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SERRE CLASS AND THE DIRECT SUMS OF MODULES

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Abstract. The purpose of this paper is to further the study of weakly injective and weakly tight modules a generalization of injective modules. For a Serre class $\mathcal{K}$ of modules, we study when direct sums of modules from $\mathcal{K}$ satisfies a property $\mathbb{P}$ in $\mathcal{K}$. In particular, we get characterization of locally $q.f.d.$ modules in terms of weak tightness.

1. Introduction

Throughout this paper all rings are associative with identity and all modules are unitary. We denote the category of all right R-modules by Mod-R and for any $M \in$ Mod-R, $\sigma[M]$ stands for the full subcategory of Mod-R whose objects are submodules of M-generated modules (see [28]). A class $\mathcal{K}$ of modules is called a Serre class if it is closed under submodules, factor modules and extensions [9], [4]. The class $\sigma[M]$ is a Serre class [4]. Given a module X_R, the injective hull of X in Mod-R (resp., in $\sigma[M]$, in $\mathcal{K}$) is denoted by $E(X)$ (resp., $\widehat{X}$, $E_{\mathcal{K}}(X)$). The M-injective hull $\widehat{X}$ is the trace of M in $E(X)$, i.e. $\widehat{X} = \sum\{f(M), f \in Hom(M, E(X))\}$ [3], [28].

The purpose of this paper is to further the study of the concepts of weak injectivity, tightness and weak tightness in a Serre class $\mathcal{K}$ a generalization of $\sigma[M]$ studied in [4], [8], [21], [22], [23], [26], [29], [30]. In view of Theorem 2.9 every module X in $\mathcal{K}$ is a direct summand of a weakly injective module in $\mathcal{K}$ a result that generalizes 2.12, , 2.13, 2.14, in [16], 2.1, 2.2., and 2.3 in [18]. For a Serre class $\mathcal{K}$, we study when direct sums of modules from $\mathcal{K}$ are weakly tight in $\mathcal{K}$. In particular, we get necessary and sufficient conditions for $\sum$-weak tightness of

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the injective hull of a simple module. As a consequence, we get characterizations of *q.f.d.* rings by means of weakly injective (tight) modules given by A. Al-Huzali, S.K. Jain and S.R. López-Permouth.

Given two modules Q and $N \in \mathcal{K}$, we call Q *weakly N-injective* in $\mathcal{K}$ if for every homomorphism $\varphi : N \to E_{\mathcal{K}}(Q)$, there exists a homomorphism $\widehat{\varphi} : N \to Q$ and a monomorphism $\sigma : Q \to E_{\mathcal{K}}(Q)$ such that $\varphi = \sigma\widehat{\varphi}$. Equivalently, there exists a submodule X of $E_{\mathcal{K}}(Q)$ such that $\varphi(N) \subset X \simeq Q$. A module $Q \in \mathcal{K}$ is called *weakly injective* in $\mathcal{K}$ if for every finitely generated submodule N of the $\mathcal{K}$-injective hull $E_{\mathcal{K}}(Q)$, N is contained in a submodule Y of $E_{\mathcal{K}}(Q)$ such that $Y \simeq Q$. Equivalently, if Q is weakly N-injective for all finitely generated modules N in $E_{\mathcal{K}}(Q)$. A module X is *N-tight* in $E_{\mathcal{K}}(Q)$ if every quotient of N which is embeddable in the $\mathcal{K}$-injective hull of X is embeddable in X. A module is *tight(R-tight)* in $\mathcal{K}$ if it is tight relative to all finitely generated (cyclic) submodules of its $\mathcal{K}$-injective hull, and Q is *weakly tight (weakly R-tight)* in $\mathcal{K}$ if every finitely generated (cyclic) submodule N of $E_{\mathcal{K}}(Q)$ is embeddable in a direct sum of copies of Q. It is clear that every weakly injective module in $\mathcal{K}$ is tight in $\mathcal{K}$, and every tight module in $\mathcal{K}$ is weakly tight in $\mathcal{K}$, but weak tightness does not imply tightness, (see [4], [30]). A module M_R is called *locally q.f.d.* [3], [7], [17] in case every finitely generated (or cyclic) module $N \in \sigma[M]$ has finite uniform dimension. A module Q is called *weakly (N-)injective (resp., tight, weakly (N-)tight)* [16], [13], [14], [15] if it is weakly (N-)injective(resp., tight, weakly (N-)tight) in $\mathcal{K}$ =Mod-R.

2. Preliminaries

The class of weak injectivity (tightness, weak tightness) in $\mathcal{K}$ is closed under finite direct sums, and essential extensions. Also, the domains of the class of weak injectivity (tightness, weak tightness) in $\mathcal{K}$ are closed under submodules, and quotients.

First, we list below some basic results on weak injectivity (tightness, weak tightness) in $\mathcal{K}$ that generalizes those known results in $\sigma[M]$ that will be needed through this paper (c.f.[4], [21], [22], [23], [29], [30]).

