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On the difference equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$ with $A < 0$

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Abstract

We find conditions for the global asymptotic stability of the unique negative equilibrium $\bar{y} = 1 + A$ of the equation

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}, \tag{0.1}$$

where $y_{-k}, y_{-k+1}, \ldots, y_0 \in (0, \infty)$, $A < 0$ and $k \in \{1, 2, 3, 4, \ldots\}$.

Keywords: Recursive sequence; Global asymptotic stability

1. Introduction

In [1] the periodicity of the difference equation

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $y_{-k}, \ldots, y_{-1}, y_0$, $A \in (0, \infty)$ and $k \in \{2, 3, 4, \ldots\}$ was studied. Our aim in this paper is to establish global asymptotic stability results for this difference equation with $A < 0$.

It was shown in [3] that for the case $k = 1$ the positive equilibrium $\bar{y} = 1 + A$ of Eq. (1.1) is globally asymptotically stable for $A > 1$. In [1], the periodicity of Eq. (1.1) was investigated. In this note, other related results of asymptotic, periodicity, and semi-cycles are investigated. We list below some definitions and basic results that will be needed in this paper (see [5,7,10]).

Definition 1.1. We say that a solution $\{y_n\}_{n=-k}^{\infty}$ of a difference equation $y_{n+1} = f(y_n, y_{n-1}, \ldots, y_{n-k})$ is *periodic* if there exists a positive integer p such that $y_{n+p} = y_n$. The smallest such positive integer p is called the prime period of the solution of the difference equation.

Definition 1.2. The equilibrium point $\bar{y}$ of the equation:

$$y_{n+1} = f(y_n, y_{n-1}, \ldots, y_{n-k}), \quad n = 0, 1, \ldots$$

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is the point that satisfies the condition

$$\bar{y} = f(\bar{y}, \bar{y}, \dots, \bar{y}).$$

Definition 1.3. Let $\bar{y}$ be an equilibrium point of Eq. (1.1). Then the equilibrium point $\bar{y}$ is called

(1) locally stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $y_{-k}, y_{-k+1}, \dots, y_0 \in I$ with $|y_{-k} - \bar{y}| + |y_{-k+1} - \bar{y}| + \cdots + |y_0 - \bar{y}| < \delta$, we have $|y_n - \bar{y}| < \epsilon$ for all $n \geqslant -1$,
(2) locally asymptotically stable if it is locally stable and if there exists $\gamma > 0$ such that for all $y_{-k}, y_{-k+1}, \dots, y_0 \in I$ with $|y_{-k} - \bar{y}| + |y_{-k+1} - \bar{y}| + \cdots + |y_0 - \bar{y}| < \gamma$, we have $\lim_{n\to\infty} y_n = \bar{y}$,
(3) a global attractor if for all $y_{-k}, y_{-k+1}, \dots, y_0 \in I$, we have $\lim_{n\to\infty} y_n = \bar{y}$,
(4) globally asymptotically stable if $\bar{y}$ is locally stable and $\bar{y}$ is a global attractor.

The linearized equation of Eq. (1.1) about the negative equilibrium $\bar{y} = 1 + A$ is

$$z_{n+1} - \frac{1}{1+A} z_n + \frac{1}{1+A} z_{n-k} = 0, \quad n = 0, 1, \dots \tag{1.2}$$

The following result is a consequence of the conditions given in [6, page 12], see also [8,9].

Lemma 1.4. *Assume that $a, b \in \mathbb{R}$ and $k \in \{1, 2, \dots\}$. Then*

$$|a| + |b| < 1 \tag{1.3}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+1} + a y_n + b y_{n-k} = 0, \quad n = 0, 1, \dots \tag{1.4}$$

Suppose in addition that one of the following two cases holds.

(*a*) *k odd and $b < 0$.*
(*b*) *k even and $ab < 0$.*

Then (1.3) is also a necessary condition for the asymptotic stability of Eq. (1.4).

Lemma 1.5. *Assume that $a, b \in \mathbb{R}$. Then*

$$|a| < b + 1 < 2$$

is a necessary and sufficient condition for the asymptotic stability of the difference equation

$$y_{n+1} + a y_n + b y_{n-k} = 0, \quad n = 0, 1, \dots \tag{1.5}$$

Lemma 1.6. *The difference equation*

$$y_{n+1} - b y_n + b y_{n-k} = 0, \quad n = 0, 1, \dots \tag{1.6}$$

is asymptotically stable iff $0 < |b| < 1/2 \cos\left(\frac{k\pi}{k+2}\right)$.

