+3}{4} = 0. \tag{19}$$

From Equation 18, we get

$$\cos\theta = \frac{\beta}{2(\beta+2)}.$$

Note that this is a root of Equation 19. At $A = \frac{4(\beta+1)}{(\beta+2)^2}$, let $\theta_0 = \arccos\left(\frac{\beta}{2(\beta+2)}\right)$. Then, $0 < \cos\theta_0 < \frac{1}{2}$ and $\theta_0 \in (0, \frac{\pi}{2})$. Note that $\theta_0 \neq 0, \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pi$, it follows that $e^{ik\theta_0} \neq 1$ for $k \in \{1, 2, 3, 4\}$.

For the transversality condition, we show that $\frac{d|\lambda|^2}{dA}|_{A^*,\theta_0} \neq 0$

$$\begin{aligned}
\frac{d|\lambda|^2}{dA} &= \lambda\left(\frac{\partial P(\bar\lambda)}{\partial A}.\quad \frac{\partial\bar\lambda}{\partial P(\bar\lambda)}\right) + \bar\lambda\left(\frac{\partial P(\lambda)}{\partial A}.\quad \frac{\partial\lambda}{\partial P(\lambda)}\right)\\
&= \lambda\left(\frac{-\bar\lambda^2}{\beta+1}.\frac{1}{4\bar\lambda^3 - \frac{3\beta}{\beta+1}\bar\lambda^2 + \frac{2(\beta-A+1)}{\beta+1}\bar\lambda}\right) + \bar\lambda\left(\frac{-(\lambda)^2}{\beta+1}.\frac{1}{4\lambda^3 - \frac{3\beta}{\beta+1}\lambda^2 + \frac{2(\beta-A+1)}{\beta+1}\lambda}\right)\\
&= \frac{-\bar\lambda\left[4(\beta+1)\lambda^3 - 3\beta\lambda^2 + 2(\beta-A+1)\lambda\right] + (-\lambda)\left[4(\beta+1)\bar\lambda^3 - 3\beta\bar\lambda^2 + 2(\beta-A+1)\bar\lambda\right]}{\left[4(\beta+1)(\bar\lambda)^3 - 3\beta(\bar\lambda)^2 + 2(\beta-A+1)\bar\lambda\right]\left[4(\beta+1)(\lambda)^3 - 3\beta(\lambda)^2 + 2(\beta-A+1)\lambda\right]}.
\end{aligned}$$

Finally, we find that

$$\frac{d|\lambda|^2}{dA}|_{\theta_0,A^*} = \frac{-16(\beta+1)\cos^2\theta_0 + 6\beta\cos\theta_0 + 4(\beta+A^*+1)}{L},$$

where

$$L = 16(\beta+1)^2 + 9\beta^2 + 4(\beta-A^*+1)^2 - (12\beta(\beta+1) + 6\beta(\beta-A^*+1))(2\cos\theta_0) +$$

$$8(\beta-A^*+1)(\beta+1)(2\cos^2\theta_0 - 1).$$

It can be shown that $d|\lambda^2|/dA|_{\theta_0,A^*} \neq 0$ since

$$-16(\beta+1)\cos^2\theta_0 + 6\beta\cos\theta_0 + 4(\beta+A^*+1) = \frac{7\beta^2+12\beta+8}{(\beta+2)^2} > 0.$$

This completes the proof. □

We have shown that system (3) undergoes a Neimark-Sacker bifurcation. Now, we determine the direction of stability of the invariant closed curve that bifurcates from the positive fixed point. We follow the normal form theory of Neimark-Sacker bifurcation as in Kuznetsov.[7] Shift the fixed point to the origin by taking $x_n = u_n - u^*, y_n = u_n - u^*, t_n = v_n - v^*$ and $w_n = z_n - z^*$. Then Equation 6 becomes

$$\begin{pmatrix} x_{n+1}\\ y_{n+1}\\ t_{n+1}\\ w_{n+1}\end{pmatrix} = \begin{pmatrix} \frac{\beta(x_n+X^*)+w_n+X^*}{A+y_n+X^*} - X^*\\ x_n\\ y_n\\ t_n\end{pmatrix}. \tag{20}$$