Lemma 2.1. *Given modules $N, X \in \mathcal{K}$. If X is quasi-injective and weakly N-injective in $\mathcal{K}$, then X is N-injective in $\mathcal{K}$.*

Proof. Let $\phi : N \to E_{\mathcal{K}}(X)$. It is enough to show that $\phi(N) \subseteq X$. By weak injectivity of X, there exist a homomorphism $\varphi : N \to X$ and a monomorphism $\sigma : X \to E_{\mathcal{K}}(X)$ such that $\phi = \sigma\varphi$. Also Since X is quasi-injective, $\sigma(X) \subseteq X$ and thus $\phi(N) \subseteq X$, proving that X is N-injective □

Corollary 2.2. *A quasi-injective and weakly R-injective module is injective.*

Remark 2.3. *Weak injectivity in Lemma 2.1 and Corollary 2.2 could be replaced by tightness but we don't know if it could be replaced by weak tightness.*

Lemma 2.4. *Given $X, N \in \mathcal{K}$, X is weakly N-injective in $\mathcal{K}$ iff*

(i) for every submodule K of N and for every monomorphism $\varphi : N/K \to E_{\mathcal{K}}(X)$, there exists a monomorphism $\widehat{\varphi} : N/K \to X$ and a monomorphism $\sigma : X \to E_{\mathcal{K}}(X)$ such that $\varphi = \sigma\widehat{\varphi}$; and

(ii) for every complement C of $\widehat{\varphi}(N/K)$ in X there exists $C' \subseteq E_{\mathcal{K}}(X)$ such that $C' \cap \sigma(N/K)$=0 and $C' \simeq C$.

Proof. (i) is clear from the definition. To prove (ii) let C be a complement of $\widehat{\varphi}(N/K)$ in X. Then $C' = \sigma(C)$ is isomorphic to C and independent from $\sigma(N/K)$, proving (ii). Conversely, assume that (i) and (ii) hold and let $\phi : N/K \to E_{\mathcal{K}}(X)$ be a monomorphism. By (i) there exists a monomorphism $\widehat{\varphi} : N/K \to X$. Let C be a complement of $\widehat{\varphi}(N/K)$ in X. Using (ii), we get a monomorphism $\alpha : \widehat{\varphi}(N/K) \bigoplus C \to E_{\mathcal{K}}(X)$. Since $\widehat{\varphi}(N/K) \bigoplus C \subseteq' X$, we may extend α to a monomorphism $\beta : X \to E_{\mathcal{K}}(X)$. It follows that $\beta\widehat{\varphi} = \sigma$, proving that X is weakly N-injective in $\mathcal{K}$. □

Corollary 2.5. *A uniform module $X \in \mathcal{K}$ is tight in $\mathcal{K}$ iff X is weakly injective in $\mathcal{K}$.*

Lemma 2.6. *A finite direct sums of weakly injective (tight, weakly tight) in $\mathcal{K}$ is weakly injective (tight, weakly tight) in $\mathcal{K}$, and an essential extension of a weakly injective (tight, weakly tight) module in $\mathcal{K}$ is weakly injective (tight, weakly tight) in $\mathcal{K}$.*

Lemma 2.7. *A uniform module $X \in \mathcal{K}$ is weakly tight (weakly R-tight) in $\mathcal{K}$ iff X is weakly injective (weakly R-injective) in $\mathcal{K}$.*

Proof. Let X be uniform and weakly tight in $\mathcal{K}$, and let N be a finitely generated submodule of $E_{\mathcal{K}}(X)$. Then N is embeddable in $X^{(\alpha)}$ via a monomorphism, say, ϕ. Let $\pi_i : X^{(\alpha)} \to X$ be the ith projection map. Then $\cap_{i\in\alpha} ker(\pi_i\phi) \subseteq \ker\phi = 0$. Since X is uniform then $ker(\pi_i\phi) = 0$, and thus N embeds in X, proving that X is tight. By Corollary 2.5, X is weakly injective in $\mathcal{K}$. The other case is similar. □

Lemma 2.8. *Let $\mathcal{K}$ be a Serre class. Then every tight module in $\mathcal{K}$ is weakly injective in $\mathcal{K}$.*

Proof. The proof follows from [16, Theorem 2.8]. □

In [18], it is shown that any semisimple module is a direct summand of a weakly injective module, the next lemma shows that in fact any module is a direct summand of a weakly injective module.

Theorem 2.9. *Every module in $\mathcal{K}$ is a direct summand of a weakly injective module in $\mathcal{K}$.*

Proof. For any module $X \in \mathcal{K}$, $L = X \oplus (E_{\mathcal{K}}(X))^{(\alpha)}$, where α is an infinite cardinal, is weakly injective in $\mathcal{K}$. Let N be a finitely generated submodule of $E_{\mathcal{K}}(L) = (E_{\mathcal{K}}(X))^{(\alpha)}$. Then N is contained in $(E_{\mathcal{K}}(X))^{(n)}$, for some finite n. Since $X \subseteq E_{\mathcal{K}}(X)$, there exists $M \subseteq E_{\mathcal{K}}(X)$ such that $X \simeq M$. Let $Y = (E_{\mathcal{K}}(X))^{(\alpha)} \oplus M \oplus (E_{\mathcal{K}}(X))^{(n)} \subseteq L$. Then $N \subseteq Y \simeq L$, proving that L is weakly injective. $\square$

Theorem 2.9 answers an open question in [16] and generalizes 2.12, 2.13, 2.14, in [16], 2.1, 2.2., and 2.3 in [18].

Example 2.10. *(i)* [16, Example 2.11], [18]. *Let R be the ring of endomorphisms of an infinite dimensional vector space V over a field F. Then $M = Soc(R_R) \oplus R$ is tight but not weakly injective.*

(ii) [4]. *Let $R = Z$ and $X = (Q/Z) \oplus (Z/pZ)$. Then X is weakly tight in $\sigma[M]$ but not tight.*

(iii) [16, Example 4.4(d)]. *Let F be a field. Then $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ is weakly injective but the summand $S = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ as an R-module is not weakly injective.*

The above results show that the classes of weakly injective, tight, and weakly tight modules are quite large.

