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ON Q.F.D. MODULES AND Q.F.D. RINGS

MOHAMMAD SALEH

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Abstract. Given a right R-module M, a module $N \in \sigma[M]$ is said to be weakly tight (weakly injective) in $\sigma[M]$ if every finitely generated submodule Y of the M-injective hull $\widehat{N}$ is embeddable in $N^{(\aleph_0)}$(there exists $X \subseteq \widehat{N}$ such that $Y \subseteq X \cong N$). For some classes $\mathcal{M}$ of modules in $\sigma[M]$ we study when direct sums of modules from $\mathcal{M}$ are weakly tight in $\sigma[M]$. In particular, we get necessary and sufficient conditions for $\sum$-weak tightness of the injective hull of a simple module. As a consequence, we get characterizations of *q.f.d.* rings by means of weakly injective (tight) modules given by S.K. Jain and S.López-Permouth.

1. Introduction

Throughout this paper all rings are associative with identity and all modules are unitary. We denote the category of all right R-modules by Mod-R and for any $M \in$ Mod-R, $\sigma[M]$ stands for the full subcategory of Mod-R whose objects are submodules of M-generated modules (see Wisbauer [23]). Given a module X_R the injective hull of X in Mod-R (resp., in $\sigma[M]$) is denoted by $E(X)$ (resp., $\widehat{X}$). The M-injective hull $\widehat{X}$ is the trace of M in $E(X)$, i.e. $\widehat{X} = \sum\{f(M), f \in Hom(M, E(X))\}$.

The purpose of this paper is to give several characterizations of *q.f.d.* modules in terms of weak injectivity (tightness, weak tightness) in $\sigma[M]$ a generalization of *q.f.d.* rings given in [2],[14].

Given two modules Q and $N \in \sigma[M]$, we call Q *weakly N-injective* in $\sigma[M]$ if for every homomorphism $\varphi : N \to \widehat{Q}$, there exists a homomorphism $\widehat{\varphi} : N \to Q$ and a monomorphism $\sigma : Q \to \widehat{Q}$ such that $\varphi = \sigma\widehat{\varphi}$. Equivalently, there exists a submodule X of $\widehat{Q}$ such that $\varphi(N) \subset X \simeq Q$. A module $Q \in \sigma[M]$ is called *weakly*

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injective in $\sigma[M]$ if for every finitely generated submodule N of the M-injective hull $\widehat{Q}$, N is contained in a submodule Y of $\widehat{Q}$ such that $Y \simeq Q$. Equivalently, if Q is weakly N -injective for all finitely generated modules N in $\sigma[M]$. A module X is *N-tight* in $\sigma[M]$ if every quotient of N which is embeddable in the M-injective hull of X is embeddable in X. A module is *tight(R-tight)* in $\sigma[M]$ if it is tight relative to all finitely generated (cyclic) submodules of its M-injective hull, and Q is *weakly tight (weakly R-tight)* in $\sigma[M]$ if every finitely generated (or cyclic) submodule N of $\widehat{Q}$ is embeddable in a direct sum of copies of Q. It is clear that every weakly injective module in $\sigma[M]$ is tight in $\sigma[M]$, and every tight module in $\sigma[M]$ is weakly tight in $\sigma[M]$, but weak tightness does not imply tightness, (see [4],[25]).

A module M_R is called *locally q.f.d. (l.q.f.d.)* in case every finitely M-generated (M-cyclic) module $N \in \sigma[M]$ has finite uniform dimension. A module Q is called *weakly (N-)injective (resp., (R-)tight, weakly (R-)tight)* [11],[12],[13],[14], if it is weakly (N-)injective (resp., (R-)tight, weakly (R-)tight) in $\sigma[R_R] = Mod - R$.

For a module X_R and a module property $I\!P$, X is said to be $\sum - I\!P$ in case every direct sum of copies of X enjoys the property $I\!P$. Also we call X *locally $I\!P$* in case every finitely generated submodule of X enjoys the property $I\!P$ (see [1], [3],[15]).

