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ORIGINAL RESEARCH

Global asymptotic stability of the higher order equation $x_{n+1} = \frac{ax_n+bx_{n-k}}{A+Bx_{n-k}}$

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Abstract In this paper, we investigate the local and global stability and the period two solutions of all nonnegative solutions of the difference equation,

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}}$$

where a, b, A, B are all positive real numbers, $k \geq 1$ is a positive integer, and the initial conditions $x_{-k}, x_{-k+1}, ..., x_0$ are nonnegative real numbers. It is shown that the zero equilibrium point is globally asymptotically stable under the condition $a+b \leq A$, and the unique positive solution is also globally asymptotically stable under the condition $a - b \leq A \leq a + b$. By the end, we study the global stability of such an equation through numerically solved examples.

Keywords Difference equation · Global asymptotic stability · Equilibrium point · Semi-cycles

1 Introduction

The equation

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_n} \tag{1.1}$$

was studied in [5] by Li and Zhao.

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Our goal in this paper is to study the rational higher order difference equation

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}} \tag{1.2}$$

where a, b, A, B are all positive real numbers, $k \geq 1$ is a positive integer, and the initial conditions $x_{-k}, x_{-k+1}, ..., x_0$ are nonnegative real numbers.

Here, we recall some basic notations, definitions, and results that will be useful in this paper.

Definition 1.1 The equilibrium point $\bar{y}$ of the equation

$$y_{n+1} = f(y_n, y_{n-1}, ..., y_{n-k}), n = 0, 1, ... \tag{1.3}$$

is the point that satisfies the condition

$$\bar{y} = f(\bar{y}, \bar{y}, ..., \bar{y}).$$

Definition 1.2 Let $\bar{y}$ be an equilibrium point of Eq.1.3. Then the equilibrium point $\bar{y}$ is called

(1) locally stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $y_{-k}, y_{-k+1}, ...,$ $y_0 \in I$ with $|y_{-k} - \bar{y}| + |y_{-k+1} - \bar{y}| + ... + |y_0 - \bar{y}| < \delta$, we have $|y_n - \bar{y}| < \epsilon$ for all $n \geq -k$,
(2) locally asymptotically stable if it is locally stable and if there exists $\gamma > 0$ such that for all $y_{-k}, y_{-k+1}, ..., y_0 \in I$ with $|y_{-k} - \bar{y}| + |y_{-k+1} - \bar{y}| + ... + |y_0 - \bar{y}| < \gamma$, we have $lim_{n\to\infty} y_n = \bar{y}$.
(3) a global attractor if for all $y_{-k}, y_{-k+1}, ..., y_0 \in I$, we have $lim_{n\to\infty} y_n = \bar{y}$.
(4) globally asymptotically stable if $\bar{y}$ is locally stable and $\bar{y}$ is a global attractor.

Theorem 1.1 ***(Linearized Stability)***. *Consider the difference equation*

$$y_{n+1} = py_n + qy_{n-k}, \quad n = 0, 1, ...$$

(a) *If both roots of the equation have absolute values less than one, then the equilibrium $\bar{y}$ of the equation is locally asymptotically stable.*
(b) *If at least one of the roots of the equation has an absolute value greater than one, then $\bar{y}$ is unstable .*

Theorem 1.2 *Assume that $a, b \in \mathbb{R}$ and $k \in \{1, 2, ...\}$. Then*

$$|a| + |b| < 1 \tag{1.4}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_n + ay_{n-k} + by_{n-l} = 0, \ n = 0, 1, ... \tag{1.5}$$

Suppose in addition that one of the following two cases holds.

(a) *k odd and $b < 0$.*
(b) *k even and $ab < 0$.*

Then 1.4 is also a necessary condition for the asymptotic stability of Eq.1.5.

