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The Number of Ring Homomorphisms from $Z_{m_1} \times \cdots \times Z_{m_r}$ into $Z_{k_1} \times \cdots \times Z_{k_s}$

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Source: *The American Mathematical Monthly*, Vol. 105, No. 3 (Mar., 1998), pp. 259-260

Published by: Mathematical Association of America

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(ii) \Leftrightarrow (iii). Multiply both sides of (ii) by $T^{-1/2}$ to obtain the equivalent inequality $\lambda + (1 - \lambda)T^{-1} \leq T^{\lambda-1}$ for any $\lambda > 1$. Now set $\mu = 1 - \lambda < 0$ and $S = T^{-1}$. Then $\mu S + (1 - \mu) \leq S^\mu$. Thus (ii) implies (iii), and similarly (iii) implies (ii).

Theorem 3. *Let A and B be positive invertible operators on a Hilbert space. Then the following hold and are mutually equivalent:*

- (i) *If $1 \geq \lambda \geq 0$, then $(1 - \lambda)A + \lambda B \geq A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}}$;*
- (ii) *if $\lambda > 1$, then $(1 - \lambda)A + \lambda B \leq A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}}$;*
- (iii) *if $\lambda < 0$, then $(1 - \lambda)A + \lambda B \leq A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}}$.*

Proof: In Theorem 2, we have only to put $T = A^{-(1/2)}BA^{-(1/2)}$ and multiply by $A^{1/2}$ on both sides.

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The Number of Ring Homomorphisms From $Z_{m_1} \times \cdots \times Z_{m_r}$ into $Z_{k_1} \times \cdots \times Z_{k_s}$

Mohammad Saleh and Hasan Yousef

The purpose of this note is to compute the number of ring homomorphisms from $Z_{m_1} \times \cdots \times Z_{m_r}$ into $Z_{k_1} \times \cdots \times Z_{k_s}$, a result that generalizes [1] (Z_k denotes the ring of integers mod k). If A and B are rings, we use $Hom(A, B)$ to denote the set of all ring homomorphisms from A into B , and $h(A, B)$ to denote the cardinality of $Hom(A, B)$. First of all notice that

$$Hom(Z_{m_1} \times \cdots \times Z_{m_r}, Z_{k_1} \times \cdots \times Z_{k_s}) \cong \prod_{j=1}^{j=s} Hom(Z_{m_1} \times \cdots \times Z_{m_r}, Z_{k_j})$$

as abelian groups. Thus

$$h(Z_{m_1} \times \cdots \times Z_{m_r}, Z_{k_1} \times \cdots \times Z_{k_s}) = \prod_{j=1}^{j=s} h(Z_{m_1} \times \cdots \times Z_{m_r}, Z_{k_j}).$$

Let $k = p_1^{t_1} p_2^{t_2} \cdots p_s^{t_s}$ be the prime-power decomposition of k in Z . By the Chinese Remainder Theorem, it follows that Z_k is naturally ring-isomorphic to $Z_{p_1^{t_1}} \times \cdots \times Z_{p_s^{t_s}}$. Thus, we need only to compute $h(Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r}, Z_{p^k})$, where p is a prime.

Theorem 1. *The number of ring homomorphisms from $Z_{m_1} \times \cdots \times Z_{m_r}$ into Z_{p^k} is given by*

$$1 + N_{p^k}(m_1, \dots, m_r),$$

where $N_{p^k}(m_1, \dots, m_r)$ is the number of elements in the set $\{m_1, \dots, m_r\}$ that are divisible by p^k .

Proof: Let $\varphi: Z_{m_1} \times \cdots \times Z_{m_r} \rightarrow Z_{p^k}$ be a ring homomorphism. Then φ is completely determined by $\varphi(e_1), \dots, \varphi(e_r)$ where e_i is the r -tuple with 1 in the i th component and 0's elsewhere. These are idempotent in Z_{p^k} and hence each must be either 0 or 1. Also, if $\varphi(e_i) = \varphi(e_j) = 1$ for $i \neq j$, then one obtains the contradiction

$$0 = \varphi(0) = \varphi(e_i e_j) = \varphi(e_i) \varphi(e_j) = 1 \cdot 1 = 1.$$

Thus if φ is not the zero homomorphism, then $\varphi(e_i) = 1$ for exactly one value i , and moreover for that i , p^k must divide m_i . Thus, $h(Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r}, Z_{p^k}) = 1 + N_{p^k}(m_1, m_2, \dots, m_r)$, where $N_{p^k}(m_1, m_2, \dots, m_r)$ is the number of elements in the set $\{m_1, m_2, \dots, m_r\}$ that are divisible by p^k .

Theorem 2. *The number of ring homomorphisms from $Z_{m_1} \times \cdots \times Z_{m_r}$ into $Z_{p_1^{k_1}} \times \cdots \times Z_{p_s^{k_s}}$, where $p_i, 1 \leq i \leq s$, are primes not necessarily distinct, is*

$$\prod_{i=1}^s (1 + N_{p_i^{k_i}}(m_1, m_2, \dots, m_r)).$$

Formulas for the number of ring homomorphisms from rings of the form $Z_m[w]$ into $Z_n[w]$, where w is a primitive root of unity, are given in [2] and [3].

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A Simple Proof of a Theorem of Schur

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In 1905, I. Schur [3] proved that the maximum number of mutually commuting linearly independent complex matrices of order n is $\lfloor n^2/4 \rfloor + 1$. Forty years later, Jacobson [2] gave a simpler derivation of Schur's Theorem and extended it from algebraically closed fields to arbitrary fields. We present a simpler proof of this theorem.