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ON AN INVERSE DIFFUSION PROBLEM*

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Abstract. In many applications, such as the heat conduction and hydrology, there is a need to recover the (possibly discontinuous) diffusion coefficient a from boundary measurements of solutions of a parabolic equation. The complete inverse problem is ill posed and nonlinear, so numerical solution is quite difficult, and we linearize the problem around constant a. We study and solve numerically the linear ill-posed problem by using regularization.

Key words. parabolic inverse problems, inverse heat transfer, inverse hydraulic problem, regularization of ill-posed problems

AMS subject classifications. 35R30, 65R, 76S05, 80A23

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Introduction. We consider the inverse problem of finding the pair (u, a) for the Cauchy problem

$$u_t - \operatorname{div}(a\nabla u) = \delta(x - x^*)\,\delta(t) \quad \text{on } \mathbb{R}^n \times (0,T), u \text{ bounded}, \tag{0.1}$$

$$u = 0 \quad \text{on } \mathbb{R}^n \times \{0\}, \tag{0.2}$$

where $a = a(x)$ is bounded and measurable and δ is the Dirac delta function. We assume that $a = 1 + f$, where $f = 0$ outside a bounded region $\Omega \subset \mathbb{R}^n$ with piecewise C^2-smooth boundary $\partial\Omega$. As additional data we consider the solution $u(x, t; x^*)$ given for x, $x^* \in \Omega^*$, $t \in (0, T)$. Here Ω^* is a bounded domain in $\mathbb{R}^n$ whose closure does not intersect $\bar{\Omega}$.

In applications, x^* is the source position and x the receiver/sensor position. In addition, in many applications instead of the Cauchy problem with changing source, a more natural model is a parabolic initial boundary value problem in domain $\Omega^\bullet$ when they prescribe various Dirichlet (or Neumann) data on a lateral boundary $\partial\Omega^\bullet \times (0, T)$ and measure the lateral Neumann (or Dirichlet) data. In other words, for the homogeneous equation (0.1) one is given the lateral Dirichlet-to-Neumann map. Uniqueness of solution of this inverse problem is discussed in the paper [I2] and Elayyan and Isakov [EI] study discontinuous a in detail. Since one of our goals is to suggest an efficient numerical algorithm and since we feel that a basic obstruction for efficient numerics is (apparent) severe ill-posedness of this inverse problem, we try to simplify the nonlinear inverse problem in two ways: (1) linearizing it around $a = 1$ and (2) replacing a finite domain Ω (where a is actually unknown) by $\mathbb{R}^n$. The linearization is used in practical ways in many elliptic inverse problems (say, in electrical impedance tomography [CI], [SV]), and we give some mathematical justification of it in section 1. On the other hand, in section 4 we will show that the lateral Dirichlet-to-Neumann map completely determines the data of our problem (for all $x^* \in \Omega^*$) in a stable and constructive way. So we can split the solution to this ill-posed problem into a simpler linear ill-posed part and a well-posed part. In section 2 we prove uniqueness

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for the linearized inverse problem, and in section 3 we describe results of our numerical experiments. These experiments show that our scheme (with possible technical modifications) can be used, for example, in a very important inverse problem of underground hydraulics [Y], which is solved at present by extremely expensive methods including boring holes in the ground.

In the paper C will denote (possibly) different constants depending only on n, Ω, and T. ν denotes the exterior unit normal to the boundary and $\| \|_p(E)$ the norm in $L_p(E)$. Later on we let $Q = \mathbb{R}^n \times (0,T)$.

1. Linearization. Assume $a = 1 + f$, $f = 0$ in $\mathbb{R}^n \backslash \Omega$, where f is "ϵ-small" as described in Lemma 1.1. Let u_ϵ be the solution to (0.1), (0.2). Letting $v_\epsilon = u_\epsilon - u_0$ and substituting for u_ϵ in (0.1), (0.2), we get

$$u_{0t} + v_{\epsilon t} - \operatorname{div}((1+f)\nabla(u_0 + v_\epsilon)) = \delta(x - x^*)\,\delta(t-0) \quad \text{on } Q, \tag{1.1}$$

$$v_\epsilon = 0 \quad \text{on } \mathbb{R}^n \times \{0\}. \tag{1.2}$$

Let u_0 be a solution of the unperturbed problem

$$u_{ot} - \Delta u_o = \delta(x - x^*)\,\delta(t-0) \quad \text{on } Q, \tag{1.3}$$

$$u_o = 0 \quad \text{on } \mathbb{R}^n \times \{0\}. \tag{1.4}$$

Then

$$v_{\epsilon t} - \Delta v_\epsilon = \operatorname{div}(f\nabla u_0) + \operatorname{div}(f \nabla v_\epsilon) \quad \text{on } Q, \tag{1.5}$$