2. Characterizations of q.f.d. modules and q.f.d. rings

The class of weakly injective (tight, weakly tight) modules in $\sigma[M]$ is closed under finite direct sums, but fails under infinite direct sums and direct summands. Also, the domains of the class of weakly injective (tight, weakly tight) modules in $\sigma[M]$ are closed under submodules, and quotients. We list below some known lemmas that will be used in this paper.

Lemma 2.1. [25, Lemma 2.2]. *Given modules $N, Q \in \sigma[M]$. If Q is uniform then Q is weakly N-tight in $\sigma[M]$ iff Q is weakly N-injective in $\sigma[M]$.*

Lemma 2.2. [25, Lemma 2.2], [19, Lemma 2.2]. *An essential extension of weakly injective (tight, weakly tight) module in $\sigma[M]$ is weakly injective (tight, weakly tight) in $\sigma[M]$.*

In [16], it is shown that any semisimple module is a direct summand of a weakly injective module, the next lemma shows that in fact any module is a direct summand of a weakly injective module.

Lemma 2.3. *Every module X in $\sigma[M]$ is a direct summand of a weakly injective module in $\sigma[M]$.*

PROOF. For any module X in $\sigma[M]$, $X \oplus \widehat{(X)^{(\alpha)}}$, where α is an infinite cardinal number, is weakly injective in $\sigma[M]$. $\square$

The above result generalizes 2.12, , 2.13, 2.14, in [14], and 2.1, 2.2., 2.3 in [16].

Example 2.4. *(i)* [14, Example 2.11], [16]. *Let R be the ring of endomorphisms of an infinite dimensional vector space V over a field F. Then $M = Soc(R_R) \oplus R$ is tight but not weakly injective.*

(ii) [4]. *Let $R = Z$ and $X = (Q/Z) \oplus (Z/pZ)$. Then X is weakly tight in $\sigma[M]$ but not tight.*

(iii) [14, Example 4.4(d)]. *Let F be a field. Then $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ is weakly injective but the summand $S = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ as an R-module is not weakly injective.*

Theorem 2.5. *For a module M_R, the following implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f)$ always hold.*

(a) every direct sum $\bigoplus_\Lambda E_\lambda$ of injective modules in $\sigma[M]$ is weakly injective in $\sigma[M]$;
(b) every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injective modules in $\sigma[M]$ is weakly injective in $\sigma[M]$;
(c) every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injective modules in $\sigma[M]$ is tight in $\sigma[M]$;
(d) every direct sum $\bigoplus_\Lambda M_\lambda$ of tight modules in $\sigma[M]$ is tight in $\sigma[M]$;
(e) every direct sum $\bigoplus_\Lambda M_\lambda$ of tight modules in $\sigma[M]$ is weakly tight in $\sigma[M]$;
(f) every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly tight modules in $\sigma[M]$ is weakly tight in $\sigma[M]$.

PROOF. $(a) \Rightarrow (b)$. Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of weakly injective modules in $\sigma[M]$. Let N be a finitely generated submodule of $\widehat{X}$. By (a), the direct sum $\bigoplus_\Lambda \widehat{M_\lambda}$ is weakly injective in $\sigma[M]$ and $X = \bigoplus_\Lambda M_\lambda \subseteq' \bigoplus_\Lambda \widehat{M_\lambda}$ $\subseteq' \widehat{\bigoplus_\Lambda \widehat{M_\lambda}}$. Thus by (a), there exists a submodule $Y \subseteq \widehat{\bigoplus_\Lambda \widehat{M_\lambda}}$ such that $N \subseteq Y \cong \bigoplus_\Lambda \widehat{M_\lambda}$. Write $Y = \bigoplus_\Lambda \widehat{Y}_\lambda$, where $Y_i \cong M_i, i \in \Lambda$. Since N is finitely generated, there exists a finite subset $\Gamma = \{\lambda_1, ..., \lambda_m\} \subseteq \Lambda$ such that $N \subseteq \bigoplus_\Gamma \widehat{Y_\lambda} = \widehat{\bigoplus_\Gamma Y_\lambda}$. Since $Y_{\lambda_1}, \cdots, Y_{\lambda_m}$ are weakly injective in $\sigma[M]$, the finite direct sum $Y_{\lambda_1} \oplus \cdots \oplus Y_{\lambda_m}$ is weakly injective in $\sigma[M]$. Therefore, there exists $X_1 \cong \bigoplus_\Gamma Y_\lambda \cong \bigoplus_\Gamma M_\lambda$ such that $N \subseteq X_1 \subseteq \widehat{\bigoplus_\Gamma Y_\lambda}$. Thus $N \subseteq X_1 \oplus \bigoplus_{\lambda \notin \Gamma} Y_\lambda \simeq X$, proving that X is weakly injective.