Theorem 1.3 *Assume that $a, b \in \mathbb{R}$. Then $|a| < b + 1 < 2$ is a necessary and sufficient condition for the asymptotic stability of the difference equation*

$$y_n + ay_{n-k} + by_{n-l} = 0, \ n = 0, 1, \ldots . \tag{1.6}$$

Theorem 1.4 *Consider the difference equation*

$$x_{n+1} = \sum\nolimits_{i=0}^{k} x_{n-i} F_i(x_n, x_{n-1}, \ldots, x_{n-k}) n = 0, 1, \ldots \tag{1.7}$$

with initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in [0, \infty)$, where

(1) *$k \in \{1, 2, \ldots\}$;*
(2) *$F_0, F_1, \ldots, F_k \in C\ [[0, \infty)^{k+1}, [0, 1)]$;*
(3) *$F_0, F_1, \ldots, F_k$ are nonincreasing in each argument;*
(4) *$\sum_{i=0}^{k} F_i(y_0, y_1, \ldots, y_k) < 1$ for all $(y_0, y_1, \ldots, y_k) \in (0, 1)^k$;*
(5) *$F_0(y, y, \ldots, y) > 0$ for all $y \geq 0$.*

Then, $\bar{x} = 0$ is globally asymptotically stable for such equation.

Theorem 1.5 *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}); n = 0, 1, \ldots \tag{1.8}$$

where $k \in \{1, 2, \ldots\}$. Let $I = [a, b]$ be some interval of real numbers and assume that

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) *$f(u, v)$ is nondecreasing in u and nondecreasing in v.*
(b) *If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system*

$$m = f(m, m) \text{ and } M = f(M, M),$$

then $m = M$. Then Eq.1.8 has a unique equilibrium $\bar{y}$ and every solution of Eq.1.8 converges to $\bar{y}$.

Theorem 1.6 *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}); n = 0, 1, \ldots \tag{1.9}$$

where $k \in \{1, 2, \ldots\}$. Let $I = [a, b]$ be some interval of real numbers and assume that

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) *$f(u, v)$ is nondecreasing in u and nonincreasing in v.*
(b) *If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system*

$$m = f(m, M) \text{ and } M = f(M, m),$$

then $m = M$. Then Eq.1.9 has a unique equilibrium $\bar{y}$ and every solution of Eq.1.9 converges to $\bar{y}$.

2 The equilibrium points of Eq.1.2

Next, we investigate the equilibrium points of our rational difference equation,

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}} \tag{2.1}$$

where a, b, A, B are all positive real numbers, $k \geq 1$ is a positive integer, and the initial conditions $x_{-k}, x_{-k+1}, ..., x_0$ are nonnegative real numbers.

The equilibrium points of Eq.2.1 are the positive solutions of the equation

$$\begin{aligned} \bar{x} &= \frac{a\bar{x} + b\bar{x}}{A + B\bar{x}} \\ &= \frac{(a+b)\bar{x}}{A + B\bar{x}} \end{aligned}$$

Then,

$$\begin{aligned} \bar{x}(A + B\bar{x}) &= (a+b)\bar{x} \\ \bar{x}(A + B\bar{x} - a - b) &= 0 \end{aligned}$$

So, $\bar{x} = 0$ is always an equilibrium point of Eq.2.1, and when $a + b > A$, Eq.2.1 has another positive equilibrium point , $\bar{x} = \frac{a+b-A}{B}$.

To find the linearization of our problem, consider

$$f(u, v) = \frac{au + bv}{A + Bv}$$

now,

$$\begin{aligned} \frac{\partial f}{\partial u} &= \frac{a}{A + Bv} \\ \frac{\partial f}{\partial v} &= \frac{(A + Bv)b - (au + bv)B)}{(A + Bv)^2} \end{aligned}$$