A Serre class $\mathcal{K}$ is called *weakly semisimple (weakly R-semisimple)* iff every module $N \in \mathcal{K}$ is weakly injective (weakly R-injective) in $\mathcal{K}$. As a direct applications of the above results, we state the following characterizations of semisimple and weakly (R-)semisimple modules in terms of weak injectivity, tightness, and weak tightness without proof.

Theorem 2.11. *For a Serre class $\mathcal{K}$, the following are equivalent:*

(a) $\mathcal{K}$ is semisimple;
(b) every weakly injective module in $\mathcal{K}$ is (quasi-)discrete;
(c) every weakly injective module in $\mathcal{K}$ is (quasi-)continuous;

(d) every direct summand of a weakly injective module in $\mathcal{K}$ is quasi-injective in $\mathcal{K}$.

Theorem 2.12. *For a Serre class $\mathcal{K}$, the following are equivalent:*

(a) $\mathcal{K}$ is weakly semisimple (resp., weakly R-semisimple);

(b) every direct summand of a weakly injective (or tight, weakly tight) (resp., weakly R-injective) (or R-tight, weakly R-tight) module in $\mathcal{K}$ is weakly injective (or tight, weakly tight) (resp., weakly R-injective) (or R-tight, weakly R-tight) in $\mathcal{K}$.

In case $\mathcal{K} = R$ in the above two theorems we get characterizations of semisimple, weakly semisimple, and weakly R-semisimple rings.

3. Direct sums of classes of modules.

For a module X_R and a module property $I\!\!P$, X is said to be $\sum - I\!\!P$ in case every direct sum of copies of X enjoys the property $I\!\!P$. A class $\mathcal{K}$ of modules is called a Serre class if it is closed under submodules, factor modules and extensions. The class $\sigma[M]$ is a Serre class [4].

Theorem 3.1. *For a Serre class $\mathcal{K}$, the following implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f)$ always hold.*

(a) every direct sum $\bigoplus_\Lambda E_\lambda$ of injectives in $\mathcal{K}$ is weakly injective in $\mathcal{K}$;

(b) every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injective modules in $\mathcal{K}$ is weakly injective in $\mathcal{K}$;

(c) every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injective modules in $\mathcal{K}$ is tight in $\mathcal{K}$;

(d) every direct sum $\bigoplus_\Lambda M_\lambda$ of tight modules in $\mathcal{K}$ is tight in $\mathcal{K}$;

(e) every direct sum $\bigoplus_\Lambda M_\lambda$ of tight modules in $\mathcal{K}$ is weakly tight in $\mathcal{K}$;

(f) every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly tight modules in $\mathcal{K}$ is weakly tight in $\mathcal{K}$.

Proof. $(a) \Rightarrow (b)$. Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of weakly injective modules in $\mathcal{K}$. Let N be a finitely generated submodule of $E_\mathcal{K}(X)$. By (a), the direct sum $\bigoplus_\Lambda E_\mathcal{K}(M_\lambda)$ is weakly injective in $\mathcal{K}$ and $X = \bigoplus_\Lambda M_\lambda \subseteq' \bigoplus_\Lambda E_\mathcal{K}(M_\lambda) \subseteq' E_\mathcal{K}(\bigoplus_\Lambda E_\mathcal{K}(M_\lambda))$. Thus by (a) there exists a submodule $Y \subseteq E_\mathcal{K}(\bigoplus_\Lambda E_\mathcal{K}(M_\lambda))$ such that $N \subseteq Y \cong \bigoplus_\Lambda E_\mathcal{K}(M_\lambda)$. Write $Y = \bigoplus_\Lambda E_\mathcal{K}(Y_\lambda)$, where $Y_i \cong M_i, i \in \Lambda$. Since N is finitely generated, there exists a finite subset $\Gamma = \{\lambda_1, ..., \lambda_m\} \subseteq \Lambda$ such that $N \subseteq \bigoplus_\Gamma E_\mathcal{K}(Y_\lambda) = E_\mathcal{K}(\bigoplus_\Gamma (Y_\lambda))$. Since $Y_{\lambda_1}, \cdots, Y_{\lambda_m}$ are weakly injective in $\mathcal{K}$, the finite direct sum $Y_{\lambda_1} \oplus \cdots \oplus Y_{\lambda_m}$ is weakly injective in $\mathcal{K}$. Therefore, there exists $X_1 \cong \bigoplus_\Gamma Y_\lambda \cong \bigoplus_\Gamma M_\lambda$ such that $N \subseteq X_1 \subseteq E_\mathcal{K}(\bigoplus_\Gamma Y_\lambda)$. Thus $N \subseteq X_1 \oplus \bigoplus_{\lambda \notin \Gamma} Y_\lambda \simeq X$, proving that X is weakly injective in $\mathcal{K}$.

$(c) \Rightarrow (d)$. Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of tight modules in $\mathcal{K}$. Let N be a finitely generated submodule of $E_\mathcal{K}(X) = E_\mathcal{K}(\bigoplus_\Lambda E_\mathcal{K}(M_\lambda))$. By (c) the direct sum $\bigoplus_\Lambda E_\mathcal{K}(M_\lambda)$ is tight in $\mathcal{K}$. Thus N embeds in $\bigoplus_\Lambda E_\mathcal{K}(M_\lambda)$ via a monomorphism, say, φ. Also $\varphi(N)$ is finitely generated and thus $N \subset E_\mathcal{K}(M_{\lambda_1}) \oplus \cdots \oplus E_\mathcal{K}(M_{\lambda_m}) = \bigoplus_{\lambda=1}^{\lambda=m} E_\mathcal{K}(M_\lambda)$ for some finite $\{\lambda_1, ..., \lambda_m\} \subseteq \Lambda$. Since $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$ is tight then $N \simeq \varphi(N)$ embeds in the finite direct sums $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$, proving that X is tight in $\mathcal{K}$.

