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ON FAINT AND QUASI θ-CONTINUITIES

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Abstract

In this article, we investigate some properties of quasi θ-continuity and faint continuity and their graphs. The notions of θ-Hausdorffness and θ-compactness are introduced.

1. Introduction

The concepts of θ-closure and θ-interior operators were first introduced by Velicko. These operators have since been studied intensively by many authors. Although θ-closure operators are not idempotents, the collection of all θ-open sets in a topological space (X, Γ) forms a topology Γ_θ on X, weaker than Γ. So far, numerous applications of such operators have been found in studying different types of continuous like maps, separation of axioms, and above all, to many important types of compact like properties. In 1982, Long and Herrington [5] introduced faintly continuous maps as a generalization of weakly continuous maps. Later in 1990, Noiri and Popa [8] introduced quasi θ-continuity as a generalization of θ-continuity.

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The purpose of this paper is to study further these concepts. In Section 2, we give some basic properties of these maps. Among other results, it is shown that a function $f : X \to Y$ is faintly continuous iff its graph mapping $g = (x, f(x))$ is faintly continuous improving a result by Long and Herrington. Similarly, for quasi θ-continuity. In Section 3, we introduce the notion of θ-Hausdorffness and θ-compactness. It is shown that the preimage of a θ-Hausdorff injective quasi θ-continuous is θ-Hausdorff. We get similar results to some of those in [3, 4, 5, 6, 7, 8, 9, 10, 11] applied to faint and quasi θ-continuities.

For a set A in a space X, let us denote by $Int(A)$ and $\overline{A}$, the interior and the closure of A in X, respectively. Following Velicko [12], a point x of a space X is called a θ-*adherent point* of a subset A of X iff $\overline{U} \cap A \neq \varnothing$, for every open set U containing x. The set of all θ-adherent points of A is called the θ-*closure* of A, denoted by $cls_\theta A$. A subset A of a space X is called θ-*closed* iff $A = cls_\theta A$. The complement of a θ-closed set is called θ-*open*. Similarly, the θ-interior of a set A in X, written $Int_\theta A$, consists of those points x of A such that for some open set U containing x, $\overline{U} \subseteq A$. A set A is θ-open iff $A = Int_\theta A$, or equivalently, $X - A$ is θ-closed. One of the most interesting weaker forms of compactness is closure compactness. A subset A of a space X is called a *closure compact subset* or *quasi-H-closed* (QHC) if every open cover has a finite subcollection whose closures cover A. A closure compact Hausdorff space is called *H-closed*, first defined by Alexandroff and Urysohn [1]. A function $f : X \to Y$ is *closure* or θ-*continuous* (resp., *weakly*, *strongly*) continuous if given any open set V in Y, there exist an open set U in X such that $f(\overline{U}) \subseteq \overline{V}$ (resp., $f(U) \subseteq \overline{V}$, $f(\overline{U}) \subseteq V$). A space X is called *Urysohn* if for every $x \neq y \in X$, there exists an open set U containing x and an open set V containing y such that $\overline{U} \cap \overline{V} = \varnothing$. A function $f : X \to Y$ is said to be *faintly continuous* (*f.c.*) (resp., quasi θ-continuous (*q.θ.c.*)) if the inverse image of every θ-open set is open (θ-open).

2. Basic Results

In this section, we give some basic results of *faint* and *quasi* θ-*continuities* that will be needed in this paper.

It is clear that a space X is Hausdorff iff $\{x\}$ is θ-closed, for every $x \in X$. Moreover, a space X is Hausdorff iff every compact subset is θ-closed.

Lemma 2.1. *Let $f : X \to Y$ be weakly (resp., closure) continuous. Then the inverse image of every θ-open set is open (θ-open).*

Corollary 2.1. *Every weakly (resp., closure) continuous function is faintly (resp., quasi θ-continuous) continuous.*

Corollary 2.2. *Let $f : X \to Y$ be faintly (resp., quasi θ-continuous) continuous, where Y is a Hausdorff-space. Then f has closed (θ-closed) point inverses.*

As a consequence of Corollary 2.2, we get Theorem 6 in [2]. A quasi θ-continuous need not be weakly continuous as it is shown in the next example.

Example 2.1. Let $X = R$ with the cocountable topology $\Im_c$, $Y = \{0, 1, 2\}$ with $\Im = \{\varnothing, \{0\}, \{1\}, \{0, 1\}, Y\}$. Define $f : X \to Y$ as f (*rationals*) $= 0$, f (*irrationals*) $= 1$. Then f is quasi θ-continuous but not weakly continuous.