$$v_\epsilon = 0 \quad \text{on } \mathbb{R}^n \times \{0\}. \tag{1.6}$$

We will show that for small ϵ the nonlinear term in (1.5) (which is the second term in the right-hand side) is small relative to the first term. By dropping it we get the linearized equation

$$v_t - \Delta v = \operatorname{div}(f \nabla u_0) \quad \text{on } Q, \tag{1.7}$$

$$v = 0 \quad \text{on } \mathbb{R}^n \times \{0\}. \tag{1.8}$$

We will make use of the notation in [LSU]

$$|u| = \text{ess sup}\, \|u(\ ,t)\|_2(\Omega) + \|\nabla_x u\|_2(Q),$$

where the sup is taken over $0 \le t \le T$.

LEMMA 1.1. *Let $\|f\|_\infty(\Omega) \le \epsilon$. Then $|v| \le C\epsilon$ and $|v - v_\epsilon| \le C\epsilon^2$.*

Proof. Subtracting (1.3), (1.4) from (1.1), (1.2), we get

$$v_{\epsilon t} - \operatorname{div}(a \nabla v_\epsilon) = \operatorname{div}(f \nabla u_0) \quad \text{on } Q, \tag{1.9}$$

$$v_\epsilon = 0 \quad \text{on } \mathbb{R}^n \times \{0\}. \tag{1.10}$$

As is well known, the solution u_o to the problem (1.3), (1.4) is given by the formula

$$u_0(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x^* - x|^2}{4t}\right). \tag{1.11}$$

Since $x^* \in \Omega^*$, u_0 is smooth and continuous on $\overline{\Omega} \times [0,T]$, so

$$\|f\nabla u_0\|_2(Q) \le C\|f\|_\infty(\Omega) \le C\epsilon.$$

Using [LSU, Theorem 2.1, p. 143] (where $\psi_o = 0$, $f = 0$, f_f have to be replaced by $f\partial_j u_o$), we get

$$|v_\epsilon| \leq C\|f\nabla u_0\|_2(Q) \leq C\epsilon. \tag{1.12}$$

Now subtracting (1.7), (1.8) from (1.5), (1.6), we get

$$(v_\epsilon - v)_t - \nabla(v_\epsilon - v) = \operatorname{div}(f\nabla v_\epsilon) \quad \text{on } Q, \tag{1.13}$$

$$v_\epsilon - v = 0 \quad \text{on } \mathbb{R}^n \times \{0\}. \tag{1.14}$$

Using the same theorem from [LSU] we obtain

$$|v_\epsilon - v| \leq C\|f\nabla v_\epsilon\|_2(Q) \leq C\epsilon^2,$$

where we did use (1.12).

To complete the proof we observe that

$$|v| \leq |v_\epsilon| + |v - v_\epsilon| \leq C\epsilon.$$

Remark 1.2. Some justification of the linearization is available for more special perturbations that are not uniformly small and that are more natural in those applications when one is looking for shapes of unknown inclusions of relatively small volume.

More precisely, let $f = \mu\chi(D)$, where D is an open subset of Ω and μ is a function that is C^2-smooth on D and assume that

$$\text{area } \partial D \leq C, \|\mu\|(C^1(\Omega)) \leq C, \quad \text{vol } D = \epsilon \text{ is small} \tag{1.15}$$

and

$$\operatorname{dist}(D, \partial\Omega) \geq \frac{1}{C}.$$

Again, we have the first inequality (1.12). But now

$$\|f\nabla u_0\|_2^2(Q) = \int_Q |f\nabla u_0|^2 \leq C\int_0^T \int_D |\mu|^2\, dx\, dt \leq C \text{ vol } D \leq C\epsilon,$$

so

$$|v_\epsilon| \leq C\epsilon^{1/2}.$$

The classical integral representation of a solution to the Cauchy problem (1.5), (1.6) (explained in more detail below) gives

$$v_\epsilon(x,t) = -\int_0^t \int_D \mu\nabla_y\Gamma(x,t;y,\tau)\cdot\nabla(u_0(y,\tau) + v_\epsilon(y,\tau))\, dy\, d\tau.$$

Using the above estimate and the Hölder inequality for integrals over $(0,t)\times D$, one can show that $\|v_\epsilon\|_\infty((0,t)\times D) \leq C\epsilon$. Integrating by parts we obtain

$$v_\epsilon(x,t) = \int_0^t \int \nabla_D(\mu\nabla_y\Gamma(x,t;y,\tau))(u_0(y,\tau) + v_\epsilon(y,\tau))\, dy\, d\tau$$
$$- \int_0^t \int_{\partial D} \mu\partial_\nu\Gamma(x,t;y,\tau)(u_0(y,\tau) + v_\epsilon(y,\tau))\, dy\, d\tau.$$

According to formula (1.11), $u_o(x,t) > \epsilon(t) > 0$. So the integrals above that involve v_ϵ are much smaller than the integrals involving u_o.

