The Number of Homomorphisms from $Z_{m_1} \times Z_{m_2} \times \ldots \times Z_{m_r}$ into $Z_{k_1} \times Z_{k_2} \times \ldots \times Z_{k_s}$

M. Saleh, H. Yousef, and J. Abu Hlail*

ABSTRACT

In this paper, we generalize results by J. Gallian and J. Van Buskirk, on the number of group and ring Homomorphisms. For the group case, we compute the number of group Homomorphisms from a finite abelian group into a finite abelian group. For the ring case, we compute the number of ring Homomorphisms from:

$Z_{m_1} \times Z_{m_2} \times \ldots \times Z_{m_r}$ into $Z_{k_1} \times Z_{k_2} \times \ldots \times Z_{k_s}$

INTRODUCTION

The purpose of this article is to generalize results given in (Gallian and Van Buskirk, 1984). We compute the number of group homomorphisms from a finite abelian group into a finite abelian group, which is a generalization of [2, First Theorem]. For the ring case, we compute the number of ring homomorphisms from $Z_{m_1} \times \ldots \times Z_{m_r}$ into $Z_{k_1} \times \ldots \times Z_{k_s}$, which is a generalization of [2, Second Theorem]. Let A and B be two groups (respectively, two rings), *Hom* (A, B) denotes the set of all group homomorphisms (respectively, ring homomorphisms) from A into B, and h (A, B) denotes the cardinality of Hom (A, B).

The first theorem is a generalization of [2, First Theorem]. Let G and H be finite abelian groups of orders m and n, respectively. By the fundamental theorem of finite abelian groups, there exist $m_1, m_2, \ldots, m_r$ and $n_1, n_2, \ldots, n_s$ such that G is isomorphic to $Z_{m_1} \times \ldots \times Z_{m_r}$ and H is isomorphic to $Z_{n_1} \times \ldots \times Z_{n_s}$. Thus, it is enough to compute the number of homomorphisms from $Z_{m_1} \times \ldots \times Z_{m_r}$ into $Z_{n_1} \times \ldots \times Z_{n_s}$.

Theorem 1. The number of group homomorphisms from $Z_{m_1} \times \ldots \times Z_{m_r}$ into $Z_{k_1} \times \ldots \times Z_{k_s}$ is:

$$\prod_{i=1, j=1}^{i=r, j=s} \gcd(m_i, k_j)$$

Proof. The proof follows directly from the fact that as groups $Hom\,(Z_{m_1} \times \ldots \times Z_{m_r}, Z_{k_1} \times \ldots \times Z_{k_s}) \cong \prod_{i=1, j=1}^{i=r, j=s} Hom\,(Z_{m_i}, Z_{k_j})$.

Now, we are in a position to prove our main result. First of all, notice that it is well-known that if R, T_i, $i \in I$ are rings with identities, then $Hom\,(R, \prod_{i \in I} T_i) \cong \prod_{i \in I} Hom\,(R, T_i)$ (see, [1, Proposition 16.4], [5, 11.10(2)]).

Consequently, $Hom\,(Z_{m_1} \times Z_{m_2} \times \ldots \times Z_{m_r}, Z_{k_1} \times \ldots \times Z_{k_s}) \cong \prod_{j=1}^{j=s} Hom\,(Z_{m_1} \times Z_{m_2} \times \ldots \times Z_{m_r}, Z_{k_j})$ and thus $h\,(Z_{m_1} \times Z_{m_2} \times \ldots \times Z_{m_r}, Z_{k_1} \times \ldots \times Z_{k_s}) = \prod_{j=1}^{j=s} h\,(Z_{m_1} \times Z_{m_2} \times \ldots \times Z_{m_r}, Z_{k_j})$.

Let $k = p_1^{t_1} p_2^{t_2} \ldots p_s^{t_s}$ be the prime- power decomposition of k in Z. By the Chinese remainder Theorem, it follows that Z_k is naturally ring-isomorphic to $Z_{p_1^{t_1}} \times \ldots \times Z_{p_s^{t_s}}$. Thus, we need only to compute $h\,(Z_{m_1} \times Z_{m_2} \times \ldots \times Z_{m_r}, Z_{p^e})$, where p is a prime.

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*Mathematics Department, Birzeit University, P.O. Box 14, Birzeit, West Bank, Palestine. Received on 11/12/1995 and Accepted for Publication on 26/11/1997.

Theorem 2. The number of ring homomorphisms from $Z_{m_1} \times Z_{m_2} \times ... \times Z_{m_r}$ into Z_{p^e} is given by:

$$1 + N_{p^e}(m_1, m_2, ..., m_r),$$

where $N_{p^e}(m_1, m_2, ..., m_r)$ is the number of elements in the sequence $m_1, m_2, ..., m_r$ that are divisible by p^e.