The proofs of the following results follow easily from the definitions.

Theorem 2.1. *Let $f : X \to Y$ be faintly continuous and let $g : Y \to Z$ be quasi θ-continuous (resp., strongly continuous). Then $g \circ f : X \to Z$ is faintly continuous (resp., continuous).*

Theorem 2.2. *Let $f : X \to Y$ be continuous and let $g : Y \to Z$ be faintly continuous. Then $g \circ f : X \to Z$ is faintly continuous.*

Theorem 2.3. *Let $f : X \to Y$ be a quasi θ-continuous and let $g : Y \to Z$ be quasi θ-continuous (resp., strongly continuous). Then $g \circ f : X \to Z$ is quasi θ-continuous (resp., strongly continuous).*

In [3, 6, 10] it is shown that a function f is almost (resp., weakly, closure) continuous iff its graph mapping g, where $g(x) = (x, f(x))$, is almost (resp., weakly, closure) continuous. Also, in [5, Theorem 14], it is shown that if f is weakly continuous, then the graph mapping is faintly continuous. Next, we improve this result and get similar results for quasi θ-continuities.

Lemma 2.2. *A function $f : X \to Y$ is q.θ.c.(f.c.) at $x \in X$ iff for every θ-open set V containing $f(x)$ in Y there exists an open set U containing x in X such that $f(\overline{U}) \subseteq V$ ($f(U) \subseteq V$).*

Theorem 2.4. *Let $f : X \to Y$ be a mapping and let $g : X \to X \times Y$ be the graph mapping of f given by $g(x) = (x, f(x))$ for every point $x \in X$. Then $g : X \to X \times Y$ is f.c. (resp., q.θ.c.) iff $f : X \to Y$ is f.c. (resp., q.θ.c.).*

Proof. We will give the proof for f.c. maps only. Suppose g is f.c. Theorem 2.2 implies that f is f.c. Conversely, assume f is f.c. Let $x \in X$, and let V be a θ-open subset of $X \times Y$ containing $g(x)$. Then there exist θ-open sets U, W in X, Y, respectively, containing x, $f(x)$ such that $g(x) \in U \times W \subseteq V$, since $g(x)$ is the graph mapping of f. By f.c. of f, there exists an open set A of x such that $f(A) \subseteq W$. Let $B = U \cap A$. Then $g(B) \subseteq U \times W \subseteq V$, proving that g is f.c.

The next theorem gives characterizations of f.c. (resp., q.θ.c.) using nets and filters.

Theorem 2.5. *Let $f : X \to Y$. Then the following are equivalent:*

(a) *f is f.c. (resp., q.θ.c.).*

(b) *For each $x \in X$ and for each filter base $\mathcal{F}$ converges (resp., θ-converges) to x, $f(\mathcal{F})$ θ-converges to $f(x)$.*

(c) *For each $x \in X$ and for each net $\{x_\alpha\}_{\alpha \in D}$ converges (resp., θ-converges) to x, $\{f(x_\alpha)\}_{\alpha \in D}$ θ-converges to $f(x)$.*

Similar to δ-continuity [7, Theorems 3.3 and 3.4] and following similar arguments as in [4, Theorems 6 and 7], we get the following results.

Theorem 2.6. *Let $f : X \to \prod_{\alpha \in I} X_\alpha$ be given. Then f is q.θ.c. (resp., f.c.) iff the composition with each projection π_α is q.θ.c. (resp., f.c.).*

Theorem 2.7. *Define $\prod_{\alpha \in I} f_\alpha : \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ by $\{x_\alpha\} \to \{f_\alpha(x_\alpha)\}$. Then $\prod f_\alpha$ is q.θ.c. (resp., f.c.) iff each $f_\alpha : X_\alpha \to Y_\alpha$ is q.θ.c. (resp., f.c.).*

3. Applications

Definition 3.1. A space X is said to be θ-*Hausdorff* if for every $x \neq y \in X$, there exist θ-open sets U_x, V_y such that $U_x \cap V_y = \varnothing$.

It is clear that every θ-Hausdorff is Urysohn and every regular T_1-space is θ-Hausdorff, but a θ-Hausdorff space need not be regular. Also, a Urysohn space is not necessarily a θ-Hausdorff space as shown in the following example.

Example 3.1. Let X be the reals with the topology $\Im$ whose basis is generated by the sets of the form (a, b) and $(a, b) - K$, where $K = \left\{\frac{1}{n} : n \in Z^+\right\}$. Then $\Im$ is θ-Hausdorff but not regular.