Which can be written as

$$Y_{n+1} = JY_n + G(Y_n), \tag{21}$$

where $G(Y)=\frac{1}{2}B(Y,Y)+\frac{1}{6}C(Y,Y,Y)+O(||Y||^4)$, $Y_n=\begin{pmatrix} x_n\\ y_n\\ t_n\\ w_n\end{pmatrix}$,

$$B(Y,Y)=\begin{pmatrix} B_1(Y,Y)\\ 0\\ 0\\ 0\end{pmatrix}, \qquad \text{and} \quad C(Y,Y,Y)=\begin{pmatrix} C_1(Y,Y,Y)\\ 0\\ 0\\ 0\end{pmatrix},$$

such that

$$B_i(x,y)=\sum_{j,k=1}^{n}\frac{\partial^2 Y_i(\xi)}{\partial\xi_j\partial\xi_k}|_{\xi=0}(x_j y_k),$$

and

$$\begin{aligned}
C_i(x,y,z)&=\sum_{j,k,l=1}^{n}\frac{\partial^3 Y_i(\xi)}{\partial\xi_j\partial\xi_k\partial\xi_l}|_{\xi=0}(x_j y_k z_l),\\
B_1(\phi,\psi)&=\frac{-\beta}{(\beta+1)^2}(\phi_2\psi_1+\phi_1\psi_2)+2\frac{\beta-A+1}{(\beta+1)^2}\phi_2\psi_2+\frac{-1}{(\beta+1)^2}(\phi_2\psi_4+\phi_4\psi_2),\\
C_1(\phi,\psi,\eta)&=\frac{-6(\beta-A+1)}{(\beta+1)^3}\phi_2\psi_2\eta_2+\frac{2\beta}{(\beta+1)^3}(\phi_2\psi_2\eta_1+\phi_1\psi_2\eta_2+\phi_2\psi_1\eta_2)\\
&\quad+\frac{2}{(\beta+1)^3}(\phi_2\psi_2\eta_4+\phi_4\psi_2\eta_2+\phi_2\psi_4\eta_2).
\end{aligned}$$

Let $Jq = e^{i\theta_0}q$, $J^Tp^* = e^{-i\theta_0}p^*$ where q and p^* are the eigenvectors corresponding to the eigenvalues $e^{i\theta_0}$ and $e^{-i\theta_0}$, respectively. We obtain $q\sim\left(1,e^{-i\theta_0},e^{-2i\theta_0},e^{-3i\theta_0}\right)^T$. Similarly, the eigenvector p^* of J^T is given by

$$p^*\sim\left(1,-\frac{\beta}{\beta+1}+e^{-i\theta_0},\frac{e^{2i\theta_0}}{\beta+1},\frac{e^{i\theta_0}}{\beta+1}\right)^T.$$

To normalize p^* and q so that $\langle p^*,q\rangle=1$, where $\langle .,.\rangle$ is the standard scalar product in $\mathbb{C}^3$. Let

$$\eta=\langle p^*,q\rangle=2-\frac{\beta}{\beta+1}e^{-i\theta_0}+2\frac{e^{-4i\theta_0}}{\beta+1}.$$

So let $p=\eta^{-1}p^*$, where $\eta^{-1}=1/\eta$. The critical real eigenspace T^c corresponding to $\lambda_{1,2}$ is 2-dimensional and is spanned by $\{Re(q),Im(q)\}$. The real eigenspace T^s corresponding to the real eigenvalues of J is 2-dimensional. Any vector $x\in\mathbb{R}^4$ may be decomposed as

$$x=zq+\bar{z}\bar{q}+y,$$

where $z\in\mathbb{C}^1$, and $\bar{z}\bar{q}\in T^c$, $y\in T^s$. The complex variable z is a coordinate on T^c. We have

$$\begin{aligned}
z&=\langle p,x\rangle,\\
y&=x-\langle p,x\rangle q-\langle\bar{p},x\rangle\bar{q}.
\end{aligned}$$