$(e) \Rightarrow (f)$. Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of weakly tight modules in $\mathcal{K}$. Let N be a finitely generated submodule of $E_\mathcal{K}(X) = E_\mathcal{K}(\bigoplus_\Lambda E_\mathcal{K}(M_\lambda))$. By (e) the direct sum $\bigoplus_\Lambda E_\mathcal{K}(M_\lambda)$ is weakly tight in $\mathcal{K}$. Thus N embeds in $(\bigoplus_\Lambda E_\mathcal{K}(M_\lambda))^{(\aleph_0)}$ via a monomorphism, say, φ. Also $\varphi(N)$ is finitely generated and thus $N \subset E_\mathcal{K}(M_{\lambda_1}) \oplus \cdots \oplus E_\mathcal{K}(M_{\lambda_m}) = \bigoplus_{\lambda=1}^{\lambda=m} E_\mathcal{K}(M_\lambda)$ for some finite $\{\lambda_1, ..., \lambda_m\} \subseteq \Lambda$. Since $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$ is weakly tight then $N \simeq E_\mathcal{K}(N)$ embeds in a direct sums of copies of $(M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m})$ and thus embeds in a direct sums of X, proving that X is weakly tight in $\mathcal{K}$.

Clearly, $(b) \Rightarrow (c)$ and $(d) \Rightarrow (e)$. □

The next theorem provides several characterizations of a Serre class $\mathcal{K}$ which extends the main results in [24, Theorem 2.7, Corollary 2.8], [25, Theorem 1.10, Theorem 1.11]. Consequently, we get the main result in [2, Theorem] as a corollary to the main results of this section.

Theorem 3.2. *For a Serre class $\mathcal{K}$, the following conditions are equivalent:*

(a) every cyclic module in $\mathcal{K}$ has finite uniform dimension;
(b) every finitely generated module in $\mathcal{K}$ has finite uniform dimension;
(c) every direct sum $\bigoplus_\Lambda E_\lambda$ of injectives in $\mathcal{K}$ is weakly injective in $\mathcal{K}$;
(d) every direct sum $\bigoplus_\Lambda E_\lambda$ of tight modules in $\mathcal{K}$ is tight in $\mathcal{K}$;
(e) every direct sum $\bigoplus_\Lambda E_\lambda$ of weakly tight modules in $\mathcal{K}$ is weakly tight in $\mathcal{K}$;
(f) every direct sum $\bigoplus_\Lambda E_\lambda$ of weakly tight modules in $\mathcal{K}$ is weakly N-tight $\mathcal{K}$, for every cyclic module N in $\mathcal{K}$;
(g) every direct sum $\bigoplus_\Lambda E_\mathcal{K}(P_\lambda)$, where P_λ is simple in $\mathcal{K}$, is weakly N-tight in $\mathcal{K}$, for every cyclic module N in $\mathcal{K}$.

Proof. $(a) \Rightarrow (b)$. Follows by an argument similar to that in [7].

$(b) \Rightarrow (c)$. Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of injective modules in $\mathcal{K}$. Let N be a finitely generated submodule of $E_\mathcal{K}(X)$. By (b), N contains as an essential submodule a finite direct sum of uniform submodules $\bigoplus_\Lambda U_\lambda$. Since X is essential in $E_\mathcal{K}(X)$, for each i, choose $0 \neq x_i \in U_i \cap X$. Then $\bigoplus_{i=1}^{i=n} x_i R \subseteq \bigoplus_{i=1}^{i=n} M_{\lambda_i}$ for some λ_i's and $\bigoplus_{i=1}^{i=n} x_i R \subseteq' \bigoplus_\Lambda U_\lambda \subseteq' N$. It follows that $\bigoplus_{i=1}^{i=n} M_{\lambda_i}$ contains an $\mathcal{K}$-injective hull E of $\bigoplus_{i=1}^{i=n} x_iR$. Since E is injective in $\mathcal{K}$ and contained in X, we may write $X = E \oplus K$, for some submodule K of X. On the other hand, let $E_{\mathcal{K}}(N)$ be an $\mathcal{K}$-injective hull of N in $E_{\mathcal{K}}(X)$. Then $E_{\mathcal{K}}(N) = E_{\mathcal{K}}(\bigoplus_{i=1}^{i=n} x_iR \cong E)$. Since $\bigoplus_{i=1}^{i=n} x_iR$ is essential in $E_{\mathcal{K}}(N)$, it follows that $E_{\mathcal{K}}(N) \cap K = 0$. So let $Y = E_{\mathcal{K}}(N) \oplus K \cong E \oplus K = X$. Then $N \subseteq Y \cong X$, proving that X is weakly injective in $\mathcal{K}$.

$(c) \Rightarrow (d)$. Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of tight modules in $\mathcal{K}$. Let N be a finitely generated submodule of $E_{\mathcal{K}}(X)$. By (c) the direct sum $\bigoplus_\Lambda E_{\mathcal{K}}(M_\lambda)$ is tight in $\mathcal{K}$ and thus N embeds in $\bigoplus_\Lambda E_{\mathcal{K}}(M_\lambda)$ via a monomorphism, say, φ. Also $\varphi(N)$ is finitely generated and thus $N \subset E_{\mathcal{K}}(M_{\lambda_1}) \oplus \cdots \oplus E_{\mathcal{K}}(M_{\lambda_m})$ for some finite $\{\lambda_1, ..., \lambda_m\} \subseteq \Lambda$. Since $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$ is tight then $N \simeq \varphi(N)$ embeds in $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$ and thus embeds in X, proving that X is tight $\mathcal{K}$.