Example 3.2. Let w be the first infinite ordinal and w_1 be the first uncountable ordinal both with the order topology. Let $R = \{(w_1 + 1) \times (w + 1) \setminus (w_1, w)\}$ with the product topology. The space R is Tychonoff and it is called the *Tychonoff plank*. Let $Rn = R \times n$ and $E = R1 \cup R2 \cup R3$ with (w_1, y, n) identified with $(w_1, y, n+1)$ whenever n is odd and (x, w, n) identified with $(x, w, n+1)$ whenever n is even; place quotient topology on E. Let X be the space of E union two points p and q. A subset U of X is defined to be open if (1) U intersect E is open in E, (2) p in U implies there exist some t less than w_1 and s less than w

such that $(x, y, 1)$ is in U whenever t less than y and y is less than w_1 and s is less than x and x is less or equal to w, (3) q in U implies there exist some t less than w_1 and s less than w such that $(x, y, 3)$ is in U whenever t less than y and y is less than or equal to w_1 and s is less than x and x is less than w. The space X is Urysohn but not θ-Hausdorff since the θ-open sets containing p and q intersect.

By a faint retraction (resp., quasi θ-retraction, strong retraction) we mean an *f.c.* (resp., *q.θ.c*, *sc.*) function $f : X \to A$ where $A \subseteq X$ and $f|_A$ is the identity function on A.

Theorem 3.1. *Let $A \subseteq X$ and let $f : X \to A$ be a faint retraction of X onto A. If X is a θ-Hausdorff space, then A is a closed subset of X.*

Proof. Suppose not, then there exists a point $x \in \overline{A} \setminus A$. Since f is a faint retraction, $f(x) \neq x$. Since X is θ-Hausdorff, there exist θ-open sets U and V containing x and $f(x)$, respectively, such that $U \cap V = \emptyset$. Now, let W be any open set in X containing x. Then $U \cap W$ is an open set containing x and hence $U \cap W \cap A \neq \emptyset$, since $x \in \overline{A}$. Therefore, there exists a point $y \in U \cap W \cap A$. Since $y \in A$, $f(y) = y \in U$ and hence $f(y) \notin V$. This shows that $f(W)$ is not contained in V. This contradicts the hypothesis that f is faintly continuous. Thus A is closed as claimed.

Theorem 3.2. *Let $f : X \to Y$ be a f.c. and injective function. If Y is θ-Hausdorff, then X is Hausdorff.*

Proof. For any distinct points $x_1, x_2 \in X$, since f is one-to-one, we have $f(x_1) \neq f(x_2)$. Since Y is a θ-Hausdorff, there exist θ-open sets V_1, V_2 containing $f(x_1)$ and $f(x_2)$, respectively, such that $V_1 \cap V_2 = \emptyset$. But since f is *f.c.* there exist open sets U_1, U_2 containing x_1, x_2, respectively, such that $f(U_1) \subseteq V_1$, and $f(U_2) \subseteq V_2$. Thus $U_1 \cap U_2 = \emptyset$, proving that X is Hausdorff.

Now, we list following straightforward results.

Theorem 3.3. *Let $A \subseteq X$ and let $f : X \to A$ be a quasi θ-retraction of X onto A. If X is θ-Hausdorff, then A is a θ-closed subset of X.*

Theorem 3.4. *Let $f : X \to Y$ be a quasi θ-continuous and injective function. If Y is θ-Hausdorff, then X is θ-Hausdorff.*

Theorem 3.5. *Let $f : X \to Y$ be a strong retraction of X onto A. If X is Hausdorff, then A is a θ-closed subset of X.*

Theorem 3.6. *Let $f : X \to Y$ be a strongly continuous and bijective function. If Y is Hausdorff, then X is a θ-Hausdorff space.*

Similar to *a.c.* and weakly continuous functions, *f.c.* (resp., *q.θ.c.*) maps are not connected (do not preserve connectedness), since if $f : X \to Y$ is *f.c.* (resp., *q.θ.c.*), then it need not be true that $f : X \to f(X)$ is *f.c.* (resp., *q.θ.c.*) as we show in the next example. However, the restricted map of *f.c.* (resp., *q.θ.c.*) in the domain is *f.c.* (resp., *q.θ.c.*).

Example 3.3. Let $X = R$ be the reals, U be the usual topology, $\Im_c$ be the cocountable topology. Define $f : (R, U) \to (R, \Im_c)$ as f *(rationals)* $= 0$, f *(irrationals)* $= 1$, then f is closure continuous, but $f : (R, U) \to f(X) = \{0, 1\}$ is not even *f.c.* It is clear that f is not connected.

