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CHARACTERIZING DEMAND FUNCTIONS WITH PRICE DEPENDENT INCOME

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ABSTRACT. We consider the demand function of consumer whose wealth depends on prices. This extends the two traditional cases when the consumer holds a goods bundle, so that his wealth depends linearly on prices, and when his wealth is prescribed, independently of prices. We extend the Slutsky relations to this general case, and we show that they fully characterize the demand functions, as in the traditional cases.

Keywords: Slutsky matrix, Demand function, Indirect utility, Mathematical integration, Economic integration.

Part 1. Introduction

The economics of efficient group behavior has recently attracted renewed attention from economists.[1] One basic framework can be described as follows. Consider a K-person group in a n-commodity framework. The group is endowed with some aggregate income y, which by homogeneity can be normalized to be 1; this income can be used to purchase some (aggregate) bundle X under the budget constraint $p'X = 1$. All commodities are privately consumed; following the standard, 'collective' approach, we assume that the internal decision process always generates efficient outcomes. What does theory predict regarding the group's behavior? Specifically, is the efficiency assumption testable? More generally, is it possible to find necessary and sufficient conditions for efficiency?

The answer to that question obviously depends on the type of data available. One polar case obtains when *only the group's aggregate demand* $X(p)$ is observable. In this case, a result by Chiappori and Ekeland (2009) states that the efficiency assumption is testable if and only if the size of the group is smaller than the number of commodities; the authors derive conditions on $X(p)$ that fully characterize the efficiency assumption. This result has been widely applied empirically, in particular to the analysis of household behavior.However, an increasing number of data sets provide much richer information than the sole aggregate consumption at the group (or household) level. An opposite, and increasingly interesting polar case is therefore one in which individual consumptions within the group are fully recorded; i.e., one can observe the vector $x^k(p)$ for $k = 1, ..., K$. In such a framework, what are the implications of efficiency? For instance, in an efficient household, what properties (if any) should *individual* demand functions satisfy?

Surprisingly enough, the answer to that question is still an open problem. The difficulty, here, comes from the following issue. From the second welfare theorem,

I would like to thank I. Ekeland and G. Carlier for helpful discussions during my visit to the University Paris-Dauphine in September 2010 where the major part of this article was written.

[1]See [7] for a recent survey.

any efficient allocation can be decentralized: there exists functions $w^1(p), ..., w^K(p)$, with $\sum_k w^k(p) = 1$, such that $x^k(p)$ maximizes member k's utility under the budget constraint $p'x^k(p) = w^k(p)$. We are thus left with the following question: can we find necessary and sufficient conditions for a function $x^k(p)$ to solve a maximization problem of this type? In the particular case in which $w^k(p)$ is constant, the answer has been known since the pioneering work of Antonelli [3] and Slutsky [11]: individual demand functions are fully characterized by the fact that the associated Slutsky matrices are symmetric and negative definite. This is quite an important property. On the one hand, since demand functions can be observed, the Slutsky relations provide a testable consequence of utility maximization; empirical tests have been carried out, and provide a validation of current theory (see [4]). On the other, the preference relation can be recovered from the demand function, and this has obvious policy and welfare implications. Another particular case, which has been recently studied, obtains when $w^k(p)$ is linear - i.e., $w^k(p) = p'\omega^k$ where ω^k is a positive initial endowment. Then $x^k(p)$ is an individual excess demand function, whose properties have been fully characterized by Chiappori and Ekeland [8].

The problem is that these cases are extremely specific. Assume that the intrahousehold allocation of income - the $w^k(p)$ functions - stem from some (efficient) bargaining process. Then it is very unlikely that the outcome will involve either fixed or linear allocations; actually, the empirical literature suggests much more complex forms for the sharing rule. The problem is particularly acute when intragroup production is taken into account - an aspect which is important in developed economies and absolutely crucial in developing ones. Generally speaking, whenever production is involved, individuals receive income from production activities; since production functions are typically nonlinear, this will result in nonlinear effects, and individual shares will be nonlinear functions of prices.

In such a general context, where $w^k(p)$ is a known but arbitrary function of prices, little is known about the structure of individual demands. The goal of the present note is precisely to fill this gap, and to study the properties of individual demands stemming from the maximization of individual utility when the individual budget set is given by:

$$B_g(p) := \{x \geq 0 \mid px \leq w(p)\}$$

where $w(p)$ is a prescribed function of the prices p.

The rest of the article is organized as follows. The model is introduced in the next section. In section 2, some preliminary results are given. The main results are given in section 3. In section 4, the problem of characterizing homogeneous demand functions is considered. In the last section, two particular cases are given in which the individual income is price dependent. Namely, we consider an exchange economy and an economy with production.

1. The model

We consider the individual problem in which the agent's income is a function of the price vector $p \in \mathbb{R}^n_{++}$. The consumer faces a problem of the following form

$$(\mathcal{P}) \begin{cases} \max\limits_x \ U(x) \\ p'x = w(p) \end{cases}$$

This kind of problems arise in many economic contexts. Two examples of such models will be given in the last section. We need to adopt some assumptions to insure existence and differentiability of demand functions. We suppose that the utility function is smooth, increasing and concave in a strong sense, namely:

Assumption 1. *We assume that the function $U(x)$ satisfies the following conditions:*

- *the function U is of class C^3 on R^n_{++}.*
- *the gradient $\nabla U(x)$ belongs to R^n_{++}.*
- *the Hessian matrix of U is negative definite on $\{\nabla U\}^{\perp}$.*