so

$$\frac{\partial f}{\partial v} = \frac{Ab - au}{(A + Bv)^2}$$

Hence, for $\bar{x} = 0$,

$$\frac{\partial f}{\partial u}(\bar{x}, \bar{x}) = \frac{a}{A}$$

also

$$\frac{\partial f}{\partial v(\bar{x}, \bar{x})} = \frac{b}{A}$$

So, the linearized equation about the zero equilibria is

$$z_{n+1} = \frac{a}{A} z_n + \frac{b}{A} z_{n-k} \tag{2.2}$$

and its characteristic equation is

$$\lambda^{k+1} - \frac{a}{A}\lambda^k - \frac{b}{A} = 0 \tag{2.3}$$

For the positive equilibria $\bar{x} = \frac{a+b-A}{B}$,

$$\frac{\partial f}{\partial u}(\bar{x}, \bar{x}) = \frac{a}{a+b}$$

also

$$\frac{\partial f}{\partial v(\bar{x}, \bar{x})} = \frac{A-a}{a+b}$$

So, the linearized equation about the positive equilibria is

$$z_{n+1} = \frac{a}{a+b} z_n + \frac{A-a}{a+b} z_{n-k} \tag{2.4}$$

and its characteristic equation is

$$\lambda^{k+1} - \frac{a}{a+b}\lambda^k - \frac{A-a}{a+b} = 0 \tag{2.5}$$

3 The local stability of the equilibrium points

3.1 The local stability of the zero equilibrium point

Our equation posses the equilibrium point $\bar{x} = 0$ always in all of the following cases:

(1) $a + b < A$
(2) $a + b = A$

(3) $a + b > A$

We will study the stability of $\bar{x} = 0$ in all of the cases.

3.1.1 The Case $a + b < A$

Theorem 3.1 *The zero equilibrium point $\bar{x} = 0$ will be locally asymptotically stable when $a + b < A$.*

Proof The characteristic equation of the zero equilibrium point is

$$\lambda^{k+1} - \frac{a}{A}\lambda^k - \frac{b}{A} = 0 \tag{3.1}$$

where a, b, A, B are all positive real numbers.

By applying 1.2, and as we assume $a + b < A$, then it is easy then to show that $\bar{x} = 0$ is asymptotically equilibrium point. This completes the proof. □

3.1.2 The case $a + b = A$

Theorem 3.2 *The zero equilibrium point $\bar{x} = 0$ will be locally stable when $a+b = A$.*

Proof Our difference equation is

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}} \tag{3.2}$$

Let $\{x_n\}_{n=-k}^{\infty}$ be a nonnegative solution of our equation with the initial points $x_{-k}, \ldots\ldots, x_0$ are to be nonnegative.
Let $\epsilon > 0$ and

$$|x_{-k} - \bar{x}| < \epsilon, \ldots\ldots, |x_0 - \bar{x}| < \epsilon$$

So,and since $\bar{x} = 0$,

$$0 \leq x_{-k} < \epsilon, \ldots\ldots, 0 \leq x_0 < \epsilon$$

Then, when $a + b = A$

$$\begin{aligned} x_1 &= \frac{ax_0 + bx_{-k}}{A + Bx_{-k}} \\ &< \frac{ax_0 + bx_{-k}}{A} \\ &< \frac{(a+b)\epsilon}{A} \\ &< \epsilon \end{aligned}$$

So, $0 \leq x_1 < \epsilon$, and by Mathematical Induction we get

$$0 \leq x_n < \epsilon, \quad \forall n \geq -k$$

Then,

$$|x_n - \bar{x}| < \epsilon, \quad \forall n \geq -k$$

By definition, then $\bar{x} = 0$ is a locally stable equilibrium point when $a + b = A$. This completes the proof. □

3.1.3 The Case $a + b > A$

Theorem 3.3 *The zero equilibrium point $\bar{x} = 0$ is unstable when $a + b > A$*

Proof The characteristic polynomial of the zero equilibrium point

$$f(\lambda) = \lambda^{k+1} - \frac{a}{A}\lambda^k - \frac{b}{A} \tag{3.3}$$

is a continuous function for all λ.

For $\lambda = 1$,

$$f(1) = 1 - \frac{a}{A} - \frac{b}{A} = \frac{A - (a + b)}{A}$$

and since $a + b > A$, $f(1) < 0$.