In these coordinates, the map (21) takes the form

$$\begin{aligned}
\tilde{z}&=e^{i\theta_0}z+\langle p,G(zq+\bar{z}\bar{q}+y)\rangle,\\
\tilde{y}&=Jy+G(zq+\bar{z}\bar{q}+y)-\langle p,G(zq+\bar{z}\bar{q}+y)\rangle q-\langle\bar{p},G(zq+\bar{z}\bar{q}+y)\rangle\bar{q}.
\end{aligned}$$

The previous system can be written in the form

$$\begin{aligned}
\tilde{z}&=e^{i\theta_0}z+\frac{1}{2}G_{20}z^2+G_{11}z\bar{z}+\frac{1}{2}G_{02}\bar{z}^2+\frac{1}{2}G_{21}z^2\bar{z}+\langle G_{10},y\rangle z+\langle G_{01},y\rangle\bar{z},\\
\tilde{y}&=Jy+\frac{1}{2}H_{20}z^2+H_{11}z\bar{z}+\frac{1}{2}H_{02}\bar{z}^2+\frac{1}{2}H_{21}z^2\bar{z}.
\end{aligned}$$

Where $G_{20} = \langle p, B(q,q)\rangle, G_{11} = \langle p, B(q,\bar{q})\rangle, G_{02} = \langle p, B(\bar{q},\bar{q})\rangle, G_{21} = \langle p, C(q,q,\bar{q})\rangle$, and

$$H_{20} = B(q,q) - \langle p, B(q,q)\rangle q - \langle \bar{p}, B(q,q)\rangle \bar{q}, H_{11} = B(q,\bar{q}) - \langle p, B(q,\bar{q})\rangle q - \langle \bar{p}, B(q,\bar{q})\rangle \bar{q},$$

with

$$\langle G_{10}, y\rangle = \langle p, B(q,y)\rangle, \langle G_{01}, y\rangle = \langle p, B(\bar{q},y)\rangle,$$

and the scalar product in $\mathbb{C}^3$ is used. From the center manifold theorem, there exists a center manifold W^c, which can be approximated as

$$Y = V(z,\bar{z}) = \frac{1}{2}w_{20}z^2 + w_{11}z\bar{z} + \frac{1}{2}w_{02}\bar{z}^2,$$

where $\langle q, w_{ij}\rangle = 0$. The vectors $w_{ij} \in \mathbb{C}^3$ can be found from the linear equations

$$\begin{aligned} w_{20} &= (e^{2i\theta_0}I_3 - J)^{-1}H_{20}, \\ w_{11} &= (I_3 - J)^{-1}H_{11}, \\ w_{02} &= (e^{-2i\theta_0}I_3 - J)^{-1}H_{02}. \end{aligned}$$

So z can be expressed as

$$\tilde{z} = e^{i\theta_0}\bar{z} + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2$$

$$+\frac{1}{2}(G_{21} + 2\langle p, B(q,(I-J)^{-1}H_{11})\rangle + \langle p, B(\bar{q},(e^{2i\theta_0}I-J)^{-1}H_{20})\rangle)z^2\bar{z}.$$

Taking into account the identities

$$(I-J)^{-1}q = \frac{1}{1-e^{i\theta_0}}q, \qquad (e^{2i\theta_0}I - J)^{-1}q = \frac{e^{-i\theta_0}}{e^{i\theta_0}-1}q,$$

and

$$(I-J)^{-1}\bar{q} = \frac{1}{1-e^{i\theta_0}}\bar{q}, \qquad (e^{2i\theta_0}I - J)^{-1}\bar{q} = \frac{e^{-i\theta_0}}{e^{i\theta_0}-1}\bar{q}.$$

We can express z using the map

$$\tilde{z} = e^{i\theta_0}z + \sum_{k+l\geq 2}\frac{1}{k!j!}g_{k_j}z^k\bar{z}^j,$$

where $g_{20} = \langle p, B(q,q)\rangle, g_{11} = \langle p, B(q,\bar{q})\rangle$, and $g_{02} = \langle p, B(\bar{q},\bar{q})\rangle$.