Now we will discuss an integral representation of a solution to the Cauchy problem (1.7), (1.8). If f is a (C^2-)smooth compactly supported function, then according to [F],

$$v(x,t) = \int_0^t \int_{R^n} \Gamma(x,t;y,\tau) \operatorname{div} f\nabla u_o(y,\tau)\, dy\, d\tau,$$

where

$$\Gamma(x,t;y,\tau) = \frac{1}{(4\pi(t-\tau))^{n/2}} \exp\left(-\frac{|x-y|^2}{4(t-\tau)}\right), \quad x \in \mathbb{R}^n, \quad t > \tau \tag{1.16}$$

is a fundamental solution of the heat equation. Integrating by parts we obtain

$$v(x,t) = -\int_0^t \int_{R^n} f(y,\tau)\nabla_y\Gamma(x,t;y,\tau)\cdot\nabla_y u_o(y,\tau)\, dy\, d\tau. \tag{1.17}$$

If $f \in L_\infty(\Omega)$, we can approximate it (in $L_1(\Omega)$) by smooth unformly bounded f; use (1.17) and extend it to our less regular f by passing to the limit. Using the formula (1.16) for Γ and the formula (1.11) for u_o, we obtain the following integral representation

$$\begin{aligned} &v(x,t;x^*) \\ &= -\frac{1}{(4)^{n+1}\pi^n}\int_0^t\int_\Omega f(y)\frac{(x-y)\cdot(x^*-y)}{(\tau(t-\tau))^{n/2+1}}\exp\left(-\frac{|x-y|^2}{4(t-\tau)} - \frac{|x^*-y|^2}{4\tau}\right) dy\, d\tau. \end{aligned} \tag{1.18}$$

Since our unknown function f is of n variables, we'd like to reduce the overdeterminancy of the inverse problem by letting $x = x^*$ and $t = T$. Finally, we formulate the following problem.

THE LINEARIZED INVERSE PROBLEM. Find a function $f \in L_\infty(\Omega)$ given the function $F(x^*) = v(x^*,T;x^*)$, $x^* \in \Omega^*$.

This formulation is motivated also by backscattering and is very reasonable physically. It means that measurements are implemented only at the point x^* where a source has been applied but at some later point in time. The physical reduction of overdeterminancy is important because less (sometimes expensive) measurements are needed. Mathematically, this reduction is especially desirable because due to ill-posedness of the inverse diffusion problem one has to minimize propagation of errors.

As follows from representation (1.18), this inverse problem is equivalent to the following integral equation

$$Af(x) = F(x), \qquad x \in \Omega^*, \tag{1.19}$$

where $Af(x) = \int_\Omega k(x-y)f(y)\,dy$, and the kernel k is defined as

$$k(x) = -\frac{1}{4(4\pi)^n}\int_0^T \frac{|x|^2}{(\tau(T-\tau))^{n/2+1}}\exp\left(-\frac{|x|^2 T}{4\tau(T-\tau)}\right) d\tau. \tag{1.20}$$

A is considered as an operator from $L_2(\Omega)$ into $L_2(\Omega^*)$. The above equation is a Fredholm integral equation of the first kind, which represents an (strongly) ill-posed problem because A maps any Sobolev space $H_{(k)}(\Omega)$ (with positive or negative k) into the space of functions analytic in a neighborhood of $\overline{\Omega}^*$. In order to get a stable solution, we will use regularization schemes, but first we will show uniqueness for f.

A similar linearization of the inverse electrical conductivity problem was suggested by Engl and Isakov [EnI].

2. Uniqueness. We write equation (1.19) as

$$F(x) = \int_\Omega k(x-y)f(y)\,dy, \qquad x \in \Omega^*. \tag{2.1}$$

Before proving uniqueness we calculate the Fourier transform $\hat{k}$ of k.

Referring to [H, section 7.1] we remind the reader that the Fourier transform $\hat{u}$ of a (tempered) distribution u has the following property: the Fourier transform of $x_j u(x)$ is $i\partial_j u$. So

$$|x|^2\hat{\ } u(x) = -\Delta u(\xi).$$

It is known [H, section 7.6] that the Fourier transform of the function $u(x) = \exp(-\lambda|x|^2/2)$ $(\lambda > 0)$ is $(2\pi)^{n/2}\lambda^{-n/2}\exp(-|\xi|^2/2\lambda)$. Therefore, the Fourier transform of the function $|x|^2\exp(-\lambda|x|^2/2)$ is

$$-(2\pi)^{n/2}\lambda^{-n/2}\Delta_\xi \exp(-|\xi|^2/2\lambda) = -(2\pi)^{n/2}\lambda^{-n/2-1}(\lambda^{-1}|\xi|^2 - n)\exp(-|\xi|^2/2\lambda).$$