It is well-known that the image of compact is closure compact under weakly continuous functions, the image of closure compact is compact under strongly continuous functions and the image of closure compact is closure compact under closure continuous functions. The following results are similar to that applied to faint and quasi θ-continuities.

Definition 3.2. A subset A of a space X is said to be *theta compact* (briefly, *θ-compact*) if every cover of θ-open sets of A has a finite subcover, or equivalently A is compact in Γ_θ.

It is clear that every closure compact subset is θ-compact but not conversely as in Example 3.2, if U is a θ-open set containing p and V is a θ-open set containing q, then $X \setminus (U \cup V)$ is compact. But the space X is not H-closed.

Lemma 3.1. *A function* $f : (X, \Gamma) \to (Y, \Im)$ *is faintly continuous iff* $f : (X, \Gamma) \to (Y, \Im_\theta)$ *is continuous and* $f : (X, \Gamma) \to (Y, \Im)$ *is quasi* θ*-continuous iff* $f : (X, \Gamma_\theta) \to (Y, \Im_\theta)$ *is continuous. Also,* $f : (X, \Gamma) \to (Y, \Im)$ *is strongly continuous iff* $f : (X, \Gamma_\theta) \to (Y, \Im)$ *is continuous.*

Lemma 3.2. *A subset* $K \subset (X, \Gamma)$ *is a* θ*-compact subset iff* $K \subset (X, \Gamma_\theta)$ *is compact. Also, a space* (X, Γ) *is* θ*-Hausdorff iff* (X, Γ_θ) *is Hausdorff. A subset* $A \subseteq X$ *is* θ*-closed in* (X, Γ) *iff* A *is closed in* (X, Γ_θ).

The proofs of the following results are straightforward from Lemmas 3.1 and 3.2.

Theorem 3.7. *Let* $f : X \to Y$ *be f.c. and let K be a compact subset of X. Then* $f(K)$ *is a* θ*-compact subset of Y.*

Theorem 3.8. *Let* $f : X \to Y$ *be q.*θ*.c. and let K be a* θ*-compact subset of X. Then* $f(K)$ *is a* θ*-compact subset of Y.*

Theorem 3.9. *Let* $f : X \to Y$ *be strongly continuous and let K be a* θ*-compact subset of X. Then* $f(K)$ *is a compact subset of Y.*

Theorem 3.10. *A* θ*-compact subset of a* θ*-Hausdorff space is* θ*-closed.*

Theorem 3.11. *Every* θ*-closed subset of a* θ*-compact space is* θ*-compact.*

Theorem 3.12. *The product of* θ*-compact spaces is* θ*-compact.*

Theorem 3.13. *A finite union of* θ*-compact is* θ*-compact.*

Theorem 3.14. *Let* f, g *be f.c. from a space X into a* θ*-Hausdorff space Y. Then the set* $A = \{x \in X : f(x) = g(x)\}$ *is a closed set.*

Theorem 3.15. *Let* f, g *be q.*θ*.c. from a space X into a* θ*-Hausdorff space Y. Then the set* $A = \{x \in X : f(x) = g(x)\}$ *is a* θ*-closed set.*

Definition 3.3. A function *f* is said to be θ*-open* if the image of every open set is θ-open. Similarly, a function *f* is said to be θ*-closed* if the image of every closed set is θ-closed.

Theorem 3.16. *Let* $f : X \to Y$ *be f.c.* $1-1$*, onto. If X is compact and Y is* θ*-Hausdorff, then f is* θ*-open.*

Proof. Let U be an open subset of X, and thus $X \setminus U$ is a closed subset of X. Hence, $X \setminus U$ is compact. Since *f* is *f.c.*, Theorem 3.7 implies that $f(X \setminus U)$ is θ-compact. Therefore, Theorem 3.10 implies that $f(X \setminus U) = Y \setminus f(U)$ is θ-closed, and thus $f(U)$ is θ-open.

The following results follow easily.

Theorem 3.17. *Let* $f : X \to Y$ *be f.c. If X is compact and Y is* θ*-Hausdorff, then f is* θ*-closed.*

Theorem 3.18. *Let* $f : X \to Y$ *be q.*θ*.c.* $1-1$*, onto. If X is* θ*-compact and Y is* θ*-Hausdorff, then the image of* θ*-open is* θ*-open.*

Theorem 3.19. *Let* $f : X \to Y$ *be q.*θ*.c. If X is* θ*-compact and Y is* θ*-Hausdorff, then the image of* θ*-closed is* θ*-closed.*

Questions. (1) Does there exists a faintly continuous map which is not quasi θ-continuous?

(2) Is the composition of two faintly continuous maps faintly continuous?

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