$$g_{21} = \langle p, C(q,q,\bar{q})\rangle + 2\langle p, B(q,(I-J)^{-1}B(q,\bar{q}))\rangle + \langle p, B(\bar{q},(e^{2i\theta_0}I-J)^{-1}B(q,q))\rangle + \cdots.$$

Equivalently, $\tilde{z}$ can be written as

$$\tilde{z} = e^{i\theta_0}z(1 + d(A^*))|z^2|,$$

where the real number $\beta(A^*) = Re(d(A^*))$ that determines the direction of bifurcation of a closed invariant curve can be computed via

$$\beta(A^*) = \mathrm{Re}(\frac{e^{-i\theta_0}g_{21}}{2}) - \mathrm{Re}(\frac{(1-2e^{i\theta_0})e^{-2i\theta_0}}{2(1-e^{i\theta_0})}g_{20}g_{11}) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2 \tag{22}$$

$$g_{20} = \langle p, B(q,q)\rangle \text{ and } B(q,q) = \begin{pmatrix} \frac{-2\beta e^{-i\theta_0} + 2(\beta - A + 1)e^{-2i\theta_0} - 2e^{-4i\theta_0}}{(\beta+1)^2} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It follows that $\langle p, B(q,q)\rangle = \frac{-2\beta e^{-i\theta_0}+2(\beta-A+1)e^{-2i\theta_0}-2e^{-4i\theta_0}}{(\beta+1)(2(\beta+1)-\beta e^{-i\theta_0}+2e^{-4i\theta_0})}$.Moreover,

$$B(q,\bar{q}) = \begin{pmatrix} \frac{-2\beta\cos\theta_0+2(\beta-A+1)-2\cos 2\theta_0}{(\beta+1)^2} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

So

$$g_{11} = \langle p, B(q,\bar{q})\rangle = \frac{-2\beta\cos\theta_0+2(\beta-A+1)-2\cos 2\theta_0}{(\beta+1)(2(\beta+1)-\beta e^{-i\theta_0}+2e^{-4i\theta_0})}.$$

On the other hand,

$$B(\bar{q},\bar{q}) = \begin{pmatrix} \frac{-2\beta e^{i\theta_0}+2(\beta-A+1)e^{2i\theta_0}-2e^{4i\theta_0}}{(\beta+1)^2} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then,

$$g_{02} = \langle p, B(\bar{q},\bar{q})\rangle = \frac{-2\beta e^{i\theta_0}+2(\beta-A+1)e^{2i\theta_0}-2e^{4i\theta_0}}{(\beta+1)(2(\beta+1)-\beta e^{-i\theta_0}+2e^{-4i\theta_0})}.$$

Now, we find the terms in the formula for g_{21} that are required to calculate $\beta(A^*)$.

$$C(q,q,\bar{q}) = \begin{pmatrix} \frac{-6(\beta-A+1)e^{-i\theta_0}+2\beta(2+e^{-2i\theta_0})+2(e^{i\theta_0}+2e^{-3i\theta_0})}{(\beta+1)^3} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\langle p, C(q,q,\bar{q})\rangle = \frac{-6(\beta-A+1)e^{-i\theta_0}+2\beta(2+e^{-2i\theta_0})+2(e^{i\theta_0}+2e^{-3i\theta_0})}{(\beta+1)^2(2(\beta+1)-\beta e^{-i\theta_0}+2e^{-4i\theta_0})}.$$

To find $\langle p, B(q,(I-J)^{-1}B(q,\bar{q}))\rangle$

$$(I-J)^{-1} = \begin{pmatrix} \frac{\beta+1}{\beta-A+1} & \frac{A-\beta}{\beta-A+1} & \frac{1}{\beta-A+1} & \frac{1}{\beta-A+1} \\ \frac{\beta+1}{\beta-A+1} & \frac{1}{\beta-A+1} & \frac{1}{\beta-A+1} & \frac{1}{\beta-A+1} \\ \frac{\beta+1}{\beta-A+1} & \frac{1}{\beta-A+1} & \frac{\beta-A+2}{\beta-A+1} & \frac{1}{\beta-A+1} \\ \frac{\beta+1}{\beta-A+1} & \frac{1}{\beta-A+1} & \frac{\beta-A+2}{\beta-A+1} & \frac{\beta-A+2}{\beta-A+1} \end{pmatrix},$$