Lemma 1.7. *Consider Eq. (1.1). If $A < -2\cos\left(\frac{\pi}{k+2}\right) - 1$ then the unique negative equilibrium $\bar{y} = 1 + A$ of Eq. (1.1) is locally asymptotically stable, while if $A > -2\cos\left(\frac{\pi}{k+2}\right) - 1$ then the positive equilibrium is unstable.*

Proof. The proof is a direct consequence of the conditions in Lemma 1.6. □

The above lemmas lead to parts (a), (b) of the next theorem and the proof of part (c) is straightforward.

Theorem 1.8. *The following statements are true*:

(*a*) *The equilibrium point* $A+1$ *of Eq.* (1.1) *is locally asymptotically stable iff* $A<-3$.
(*b*) *The equilibrium point* $A+1$ *of Eq.* (1.1) *is unstable if* $0 \geqslant A \geqslant -3$.
(*c*) *If a solution of Eq.* (1.1) *is eventually constant then* $y_n = A+1$, $n=-k,-k+1,\dots$

2. Analysis of the global stability, boundedness and the semi-cycles of solutions of Eq. (1.1)

In this section, we show that every negative solution of Eq. (1.1) is globally asymptotically stable and thus get as a corollary the boundedness and persistence of solutions.

We say that a solution $\{y_n\}$ of a difference equation $y_{n+1}=f(y_n,y_{n-1},\dots,y_{n-k})$ is *bounded and persists* if there exist positive constants P and Q such that

$$P \leqslant x_n \leqslant Q, \quad \text{for } n=-1,0,\dots$$

A positive semi-cycle of a solution $\{y_n\}$ of Eq. (1.1) consists of a "string" of terms $\{y_l,y_{l+1},\dots,y_m\}$, all greater than or equal to the equilibrium $\bar{y}$, with $l \geqslant -k$ and $m \leqslant \infty$ and such that

$$\text{either } l=-k \quad \text{or} \quad l>-k \text{ and } y_{l-1}<\bar{y},$$

and

$$\text{either } m=\infty \quad \text{or} \quad m<\infty \text{ and } y_{m+1}<\bar{y}.$$

A negative semi-cycle of a solution $\{y_n\}$ of Eq. (1.1) consists of a "string" of terms $\{y_l,y_{l+1},\dots,y_m\}$, all less than the equilibrium $\bar{y}$, with $l \geqslant -k$ and $m \leqslant \infty$ and such that

$$\text{either } l=-k \quad \text{or} \quad l>-k \text{ and } y_{l-1}\geqslant\bar{y},$$

and

$$\text{either } m=\infty \quad \text{or} \quad m<\infty \text{ and } y_{m+1}\geqslant\bar{y}.$$

The first semi-cycle of a solution starts with the term y_{-k} and is positive if $y_{-k}\geqslant\bar{y}$ and negative if $y_{-k}<\bar{y}$.

A solution $\{y_n\}$ of Eq. (1.1) is called *nonoscillatory* if there exists $N\geqslant -k$ such that $y_n>\bar{y}$ for all $n\geqslant N$ or $y_n<\bar{y}$ for all $n\leqslant N$.

And a solution $\{y_n\}$ is called *oscillatory* if it is not nonoscillatory.

Theorem 2.1. *Eq.* (1.1) *has no solution of prime period* 2 *if* $A\neq -1$ *or* k *is even.*

Proof. If k is even then $\Phi=\Psi=A+1$ in which case $p\neq 2$.

If n is odd, then $\Phi=A+\frac{\Phi}{\Psi}$ and $\Psi=A+\frac{\Psi}{\Phi}$. It follows that $\frac{\Phi}{\Psi}=\Phi-A$ and $\frac{\Psi}{\Phi}=\Psi-A$. Multiplying the last two equations, we get $(\Psi-A)(\Phi-A)=1$. Thus, $\Phi\neq A$ and $\Psi\neq A$.

Moreover, we conclude that $\Psi=\frac{1}{\Phi-A}+A$. But on the other hand, we have $\frac{1}{\Psi}-\frac{1}{\Phi}=\frac{1}{\Psi^2}-\frac{1}{\Phi^2}$. Therefore, we get $\frac{1}{\Psi}+\frac{1}{\Phi}=-1$. Solving for Ψ, we get $\Psi=\frac{-\Phi}{1+\Phi}$.

The last two equations lead to $A=-1$. We conclude that the period 2 solution takes the form $\dots,\Phi,\frac{-\Phi}{1+\Phi},\Phi,\frac{-\Phi}{1+\Phi},\dots$ This completes the proof. □

Notice that the solution oscillates about the steady state $y=0$ when $A=-1$. Every semi-cycle is of length one.