These assumptions guarantee the existence, uniqueness and differentiability of the solution $x(p) \in R^n_{++}$. The constraint is binding:

$$p'x(p) = w(p)$$

and there is a Lagrange multiplier $\lambda(p) \geq 0$, so that the problem $(\mathcal{P})$ can be restated as:

$$\max_x \{U(x) + \lambda(p)(w(p) - p'x)\}$$

We prove now the following preliminary result

Lemma 1. *If the utility function $U(x)$ satisfies the conditions of Assumption (1) and if $w(p)$ is of class C^2 then the map $p \to x(p)$ and the function $p \to \lambda(p)$ are of class C^2.*

Proof. We apply the implicit function theorem. The functions $x(p)$ and $\lambda(p)$ are defined implicitly by the $n+1$ conditions

$$\begin{aligned} D_x U(x) - \lambda p &= 0 \\ p'x - w(p) &= 0 \end{aligned}$$

Let $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n+1}$ be defined by $F(p, x, \lambda) = (F_1(p, x, \lambda), F_2(p, x, \lambda))$ where $F_1(p, x, \lambda) = D_x U(x) - \lambda p \in \mathbb{R}^n$ and $F_2(p, x, \lambda) = p'x - w(p) \in \mathbb{R}$. To apply the implicit function theorem, it suffices to show that the matrix

$$D_{x,\lambda}F = \begin{pmatrix} D_x F_1 & D_\lambda F_1 \\ D_x F_2 & D_\lambda F_2 \end{pmatrix} = \begin{pmatrix} D^2_{xx}U(x) & -p \\ p' & 0 \end{pmatrix}$$

is nonsingular. Let $\zeta = (\zeta^n, \zeta^1) \in \mathbb{R}^n \times \mathbb{R}$. We will show that the linear system $(D_{x,\lambda}F)\zeta = 0$ has only the zero solution. This system can be written in an equivalent form as

$$\begin{aligned} D^2_{xx}U(x)\zeta^n - p\zeta^1 &= 0 \\ p'\zeta^n &= 0 \end{aligned}$$

It follows that $\zeta^n \in \{p\}^{\perp} = \{\nabla U\}^{\perp}$. Multiply the first equality by $\zeta^{n\prime}$ we get

$$\zeta^{n\prime}(D^2_{xx}U(x))\zeta^n = 0$$

Since $\zeta^n \in \{\nabla U\}^{\perp}$ and $D^2_{xx}U$ is negative definite on this subspace, we conclude that $\zeta^n = 0$ and therefore $\zeta = 0$ is the only solution. So the matrix $D_{x,\lambda}F$ is nonsingular and we can apply the implicit function theorem which guarantees that $x(p)$ and $\lambda(p)$ are of class C^2. The proof is complete.

□

Introduce the indirect utility function

$$V(p) = U\left(x(p)\right) + \lambda\left(p\right)\left(w\left(p\right) - p'x(p)\right) \tag{1}$$

The envelope theorem implies that the derivative of the function V with respect to p_i is given by

$$\frac{\partial V}{\partial p_i} = \lambda(p)\left(\frac{\partial w}{\partial p_i} - x^i(p)\right), \quad i = 1, \cdots, n \tag{2}$$

Some properties of the indirect utility function V is given by the following theorem

Theorem 1. *Let $V(p)$ be the indirect utility function defined by (1). Then $V(p)$ has the following properties:*

- **a:** Positively homogenous of degree zero if $w(p)$ is positively homogeneous of degree one.
- **b:** Quasiconvex if $w(p)$ is convex.

Proof. (a) Suppose $w(p)$ is positively homogenous of degree one. Then, for all $t \geq 0$, changing p to tp does not change the budget set in problem $(\mathcal{P})$, so that $x\left(tp\right) = x\left(p\right)$ and $V\left(p\right) = U\left(x\left(p\right)\right)$ is unchanged. To prove (b), we argue as in Varian [12]. Suppose that $V(\hat{p}) \leq u$ and $V(\bar{p}) \leq u$. Let $\tilde{p} = t\hat{p} + (1-t)\bar{p}$. We want to show that $V(\tilde{p}) \leq \max\{V(\hat{p}), V(\bar{p})\}$. Introduce the following sets

$$\hat{S} = \{x \mid \hat{p}'x \leq w(\hat{p})\}, \quad \bar{S} = \{x \mid \bar{p}'x \leq w(\bar{p})\}, \quad \tilde{S} = \{x \mid \tilde{p}'x \leq w(\tilde{p})\}$$

We claim that $\tilde{S} \subset \hat{S} \cup \bar{S}$. Indeed, if this is not the case then there exists x such that $\hat{p}'x > w(\hat{p})$ and $\bar{p}'x > w(\bar{p})$ whereas $\tilde{p}'x \leq w(\tilde{p})$. It follows that for any $t \in (0,1)$, $t\hat{p}'x > tw(\hat{p})$ and $(1-t)\bar{p}'x > (1-t)w(\bar{p})$. Adding up the last two inequalities and using the convexity of w, we get

$$\tilde{p}'x = (t\hat{p}' + (1-t)\bar{p}')x > tw(\hat{p}) + (1-t)w(\bar{p}) \geq w(t\hat{p} + (1-t)\bar{p}) = w(\tilde{p})$$

Hence $\tilde{p}'x > w(\tilde{p})$ which is a contradiction, so $\tilde{S} \subset \hat{S} \cup \bar{S}$ as announced. This result implies that

$$V(\tilde{p}) = \max_{x \in \tilde{S}} U(x) \leq \max_{x \in \hat{S} \cup \bar{S}} U(x) = \max\{V(\hat{p}), V(\bar{p})\}$$

Which means that $V(p)$ is quasiconvex. The proof is complete. □

The indirect utility function can be written as $V(p) = V(p, w(p))$. The envelope theorem implies that

$$\frac{\partial V}{\partial w}(p, w(p)) = \lambda(p) \tag{3}$$

Using equation (2) and the fact that $D_p w(p) - x(p) = p' D_p x(p)$, we get

$$\frac{\partial V(p, w(p))/\partial p_i}{\partial V(p, w(p))/\partial w} = p' D_{p_i} x(p)$$

which is a generalization of Roy's identity in the standard individual model. If the income is price independent, $w(p) = y$, then $p' D_p x(p, y) = -x(p, y)$, so we get Roy's identity. The above equality means that a change in p_i by dp_i can be compensated, to keep individual's utility constant, by a change in income which equals to $dw = p'(D_{p_i} x(p)) dp_i$.