Using this formula with $\lambda = T/(2\tau(T-\tau))$, applying the Fourier transform to the formula (1.20), and commuting the Fourier transform and the integration with respect to τ, we obtain

$$\hat{k}(\xi) = c_n \int_0^T (2\tau(T-\tau)/T|\xi|^2 - n)\exp(-|\xi|^2\tau(T-\tau)/T)\,d\tau, \tag{2.2}$$

where $c_n = 2^{-1-n}T^{-n/2-1}\pi^{-n/2}$. Observe that commutativity of the integration and of the Fourier transform follows from the uniform convergence of integrals and the Fubini theorem.

THEOREM 2.1. *A solution $f \in L_2(\Omega)$ to equation* (2.1) *is unique.*

Proof. Since the integral equation is linear, to prove uniqueness it suffices to assume that $F = 0$ on Ω^* and to prove that $f = 0$. Suppose $f \neq 0$.

Using the Fourier transform of convolutions, we obtain

$$\hat{F}(\xi) = \hat{k}(\xi)\hat{f}(\xi). \tag{2.3}$$

Since f, F have compact supports, by the Payley–Wiener theorem $\hat{f}$, $\hat{F}$ are entire analytic functions in order 1; in particular, they are well defined for $\xi = \zeta \in \mathbb{C}^n$, and there is constant C such that

$$|\hat{f}(\zeta)| \le Ce^{C|\zeta|} \quad \text{and} \quad |\hat{F}(\zeta)| \le Ce^{C|\zeta|}, \qquad \zeta \in \mathbb{C}^n. \tag{2.4}$$

LEMMA 2.2. *For $\zeta = z\xi_o$, $\xi_o \in \mathbb{R}^n$, $z = iR$, $R > 1$, there is $\epsilon > 0$ such that*

$$|\hat{k}(\zeta)| \ge \epsilon e^{\epsilon|\zeta|^2}.$$

Proof. Substituting $\xi = \zeta = iR\xi_o$ into formula (2.2), incorporating the fact that the two terms of the integrand have the same sign, and dropping the first term, we obtain

$$|\hat{k}(\zeta)| \geq c_n n \int_0^T \exp(|\zeta|^2 \tau(T-\tau)/T)\, d\tau$$
$$\geq c_n n \int_{T/4}^{T/2} \exp(|\zeta|^2 3T/16)\, d\tau = c_n nT/4 \exp(3T/16|\zeta|^2),$$

where we shrunk the integration interval in the first integral and used that $3T/16 \leq \tau(T-\tau)$ when $T/4 < \tau < T/2$. Now the choice of ϵ is obvious, and we complete the proof. □

We return to the proof of Theorem 2.1.

If $f \neq 0$, then $f(\xi_o) \neq 0$ for some nonzero $\xi_o \in \mathbb{R}^n$, and, therefore, the entire function $\phi(z) = f(z\xi_o)$ of one complex variable is of order one and not identically zero. From now on we consider only $\zeta = z\xi_o$. According to the known results from complex variables ([T, p. 277] or, more precisely, by the Littlewood theorem), there are a constant C and a sequence $r_j \to \infty$ such that $\min|\phi(z)| > e^{-Cr_j}$, where the minimum is over $|z| = r_j$. From (2.3) and (2.4) it follows now that

$$|\hat{k}(\zeta)| \leq Ce^{2Cr_j}, \qquad |\zeta| = r_j,$$

which contradicts Lemma 2.2.

The contradiction shows that $f = 0$. The proof of Theorem 2.1 is complete. □

Stability is an open question here. While we expect a logarithmic one, we have no proof of it yet.

3. A numerical solution to equation (1.19). To solve the ill-posed problem (1.19), we will use the Tikhonov regularization, replacing the original integral equation (1.19) by the following one

$$(\alpha I + A^* A) f_\alpha = F^*, \tag{3.1}$$

where $F^* = A^*F$ and α is a regularization parameter. A known theory of regularization (e.g., [I1], [EnHN]) guarantees existence of its solution f_α and its convergence to the solution f of the original equation (1.19) when $\alpha \to 0$, provided f does exist and one had proven uniqueness for the original equation. Then we will discretize the regularized normal equation (3.1) and solve it numerically.