$(d) \Rightarrow (e)$. Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of weakly tight modules in $\mathcal{K}$. Let N be a finitely generated submodule of $E_{\mathcal{K}}(X)$. By (d) the direct sum $\bigoplus_\Lambda E_{\mathcal{K}}(M_\lambda)$ is tight in $\mathcal{K}$. Thus N embeds in $\bigoplus_\Lambda E_{\mathcal{K}}(M_\lambda)$ via a monomorphism, say, φ. Also $\varphi(N)$ is finitely generated and thus $N \subset E_{\mathcal{K}}(M_{\lambda_1}) \oplus \cdots \oplus E_{\mathcal{K}}(M_{\lambda_m})$ for some finite $\{\lambda_1, ..., \lambda_m\} \subseteq \Lambda$. Since $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$ is weakly tight then $N \simeq \varphi(N)$ embeds in a finite direct sums of $(M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m})$ and thus embeds in a finite direct sums of X, proving that X is weakly tight in $\mathcal{K}$.

Clearly, $(e) \Rightarrow (f) \Rightarrow (g)$.

$(g) \Rightarrow (a)$. Let C be a cyclic module in $\mathcal{K}$, and suppose C is not finite dimensional. Then C contains an essential submodule which is a direct sum of infinitely many nonzero uniform submodules $\bigoplus_\Lambda U_\lambda$. Then $E_{\mathcal{K}}(C) = \bigoplus_\Lambda E_{\mathcal{K}}(U_\lambda)$ $= E_{\mathcal{K}}(\bigoplus_\Lambda E_{\mathcal{K}}(U_\lambda))$. By (g), $\bigoplus_\Lambda E_{\mathcal{K}}(U_\lambda)$ is weakly C-tight. Therefore, C is embeddable in $(\bigoplus_\Lambda E_{\mathcal{K}}(U_\lambda))^{(\aleph_0)}$. Then C is embeddable in $\bigoplus_\Gamma E_{\mathcal{K}}(U_\lambda)$, for some finite subset $\Gamma = \{\lambda_1, ..., \lambda_m\} \subseteq \Lambda$. Thus $E_{\mathcal{K}}(C)$ embeds in $\bigoplus_\Gamma E_{\mathcal{K}}(U_\lambda)$, showing that $U_\lambda = 0$ for all $\lambda \notin \Gamma$, a contradiction. $\square$

Combining Theorem 3.1 and Theorem 3.2 we get the following

Theorem 3.3. *For a Serre class $\mathcal{K}$, the following conditions are equivalent:*

(a) every cyclic module in $\mathcal{K}$ has finite uniform dimension;
(b) every finitely generated module in $\mathcal{K}$ has finite uniform dimension;
(c) every direct sum $\bigoplus_\Lambda E_\lambda$ of injectives in $\mathcal{K}$ is weakly injective (or tight, weakly tight, Weakly R-injective, R-tight, weakly R-tight) in $\mathcal{K}$;
(d) every direct sum $\bigoplus_\Lambda E_\lambda$ of weakly injectives in $\mathcal{K}$ is weakly injective (or tight, weakly tight, weakly R-injective, R-tight, weakly R-tight) in $\mathcal{K}$;

(e) every direct sum $\bigoplus_\Lambda M_\lambda$ of tight modules in $\mathcal{K}$ is tight (or weakly tight, R-tight, weakly R-tight) in $\mathcal{K}$;
(f) every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly tight modules in $\mathcal{K}$ is weakly tight (or weakly R-tight) in $\mathcal{K}$;
(g) every direct sum $\bigoplus_\Lambda E_\mathcal{K}(P_\lambda)$, where P_λ is simple in $\mathcal{K}$, is weakly N-tight for every cyclic module N in $\mathcal{K}$;
(h) every direct sum $\bigoplus_\Lambda E_\mathcal{K}(P_\lambda)$, where P_λ is simple in $\mathcal{K}$, is weakly R-tight in $\mathcal{K}$.

Taking $\mathcal{K} = \sigma[M]$, M-singular modules in $\sigma[M]$ in Theorem 3.3 we get [24, Theorem 2.7, Corollary 2.8], [25, Theorem 1.10] as a corollary. Recall that a module M is called $q.f.d.(g.q.f.d.)$ if every finitely M-generated (every finitely M-generated M-singular) module in $\sigma[M]$ has finite uniform dimension.

Corollary 3.4. [24, Theorem 2.7] *For a module M_R, the following conditions are equivalent:*
(a) M is q.f.d.;
(b) every direct sum $\bigoplus_\Lambda E_\lambda$ of injectives in $\sigma[M]$ is weakly injective (or tight, weakly tight, weakly R-tight, R-tight, weakly R-tight) in $\sigma[M]$;
(c) every direct sum $\bigoplus_\Lambda E_\lambda$ of weakly injective in $\sigma[M]$ is weakly injective (or tight, weakly tight, weakly R-tight, R-tight, weakly R-tight) in $\sigma[M]$;
(d) every direct sum of tight modules in $\sigma[M]$ is tight (or weakly tight, R-tight, weakly R-tight) in $\sigma[M]$;
(e) every direct sum of weakly tight modules in $\sigma[M]$ is weakly tight (or weakly R-tight) in $\sigma[M]$;
(f) every direct sum $\bigoplus_\Lambda \widehat{P_\lambda}$, where each P_λ is simple, is weakly N-tight for every cyclic module N in $\sigma[M]$;
(g) every direct sum $\bigoplus_\Lambda \widehat{P_\lambda}$, where each P_λ is simple, is weakly R-tight in $\sigma[M]$.