But also, as $\lambda \to \infty$

$$\lim_{\lambda \to \infty} f(\lambda) = \infty > 0$$

By Roll's Theorem, we can say that $f(\lambda)$ has a zero solution λ_0, where $f(\lambda_0) = 0$ and $\lambda_0 \in (1, \infty)$.

Then there exists a solution λ_0, where λ_0 lies outside the unitary disk.

By Theorem1.1, $\bar{x} = 0$ is unstable equilibrium point when $a + b > A$. This completes the proof. □

3.2 The local stability of the positive equilibria

Our equation posses a positive equilibrium $\bar{x} = \frac{a+b-A}{B}$ under the condition $a+b > A$. The linearized equation of this equilibrium point is

$$z_{n+1} = \frac{a}{a+b} z_n + \frac{A-a}{a+b} z_{n-k} \tag{3.4}$$

and its characteristic equation is

$$\lambda^{k+1} - \frac{a}{a+b}\lambda^k - \frac{A-a}{a+b} = 0 \tag{3.5}$$

Lets apply Theorem1.2 under the both cases $A \geq a$ and $A < a$.

(1) **The case** $A \geq a$:

$$\begin{aligned}\left|\frac{a}{a+b}\right| + \left|\frac{A-a}{a+b}\right| &= \frac{a+|A-a|}{a+b}\\ &= \frac{a+A-a}{a+b}\\ &= \frac{A}{a+b}\end{aligned}$$

Then the necessary condition

$$\frac{A}{a+b} < 1$$

can be written as

$$a \leq A < a+b$$

(2) **The case** $A < a$:

$$\begin{aligned}\left|\frac{a}{a+b}\right| + \left|\frac{A-a}{a+b}\right| &= \frac{a+|A-a|}{a+b}\\ &= \frac{a+a-A}{a+b}\\ &= \frac{2a-A}{a+b}\end{aligned}$$

Then the necessary condition

$$\frac{2a-A}{a+b} < 1$$

can be written as

$$a-b < A < a$$

If we combine the both cases, we get the following theorem.

Theorem 3.4 *If k is odd or if k is even, the unique positive equilibrium point $\bar{x} = \frac{a+b-A}{B}$ will be locally asymptotically stable if and only if $a-b < A < a+b$.*

4 The periodic two solution of Eq.1.2

We study here the periodic solution of our equation,

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}} \tag{4.1}$$

Lets assume that the two periodic nonnegative solution of our equation will be in the form

$$...., \psi, \phi, \psi, \phi, ...$$

If k is odd then,

$$x_{n+1} = x_{n-k} \tag{4.2}$$

By so we get,

$$\psi = \frac{a\phi + b\psi}{A + B\psi} \tag{4.3}$$

$$\phi = \frac{a\psi + b\phi}{A + B\phi} \tag{4.4}$$

This yields to

$$\psi(A + b + B\psi) = a\phi$$
$$\phi(A + b + B\phi) = a\psi$$

By subtract the second equation from the first we get the equation

$$(A + a + b)(\psi - \phi) + B(\psi^2 - \phi^2) = 0 \tag{4.5}$$

$$(\psi - \phi)(A + a + b + B(\psi + \phi)) = 0 \tag{4.6}$$

Then, either $\psi = \phi$ or $(\psi + \phi) = -\frac{A+a+b}{B}$ which is impossible since both ϕ and ψ are nonnegative. Then in this case there is no two periodic nonnegative solution for our equation.

Lets now take k to be even and see what we will get.

If k is even then,

$$x_n = x_{n-k} \tag{4.7}$$

So,

$$\psi = \frac{a\phi + b\phi}{A + B\phi} \tag{4.8}$$

$$\phi = \frac{a\psi + b\psi}{A + B\psi} \tag{4.9}$$

We get that

$$\phi(A + B\psi) = (a + b)\psi$$
$$\psi(A + B\phi) = (a + b)\phi$$

By subtract both equations we get

$$(\phi - \psi)(A + a + b) = 0 \tag{4.10}$$

Then either $\phi = \psi$ or $A = -(a+b)$ which i also impossible. Then also in this case there exists no two periodic nonnegative solution for our equation. We can conclude the following thus.