2. Mathematical tools

In this section, we give some theorems from exterior differential calculus (see [7], part 2, for a primer, and [5] for a full exposition). The question to be answered is the following. Given a vector field in a space of dimension n, when can it be decomposed as a linear combination of k gradients, with $k < n$? This question is best rephrased in the language of exterior differential calculus: given a 1-form

$$\omega = \sum_{i=1}^{n} \omega^i dp_i$$

on a space of dimension n, when can one find functions $u^j(p)$ and $v_j(p)$, with $1 \leq j \leq k < n$ such that:

$$\omega = \sum_{j=1}^{k} u^j dv_j$$

We refer to [7] and the literature therein, notably [1], [2], [8] for the mathematics and the economics of this question. The answer is provided by the following theorems, which go back to Darboux:

Theorem 2. *Let ω be a 1-form defined in a neighbourhood $\mathcal{U}$ of $\bar{p}$ in R^n. Suppose that, we have*

$$\omega \wedge (d\omega)^{k-1} \neq 0, \quad (d\omega)^k = 0 \quad \text{on } \mathcal{U} \tag{4}$$

Then there is a neighbourhood $\mathcal{V} \subset \mathcal{U}$ of $\bar{p}$ and smooth functions v_1 and $\left(u^j, v_j\right)$, $2 \leq j \leq k$, such that the dv_j and the du^j do not vanish, and

$$\omega = dv_1 + \sum_{j=2}^{k} u^j dv_j \quad \text{on } \mathcal{V} \tag{5}$$

Conversely, if ω decomposes in the form (5) on $\mathcal{U}$, and the dv_j and the du^j do not vanish, then it satisfies condition (4) on $\mathcal{U}$.

Proof. The case $k = 1$ states that $\omega = dv_1$ if and only if $d\omega = 0$, which is the well-known Poincaré Lemma. For $k > 1$, the converse is easy to prove. Indeed, if ω decomposes in the form (5), we have:

$$d\omega = \sum_{j=2}^{k} du^j \wedge dv_j$$

and

$$(d\omega)^{k-1} = (k-1)! du^2 \wedge dv_2 \wedge ... \wedge du^k \wedge dv_k$$

Then $(d\omega)^k = 0$ follows immediately (note, however, that $\omega \wedge (d\omega)^{k-1} \neq 0$). For the direct part, we refer to [5]. □

We have also the following problem

Theorem 3. *Let ω be a 1-form defined in a neighbourhood $\mathcal{U}$ of $\bar{p}$ in R^n. Suppose that, on $\mathcal{U}$, we have*

$$\omega \wedge (d\omega)^{k-1} \neq 0, \quad (d\omega)^k \neq 0, \quad \omega \wedge (d\omega)^k = 0 \tag{6}$$

Then there is a neighbourhood $\mathcal{V} \subset \mathcal{U}$ and smooth functions $\left(u^j, v_j\right)$, $1 \leq j \leq k$, such that the dv_j and the du^j do not vanish, and:

$$\omega = \sum_{j=1}^{k} u^j dv_j \quad \text{on } \mathcal{V} \tag{7}$$

Conversely, if ω decomposes in the form (7) on $\mathcal{U}$, and the dv_j and the du^j do not vanish, then it satisfies condition (6) on $\mathcal{U}$.

Proof. Again, the converse is easy. If the decomposition (7) holds, then:

$$(d\omega)^k = k! du^1 \wedge dv_1 \wedge ... \wedge du^k \wedge dv_k$$

and $\omega \wedge (d\omega)^k = 0$. For the direct part, we refer to [5] □

Another result, due to Ekeland and Chiappori in the analytic case (the coefficients ω^i of the 1-form ω are supposed to be analytic functions of p), and to Ekeland and Nirenberg [9] in the general case, addresses the problem of finding such decompositions when the functions v_j are required to be convex and the u^i positive. We state it as follows:

Theorem 4. *Let ω be a smooth 1-form in the neighbourhood of some point $\bar{p}$. There exist $2k$ functions $u^1, ..., u^k, v_1, ..., v_k$ such that ω can be decomposed as $\omega = u^1 dv_1 + u^2 dv_2 + \cdots + u^k dv_k$ where the functions u^i are positive and the v_i are convex if and only if*

(1) $\omega \wedge (d\omega)^k = 0$.

(2) There is a k-dimensional subspace S of $\mathcal{I} = \{\alpha \mid \alpha \wedge \omega \wedge (d\omega)^{k-1} = 0\}$ containing $\omega(\bar{p})$ such that on $S^{\perp}$, the matrix $\omega_{ij}(\bar{p})$ is symmetric and positive definite.

3. The non-homogeneous case

In this section, we assume that the income function $w(p)$ is not positively homogeneous of degree one which means that the demand function and the indirect utility function are not homogenous of degree zero. In particular, $w(p)$ cannot be constant. The case of homogeneous demand functions will be treated in the next section.

We are given a smooth map $x(p)$ from a neighbourhood of $\bar{p}$ in R_+^n to a neighbourhood of $x(\bar{p})$ in R_{++}^n, and a smooth function $w(p) = p'x(p)$, which is not 1-homogeneous. We ask whether it is an individual demand function, that is, whether there exists $U(x)$ such that $x(p)$ is the solution of problem $(\mathcal{P})$.