$(c) \Rightarrow (d)$. Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of tight modules in $\sigma[M]$. Let N be a finitely generated submodule of $\widehat{X} = \widehat{\bigoplus_\Lambda \widehat{M_\lambda}}$. By (c), the direct sum $\bigoplus_\Lambda \widehat{M_\lambda}$ is tight in $\sigma[M]$. Thus N embeds in $\bigoplus_\Lambda \widehat{M_\lambda}$ via a monomorphism, say, φ. Also $\varphi(N)$ is finitely generated and thus $\varphi(N) \subset \widehat{M}_{\lambda_1} \oplus \cdots \oplus \widehat{M}_{\lambda_m} = \bigoplus_{\lambda=1}^{\lambda=m} \widehat{M_\lambda}$ for some finite $\{\lambda_1, ..., \lambda_m\} \subseteq \Lambda$. Since $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$ is tight then $N \simeq \varphi(N)$ embeds in the finite direct sums $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$, proving that X is tight.

$(e) \Rightarrow (f)$. Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of weakly tight modules in $\sigma[M]$. Let N be a finitely generated submodule of $\widehat{X} = \widehat{\bigoplus_\Lambda \widehat{M_\lambda}}$. By (e), the direct sum $\bigoplus_\Lambda \widehat{M_\lambda}$ is weakly tight in $\sigma[M]$. Thus N embeds in $(\bigoplus_\Lambda \widehat{M_\lambda})^{(\aleph_0)}$ via a monomorphism, say, φ. Also $\varphi(N)$ is finitely generated and thus $N \subset \widehat{M}_{\lambda_1} \oplus \cdots \oplus \widehat{M}_{\lambda_m} = \bigoplus_{\lambda=1}^{\lambda=m} \widehat{M_\lambda}$ for some finite $\{\lambda_1, ..., \lambda_m\} \subseteq \Lambda$. Since $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$ is weakly tight then $N \simeq \varphi(N)$ embeds in a direct sums of copies of $(M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m})$ and thus embeds in a direct sums of X, proving that X is weakly tight.

Clearly, $(b) \Rightarrow (c)$,$(d) \Rightarrow (e)$. $\square$

In the next result we provide other characterization of *locally q.f.d.* modules using tightness and weak tightness, a generalization of the characterization given in [4, Proposition 10].

Theorem 2.6. *For a module M_R, the following conditions are equivalent:*

(a) M is a locally q.f.d. module;
(b) every direct sum $\bigoplus_\Lambda E_\lambda$ of injective modules in $\sigma[M]$ is weakly injective in $\sigma[M]$;
(c) every direct sum $\bigoplus_\Lambda M_\lambda$ of injective modules in $\sigma[M]$ is tight in $\sigma[M]$;
(d) every direct sum $\bigoplus_\Lambda M_\lambda$ of injective modules in $\sigma[M]$ is weakly tight in $\sigma[M]$;
(e) every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly tight modules in $\sigma[M]$ is weakly tight in $\sigma[M]$;
(f) every direct sum $\bigoplus_\Lambda M_\lambda$ of tight modules in $\sigma[M]$ is weakly tight in $\sigma[M]$;
(g) every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injective modules in $\sigma[M]$ is weakly tight in $\sigma[M]$;
(h) every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injective modules in $\sigma[M]$ is weakly N-tight for every cyclic module N in $\sigma[M]$;