Theorem 4.1 *There exists no two periodic nonnegative solution for the equation*

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}}$$

under any condition.

5 The global stability of Eq.1.2

Here also we will consider the two equilibrium points separately.

5.1 The global stability of the zero equilibria

We study the global stability of the zero equilibrium point under the condition $a+b \leq A$.

Our equation

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}} \tag{5.1}$$

can be written as

$$x_{n+1} = \frac{a}{A + Bx_{n-k}}x_n + \frac{b}{A + Bx_{n-k}}x_{n-k} \tag{5.2}$$

Lets apply Theorem 1.4 now. We can consider $F_0 = \frac{a}{A+Bx_{n-k}}$ and $F_1, ...F_{k-1} = 0$ and $F_k = \frac{b}{A+Bx_{n-k}}$. Then it is obvious that that theorem could be applied so easily since:

(1) $k \in \{1, 2, ...\}$;
(2) $F_0, F_k \in \mathrm{C}\left[[0, \infty)^{k+1}, [0, 1)\right]$;
(3) F_0, F_k are nonincreasing in each argument;

(4) $F_0 + F_k = \frac{a+b}{A+Bx_{n-k}}$, and since $a + b < A$,

$$\begin{aligned} F_0 + F_k &= \frac{a+b}{A+Bx_{n-k}} \\ &< \frac{a+b}{A} \leq 1 \\ &< 1 \end{aligned}$$

(5) $F_0(y, y, ..., y) = \frac{a}{A+By} > 0$ for all $y \geq 0$.

We can conclude now

Theorem 5.1 *The zero equilibrium point $\bar{x} = 0$ is globally asymptotically stable for the equation*

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}}$$

under the condition $a + b \leq A$.

5.2 The global stability of the positive equilibria

We study the global stability of the positive equilibria under the condition $a + b > A$.

Our equation as we said before could be written as

$$x_{n+1} = \frac{a}{A + Bx_{n-k}} x_n + \frac{b}{A + Bx_{n-k}} x_{n-k} \tag{5.3}$$

So, take

$$f(u, v) = \frac{a}{A + Bv} u + \frac{b}{A + Bv} v \tag{5.4}$$

So, by returning back to the main theorems of global stability, Theorem 1.5 and Theorem 1.6, and as out equation has no period two solution, we can get in to the following theorem for the global stability of the positive equilibrium point.

Theorem 5.2 *The positive equilibria $\bar{x} = \frac{a+b-A}{B}$ is a globally asymptotically stable point of the equation*

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}}$$

under the condition $a + b > A$.

6 Numerical discussion

In this section, we investigate some examples that include all the case that the two equilibrium points are globally asymptotically stable by theory in order to illustrate the results we got. The examples were carried on MATLAB 6.5.

Example 1 Assume that Eq. 1.2 holds, take $k = 2$, $A = 4$, $B = 3$, $a = 1$, $b = 2$. So the equation will be reduced to the following:

$$x_{n+1} = \frac{x_n + 2x_{n-2}}{4 + 3x_{n-2}} \tag{6.1}$$

In this case, $a + b = 3 < A = 4$ We assumed that the initial points $x_{-2}, x_{-1}, x_0 \in [0, \infty)$ are to be respectively $\{0.7, 0, 1.2\}$.

By theory, the zero equilibrium point under the condition $a + b < A$ is globally asymptotically stable as it is also obvious from Fig. 1.

Example 2 Assume that Eq. 1.2 holds, take $k = 3$, $A = 6$, $B = 2$, $a = 4$, $b = 2$. So the equation will be reduced to the following:

$$x_{n+1} = \frac{4x_n + 2x_{n-3}}{6 + 2x_{n-3}} \tag{6.2}$$

In this case, $a + b = 6 = A$ We assumed that the initial points $x_{-3}, x_{-2}, x_{-1}, x_0 \in [0, \infty)$ are to be respectively $\{0.5, 1.2, 1.9, 2.4\}$.