Define the differential 1-form ω as follows:

$$\omega = \sum_{i=1}^{n} x^i(p) dp_i \tag{8}$$

Using equation (2), the 1-form ω can be decomposed as

$$\omega = \mu dV + dw \tag{9}$$

where

$$\mu(p) = -\frac{1}{\lambda(p)} < 0$$

Notice that ω has a decomposition of the form (5) and that $d\omega = d\mu \wedge dV$, $\omega \wedge d\omega = dw \wedge d\mu \wedge dV \neq 0$ and $d\omega \wedge d\omega = 0$. Therefore, the necessary and sufficient condition for this decomposition is fulfilled. However, in our setting the income function $w(p)$ can be found from $w(p) = p'x(p)$ once $x(p)$ is given.

The problem of *mathematical integration* consists in finding $\mu(p)$ and $V(p)$ such that the decomposition (9) holds. The problem of economic integration consists in finding such a decomposition with $\mu(p) < 0$ and $V(p)$ quasi-convex (see [7]).

3.1. **Mathematical integration.** A necessary condition can be obtained directly from equation (2).

Lemma 2. *Suppose $x(p)$ is an individual demand function. Then $D_p x(p)$ is the sum of a symmetric matrix and a matrix of rank one.*

Proof. Recall that

$$x(p) = \mu(p) D_p V(p) + D_p w(p)$$

Differentiating both sides of this equation with respect to p, the matrix $D_p x(p)$ can be decomposed as

$$D_p x(p) = \mu(p) D^2_{pp} V(p) + D^2_{pp} w(p) + D_p \mu(p) (D_p V(p))'$$

Using the fact that $D_p V(p) = \lambda(p)(D_p w(p) - x(p))$, we get

$$D_p x(p) = \mu(p) D^2_{pp} V(p) + D^2_{pp} w(p) + \frac{D_p \mu(p)}{\mu(p)} (x(p) - D_p w(p))' \tag{10}$$

The first two terms are symmetric while the last term is a rank one matrix as required. □

It follows from the last result that the restriction of $D_p x(p)$ to a subspace of codimension one is symmetric. More precisely, let $\eta \in \{D_p w(p) - x(p)\}^{\perp}$ then:

$$\eta' D_p x(p) \eta = -\frac{1}{\lambda(p)} \eta' D^2_{pp} V(p) \eta + \eta' D^2_{pp} w(p) \eta \tag{11}$$

which means that the restriction of $D_p x(p)$ to $\{D_p V(p)\}^{\perp}$, which is a subspace of codimension one, is symmetric.

The following result provides a necessary and sufficient condition, that is, a full characterization.

Theorem 5. *Let ω be the 1-form associated with $x(p)$ by (8) and $w(p) = p'x(p)$. Then, ω can be decomposed as $\omega = \mu dV + dw$ on some neighbourhood of $\bar{p}$ if and only if the matrix*

$$S = D_p x + \frac{1}{p'(D_p x) p} ((D_p x)' - D_p x) p p' (D_p x) \tag{12}$$

is symmetric on some neighbourhood of $\bar{p}$.

Proof. Suppose that $\omega = \mu dV + dw$, so that $\omega - dw = \mu dV$. The last condition is equivalent to:

$$(\omega - dw) \wedge d\omega = 0$$

This, in turn, is equivalent to saying that there exists some 1-form β such that

$$d\omega = \beta \wedge (\omega - dw) \tag{13}$$

We shall determine the 1-form β (mod $\omega - dw$) by applying the vector field

$$\xi = \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i} \tag{14}$$

to both sides of equation (13). This gives

$$d\omega(\xi,.) = \beta \wedge (\omega - dw)(\xi,.) \tag{15}$$

Expanding the right-hand side we find that

$$\beta \wedge (\omega - dw)(\xi,.) = <\beta,\xi> (\omega - dw) - \beta(w - p'Dw)$$

Equation (15) becomes

$$d\omega(\xi,.) = <\beta,\xi> (\omega - dw) - \beta(w - p'Dw)$$

Solving for β, we get

$$\beta = \frac{1}{w - p'Dw}(-d\omega(\xi,.) + \ <\beta,\xi> (\omega - dw))$$

Note that the denominator does not vanish, since $w\,(p)$ has been assumed not to be homogeneous. Plugging this value of β into equation (13), we obtain

$$d\omega = \frac{-1}{w - p'Dw} d\omega(\xi,.) \wedge (\omega - dw) \tag{16}$$

However, a direct computation gives:

$$d\omega = \sum_{i<j} \left(\frac{\partial x^i}{\partial p_j} - \frac{\partial x^j}{\partial p_i} \right) dp_j \wedge dp_i$$

$$d\omega(\xi,.) = \sum_{j,k} \frac{\partial x^j}{\partial p_k} p_k dp_j - \sum_{j,k} \frac{\partial x^k}{\partial p_j} p_k dp_j \tag{17}$$

Performing the exterior product in equation (16), we get:

$$d\omega = \frac{1}{w - p'Dw} \sum_{i,j} \sum_k \left(\frac{\partial x^k}{\partial p_j} p_k (x^i - \frac{\partial w}{\partial p_i}) - \frac{\partial x^j}{\partial p_k} p_k (x^i - \frac{\partial w}{\partial p_i}) \right) dp_j \wedge dp_i$$

Let $I = (w - p'D_p w(p))^{-1}$. This equation takes the equivalent form

$$\sum_{i<j} \left(\frac{\partial x^i}{\partial p_j} - \frac{\partial x^j}{\partial p_i} \right) dp_j \wedge dp_i = I \sum_{\substack{k=1 \\ i<j}}^{n} \left(\frac{\partial x^k}{\partial p_j} p_k (x^i - \frac{\partial w}{\partial p_i}) - \frac{\partial x^j}{\partial p_k} p_k (x^i - \frac{\partial w}{\partial p_i}) \right.$$
$$\left. - \frac{\partial x^k}{\partial p_i} p_k (x^j - \frac{\partial w}{\partial p_j}) + \frac{\partial x^i}{\partial p_k} p_k (x^j - \frac{\partial w}{\partial p_j}) \right) dp_j \wedge dp_i$$

We conclude from the last equality that the matrix s defined by

$$s_{ij} = \frac{\partial x^i}{\partial p_j} + \frac{1}{w - p'Dw} \sum_k \left(\frac{\partial x^k}{\partial p_i} p_k - \frac{\partial x^i}{\partial p_k} p_k \right) (x^j - \frac{\partial w}{\partial p_j}) \tag{18}$$

is symmetric.