To find A^*: $L_2(\Omega^*) \to L_2(\Omega)$, we are reminded that

$$(A\phi, \psi)_{L_2(\Omega^*)} = (\phi, A^*\psi)_{L_2(\Omega)}.$$

So

$$(A^* f^*)(y) = -\frac{1}{4(4\pi)^n} \int_o^T \int_{\Omega^*} f^*(x) \frac{|x-y|^2}{(\tau(T-\tau))^{3/2}} \exp\left(-\frac{|x-y|^2 T}{4\tau(T-\tau)}\right) dx\, d\tau, \quad y \in \Omega,$$

and we will have

$$(A^*Af)(x) = \frac{1}{16(4\pi)^{2n}} \int_{\Omega^*} \int_{\Omega} f(y)|y-z|^2|x-z|^2 K(x-z,T)K(y-z,T)\, dy\, dz,$$

where

$$K(x-z,T) = \int_0^T \frac{\exp\left(-\dfrac{|x-z|^2 T}{4\tau(T-\tau)}\right)}{(\tau(T-\tau))^{n/2+1}}\, d\tau.$$

In numerics we consider $\Omega = [0,2]^n$ and $\Omega^* = [3,5]$, $n = 1$ or $\Omega^* = [3,5] \times [0,2]$, $n = 2$.

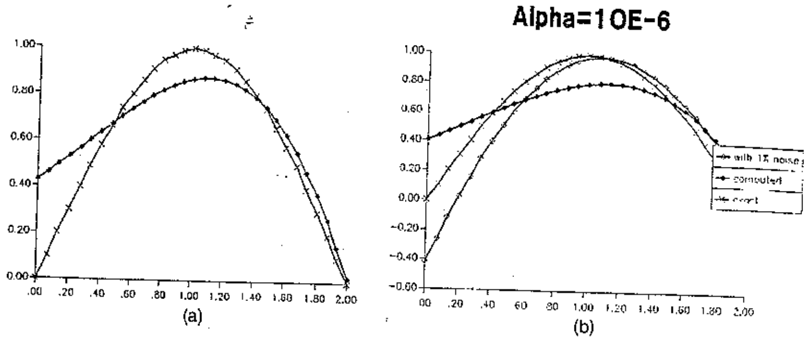

FIG. 1.

Discretizing by the trapezoid method we get $A^{d*}A^d f^d = Bf^d$, where B is the matrix calculated from the discretization and f^d is a function of discrete argument defined on the rectangular uniform $N \times N$ grid. Then the discretized equation (3.1) can be written as

$$(\alpha I + B)f^d = F^d. \tag{3.2}$$

Since an analytic calculation of the integral (1.20) is not realistic even for simplest f (say, $f = 1$), we did generate the data for the inverse problem by numerical calculation using a similar discretization.

Equation (3.1) involves several parameters ($n, \alpha, \Omega, \Omega^*$ and the time T). We will present several examples illustrating the algorithm described above. In each case we will change some parameters or f.

Assuming $n = 1$, $\Omega = (0, 2)$, $\Omega^* = (3, 5)$, $T = 4$, we recovered $f(x) = \sin(\pi x/2)$ with $\alpha = .000001$, as shown below.

Observe that the left side of Figure 1 illustrates recovery (graph with squares) of genuine f (dotted graph) from exact data generated numerically. On the right side of Figure 1 we added reconstruction from noised data (1% of relative noise) marked by small disks. Since we used the same matrix to get the data, our pictures were good for $\alpha = 0$, but they blew up for $\alpha = 0$ when we introduced a noise in the right-hand side. We get good results for the noisy right-hand side when α is small, which means that our scheme is stable. The reconstruction deteriorates near $x = 0$; in our opinion it is because the data are collected from the right side (on $(3, 5)$), opposite to $x = 0$. In the one-dimensional case the complement to Ω is not connected, so the data probably cannot propagate to $x = 0$ as in the many-dimensional case.

Next we consider the two-dimensional case letting $\Omega = (0, 2) \times (0, 2)$, $\Omega^* = (3, 5) \times (0, 2)$, $T = 4$. We give the pictures of numerical reconstruction of the function $f(x) = \sin \pi x_1$ with three values of the regularization parameter $\alpha = 10^{-9}$, 10^{-11}, 10^{-14} in Figure 2.

As illustrated by Figure 2 the reconstruction is getting better if α is smaller. A partial explanation is that the entires of the matrix in (3.2) are very small, so larger magnitude of the regularizing parameter combined with presence of corners of the square Ω can produce some deterioration. On the other hand, smaller α makes numerics very unstable. Unlike in the one-dimensional case, reconstruction is relatively good far from and close to Ω^*; we think that it is due to connectedness of $\mathbb{R}^2 \backslash \Omega$.

Figure 3 illustrates recovery of the discontinuous function $f = 1$ in $(1, 2) \times (0, 2)$ and 0 outside this set. Here $\Omega = (0, 2) \times (0, 2)$, $\Omega^* = (3, 5) \times (0, 2)$. First we show the graph of f and then its recovery with $\alpha = 10^{-8}$, 10^{-14} and with $T = 4$.