$$(I-J)^{-1}B(q,\bar{q}) = \begin{pmatrix} \frac{-2\beta\cos\theta_0+2(\beta-A+1)-2\cos 2\theta_0}{(\beta+1)(\beta-A+1)} \\ \frac{-2\beta\cos\theta_0+2(\beta-A+1)-2\cos 2\theta_0}{(\beta+1)(\beta-A+1)} \\ \frac{-2\beta\cos\theta_0+2(\beta-A+1)-2\cos 2\theta_0}{(\beta+1)(\beta-A+1)} \\ \frac{-2\beta\cos\theta_0+2(\beta-A+1)-2\cos 2\theta_0}{(\beta+1)(\beta-A+1)} \end{pmatrix}.$$

To find $\langle p, B(\bar{q},(e^{2i\theta_0}I-J)^{-1})B(q,q)\rangle$, we calculate $(e^{2i\theta_0}I-J)^{-1}$, which is given by

$$\frac{1}{D}\begin{pmatrix} e^{6i\theta_0} & \frac{1}{\beta+1}-B^*e^{4i\theta_0} & \frac{e^{2i\theta_0}}{\beta+1} & \frac{e^{4i\theta_0}}{\beta+1} \\ e^{4i\theta_0} & e^{6i\theta_0}-\frac{\beta e^{4i\theta_0}}{\beta+1} & \frac{1}{\beta+1} & \frac{e^{2i\theta_0}}{\beta+1} \\ e^{2i\theta_0} & e^{4i\theta_0}-\frac{\beta e^{2i\theta_0}}{\beta+1} & e^{6i\theta_0}-\frac{\beta e^{4i\theta_0}}{\beta+1}+B^*e^{2i\theta_0} & \frac{1}{\beta+1} \\ 1 & e^{2i\theta_0}-\frac{\beta}{\beta+1} & e^{4i\theta_0}-\frac{\beta e^{2i\theta_0}}{\beta+1}+B^* & e^{6i\theta_0}-\frac{\beta e^{4i\theta_0}}{\beta+1}+B^*e^{2i\theta_0} \end{pmatrix}.$$

Where $D = \frac{(\beta+1)e^{8i\theta_0} - \beta e^{6i\theta_0} + (\beta-A+1)e^{4i\theta_0} - 1}{\beta+1}$ and $B^* = \frac{\beta-A+1}{\beta+1}$

$$(e^{2i\theta_0}I - J)^{-1}B(q,q) = \begin{pmatrix} \frac{-2\beta e^{5i\theta_0}+2(\beta-A+1)e^{4i\theta_0}-2e^{2i\theta_0}}{(\beta+1)((\beta+1)e^{8i\theta_0}-\beta e^{6i\theta_0}+(\beta-A+1)e^{4i\theta_0}-1)} \\ \frac{-2\beta e^{3i\theta_0}+2(\beta-A+1)e^{2i\theta_0}-2}{(\beta+1)((\beta+1)e^{8i\theta_0}-\beta e^{6i\theta_0}+(\beta-A+1)e^{4i\theta_0}-1)} \\ \frac{-2\beta e^{i\theta_0}+2(\beta-A+1)-2e^{-2i\theta_0}}{(\beta+1)((\beta+1)e^{8i\theta_0}-\beta e^{6i\theta_0}+(\beta-A+1)e^{4i\theta_0}-1)} \\ \frac{-2\beta e^{-i\theta_0}+2(\beta-A+1)e^{-2i\theta_0}-2e^{-4i\theta_0}}{(\beta+1)((\beta+1)e^{8i\theta_0}-\beta e^{6i\theta_0}+(\beta-A+1)e^{4i\theta_0}-1)} \end{pmatrix},$$