We now recall that $w\,(p) = p'x\,(p)$. Differentiating with respect to p_i, we get

$$\sum_k \frac{\partial x^k}{\partial p_i} p_k + x^i = \frac{\partial w}{\partial p_i} \tag{19}$$

It follows that

$$x^i - \frac{\partial w}{\partial p_i} = -\sum_k \frac{\partial x^k}{\partial p_i} p_k = -p'D_{p_i} x \tag{20}$$

Multiplying the last equation by p_i and adding up yields

$$w - p' D_p w = -p'(D_p x)p \tag{21}$$

Using (20) and (21), we can write (18) as

$$S_{ij} = \frac{\partial x^i}{\partial p_j} + \frac{1}{p'(D_p x)p} \sum_{k=1}^{n} \left(\frac{\partial x^k}{\partial p_i} - \frac{\partial x^i}{\partial p_k} \right) p_k (p' D_{p_j} x)$$

and this matrix should be symmetric.

Conversely, let there be a function $x(p)$ such that S_{ij} is symmetric, and set $w(p) = p'x(p)$. Differentiating, we get equations (20) and (21) so that $S = S'$ is equivalent to $s = s'$. But the symmetry condition $s = s'$ is equivalent to $(\omega - dw) \wedge d\omega = 0$. Therefore, there exist two functions $\mu(p)$ and $V(p)$ such that $\omega - dw = \mu dV$. This completes the proof. □

This solves the mathematical integration problem. The necessary and sufficient condition for mathematical integration is the symmetry of the matrix

$$S = D_p x + \frac{1}{p'(D_p x)p}((D_p x)' - D_p x)pp'(D_p x) \tag{22}$$

It is the natural generalization of the Slutsky matrix. Indeed, in the case when the income is independent of prices, $w(p) = y$, the Marshallian demand has the form $x(p, y)$. Relation (19) becomes:

$$\sum_k \frac{\partial x^k}{\partial p_i} p_k + x^i = 0$$

and since $x(p, y)$ is positively homogeneous of degree zero:

$$\sum_k \frac{\partial x^i}{\partial p_k} p_k + \frac{\partial x^i}{\partial y} y = 0$$

Its follows that $p' D_p x(p) = -x(p)$ and $p' D_p x(p) p = -y$. Consequently, the symmetry condition $S_{ij} = S_{ji}$ is equivalent to

$$S_{ij} = \frac{\partial x^i}{\partial p_j} - \frac{1}{y}(-x^i + \frac{\partial x^i}{\partial y} y)(-x^j) = S_{ji} = \frac{\partial x^j}{\partial p_i} - \frac{1}{y}(-x^j + \frac{\partial x^j}{\partial y} y)(-x^i)$$

Canceling similar terms from both sides, we end up with

$$\frac{\partial x^i}{\partial p_j} + \frac{\partial x^i}{\partial y} x^j = \frac{\partial x^j}{\partial p_i} + \frac{\partial x^j}{\partial y} x^i$$

which are the standard Slutsky conditions. By comparing the matrix S with the Slutsky matrix one can argue that the second part of the matrix S given by

$$\frac{1}{p'(D_p x)p}((D_p x)' - D_p x)pp'(D_p x)$$

is the income effect. When the price of some good changes, the income effect represents the change in demand that results from the change in real income of the consumer. Notice that demand functions are homogeneous of degree zero when income is price independent. Moreover, zero homogeneity implies that the price vector belongs to the null space of the Slutsky matrix.

Demand functions that we consider in this section are not homogeneous of degree zero. Consequently, $Sp \neq 0$. More precisely

$$Sp = (D_p x(p))' p \neq 0$$

We conclude that the price vector doesn't belong to the null space to our extended Slutsky matrix unless income is price independent or demand functions are homogeneous of degree zero.

3.2. **Economic integration.** We begin, as above, by a simple necessary condition

Lemma 3. *Let $x(p)$ be an individual demand function with convex income function $w(p)$. Then, the matrix $D_p x(p) - D^2_{pp} w(p)$ is negative semidefinite on the subspace $\{D_p w(p) - x(p)\}^{\perp}$.*

Proof. Using equation (10), we can write

$$D_p x(p) - D^2_{pp} w(p) = -\frac{1}{\lambda(p)} D^2_{pp} V(p) + \frac{1}{\lambda(p)} (D_p \lambda(p))(D_p w(p) - x(p))' \tag{23}$$

Since the indirect utility function is quasiconvex, $D^2_{pp} V$ is positive semidefinite on the subspace

$$\{D_p V(p)\}^{\perp} = \{D_p w(p) - x(p)\}^{\perp}$$

It follows that for any vector $\eta \in \{D_p w(p) - x(p)\}^{\perp}$, we have

$$\eta' D_p x(p) \eta - \eta' D^2_{pp} w(p) \eta \leq 0$$

which is the desired result. □

We conclude from the last inequality that $D_p x(p)$ could be either positive or negative semidefinite on $\{D_p w(p) - x(p)\}^{\perp}$. Moreover, if the income is price independent, the last theorem states that the Jaccobian matrix $D_p x(p)$ is negative semidefinite on $\{x(p)\}^{\perp}$ which follows from the negative semidefiniteness of the Slutsky matrix.