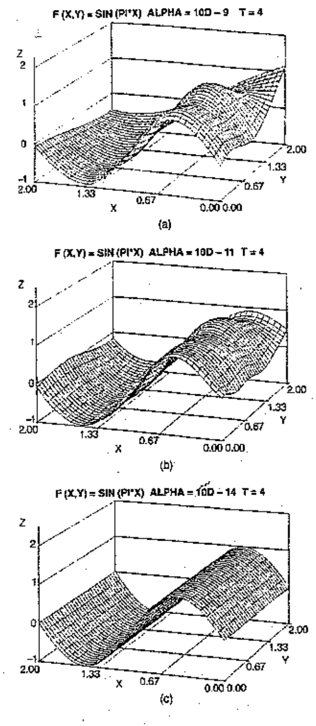
F (X,Y) = SIN (PI*X) ALPHA = 10D − 9 T = 4
(a)
F (X,Y) = SIN (PI*X) ALPHA = 10D − 11 T = 4
(b)
F (X,Y) = SIN (PI*X) ALPHA = 10D − 14 T = 4
(c)
Z
X
Y

FIG. 2.

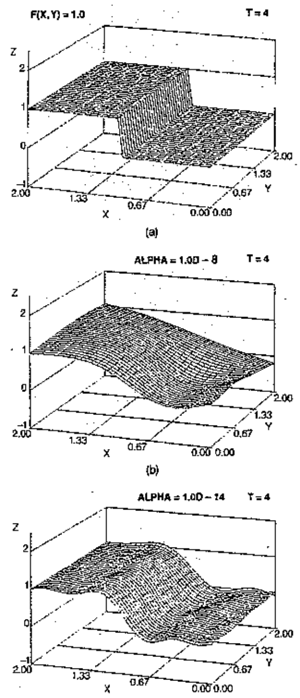
F(X, Y) = 1.0 T = 4
(a)
ALPHA = 1.0D − 8 T = 4
(b)
ALPHA = 1.0D − 14 T = 4
(c)
Z
X
Y

FIG. 3.

Looking at equation (1.19) one can realize that for very small T the kernel of our integral equation is exponentially small, and it decays (as a power) when T is getting large. So the reconstruction must be optimal for intermediate values of T, which can be found experimentally. We observed this optimal value in our numerical experiments, and the few figures given here somehow illustrate this phenomenon.

For $n = 1$ the computations were fast, but it wasn't the case for $n = 2$ because the size of the matrix B becomes very large; i.e., for $N = 32$ the size of the matrix is $(33)^2 \times (33)^2$. But using the symmetry of B helps by reducing more than half the calculations. By using more sophisticated methods (utilizing Toeplitz structure and fast Fourier transform) we hope to substantially decrease computational time, to increase resolution, and even to approach the three-dimensional case.

To conclude we observe that to our knowledge this is one of the first attempts to solve numerically a quite difficult and very important applied inverse diffusion problem. Definitely much more of work (decomposition of matrices, use of symmetries, right choice of precision, and, of course, study of the complete nonlinear inverse problem) has to be done here to enhance effectiveness of numerics. However, even at this initial stage the numerical results are quite encouraging.

4. Reduction from the Dirichlet-to-Neumann map. In this section, we will establish that the data in the problem considered are equivalent to the lateral Dirichlet-to-Neumann map of a related initial boundary value problem.

Consider the initial boundary value problem

$$U_t - \operatorname{div}(a\nabla U) = 0 \quad \text{in } \Omega^\bullet \times (0,T), \tag{4.1}$$

$$U = 0 \quad \text{on } \Omega^\bullet \times \{0\}, \tag{4.2}$$

$$U = g \quad \text{on } \Sigma = \partial\Omega^\bullet \times (0,T), \tag{4.3}$$

where $\Omega^\bullet$ is a domain with the C^2 boundary containing $\overline{\Omega}$. It is well known [LSU] that for any $g \in C^2(\Sigma)$, $g = 0$ on $\partial\Omega^\bullet \times \{0\}$ there is a unique (generalized) solution u with $\nabla_x u$ continuous near Σ. So we have the well-defined lateral Dirichlet-to-Neumann map Λ_l: $g \to \partial_\nu U$ on Σ (results of all possible lateral boundary measurements of U). Now we will show that Λ_l determines $u(x,t;x^*)$, x, $x^* \in \Omega^*$, $0 < t < T$, in a unique and constructive way.