Corollary 3.5. [25, Theorem 1.10] *For a module M_R, the following conditions are equivalent:*
(a) M is g.q.f.d.;
(b) every direct sum $\bigoplus_\Lambda E_\lambda$ of nonsingular injectives in $\sigma[M]$ is weakly injective (or tight, weakly tight, weakly R-injective, R-tight, weakly R-tight) in $\sigma[M]$;
(c) every direct sum $\bigoplus_\Lambda E_\lambda$ of nonsingular weakly injective in $\sigma[M]$ is weakly injective (or tight, weakly tight, weakly R-injective, R-tight, weakly R-tight) in $\sigma[M]$;
(d) every direct sum of nonsingular tight modules in $\sigma[M]$ is tight (or weakly tight, R-tight, weakly R-tight) in $\sigma[M]$;

(e) *every direct sum of nonsingular weakly tight modules in* $\sigma[M]$ *is weakly tight (or weakly R-tight) in* $\sigma[M]$*;*
(f) *every direct sum* $\bigoplus_\Lambda \widehat{P_\lambda}$*, where each* P_λ *is nonsingular simple, is weakly N-tight for every cyclic module N in* $\sigma[M]$*;*
(g) *every direct sum* $\bigoplus_\Lambda \widehat{P_\lambda}$*, where each* P_λ *is nonsingular simple, is weakly R-tight in* $\sigma[M]$.

In case $M = R_R$ in Corollaries 3.4, 3.5 we obtain characterizations of $q.f.d.$ rings that generalizes Theorem 2.6 and Corollary 2.7 in [29] and the main theorem in [2].

Corollary 3.6. [24, Corollary 2.8] *For a ring R, the following conditions are equivalent:*
(a) *R is q.f.d.;*
(b) *every direct sum* $\bigoplus_\Lambda E_\lambda$ *of injectives is weakly injective (or tight, weakly tight);*
(c) *every direct sum* $\bigoplus_\Lambda E_\lambda$ *of weakly injective is weakly injective (or tight, weakly tight);*
(d) *every direct sum of tight modules is tight (or weakly tight);*
(e) *every direct sum of weakly tight module is weakly tight (or weakly R-tight);*
(f) *every direct sum* $\bigoplus_\Lambda E(P_\lambda)$*, where each* P_λ *is simple, is weakly N-tight for every cyclic module N;*
(g) *every direct sum* $\bigoplus_\Lambda E(P_\lambda)$*, where each* P_λ*is simple, is weakly R-tight.*

Corollary 3.7. [25, Theorem 1.11] *For a module* M_R*, the following conditions are equivalent:*
(a) *R is g.q.f.d.;*
(b) *every direct sum* $\bigoplus_\Lambda E_\lambda$ *of nonsingular injectives is weakly injective (or tight, weakly tight, weakly R-injective, R-tight, weakly R-tight);*
(c) *every direct sum* $\bigoplus_\Lambda E_\lambda$ *of nonsingular weakly injective is weakly injective (or tight, weakly tight, weakly R-injective, R-tight, weakly R-tight);*
(d) *every direct sum of nonsingular tight modules is tight (or weakly tight, R-tight, weakly R-tight);*
(e) *every direct sum of nonsingular weakly tight modules is weakly tight (or weakly R-tight);*
(f) *every direct sum* $\bigoplus_\Lambda E(P_\lambda)$*, where each* P_λ *is nonsingular simple, is weakly N-tight for every cyclic module N;*
(g) *every direct sum* $\bigoplus_\Lambda E(P_\lambda)$*, where each* P_λ*is nonsingular simple, is weakly R-tight.*

A class $\mathcal{K}$ is called $q.f.d.$ if every cyclic module in $\mathcal{K}$ has finite uniform dimension or equivalently if it satisfies any of the equivalent conditions in Theorem 3.3.

Theorem 3.8. *A q.f.d. Serre class $\mathcal{K}$ is weakly semisimple (weakly R-semisimple) iff every uniform cyclic in $\mathcal{K}$ is weakly injective or tight, or weakly tight (resp., weakly R-injective, or R-tight, or weakly R-tight) in $\mathcal{K}$.*

Proof. Since $\mathcal{K}$ is $q.f.d.$ it follows that any X in $\mathcal{K}$ contains as an essential submodule a finite direct sums of cyclic uniforms and thus weakly injective (resp., tight, or weakly tight, weakly R-injective, R-tight, weakly R-tight) submodules and thus the sum is weakly injective (resp., tight, or weakly tight, weakly R-injective, R-tight, weakly R-tight). By Lemma 2.6, it follows that X is weakly injective (resp., tight, or weakly tight, weakly R-injective, R-tight, weakly R-tight) in $\mathcal{K}$, proving that $\mathcal{K}$ is weakly semisimple (resp., weakly R-semisimple). The converse is trivial. $\square$

A module X in $\mathcal{K}$ is called compressible (resp., R-compressible) if it is embeddable in its essential (cyclic essential) submodules. A module X in $\mathcal{K}$ is called strongly compressible (resp., strongly R-compressible) if for essential (cyclic essential) submodule N of X there exists $Y \subseteq E_{\mathcal{K}}(X)$ such that $X \subseteq Y \simeq N$.