We shall rewrite this condition under a form that involves the partial derivatives of $x(p)$ only. To this end, we define the matrix T whose ij-entry is given by

$$T_{ij}(p) = \sum_k \frac{\partial^2 x^k}{\partial p_j \partial p_i} p_k + \frac{\partial x^j}{\partial p_i} \tag{24}$$

Corollary 1. *Let $x(p)$ be an individual demand function such that the income function $w(p)$ is convex. Then the restriction of the matrix T on the subspace $\{p' D_p x(p)\}^{\perp}$ is symmetric and positive semidefinite.*

Proof. Differentiating the budget constraint we get

$$\sum_k \frac{\partial x^k}{\partial p_i} p_k + x^i = \frac{\partial w}{\partial p_i} \tag{25}$$

We write this equation in matrix form as $D_p w(p) - x(p) = p' D_p x(p)$ from which we conclude that $\{D_p w(p) - x(p)\}^{\perp} = \{p' D_p x(p)\}^{\perp}$. Differentiating both sides of equation (25) with respect to p_j, we get

$$\sum_k \frac{\partial^2 x^k}{\partial p_j \partial p_i} p_k + \frac{\partial x^j}{\partial p_i} + \frac{\partial x^i}{\partial p_j} = \frac{\partial^2 w}{\partial p_j \partial p_i}$$

Using matrix notation, this becomes:

$$T(p) + D_p x(p) = D^2_{pp} w(p)$$

It follows that we can write (23) in the following form

$$T = \frac{1}{\lambda(p)} D^2_{pp} V(p) - \frac{1}{\lambda(p)} (D_p \lambda(p))(p' D_p x(p))' \tag{26}$$

The result follows from the last equality. □

Note an immediate consequence:

Corollary 2. *The restriction of $D_p x(p)$ to $\{p' D_p x(p)\}^\perp$ is symmetric.*

To find a necessary and sufficient condition for economic integration, we apply the results of Chiappori-Ekeland and Ekeland-Nirenberg [9].

Theorem 6. *Let $x(p)$ and $w(p)$ be given functions that satisfy the following conditions:*

- **(a):** *$p' x(p) = w(p)$.*
- **(b):** *$w(p)$ is convex.*
- **(c):** *The matrix S given by (22) is symmetric.*
- **(d):** *The restriction to $\{p' D_p x(p)\}^\perp$ of the matrix T given by (24) is positive definite.*

Define $\omega = \sum x^i dp_i$. Then, locally there exist two functions $\lambda(p)$ and $V(p)$ such that $\omega = \lambda dV + dw$ where $\lambda(p)$ is positive and $V(p)$ is quasiconvex.

Proof. We apply theorem (4). Let us define the differential 1-form Ω by

$$\Omega = \omega - dw$$

By Theorem 5, $d\Omega = d\omega$ and $\Omega \wedge d\Omega = 0$. Using the notation of theorem (4), we have $k = 1$, $\mathcal{I} = \{\alpha | \alpha \wedge \Omega = 0\}$ and $S = \text{span}\{\Omega\}$. Its clear that $\Omega \in \mathcal{I}$ and that $S^\perp = \text{span}\{p' D_p x(p)\}^\perp$.

Consider the matrix:

$$\Omega_{ij} = \frac{\partial x^i}{\partial p_j} - \frac{\partial^2 w}{\partial p_j \partial p_i}$$

The restriction of (Ω_{ij}) to a subspace of codimension one is symmetric and negative definite according to (24) and hypothesis (d). It follows from the Ekeland-Nirenberg Theorem that there exist two functions $\mu(p)$ and $v(p)$ such that $\mu(p) > 0$, $v(p)$ is concave and $\Omega = \mu dv$. It follows that $\omega = \mu dv + dw$ which gives

$$x^i(p) = \mu(p) \frac{\partial v}{\partial p_i} + \frac{\partial w}{\partial p_i}$$

Setting $\lambda(p) = \frac{1}{\mu(p)} > 0$ and $V(p) = -v(p)$, $V(p)$ is convex since $v(p)$ is concave, we have:

$$\frac{\partial V}{\partial p_i} = \lambda(p) \left(\frac{\partial w}{\partial p_i} - x^i(p) \right)$$

where $V(p)$ is convex and therefore is quasiconvex. The function $V(p)$ is an indirect utility function and $\lambda(p)$ is the Lagrange multiplier. From V, we can find the direct utility function $U(x)$ for the individual whose demand function is $x(p)$. This completes the proof. □

4. The homogeneous case

In this section, we adopt the additional assumption that the income function $w(p)$ is homogeneous of degree one. Therefore, the demand function $x(p)$ and the indirect utility function $V(p)$ are homogeneous of degree zero. We proceed now as in the previous section while performing the necessary modifications implied by the homogeneity conditions. The following theorem solves the mathematical integration problem in the homogeneous case.

Theorem 7. *Let ω be a differential 1-form defined as above where $x(p)$ is a solution of problem $(\mathcal{P})$ such that $p'x(p) = w(p)$ and $w(p)$ is convex and homogeneous of degree one. Then, ω can be decomposed as $\omega = \mu dV + dw$ in the neighbourhood of a point $\bar{p}$ if and only if there is some 1-form $\beta = \sum \beta^i dp_i$ satisfying*

$$\sum p_i \beta^i(p) = 1$$

such that, for all i, j, we have:

$$\frac{\partial x^i}{\partial p_j} - \beta^i \sum_k \frac{\partial x^k}{\partial p_j} p_k = \frac{\partial x^j}{\partial p_i} - \beta^j \sum_k \frac{\partial x^k}{\partial p_i} p_k \tag{27}$$

in a neighbourhood of $\bar{p}$.