Let u_0 be a solution to the Cauchy problem (1.3), (1.4). Then the difference $w = u - u_0$ solves the following problem:

$$w_t - \Delta w = \operatorname{div}(f\nabla u) \quad \text{on } Q = \mathbb{R}^n \times (0,T), \tag{4.4}$$

$$w = 0 \quad \text{on } \mathbb{R}^n \times \{0\}. \tag{4.5}$$

Since supp $f \subset \Omega$, the function w satisfies the homogeneous heat equation outside Ω. It has zero initial data and is bounded, so it can be represented by a simple layer thermal potential

$$w(x,t) = S\rho(x,t) = \int_0^t \int_{\partial\Omega^\bullet} \rho(y,\tau)\Gamma(x-y,t-\tau)\, dS(y)\, d\tau, \tag{4.6}$$

and the density ρ is to be found. Observe that

$$\partial_\nu w^- = \partial_\nu u - \partial_\nu u_o = \Lambda_l(u_o + w) - \partial_\nu u_o = \Lambda_l S_\rho + (\Lambda_l - \partial_\nu)u_o, \tag{4.7}$$

where w^- is w outside $\Omega^\bullet \times (0,T)$. On other hand, according to the known [LSU, equation (15.9)] jump relations for the normal derivatives of single layer potentials we have

$$\partial_\nu w^-(x,t) = -\rho(x,t)/2 + W\rho(x,t),$$

where W is the normal derivative of the single layer on $\partial\Omega_1$ which can be obtained by replacing Γ in (4.6) by $\partial_{\nu x}\Gamma$. From (4.7) and the formula for $\partial_\nu w^-$ we have the following integral equation for ρ:

$$(I + (-I/2 - W + \Lambda_l S))\rho = U_o, \tag{4.8}$$

where I is the identity operator and $U_o = (\partial_\nu - \Lambda_l)u_o$ on Σ.

LEMMA 4.1. *We have*

$$\|(-I/2 - W + \Lambda_l S)\rho\|_\infty(\Sigma) \le \epsilon(T)\|\rho\|_\infty(\Sigma),$$

where $\epsilon(T) \to 0$ *as* $T \to 0$.

Proof. According to the jump relations and to the definition of Λ_l we have

$$(I/2 + W - \Lambda_l S)\rho = \partial_\nu S^+\rho - \partial_\nu u^\bullet, \tag{4.9}$$

where $+$ means that we consider a function on $\Omega^\bullet \times (0,T)$, and $u^\bullet$ is a solution to the following parabolic problem:

$$\partial_t u^\bullet - \operatorname{div}((1+f)\nabla u^\bullet) = 0 \quad \text{on } \Omega^\bullet \times (0,T),$$
$$u^\bullet = 0 \text{ on } \Omega^\bullet \times \{0\}, \quad u^\bullet = S\rho \text{ on } \Sigma.$$

The known properties of the heat potentials imply that

$$\|S\rho\|_\infty(\Omega^\bullet \times (0,T)) \le \epsilon(T)\|\rho\|_\infty(\Sigma).$$

Since $S\rho$ solves the heat equation in $\Omega^\bullet\times(0,T)$ (and remains a solution being extended as zero on $\Omega^\bullet \times (-T,0)$) from the interior estimates for parabolic equations [LSU, Theorem 10.1, p. 352] and the above bound, it follows that

$$\|\nabla S\rho\|_2(\Omega \times (0,T)) \le C\|S\rho\|_\infty(\Omega^\bullet \times (0,T)) \le \epsilon(T)\|\rho\|_\infty(\Sigma). \tag{4.10}$$

Since the single layer potential solves the heat equation outside Σ and is zero when $t=0$, the function $u^\# = u^\bullet - S\rho$ solves the parabolic problem

$$\partial_t u^\# - \operatorname{div}((1+f)\nabla u^\#) = \operatorname{div}(f\nabla S\rho) \text{ on } \Omega^\bullet \times (0,T),$$
$$u^\# = 0 \text{ on } \Omega^\bullet \times \{0\} \text{ and on } \Sigma.$$

From the basic energy estimates for parabolic equations in divergence form (with discontinuous coefficients) [LSU, Theorem 2.1, p. 143] we have

$$\|u^\#\|_2(\Omega^\bullet \times (0,T)) \le C\|f\nabla S\rho\|_2(\Omega \times (0,T)) \le \epsilon(T)\|\rho\|_\infty(\Sigma) \tag{4.11}$$

when we use (4.10).

Now we will apply to $u^\#$ interior estimates [LSU, Theorem 10.1, p. 352] once more. To do so we let Ω'' be $\Omega^\bullet\backslash\overline{\Omega}$ and Ω' be $\Omega^\bullet\backslash\overline{\Omega}^\star$ where $\Omega^\star$ is a domain containing $\overline{\Omega}$ and with $\overline{\Omega}^\star \subset \Omega^\bullet$. Extend $u^\#$ as zero onto $\Omega'' \times (-T,0]$; then $u^\#$ solves the heat equation in $Q'' = \Omega'' \times (-T,T)$ and has zero initial data and the zero Dirichlet data on $\partial\Omega^\bullet \times (-T,T)$. From the previously mentioned result of [LSU] (with $f=0$, $\Phi=0$,

$\phi = 0$, $T_o = -T$, $T_1 = 0$, $T_2 = T$) and the additional remark on p. 355 of [LSU] we have

$$\|\nabla u^{\#}\|_\infty(\Omega' \times (0,T)) \le C\|u^{\#}\|_2(Q'') \le \epsilon(T)\|\rho\|_\infty(\Sigma)$$

if we use (4.11).