(i) *every direct sum* $\bigoplus_\Lambda \widehat{P_\lambda}$, *where* P_λ *is simple module in* $\sigma[M]$, *is weakly* N-*tight, for every cyclic module* N *in* $\sigma[M]$;

(j) *every direct sum* $\bigoplus_\Lambda \widehat{P_\lambda}$, *where* P_λ *is simple module in* $\sigma[M]$, *is weakly* R-*tight in* $\sigma[M]$.

PROOF. $(a) \Rightarrow (b)$. Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of injective modules in $\sigma[M]$. Let N be a finitely generated submodule of $\widehat{X}$. By (a), N contains as an essential submodule a finite direct sum of uniform submodules $\bigoplus_\Lambda U_\lambda$. Since X is essential in $\widehat{X}$, for each i, choose $0 \neq x_i \in U_i \cap X$. Then $\bigoplus_{i=1}^{i=n} x_iR \subseteq \bigoplus_{i=1}^{i=n} M_{\lambda_i}$ for some λ_i's and $\bigoplus_{i=1}^{i=n} x_iR \subseteq' \bigoplus_\Lambda U_\lambda \subseteq' N$. It follows that $\bigoplus_{i=1}^{i=n} M_{\lambda_i}$ contains an M-injective hull E of $\bigoplus_{i=1}^{i=n} x_iR$. Since E is M-injective and contained in X, we may write $X = E \oplus K$, for some submodule K of X. On the other hand, let $\widehat{N}$ be an M-injective hull of N in $\widehat{X}$. Then $\widehat{N} = \widehat{\bigoplus_{i=1}^{i=n} x_iR} \cong E$. Since $\bigoplus_{i=1}^{i=n} x_iR$ is essential in $\widehat{N}$, it follows that $\widehat{N} \cap K = 0$. So let $Y = \widehat{N} \oplus K \cong E \oplus K = X$. Then $N \subseteq Y \cong X$, proving that X is weakly injective.

Clearly, $(b) \Rightarrow (c) \Rightarrow (d), (e) \Rightarrow (f) \Rightarrow (g) \Rightarrow (h) \Rightarrow (i) \Rightarrow (j)$.

$(d) \Rightarrow (e)$. Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of weakly tight modules in $\sigma[M]$. Let N be a finitely generated submodule of $\widehat{X}$. By (d), the direct sum $\bigoplus_\Lambda \widehat{M_\lambda}$ is weakly tight in $\sigma[M]$. Thus N embeds in $(\bigoplus_\Lambda \widehat{M}_\lambda)^{(\aleph_0)}$ via a monomorphism, say, φ. Also $\varphi(N)$ is finitely generated and thus $N \subset \widehat{M}_{\lambda_1} \oplus \cdots \oplus \widehat{M}_{\lambda_m}$ for some finite $\{\lambda_1, ..., \lambda_m\} \subseteq \Lambda$. Since $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$ is weakly tight then $N \simeq \varphi(N)$ embeds in a direct sums of $(M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m})$ and thus embeds in a direct sums of X, proving that X is weakly tight.