Proof. As above, the necessary and sufficient condition is:

$$d\omega = \beta \wedge (\omega - dw) \tag{28}$$

Let us apply the vector field $\xi = \sum p_i \frac{\partial}{\partial p_i}$ to both sides of this equation. Notice first that, since $w(p)$ is homogeneous of degree one and $x(p)$ is homogeneous of degree zero, we have from Euler's identity

$$\sum_{i=1}^n \frac{\partial w}{\partial p_i} p_i = w(p) \quad \text{and} \quad \sum_{i=1}^n \frac{\partial x^j}{\partial p_i} p_i = 0, \quad j = 1, ..., n$$

It follows that:

$$d\omega(\xi, .) =< \beta, \xi > (\omega - dw) - \beta < \omega - dw, \xi > \tag{29}$$

But

$$< \omega - dw, \xi > = < \omega, \xi > - < dw, \xi > = w - w = 0.$$

Therefore, equality (29) reduces to $d\omega(\xi, .) = < \beta, \xi > (\omega - dw)$. Recall that

$$d\omega(\xi, .) = \sum_{i,j} \frac{\partial x^i}{\partial p_j} p_j dp_i - \sum_{i,j} \frac{\partial x^i}{\partial p_j} p_i dp_j$$

The first term vanishes because $x(p)$ is homogeneous of degree zero. We end up with

$$d\omega(\xi, .) = -\sum_{i,j} \frac{\partial x^i}{\partial p_j} p_i dp_j =< \beta, \xi > (\omega - dw)$$

This equation can be written under the form

$$d\omega(\xi, .) =< \beta, \xi > \sum_j \left(x^j(p) - \frac{\partial w}{\partial p_j} \right) dp_j \tag{30}$$

Differentiating the budget constraint once, we get $x(p) - D_p w(p) = -p' D_p x(p)$. Equation (30) can be written as $d\omega(\xi, .) = < \beta, \xi > d\omega(\xi, .)$. We conclude that the differential 1-form β must satisfy $< \beta, \xi > = 1$.

We now go back to (28). Set $\beta = \sum_{j=1}^{n} \beta^j dp_j$. Using the fact that $x(p) - D_p w(p) = -p' D_p x(p)$, equation (28) can be written as

$$\sum_{i<j} \left(\frac{\partial x^i}{\partial p_j} - \frac{\partial x^j}{\partial p_i} \right) dp_j \wedge dp_i = -\sum_{i<j} \left(\beta^j \sum_k \frac{\partial x^k}{\partial p_i} p_k - \beta^i \sum_k \frac{\partial x^k}{\partial p_j} p_k \right) dp_j \wedge dp_i$$

This equality is satisfied if and only if

$$\frac{\partial x^i}{\partial p_j} - \beta^i \sum_k \frac{\partial x^k}{\partial p_j} p_k = \frac{\partial x^j}{\partial p_i} - \beta^j \sum_k \frac{\partial x^k}{\partial p_i} p_k \tag{31}$$

Where β is any differential 1-form that satisfies the condition $< \beta, \xi > = 1$. The proof is complete. □

Its clear that the price vector p belongs to the null space of the matrix obtained in theorem (7). Notice that the demand function can be written under the form $x(p, w(p))$ and the budget constraint is $p'x(p, w(p)) = w(p)$. Differentiating both sides with respect to w, we get

$$\sum_{k=1}^{n} \frac{\partial x^k}{\partial w} p_k = 1$$

Then, we can take β to be the 1-form defined by

$$\beta = \sum_{k=1}^{n} \frac{\partial x^k}{\partial w} dp_k$$

Its clear that β satisfies the condition $< \beta, \xi > = 1$. By taking $\beta^i = \frac{\partial x^i}{\partial w}$, the symmetry condition (31) takes the form

$$\frac{\partial x^i}{\partial p_j} - \frac{\partial x^i}{\partial w} \sum_k \frac{\partial x^k}{\partial p_j} p_k = \frac{\partial x^j}{\partial p_i} - \frac{\partial x^j}{\partial w} \sum_k \frac{\partial x^k}{\partial p_i} p_k \tag{32}$$

If $w(p) = y$ then the above matrix is the Slutsky matrix since

$$\sum_k \frac{\partial x^k}{\partial p_i} p_k = -x^i$$

Another possible choice for β is

$$\beta = \frac{1}{p'p} \sum_j p_j dp_j$$

Its clear that β satisfies the condition $< \beta, \xi >= 1$. For this choice of β, the symmetry condition (31) takes the form

$$\frac{\partial x^i}{\partial p_j} - \frac{p_i}{p'p} \sum_k \frac{\partial x^k}{\partial p_j} p_k = \frac{\partial x^j}{\partial p_i} - \frac{p_j}{p'p} \sum_k \frac{\partial x^k}{\partial p_i} p_k \tag{33}$$

We have the same symmetry and negative semidefiniteness as before. We state the following theorem for which we omit the proof.

Theorem 8. *If $x(p)$ is a solution of problem $(\mathcal{P})$ that is homogenous of degree zero, then $D_p x(p)$ is symmetric and negative semidefinite on the subspace $\{p' D_p x(p)\}^{\perp}$.*

Notice that all results of the previous section, except theorem (5), hold including theorem (6) that solves the economic integration problem.