By theory, the zero equilibrium point under the condition $a + b = A$ is globally asymptotically stable as it is also obvious from Fig. 2.

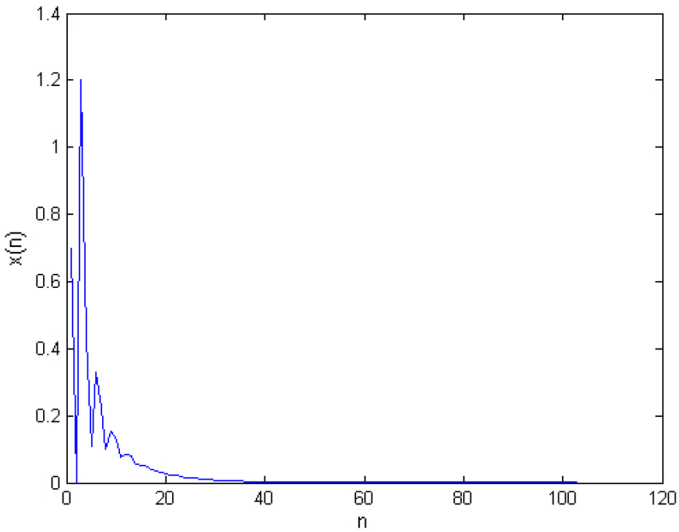

Fig. 1 The behavior of the zero equilibrium point of $x_{n+1} = \frac{x_n+2x_{n-2}}{4+3x_{n-2}}$

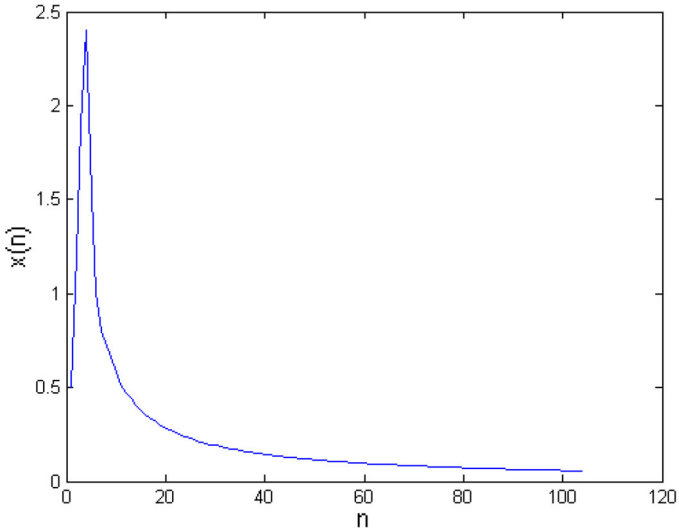

Fig. 2 The behavior of the zero equilibrium point of $x_{n+1} = \frac{4x_n+2x_{n-3}}{6+2x_{n-3}}$

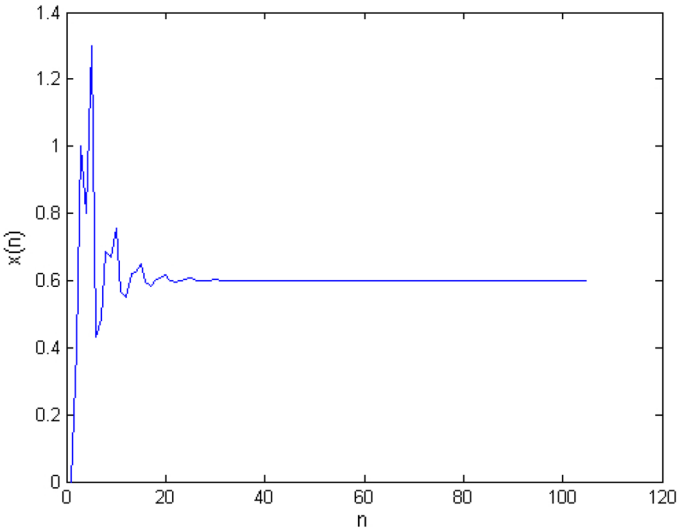

Fig. 3 The behavior of the positive equilibrium point of $x_{n+1} = \frac{x_n+5x_{n-4}}{3+5x_{n-4}}$