and $B(\bar{q}, (e^{2i\theta_0}I - J)^{-1}B(q,\bar{q})) =$

$$\begin{pmatrix} \frac{-\beta}{(\beta+1)^3}\left(\frac{-2\beta(e^{6i\theta_0}+e^{3i\theta_0})+2(\beta-A+1)(e^{5i\theta_0}+e^{2i\theta_0})-2(e^{3i\theta_0}+1)}{(\beta+1)((\beta+1)e^{8i\theta_0}-\beta e^{6i\theta_0}+(\beta-A+1)e^{4i\theta_0}-1)}\right) \\ +\frac{2(\beta-A+1)}{(\beta+1)^3}\left(\frac{-2\beta e^{4i\theta_0}+2(\beta-A+1)e^{3i\theta_0}-2e^{i\theta_0}}{(\beta+1)((\beta+1)e^{8i\theta_0}-\beta e^{6i\theta_0}+(\beta-A+1)e^{4i\theta_0}-1)}\right) \\ -\frac{1}{(\beta+1)^3}\left(\frac{-2\beta(1+e^{6i\theta_0})+2(\beta-A+1)(e^{-i\theta_0}+e^{5i\theta_0})-4\cos 3\theta_0}{(\beta+1)((\beta+1)e^{8i\theta_0}-\beta e^{6i\theta_0}+(\beta-A+1)e^{4i\theta_0}-1)}\right) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now, $\langle p, B(\bar{q}, (e^{2i\theta_0}I - J)^{-1}B(q,\bar{q}))\rangle =$

$$\frac{a_0e^{6i\theta_0} + a_1e^{5i\theta_0} + a_2e^{4i\theta_0} + a_3e^{3i\theta_0} + a_4e^{2i\theta_0} + a_5e^{-i\theta_0} + a_6e^{i\theta_0} + a_7}{(\beta+1)^2\left((\beta+1)e^{8i\theta_0} - \beta e^{6i\theta_0} + (\beta-A+1)e^{4i\theta_0} - 1\right)\left(2(\beta+1) - \beta e^{-i\theta_0} + 2e^{-4i\theta_0}\right)},$$

where $a_0 = 2\beta^2 + 2\beta, a_1 = -2\beta(\beta-A+1) - 2(\beta-A+1)a_2 = -4(\beta-A+1), a_3 = 2\beta^2 + 2\beta + 4(\beta-A+1)^2a_4 = -2\beta(\beta-A+1), a_5 = -2(\beta-A+1), a_6 = -4(\beta-A+1), a_7 = 4\beta + 4cos3\theta_0$

$$\beta(A^*) = \text{Re}\left(e^{-i\theta_0}R\right),$$

where

$$R = \frac{1}{2}\langle p, C(q,q,\bar{q})\rangle + \langle p, B(q,(I-J)^{-1})B(q,\bar{q})\rangle + \frac{1}{2}\langle p, B(\bar{q},(e^{2i\theta_0}I-J)^{-1}B(q,q))\rangle.$$

Let $R_1 = \text{Re}\left(\frac{1}{2}e^{-i\theta_0}\langle p, C(q,q,\bar{q})\rangle\right)$, then

$$R_1 = \text{Re}\left(\frac{-3(\beta-A+1)e^{-2i\theta_0} + \beta(2e^{-i\theta_0} + e^{-3i\theta_0}) + (1 + 2e^{-4i\theta_0})}{(\beta+1)^2(2(\beta+1) - \beta e^{-i\theta_0} + 2e^{-4i\theta_0})}\right).$$

Multiply and divide by the denominator's conjugate , the numerator becomes

$$2e^{4i\theta_0} + 4(\beta+1)e^{-4i\theta_0} + 4\beta e^{3i\theta_0} + (2\beta(\beta+1) - 2\beta)e^{-3i\theta_0} + (-6(\beta-A+1))e^{2i\theta_0} +$$

$$(-6(\beta+1)(\beta-A+1) - \beta^2)e^{-2i\theta_0} + 2\beta e^{i\theta_0} + (4\beta(\beta+1) + 3\beta(\beta-A+1))e^{-i\theta_0} + 2(\beta+1) - 2\beta^2 + 4.$$