Remembering that $u^{\#} = u^{\bullet} - S\rho$ and using (4.9) we complete the proof.

From Lemma 4.1 it follows that equation (4.8) in the space $C(\Sigma)$ can be solved for ρ by Picard's iterations if T is small (so that the operator $-I/2 - W + \Lambda_l S$ is a contraction in this space). Since the problem is evolutionary with respect to t, a solution for arbitrary T can be obtained by using a finite number of (small) steps in t. So there is a unique solution ρ which is completely determined by x^* and the operator Λ_l. It is quite clear that equation (4.8) represents a well-posed (stable) problem which can be solved numerically with high precision and efficiency.

A similar reduction of the elliptic Dirichlet-to-Neumann map to the scattering amplitude was used by Nachman [N].

5. Comments. We'd like to formulate some questions and possible future developments.

First, even for the linearized inverse problem one expects a substantial improvement in numerics when using preconditioners. However, it takes some time to develop reasonable ones. While exhibiting some Toeplitz structure features our problem in the many-dimensional case is at the best block Toeplitz, so it is difficult to find good preconditioners quickly, but there is some hope. Other promising directions are decomposition of matrix operators into simpler and more symmetric ones and the use of stochastization of our ill-posed problem. We hope that these ideas can reduce the amount of computations and numerical errors, so that one can solve numerically a very challenging three-dimensional problem.

Second, it is quite important to consider the version of equation (1.19) that resulted from the Green function of a half-space/plane instead of the fundamental solution in the whole space. Indeed, in many applications the domain where the diffusion occurs has the boundary, and a half-space is the simplest of such domains.

Third, the original inverse problem is nonlinear, and this is essential in many applications as well. While a proof of uniqueness in the inverse problem with the receiver equal to source can be very difficult, it is more realistic to try at least a numerical solution. Unfortunately, the existing theory of regularization of nonlinear inverse problems [EnHN] does not cover our equation, in particular because its operator is highly smoothing (maps bounded measurable functions into analytic ones). Therefore, new ideas and efforts are needed here.

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REFERENCES

[CI] M. Cheney and D. Isaacson, *An overview of inversion algorithms for impedance imaging*, Contemp. Math., 122 (1990), pp. 29–39.

[EI] A. Elayyan and V. Isakov, *On uniqueness of recovery of the discontinuous conductivity coefficient of a parabolic equation*, SIAM J. Math. Anal., 28 (1997), pp. 49–59.

[EnHN] H. Engl, M. Hanke, and A. Neubauer, *Regularization of Inverse Problems*, Kluwer, Dordrecht, The Netherlands, 1996.

[EnI] H. ENGL AND V. ISAKOV, *On the identifiability of steel reinforcement bars from magnostatic measurements*, European J. Appl. Math., 3 (1992), pp. 255–262.

[F] A. FRIEDMAN, *Partial Differential Equations of Parabolic Type*, Prentice–Hall, Englewood Cliffs, NJ, 1964.

[H] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators*, Vol. I, Springer-Verlag, Berlin, New York, 1983.

[I1] V. ISAKOV, *Inverse Source Problems, Math. Surveys and Monographs Series*, 34, AMS, Providence, RI, 1990.

[I2] V. ISAKOV, *Uniqueness and stability in inverse parabolic problems*, in Inverse Problems in Diffusion Processes, Proc. of the GAMM-SIAM Symposium, SIAM, Philadelphia, 1995, pp. 21–42.

[LSU] O.A. LADYZENSKAJA, V.A. SOLONNIKOV, N.N. URAL'CEVA, *Linear and Quasilinear Equations of Parabolic Type*, Transl. of Math. Monographs 23, AMS, Providence, RI, 1968.

[N] A. NACHMAN, *Reconstruction from boundary measurements*, Ann. Math., 128 (1988), pp. 531–576.

[SV] F. SANTOSA AND M. VOGELIUS, *A back-projection algorithm for electrical impedance imaging*, SIAM J. Appl. Math., 50 (1990), pp. 216–243.

[T] E.C. TITCHMARCH, *The Theory of Functions*, Oxford University Press, London, 1939.

[Y] W. YEH, *Review of parameter identification procedures in groundwater hydrology*, Water Resources Rev., 22 (1986), pp. 95–108.