Example 3 Assume that Eq. 1.2 holds, take $k = 4$, $A = 3$, $B = 5$, $a = 1$, $b = 5$. So the equation will be reduced to the following:

$$x_{n+1} = \frac{x_n + 5x_{n-4}}{3 + 5x_{n-4}} \tag{6.3}$$

In this case, $a + b = 6 > A = 3$ We assumed that the initial points $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 \in [0, \infty)$ are to be respectively $\{0, 0.4, 1, 0.8, 1.3\}$. Here the positive equilibrium point will be

$$\bar{x} = \frac{a + b - A}{B} = \frac{3}{5} = 0.6$$

By theory, the positive equilibrium point $\bar{x} = 0.6$ under the condition $a + b > A$ should be globally asymptotically stable as it is also obvious from Fig. 3.

So, all what we have to say now is that our theoretical discussion was satisfied with the data we get from our numerical discussion. So we have correctly illustrated our study for the equation $x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}}$.

References

1. Abu-Saris, R., DeVault, R.: Global stability of $y_{n+1} = A + \frac{y_n}{y_{n-k}}$. Appl. Math. Lett. **16**, 173–178 (2003)
2. Amleh, A., Grove, E., Ladas, G., Georgiou, G.: On the recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}$. J. Math. Anal. Appl. **533**, 790–798 (1999)
3. Dannan, F.: The Asymptotic Stability of $x_{n+k} + ax_n + bx_{n-l} = 0$. J. Differ. Equ. Appl. **10**, 589–599 (2004)
4. DeVault, R., Schultz, S.W., Ladas, G.: On the recursive sequence $x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}$. Proc. Am. Math. Soc. **126**, 3257–3261 (1998)
5. Yan, Xing-Xue, Li, Wan-Tong, Zhao, Zhu: Global asymptotic stability for a higher order nonlinear rational differnce equations. Appl. Math. Comput. **182**, 1819–1831 (2006)
6. DeVault, R., Kosmala, W., Ladas, G., Schultz, S.W.: On the Recursive Sequence Global behavior of $y_{n+1} = \frac{p + y_{n-k}}{qy_n + y_{n-k}}$. Nonlinear Anal. **47**, 4743–4751 (2001)
7. Douraki, M., Dehghan, M., Razzaghi, M.: On the higher order rational difference equation $x_n = \frac{A}{x_{n-k}} + \frac{B}{x_{n-3k}}$. Appl. Math. Comput. **173**, 710–723 (2006)
8. Elaydi, S.: An Introduction to Difference Equations, 3rd edn. Springer, New York (2005)
9. El-Owaidy, H., Ahmed, A., Mousa, M.: On asymptotic behaviour of the difference equation $x_{n+1} = \alpha + \frac{x_{n-k}}{x_n}$. Appl. Math. Comput. **147**, 163–167 (2004)
10. Kocic, V.L., Ladas, G.: Global Asymptotic Behavior of Nonlinear Difference Equations of Higher Order with Applications. Kluwer Academic Publishers, Dordrect (1993)
11. Kosmala, W., Kulenovic, M.R.S., Ladas, G., Teixeira, C.T.: On the Recursive Sequence $y_{n+1} = \frac{p + y_{n-1}}{qy_n + y_{n-1}}$. J. Math. Anal. Appl. **251**, 571–586 (2000)
12. Kuruklis, S.A.: The asymptotic stability of $x_{n+1} - ax_n + bx_{n-k} = 0$. J. Math. Anal. Appl. **188**, 719–731 (1994)
13. Papanicolaou, V.G.: On the asymptotic stability of a class of linear difference equations. Math. Mag. **69**, 34–43 (1996)
14. Saleh, M., Aloqeili, M.: On the rational difference equation $y_{n+1} = A + \frac{y_{n-k}}{y_n}$, Appl. Math. Comput. **171**(1), 862–869 (2005)