Taking the real part and denoting it by C_1

$$C_1 = b_4\cos 4\theta_0 + b_3\cos 3\theta_0 + b_2\cos 2\theta_0 + b_1\cos\theta_0 + b_0.$$

Where $b_0 = 2(\beta+1) - 2\beta^2 + 4, b_1 = 3\beta(\beta-A+1) + 4\beta(\beta+1) + 2\beta)b_2 = -6(\beta-A+1) - 6(\beta+1)(\beta-A+1) - \beta^2, b_3 = 2\beta^2 + 4\beta, b_4 = 4(\beta+1) + 2$. Multiply the denominator by its conjugate and denote it by C_2

$$C_2 = 4(\beta+1)^2 + \beta^2 + 4 + 8(\beta+1)\cos 4\theta_0 - 4\beta\cos 3\theta_0 - 4\beta(\beta+1)\cos\theta_0$$

$$R_1 = \frac{C_1}{(\beta+1)^2C_2}.$$

Let $R_2 = \mathrm{Re}(e^{-i\theta_0}\langle p, B(q,(I-J)^{-1})B(q,\bar{q})\rangle)$. It follows that

$$R_2 = \mathrm{Re}\left(\frac{-\beta L e^{-i\theta_0} + (\beta - 2A + 1)Le^{-2i\theta_0} - Le^{-4i\theta_0}}{(\beta+1)(2(\beta+1) - \beta e^{-i\theta_0} + 2e^{-4i\theta_0})}\right),$$

and let

$$R_3 = \mathrm{Re}\left(\frac{1}{2}\langle p, B(\bar{q},(e^{2i\theta_0}I - J)^{-1}B(q,q))\rangle\right).$$

Then, we have

$$\beta(A^*) = R_1 + R_2 + R_3.$$

The calculations of R_2 and R_3 are long and will be omitted, they are available upon request from the authors. We can find $\beta(A^*)$ using Equation 22. If $\beta = 1, A = 0.889$, then $\beta(A^*) = 0.01$, and the closed invariant curve is subcritical (unstable).

Theorem 3.2. *If $\beta(A^*) < 0$ (respectively, > 0), then the Neimark- Sacker bifurcation at $A = A^*$ is supercritical (respectively, subcritical) and there exists a unique invariant closed curve that bifurcates from the fixed point, which is asymptotically stable (respectively, unstable).*

4 | NUMERICAL SIMULATION

In this section, we present some numerical simulations that supports our theoretical results. In Figure 1, we have bifurcation diagram in the (A, x_{n+1}) plane. In this figure, $\beta = 1$, so the critical value of A at which Neimark-Sacker bifurcation

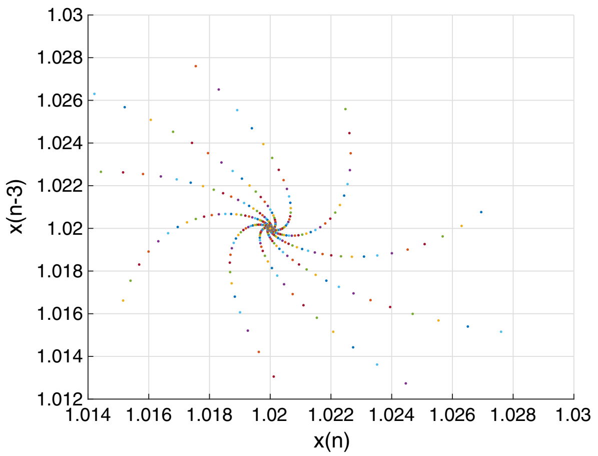

FIGURE 2 Phase portrait of Equation 5 in $(x(n), x(n-3))$ plane [Colour figure can be viewed at wileyonlinelibrary.com]