Lemma 3.9. *Let $\mathcal{K}$ be a Serre class. Then every (uniform)cyclic in $\mathcal{K}$ is weakly injective (or tight, or weakly tight) (resp., weakly R-injective or R-tight, weakly R-tight) in $\mathcal{K}$ iff every (uniform) cyclic is compressible (R-compressible).*

Proof. Let X be a uniform cyclic module and let N be an essential submodule of X. By weak tightness of X, it follows that X is embeddable in N, proving that X is compressible. Conversely, assume X is a uniform cyclic and thus a compressible module. Let xR be a cyclic submodule of $E_{\mathcal{K}}(X)$. Since $xR \cap X$ is an essential submodule of xR, xR embeds in $xR \cap X$, and thus embeds in X, proving that X is tight and thus by Lemma 2.6 it is weakly injective in $\mathcal{K}$. Similarly for the other case. $\square$

Lemma 3.10. *Let $\mathcal{K}$ be a Serre class, and let $X \in \mathcal{K}$ be finite dimensional. Then X is compressible iff X is strongly compressible.*

Proof. Let X be compressible and let N be an essential submodule of X. Then X embeds in N via, say, φ. Thus φ is extended by $\psi : E_{\mathcal{K}}(N) \to E_{\mathcal{K}}(X)$. Since X is finite dimensional, ψ is an isomorphism. Let $Y = \psi^{-1}(X) \subseteq E_{\mathcal{K}}(N)$. Then $X \subseteq Y \simeq N$, proving that X is strongly compressible. $\square$

Theorem 3.11. *For a Serre class $\mathcal{K}$, the following conditions are equivalent:*

(a) $\mathcal{K}$ *is weakly semisimple;*
(b) $\mathcal{K}$ *is q.f.d. and every finitely generated module in* $\mathcal{K}$ *is weakly injective (tight, weakly tight) in* $\mathcal{K}$*;*
(c) $\mathcal{K}$ *is q.f.d. and every cyclic module in* $\mathcal{K}$ *is weakly injective (tight, weakly tight) in* $\mathcal{K}$*;*
(d) $\mathcal{K}$ *is q.f.d. and every uniform cyclic module in* $\mathcal{K}$ *is weakly injective (tight, weakly tight) in* $\mathcal{K}$*;*
(e) $\mathcal{K}$ *is q.f.d. and every uniform cyclic module in* $\mathcal{K}$ *is compressible;*
(f) $\mathcal{K}$ *is q.f.d. and every uniform finitely generated module in* $\mathcal{K}$ *is compressible;*
(g) $\mathcal{K}$ *is q.f.d. and every uniform finitely generated module in* $\mathcal{K}$ *is strongly compressible;*
(h) $\mathcal{K}$ *is q.f.d. and every finitely generated module in* $\mathcal{K}$ *is strongly compressible.*

Proof. $(a) \Rightarrow (b)$. Follows from Theorem 3.3.

Clearly, $(b) \Rightarrow (c) \Rightarrow (d)$, and $(f) \Rightarrow (e)$, $(d) \iff (e)$ by Lemma 3.9

$(d) \Rightarrow (f)$. Let N be a finitely generated module in $\mathcal{K}$ and let $K \subset' N$. Since $\mathcal{K}$ is $q.f.d.$, N has finite uniform dimension. Thus there exists cyclic uniform submodules $U_i, i = 1, ..., n$, of N such that $\oplus_{i=1}^{i=n} U_i \subset' K \subset N$. Since each U_i is uniform it follows that each U_i is weakly injective in $\mathcal{K}$ and thus $\oplus_{i=1}^{i=n} U_i$ is weakly injective in $\mathcal{K}$. Lemma 2.6 implies that K is weakly injective in $\mathcal{K}$ and thus N embeds in K, proving that N is compressible.

$(d) \Rightarrow (a)$. Follows from Theorem 3.8. □

As a consequence of Theorem 3.11 we get Theorem 3.1 in [8].

In case $\mathcal{K} = R$ we obtain characterizations of weakly semisimple rings that generalizes those known results.

Corollary 3.12. *For a ring* R*, the following conditions are equivalent:*

(a) R *is weakly semisimple;*
(b) R *is q.f.d. and every finitely generated module is weakly injective (tight, weakly tight);*
(c) R *is q.f.d. and every cyclic module in is weakly injective (tight, weakly tight);*
(d) R *is q.f.d. and every uniform cyclic module in is weakly injective (tight, weakly tight);*
(e) R *is q.f.d. and every uniform cyclic module is compressible;*
(f) R *is q.f.d. and every uniform finitely generated module is compressible;*
(g) R *is q.f.d. and every uniform finitely generated module is strongly compressible;*
(h) R *is q.f.d. and every finitely generated module is strongly compressible.*

Following similar arguments as in Theorem 3.11, we get

Theorem 3.13. *For a Serre class $\mathcal{K}$, the following conditions are equivalent:*

(a) $\mathcal{K}$ is weakly R-semisimple;
(b) $\mathcal{K}$ is q.f.d. and every finitely generated module in $\mathcal{K}$ is weakly R-injective (R-tight, weakly R-tight) in $\mathcal{K}$;
(c) $\mathcal{K}$ is q.f.d. and every cyclic module in $\mathcal{K}$ is weakly R-injective (R-tight, weakly R-tight) in $\mathcal{K}$;
(d) $\mathcal{K}$ is q.f.d. and every uniform cyclic module in $\mathcal{K}$ is weakly R-injective (R-tight, weakly R-tight) in $\mathcal{K}$;
(e) $\mathcal{K}$ is q.f.d. and every uniform cyclic module in $\mathcal{K}$ is R-compressible;
(f) $\mathcal{K}$ is q.f.d. and every uniform finitely generated module in $\mathcal{K}$ is R-compressible;
(g) $\mathcal{K}$ is q.f.d. and every uniform finitely generated module in $\mathcal{K}$ is strongly R-compressible;
(h) $\mathcal{K}$ is q.f.d. and every finitely generated module in $\mathcal{K}$ is strongly R-compressible.