5. Two particular cases

5.1. **Case 1: An exchange economy.** In this section we consider a simple exchange economy in which the income of the consumer is simply the value of his endowment $e \in \mathbb{R}_+^n$ so that the income function $w(p) = p'e$ is a linear function of p. In this case we have $D_p w = p$ and $D_p^2 w = 0$. It follows that equality (11) becomes

$$\eta' D_p x(p)\eta = \frac{-1}{\lambda(p)}\eta' D_p^2 V(p)\eta + \eta' \frac{D_p \mu(p)}{\mu(p)}(x(p) - e)'\eta$$

This condition means that the Jacobian matrix of x is negative semidefinite on the space orthogonal to the Span$\{e - x\}$. In other words, if we define the differential 1-form ρ as

$$\rho = \sum_{i=1}^{n} (x^i(p) - e^i) dp_i$$

then the first order conditions for constrained maximum imply that $\rho = \frac{-1}{\lambda(p)} dV$ which is equivalent to $\rho \wedge d\rho = 0$. Using the constraint $p'x(p) = p'e$, we have

$$\sum_k \frac{\partial x^k}{\partial p_i} p_k = -(x^i(p) - e^i)$$

In this case, the symmetry condition (32) can be written as

$$\frac{\partial x^i}{\partial p_j} + \frac{\partial x^i}{\partial w}(x^j - e^j) = \frac{\partial x^j}{\partial p_i} + \frac{\partial x^j}{\partial w}(x^i - e^i)$$

This is indeed the Slutsky matrix in an exchange economy. It is symmetric and negative semidefinite.

In this simple economy, the initial endowment of the consumer is given which determines, in turn, the income function of the consumer. Notice again that if the consumer's income is price independent; that is, $w(p) = y$ that $e^i = 0$ for $i = 1, ..., n$ and the above symmetry condition means that the Slutsky matrix $S = D_p x + (D_y x)x'$ is symmetric.

5.2. **Case 2: An economy with production.** We give now another example of an economy in which the income of the consumer depends on prices. This is an economy with production.

Consider a private ownership economy in which consumers own shares in firms whose profits are distributed to shareholders. For any firm j, consumer i's ownership is represented by a number $0 \leq \theta^{ij} \leq 1$. We require that $\sum_i \theta^{ij} = 1$ for any firm j. We assume that there are J firms in this economy.

Let Y be the total production set of the economy. That is $Y = Y^1 + ... + Y^J$, where Y^j is the production possibility set for firm j. We assume that the total production set is closed, strictly convex and satisfies the free disposal property. The strict convexity assumption rules out constant and increasing returns to scale economies. Under these assumptions, the profit function and the net supply function are well defined and satisfy a set of important properties.

Denote by $y_j(p)$ the net supply function of firm j which is simply the function that associates to each vector p the profit maximizing net output vector at those prices. The vector-valued function $y_j(p)$ is homogeneous of degree zero, $D_p y_j$ is symmetric and positive semidefinite and $(D_p y_j)p = 0$.

It follows that the profit of firm j is $w_j(p) = p'y_j(p)$, where $y_j = (y_j^1, ..., y_j^n) \in \mathbb{R}^n$. The negative entries of the vector y_j should be interpreted as demand for inputs. The profit function $w_j(p)$ is convex, homogeneous of degree one and $D_p w_j = y_j(p)$. For more details and proofs, see Mas-Colell et al [10].

In this setting, a typical consumer's income can arise from two sources, from selling an endowment of commodities $e \in \mathbb{R}^n_+$ and from shares in the profits of any number of firms. Let $p \in \mathbb{R}^n_{++}$ be the vector of prices, the consumer's budget constraint is

$$p'x^i \leq p'e^i + \sum_j \theta^{ij} w_j(p)$$

where $x \in \mathbb{R}^n_+$ is a commodity bundle and $w_j(p)$ is the profit function of firm j. It follows that consumer i solves the following problem

$$\begin{gathered} \max_{x^i} \; U^i(x^i) \\ p'x^i = p'e^i + \textstyle\sum_j \theta^{ij} w_j(p) \end{gathered}$$

So we have an income function for consumer i that takes the form $w^i(p) = p'e^i + \sum_j \theta^{ij} w_j(p)$. This income function is homogenous of degree one and convex. If we omit the index i, the consumer's objective in this economy is to maximize the utility function $U(x)$ subject to the constraint $p'x = w(p)$ where $w(p)$ is homogeneous of degree one and convex. That is; we have an individual problem of the type $(\mathcal{P})$ for which the solution and the value function (the indirect utility function) are homogenous of degree zero and the indirect utility function is quasiconvex.

References

[1] Aloqeili, M. "*On the characterization of excess demand functions*", Economic Theory, 26, 217-225, 2005.

[2] Aloqeili, M. "*The Generalized Slutsky Relations*", Journal of Mathematical Economics, 40/1-2, pp 71-91, 2004.

[3] Antonelli, G., "*Sulla teoria matematica dell economia politica*", 1886, Pisa.

[4] Browning, M. and P.A. Chiappori, "*Efficient intra-household allocations: A general characterization and empirical tests*", Econometrica, 66, 1241-1278, 1998.

[5] Bryant, R., S. Chern, R. Gardner, H. Goldschmidt, and P. Griffiths, "*Exterior Differential Systems*". MSRI Publications (18), Springer-Verlag, 1991.

[6] Chiappori, P.A. and I. Ekeland, "*The Microeconomics of Efficient Group Behavior: Identification*", Econometrica, 77 , 2009.

[7] Chiappori, P.A. and I. Ekeland, "*The Economics and Mathematics of Aggregation: Formal Models of Efficient Group Behavior*". now Publishers Inc. Hanover, 2010.

[8] Chiappori, P.A. and I. Ekeland, "*Individual excess demands*", Journal of Mathematical Economics, 40, 41-57, 2004.

[9] Ekeland, I. and L. Nirenberg, "*A Convex Darboux Theorem*, Methods and Applications of Analysis, 9, 329-344.

[10] Mas-Colell et al.,"*Microeconomic Theory*", Oxford University Press, New York, 1995.

[11] Slutsky, E., "*Sulla teoria del bilanzio del consomatore*", Giornale Degli Economesti, 51, 1-26, 1915.

[12] Varian, H. "*Microeconomic Analysis*, 3rd Edition, W.W. Norton.

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