Now we find a global asymptotic stability result for the general case $k\in\{2,3,4,\dots\}$.

Theorem 2.2. [4] *Consider the difference equation*

$$y_{n+1}=f(y_n,y_{n-k}), \quad n=0,1,\dots, \tag{2.1}$$

where $k\in\{1,2,\dots\}$. *Let* $I=[a,b]$ *be some interval of real numbers and assume that*

$$f:[a,b]\times[a,b]\to[a,b]$$

is a continuous function satisfying the following properties:

(*a*) $f(u,v)$ *is nonincreasing in u and nondecreasing in v.*
(*b*) *If* $(m,M)\in[a,b]\times[a,b]$ *is a solution of the system*

$$m=f(M,m)\quad\text{and}\quad M=f(m,M),$$

then $m = M$. *Then Eq.* (2.1) *has a unique equilibrium* $\bar{y}$ *and every solution of Eq.* (2.1) *converges to* $\bar{y}$.

Theorem 2.3. *Let* $A<-3$. *Then the unique negative equilibrium* $\bar{y}=1+A$ *of Eq.* (1.1) *is globally asymptotically stable.*

Proof. Define $f(u,v)=A+u/v$. Then the result follows directly from Theorem 2.2. □

The global stability of the difference equation implies the boundedness of the difference equation.

Corollary 2.4. *Let* $A<-3$. *Then every solution of Eq.* (1.1) *is bounded and persists.*

We consider the following lemma about the behavior of the semi-cycles of Eq. (1.1).

Lemma 2.5. *Let* $\{y_n\}$ *be a nontrivial solution of Eq.* (1.1), $A=-1$, $k\geqslant 2$. *Then every semi-cycle has at most* 2 *terms.*

Theorem 2.6. *Let k be odd and let*

$$y_{-k},y_{-k+0},\ldots,y_{-1}\leqslant A+1,\quad 0>y_{-k+1},y_{-k+3},\ldots,y_0>A+1.$$

Then, the solution $\{y_n\}_{n=-k}^{\infty}$ *is oscillatory and every semi-cycle has length one. Moreover, every term of* $\{y_n\}_{n=-k}^{\infty}$ *is strictly greater than A with the possible exception of the first* $k+1$ *semi-cycles, no term of* $\{y_n\}_{n=1}^{\infty}$ *is ever equal to* $A+7$.

Proof. Just notice that, for any $n\geqslant 1$,

$$y_{2n+1}=A+\frac{y_{2n-k}}{y_{2n}}>A+1,$$

and

$$y_{2n}=A+\frac{y_{2n-(k+1)}}{y_{2n-1}}<A+1.$$

The result then follows. □

3. Case $-3\leqslant A<0$

As it is noticed in Theorem 1.8, in this case the equilibrium point $A+1$ is not even asymptotically stable. Also, it is shown in [2] that for the case $k=1$ not every solution is bounded and thus is not even asymptotically stable. For the case $A<1$, one might study necessary and sufficient condition on the initial conditions so that every solution is asymptotically or globally stable or even is bounded.

4. The case $k = 1$

DeVault et al. [3] studied the difference equation

$$y_{n+1}=A+\frac{y_n}{y_{n-1}},\tag{4.1}$$

with $A>0$ and strictly positive initial conditions. Now, we study the stability properties and semi-cycle behavior of this equation without positivity restrictions. Consider the equation

$$y_{n+1} = A - \frac{y_n}{y_{n-1}}, \tag{4.2}$$

where $A < 0$. Then, using the change of variables

$$y_n = -x_n.$$

Eq. (4.2) becomes

$$x_{n+1} = -A + \frac{x_n}{x_{n-1}} = \alpha + \frac{x_n}{x_{n-1}},$$

where $\alpha = -A > 0$. It follows that all the results in DeVault et al. hold for Eq. (4.2) in the following cases:

- $A < 0$, $y_{-1}, y_0 < 0$.
- $A < 0$, $y_{-1}, y_0 > 0$.
- $A > 0$, $y_{-1} > 0$, $y_0 < 0$ (without the above change of variables).

Theorem 4.1. *Let $A = -1$, $y_{-1}, y_1 \in \mathbb{R}^*$, and let $\{y_n\}_{n=1}^{\infty}$ be a solution of Eq. (4.1). Then, every positive semicycle is of length one.*

Proof. When $k = 1$. Let $y_n < 0$ and $y_{n+1} > 0$. Then

$$y_{n+2} = A + \frac{y_{n+1}}{y_n} < 0.$$

This completes the proof. □

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