The number of ring homomorphisms from Zn1 x…x Znr into Zm1 x…x Zms,

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(ii) ⇔ (iii). Multiply both sides of (ii) by $T^{-1/2}$ to obtain the equivalent inequality $\lambda + (1-\lambda)T^{-1} \le T^{\lambda-1}$ for any $\lambda > 1$. Now set $\mu = 1 - \lambda < 0$ and $S = T^{-1}$. Then $\mu S + (1-\mu) \le S^{\mu}$. Thus (ii) implies (iii), and similarly (iii) implies (ii).

Theorem 3. *Let A and B be positive invertible operators on a Hilbert space. Then the following hold and are mutually equivalent*:

(i) *If* $1 \ge \lambda \ge 0$, *then* $(1-\lambda)A + \lambda B \ge A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\lambda}A^{\frac{1}{2}}$;

(ii) *if* $\lambda > 1$, *then* $(1-\lambda)A + \lambda B \le A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\lambda}A^{\frac{1}{2}}$;

(iii) *if* $\lambda < 0$, *then* $(1-\lambda)A + \lambda B \le A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\lambda}A^{\frac{1}{2}}$.

Proof: In Theorem 2, we have only to put $T = A^{-(1/2)}BA^{-(1/2)}$ and multiply by $A^{1/2}$ on both sides.

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The Number of Ring Homomorphisms From $Z_{m_1} \times \cdots \times Z_{m_r}$ into $Z_{k_1} \times \cdots \times Z_{k_s}$

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The purpose of this note is to compute the number of ring homomorphisms from $Z_{m_1} \times \cdots \times Z_{m_r}$ into $Z_{k_1} \times \cdots \times Z_{k_s}$, a result that generalizes **[1]** (Z_k denotes the ring of integers mod k). If A and B are rings, we use $Hom(A,B)$ to denote the set of all ring homomorphisms from A into B, and $h(A,B)$ to denote the cardinality of $Hom(A,B)$. First of all notice that

$$Hom(Z_{m_1} \times \cdots \times Z_{m_r}, Z_{k_1} \times \cdots \times Z_{k_s}) \cong \prod_{j=1}^{j=s} Hom\left(Z_{m_1} \times \cdots \times Z_{m_r}, Z_{k_j}\right)$$

as abelian groups. Thus

$$h(Z_{m_1} \times \cdots \times Z_{m_r}, Z_{k_1} \times \cdots \times Z_{k_s}) = \prod_{j=1}^{j=s} h(Z_{m_1} \times \cdots \times Z_{m_r}, Z_{k_j}).$$

Let $k = p_1^{t_1} p_2^{t_2} \cdots p_s^{t_s}$ be the prime-power decomposition of k in Z. By the Chinese Remainder Theorem, it follows that Z_k is naturally ring-isomorphic to $Z_{p_1^{t_1}} \times \cdots \times Z_{p_s^{t_s}}$. Thus, we need only to compute $h(Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r}, Z_{p^k})$, where p is a prime.

Theorem 1. *The number of ring homomorphisms from* $Z_{m_1} \times \cdots \times Z_{m_r}$ *into* Z_{p^k} *is given by*

$$1 + N_{p^k}(m_1, \ldots, m_r),$$

where $N_{p^k}(m_1, \ldots, m_r)$ *is the number of elements in the set* $\{m_1, \ldots, m_r\}$ *that are divisible by* p^k.

Proof: Let $\varphi\colon Z_{m_1} \times \cdots \times Z_{m_r} \to Z_{p^k}$ be a ring homomorphism. Then φ is completely determined by $\varphi(e_1), \ldots, \varphi(e_r)$ where e_i is the r-tuple with 1 in the ith component and 0's elsewhere. These are idempotent in Z_{p^k} and hence each must be either 0 or 1. Also, if $\varphi(e_i) = \varphi(e_j) = 1$ for $i \neq j$, then one obtains the contradiction

$$0 = \varphi(0) = \varphi(e_i e_j) = \varphi(e_i)\varphi(e_j) = 1 \cdot 1 = 1.$$

Thus if φ is not the zero homomorphism, then $\varphi(e_i) = 1$ for exactly one value i, and moreover for that i, p^k must divide m_i. Thus, $h(Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r}, Z_{p^k}) = 1 + N_{p^k}(m_1, m_2, \ldots, m_r)$, where $N_{p^k}(m_1, m_2, \ldots, m_r)$ is the number of elements in the set $\{m_1, m_2, \ldots, m_r\}$ that are divisible by p^k.

Theorem 2. *The number of ring homomorphisms from* $Z_{m_1} \times \cdots \times Z_{m_r}$ *into* $Z_{p_1^{k_1}} \times \cdots \times Z_{p_s^{k_s}}$, *where* p_i, $1 \le i \le s$, *are primes not necessarily distinct, is*

$$\prod_{i=1}^{s} \left(1 + N_{p^{k_i}}(m_1, m_2, \ldots, m_r)\right).$$

Formulas for the number of ring homomorphisms from rings of the form $Z_m[w]$ *into* $Z_n[w]$, where w is a primitive root of unity, are given in [**2**] and [**3**].

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A Simple Proof of a Theorem of Schur

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In 1905, I. Schur [**3**] proved that the maximum number of mutually commuting linearly independent complex matrices of order n is $\lfloor n^2/4 \rfloor + 1$. Forty years later, Jacobson [**2**] gave a simpler derivation of Schur's Theorem and extended it from algebraically closed fields to arbitrary fields. We present a simpler proof of this theorem.