Proof. Let $\varphi : Z_{m_1} \times ... \times Z_{m_r} \to Z_{p^e}$ be a ring homomorphism. Then φ is completely determined by $\varphi(e_1), ..., \varphi(e_r)$ where e_i is the r–tuple with 0's in each component except 1 in ith component. These are idempotents in Z_{p^e}, and hence, each must be either 0 or 1. Also, if $\varphi(e_i) = \varphi(e_j) = 1$ for $i \neq j$, then one obtains a contradiction for:

$$0 = \varphi(0) = \varphi(e_i e_j) = \varphi(e_i)\,\varphi(e_j) = 1.1 = 1.$$

Thus, if φ is not the zero homomorphism, then $\varphi(e_i) = 1$ for exactly one value i, and moreover for that i we must have p^e divides m_i. Thus, $h(Z_{m_1} \times Z_{m_2} \times ... \times Z_{m_r}, Z_{p^e}) = 1 + N_{p^e}(m_1, m_2, ..., m_r)$, where $N_{p^e}(m_1, m_2, ..., m_r)$ is the number of elements in the sequence $m_1, m_2, ..., m_r$ that are divisible by p^e.

Theorem 3. The number of ring homomorphisms from $Z_{m_1} \times ... \times Z_{m_r}$ into $Z_{p_1}^{e_1} \times ... \times Z_{p_s}^{e_s}$, where p_i, $1 \leq i \leq s$, are primes not necessary distinct, is:

$$\prod_{i=1}^{i=s}(1 + N_{p_i^{e_i}}(m_1, m_2, ..., m_r))$$

We generalize [2, Second Theorem] in the following theorems using the notation found in (Gallian and Buskirk, 1984).

Theorem 4. The number of ring homomorphisms from $Z_m \times Z_n$ into Z_k is:

$$3^{\omega(k) - \omega(k/\gcd(m,n,k))} \cdot 2^{2\omega(k/\gcd(m,n,k)) - \omega(k/\gcd(m,k)) - \omega(k/\gcd(n,k))}$$

where $\omega(t)$ is the number of distinct prime divisors of t in Z.

Proof. Any ring homomorphism from $Z_m \times Z_n$ into Z_k is completely determined by its action on $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Also, since e_1 and e_2 are idempotents in $Z_m \times Z_n$, their images must be idempotents in Z_k. Let $k = p_1^{k_1} ... p_s^{k_s}$ be the prime-power decomposition of k in Z. By the Chinese remainder Theorem, it follows that Z_k is naturally ring-isomorphic to the direct product $Z_{p_1}^{k_1} \times ... \times Z_{p_s}^{k_s}$. Now, suppose that a, b are elements of Z_k where a and b are ring-homomorphic images of e_1 and e_2, respectively. In the direct sum, let $a = (a_1, ..., a_s)$ and $b = (b_1, ..., b_s)$ where each a_i, b_i is an idempotent of $Z_{p_i}^{k_i}$. If $p_i^{k_i}$ divides gcd (m, n, k), then we have three cases: a_i is 0 and b_i is 0, a_i is 0 and b_i is 1 or a_i is 1 and b_i is 0. As in [2, Second Theorem], the number of prime divisors p_i, of k such that $p_i^{k_i}$ divides gcd(m, n, k) is $\omega(k) - \omega(k / \gcd(m, n, k))$. Thus, we have $3^{\omega(k - \omega(k/\gcd(m,n,k))}$ choices. If $p_i^{k_i}$ divides gcd(m, k), but $p_i^{k_i}$ does not divide gcd(n,k), then we have two choices $a_i = 0$ or 1 and $b_i = 0$. The number of prime divisors p_i such that $p_i^{k_i}$ divides gcd (m, k) but does not divide gcd(n, k) is $\omega(k) - \omega(k / \gcd(m, k)) - [\omega(k) - \omega(k/\gcd(m, n, k))] = \omega(k/\gcd(m, n, k)) - \omega(k/\gcd(m, k))$. Similarly, if $p_i^{k_i}$ divides gcd (n ,k) but $p_i^{k_i}$ does not divide gcd (m, k), then we have two choices $a_i = 0$ and $b_i = 0$ or 1. The number of prime divisors p_i of k such that $p_i^{k_i}$ divides gcd (n, k), but does not divide gcd(m, k) is $\omega(k) - \omega(k/\gcd(n, k)) - [\omega(k) - \omega(k/\gcd(m, n, k))] = \omega(k / \gcd(m, n, k)) - \omega(k / \gcd(n, k))$.

So we have $2^{2\omega(k/\gcd(m,n,k)) - \omega(k/\gcd(m,k)) - \omega(k/\gcd(n,k))}$ choices. Thus, the number of ring homomorphisms is:

$$3^{\omega(k) - \omega(k/\gcd(m,n,k))}.$$

$$2^{2\omega(k/\gcd(m,n,k)) - \omega(k/\gcd(m,k)) - \omega(k/\gcd(n,k))}$$

Corollary 1. The number of ring homomorphisms from $Z_n \times Z_n$ into Z_k is:

$$3^{\omega(k) - \omega(k/\gcd(k,n))}$$

Theorem 5. The number of ring homomorphisms from Z_n into $Z_{k_1} \times \ldots \times Z_{k_s}$ is given by:

$$2^{\left(\sum_{i=1}^{s} \omega(k_i) - \sum_{i=1}^{s} \omega(k_i/\gcd(k_i,n))\right)}$$

Proof. The proof follows directly from the fact that as rings *Hom* $(Z_n, Z_{k_1} \times \ldots \times Z_{k_s}) \cong \prod_{j=1}^{j=s}$ *Hom* (Zn, Z_{k_j}).