$(j) \Rightarrow (a)$ Let K be a cyclic submodule of M. If $Soc(K) = 0$, we are done. Suppose $0 \neq Soc(K) = \bigoplus_\Lambda P_\lambda$. We show that $Soc(K)$ is finitely generated. For this consider the diagram

$$\begin{array}{l} 0 \to \bigoplus_\Lambda P_\lambda \xrightarrow{\gamma} K \\ \qquad \downarrow\varphi \\ \qquad \widehat{\bigoplus_\Lambda \widehat{P}_\lambda} \end{array}$$

where φ and γ are the inclusion homomorphisms. By M-injectivity of $\widehat{\bigoplus_\Lambda \widehat{P}_\lambda}$, there exists $\psi : K \to \widehat{\bigoplus_\Lambda \widehat{P}_\lambda}$ such that $\psi\gamma = \varphi$. By our hypothesis, $\bigoplus_\Lambda \widehat{P}_\lambda$ is weakly R-tight in $\sigma[M]$, hence $Im\varphi \subset Im\psi$ is embeddable in $(\bigoplus_\Lambda \widehat{P}_\lambda)^{(\aleph_0)}$. Therefore, $Soc(K)$ is embeddable in $\widehat{P}_{\lambda_1} \oplus \cdots \oplus \widehat{P}_{\lambda_m}$ for some finite $\{\lambda_1, ..., \lambda_m\} \subseteq \Lambda$. Since each $\widehat{P}_{\lambda_i}$ is uniform, $Soc(K)$ has finite uniform dimension and is therefore finitely generated. $\square$

From Lemma 2.1, Theorems 2.5 and 2.6, we get the following characterization of *locally q.f.d.* modules.

Theorem 2.7. *For a module M_R, the following conditions are equivalent:*

(a) M is a locally q.f.d.;
(b) every direct sum $\bigoplus_\Lambda E_\lambda$ of injective modules in $\sigma[M]$ is weakly injective (or tight, weakly tight, weakly R-tight) in $\sigma[M]$;
(c) every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injective modules in $\sigma[M]$ is weakly injective (or tight, weakly tight, weakly R-tight) in $\sigma[M]$;
(d) every direct sum $\bigoplus_\Lambda M_\lambda$ of tight modules in $\sigma[M]$ is tight (or weakly tight, weakly R-tight) in $\sigma[M]$;
(e) every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly tight modules in $\sigma[M]$ is weakly tight (or weakly R-tight) in $\sigma[M]$;
(f) every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly tight modules in $\sigma[M]$ is weakly N-tight, for every cyclic module N in $\sigma[M]$;
(g) every direct sum $\bigoplus_\Lambda \widehat{P_\lambda}$, where P_λ is simple module in $\sigma[M]$, is N-tight for every cyclic module N in $\sigma[M]$;
(h) every direct sum $\bigoplus_\Lambda \widehat{P_\lambda}$, where P_λ is simple module in $\sigma[M]$, is weakly R-tight in $\sigma[M]$.

In case $M = R_R$ in Theorem 2.7, we get the following characterization of *q.f.d.* rings.

Theorem 2.8. *For a ring R, the following conditions are equivalent:*

(a) R is q.f.d.;
(b) every direct sum $\bigoplus_\Lambda E_\lambda$ of injective modules is weakly injective (or tight, weakly tight, weakly R-tight);
(c) every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injective modules is weakly injective (or tight, weakly tight, weakly R-tight);
(d) every direct sum $\bigoplus_\Lambda M_\lambda$ of tight modules is tight (or weakly tight, weakly R-tight);
(e) every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly tight modules is weakly tight (or weakly R-tight);
(f) every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly tight modules is weakly N-tight, for every cyclic module N;
(g) every direct sum $\bigoplus_\Lambda E(P_\lambda)$, where P_λ is simple module, is N-tight for every cyclic module N;
(h) every direct sum $\bigoplus_\Lambda E(P_\lambda)$, where P_λ is simple module, is weakly R-tight.

Corollary 2.9. *A ring R is q.f.d. iff every essential extension of the direct sum $\bigoplus_\Lambda E_\lambda$ of injective (or tight, weakly tight) modules is weakly injective (or tight, weakly tight, weakly R-tight).*

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Department of Mathematics, Birzeit University, P.O.Box 14, West Bank, Palestine
E-mail address: msaleh@birzeit.edu; salehm@math.ohiou.edu