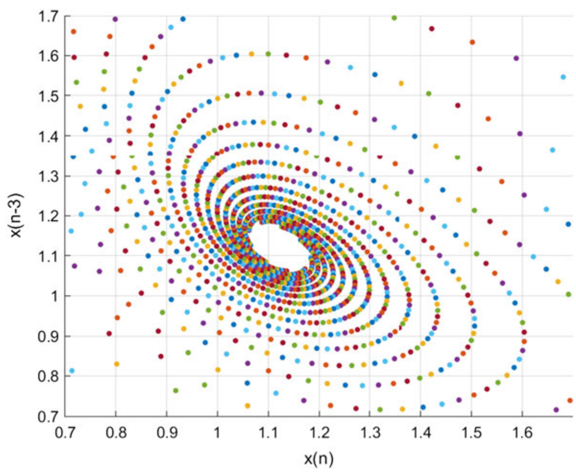

FIGURE 3 Phase portrait of Equation 5 in $(x(n), x(n-3))$ plane [Colour figure can be viewed at wileyonlinelibrary.com]

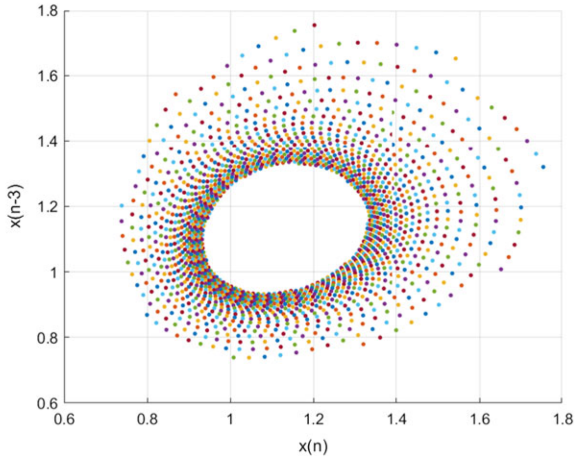

FIGURE 4 Phase portrait of Equation 5 in $(x(n), x(n-3))$ plane [Colour figure can be viewed at wileyonlinelibrary.com]

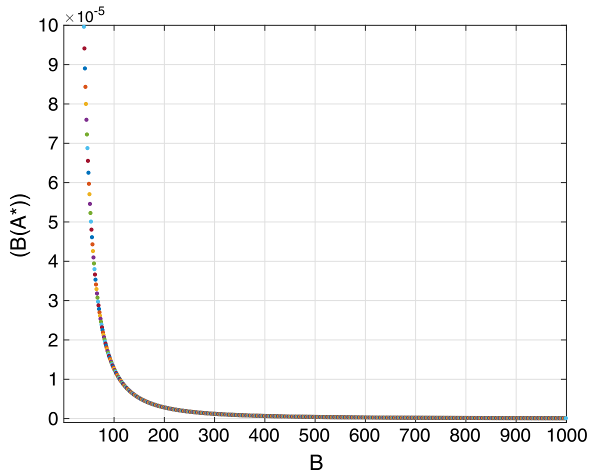

FIGURE 5 Direction of bifurcation [Colour figure can be viewed at wileyonlinelibrary.com]

occurs is $A^* = 8/9$, and the initial conditions are $X_{-3} = X_{-2} = X_{-1} = X_0 = 1$. The positive fixed point is asymptotically stable if $A > 8/9$. In Figure 2, we plot phase portrait by assigning the values $A = 0.98, \beta = 1, X_{-3} = X_{-2} = X_{-1} = X_0 = 1.6$. Note that, for this value of A, the positive equilibrium point is asymptotically stable. In Figures 3 and 4, we plot phase portraits by assigning values of A in the vicinity of the bifurcation value. In Figure 3, $A = 0.88, \beta = 1, X_{-3} = X_{-2} = X_{-1} = X_0 = 0.1$ while in Figure 4, $A = 0.889, \beta = 1, X_{-3} = X_{-2} = X_{-1} = X_0 = 0.1$. Notice the birth of the closed invariant curve that is subcritical (unstable) because the curve disappears as we move away from the fixed point. By virtue of Theorem 2.4, supported by Figure 1, the unique positive equilibrium (1.12, 1.12, 1.12, 1.12) in Figure 3 is unstable whereas in Figure 4, it is in stable. We conjecture that the closed invariant curve is subcritical (unstable), which means that $\beta(A^*) > 0$ for any $\beta > 0$. Figure 5 supports our conjecture.

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