Following similar arguments as in Theorem 4, we get the following.

Theorem 6. The number of ring homomorphisms from $Z_{m_1} \times \ldots \times Z_{m_r}$ into Z_k is:

$$J(m_1, m_2, \ldots, m_r) =$$

$$(r+1)^{\omega(k) - \omega(k_i/\gcd(k,m_1,m_2,\ldots,m_r))} L(r,i)$$

where
$$L(r,i) = \prod_{i=1}^{i=r-1} (r-i+1)$$

$$\binom{r-i+1}{i}^{\sum_{j=0}^{j=i} (-1)^j \binom{r-i+j}{j} \left(\sum_{\substack{A \subseteq \{m_1, m_2, \ldots, m_r\} \\ |A| = r-i+j}}^{s} w(k_i/\gcd(k, a \in A)) \right)}$$

and,

$\omega(d)$ is the number of distinct prime divisors of d in Z.

We illustrate the above Theorems by the following examples:

Example 1. Compute h $(Z_{10} \times Z_4, Z_{18} \times Z_2 \times Z_{16})$ using Theorems 5 and 3.

Solution. First notice that $18 = 3^2 . 2$, $16 = 2^4$.

*	3^2	2	2	2^4
10	0	1	1	0
4	0	1	1	0
$(1 + N_{p^{e_i}}(10,4,6,5)$	1	3	3	1

where $N_{p^e}(m_1, m_2, \ldots, m_r)$ is the number of elements in the sequence $m_1, m_2, \ldots, m_r$ that are divisible by p^e.

By Theorem 3, h $(Z_{10} \times Z_4, Z_{18} \times Z_2 \times Z_{16}) = 1 . 3 . 3 . 1 = 9$.

Using Theorems 4 and 5, h $(Z_{10} \times Z_4, Z_{18} \times Z_2 \times Z_{16}) =$ h $(Z_{10} \times Z_4, Z_{18})$. h $(Z_{10} \times Z_4, Z_2)$. h $(Z_{10} \times Z_4, Z_{16}) = 3^{\omega(18) - \omega(18/\gcd(10,4,18))}$

$$.2^{2\omega(18/\gcd(10,4,18)) - \omega(18/\gcd(10,18) - \omega(18/\gcd(4,18))}$$

$$\times 3^{\omega(2) - \omega(2/\gcd(10,4,2))}$$

$$.2^{2\omega(2/\gcd(10,4,2)) - \omega(2/\gcd(10,2) - \omega(2/\gcd(4,2))}$$

$$\times 3^{\omega(16) - \omega(16/\gcd(10,4,16))}$$

$$.2^{2\omega(16/\gcd(10,4,16)) - \omega(16/\gcd(10,16) - \omega(16/\gcd(4,16))}$$

$$= 3 . 2^0 . 3 . 2^0 . 3^0 . 2^0 = 9.$$

Example 2. Compute h $(Z_{10} \times Z_4 \times Z_6 \times Z_5, Z_{18} \times Z_2 \times Z_{16} \times Z_{15})$.

Solution. First notice that $18 = 3^2 . 2$, $16 = 2^4$, $15 = 3 . 5$.

*	3^2	2	2	2^4	5	3
10	0	1	1	0	1	0
4	0	1	1	0	0	0
6	0	1	1	0	0	1
5	0	0	0	0	1	0
$(1 + N_{p^{e_i}}(10,4,6,5)$	1	4	4	1	3	2

By Theorem 3, h $(Z_{10} \times Z_4 \times Z_6 \times Z_5, Z_{18} \times Z_2 \times Z_{16} \times Z_{15}) = 1.4.4.1.3.2 = 96$.

REFERENCES

Anderson, F., and K. Fuller. 1992. *Rings and Categories of Modules*, 2nd. ed., Springer-Verlag.

Gallian, J.A., and J. Van Buskirk. 1984. The Number of Homomorphisms From Z_m into Z_n, *Amer. Math. Monthly*, 91, 196–197.

Gallian, J.A. 1994. *Contemporary Abstract Algebra*, 3rd. ed., D.C. Heath and Company.

Saleh, M., and H. Yousef. 1998. Number of Ring Homomorphisms, *Amer. Math. Monthly*, forthcoming.

Wisbauer, R. 1991. *Foundations of Module and Ring Theory*, Gordon and Beach Science Publishers.

تعميم لنتائج جاليان وبسكيرك حول عدد اقترانات الزمر واقترانات الحلقات الهومومورفية

محمد صالح، حسن يوسف، و جواد ابو هليل*

ملخص

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* قسم الرياضيات، جامعة بير زيت، ص.ب. 14، بير زيت، الضفة الغربية، فلسطين.

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