In case $\mathcal{K} = R$ we obtain characterizations of weakly R-semisimple rings that generalizes those known results.

Corollary 3.14. *For a ring R, the following conditions are equivalent:*

(a) R is weakly R-semisimple;
(b) R is q.f.d. and every finitely generated module is weakly R-injective (R-tight, weakly R-tight);
(c) R is q.f.d. and every cyclic module is weakly R-injective (R-tight, weakly R-tight);
(d) R is q.f.d. and every uniform cyclic module is weakly R-injective (R-tight, weakly R-tight);
(e) R is q.f.d. and every uniform cyclic module is R-compressible;
(f) R is q.f.d. and every uniform finitely generated module is R-compressible;
(g) R is q.f.d. and every uniform finitely generated module is strongly R-compressible;
(h) R is q.f.d. and every finitely generated module is strongly R-compressible.

References

[1] T. Albu and C. Nastasescu, Relative Finiteness in Module Theory, Marcel Dekker 1984.
[2] A. Al-Huzali, S. K. Jain and S. R. López-Permouth, *Rings whose cyclics have finite Goldie dimension*, J. Algebra 153(1992) 37-40.
[3] D. Berry, *Modules whose cyclic submodules have finite dimension*, Can. Math. Bull., 19(1976) 1-6.

[4] G. Brodskii, M. Saleh, L. Thuyet, and R. Wisbauer, *On weak injectivity of direct sums of modules*, Vietnam J. Math., 26 (1998) 121-127.

[5] G. Brodskii, *Denumerable distributivity, linear compactness and the AB5* condition in modules*, Russian Acad. Sci. Dokl. Math., 53(1996) 76-77.

[6] G. Brodskii, *The Grothendieck condition AB5* and generalizations of module distributivity*, Russ. Math. 41 (1997) 1-11.

[7] V. P. Camillo, *Modules whose quotients have finite Goldie dimension*, Pacific J. Math., 69(1977) 337-338.

[8] S. Dhompong, J. Sanwong, S. Plubtieng, and H. Tansee, *On modules whose singular subgenerated modules are weakly injective*, Algebra Colloq. 8 (2001) 227- 236.

[9] C. Faith, Algebra I: Rings, Modules, and Categories, Springer-Verlag, 1981.

[10] V.K. Goel, and S. K. Jain, *π-injective modules and rings whose cyclic modules are π-injective*, Comm. Algebra, 6 (1978) 59-73.

[11] J. S. Golan and S. R. Lopéz -Permouth, *QI-filters and tight modules*, Comm. Algebra, 19(1991) 2217-2229.

[12] S. K. Jain, S. R. Lopéz -Permouth, *Rings whose cyclics are essentially embeddable in projective modules*, J. Algebra, 128(1990) 257-269.

[13] S. K. Jain, S. R. Lopéz -Permouth and M. Saleh, *On weakly projective modules*, In: Ring Theory, Proceedings, OSU-Denison conference 1992, World Scientific Press, New Jersey, 1993, 200-208.

[14] S. K. Jain, S. R. Lopéz -Permouth, K. Oshiro and M. Saleh, *Weakly projective and weakly injective modules*, Can. J. Math., 34(1994) 972-981.

[15] S. K. Jain, S. R. Lopéz -Permouth and S. Singh, *On a class of QI-rings*, Glasgow J. Math., 34(1992) 75-81.

[16] S. K. Jain, and S. R. Lopéz -Permouth, *A survey on the theory of weakly injective modules*, In: Computational Algebra, 205-233, Lecture notes in pure and applied mathematics, Marcel Dekker, Inc., New York, 1994.

[17] A. P. Kurshan, *Rings whose cyclic modules have finitely generated socle*, J. Algebra, 14(1970) 376-386.

[18] S. R. Lopéz-Permouth, *Rings characterized by their weakly injective modules*, Glasgow Math. J, 34(1992) 349-353.

[19] S. Malik and N. Vanaja, *Weak relative injective M-subgenerated modules*, Advances in Ring Theory, Birkhauser, 1997, 221-239.

[20] S. Mohamed, B. Muller, and S. Singh, *Quasi-dual continuous modules*, J. Austral, Math. Soc., Ser A **39**(1985) 287-299.

[21] M. Saleh and A. Abdel-Mohsen, *On weak injectivity and weak projectivity,* In: Proceedings of the Mathematics Conference, World Scientific Press, New Jersey, 2000, 196-207.

[22] M. Saleh, *A note on tightness*, Glasgow Math. J., 41(1999) 43-44.

[23] M. Saleh and A. Abdel-Mohsen, *A note on weak injectivity*, FJMS, 11(2)(2003) 199-206.

[24] M. Saleh, *On q.f.d. modules and q.f.d. rings*, Houston J.M., to appear.

[25] M. Saleh, S.K. Jain, ans S. R. Lopéz -Permouth, *On generalized q.f.d. modules*, to appear.

[26] N.V. Sanh, K.P. Shum, S. Dhompongsa, and S. Wongwai, *On quasi-principally injective modules*, Algebra Colloq. 6(1999) 296-276.

[27] N.V. Sanh, S. Dhompongsa, and S. Wongwai, *On generalized q.f.d. modules and rings*, Algebra and Combinatorics, (1999) 367-272.
[28] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach, 1991.
[29] Y. Zhou, *Notes on weakly semisimple rings*, Bull. Aust. Math. Soc., 52(1996) 517-525.
[30] Y. Zhou, *Weak injectivity and module classes*, Comm. Algebra, 25(1997) 2395-2407.

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