ON UNIQUENESS OF RECOVERY OF THE DISCONTINUOUS
CONDUCTIVITY COEFFICIENT OF A PARABOLIC EQUATION*

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Abstract. We prove uniqueness of a discontinuous principal coefficient of a second-order
parabolic equation of the form $a_0 + \chi(Q^*) a$ with known smooth $a_0$ and unknown $b = b(x)$ from
all possible lateral boundary measurements of solutions of this equation. In the proofs, we make use
of singular solutions of parabolic equations.

Key words. partial differential equations, inverse problems

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Introduction. We consider the problem of recovery of the coefficient $a$ of the
parabolic equation

$$u_t - \text{div}(a \nabla u) = 0 \quad \text{in } Q = \Omega \times (0, T)$$

with the initial and boundary conditions

$$u = 0 \quad \text{on } \Omega \times \{0\}, \quad u = g \quad \text{on } \partial \Omega \times [0, T]$$

when $\partial u / \partial \nu$ is given for all (regular) $g$. Here $\Omega$ is a bounded domain in $\mathbb{R}^n$, $2 \leq n$,
with the boundary $\partial \Omega \in C^2$. In this paper, we prove uniqueness of discontinuous
$a = a_0 + \chi(Q^*) a$, where $\chi(Q^*)$ is the indicator function of an open set $Q^* \subset Q$ with the
Lipschitz lateral boundary $\partial_\nu Q^*$ changing with time and $a_0 = a_0(x)$ and $b = b(x)$ are,
respectively, given and unknown $C^2(\Omega)$-functions. For elliptic equations, uniqueness
was proven by Kohn and Vogelius [8] (piecewise-analytic $a$) and Isakov [5] (Lipschitz
$Q^*$ and smooth $b$). Also for elliptic equations, when $a$ is given, and $b$ is
unknown, some partial global uniqueness results for $Q^*$ were obtained
by Friedman and Isakov [4]. Regarding parabolic equations, we can refer only to
Dellout's study [2] of local stability in the inverse problem. This inverse parabolic
problem is fundamental for groundwater search [12] in particular and important for
many engineering applications.

We introduce some notation. For standard notation, we refer to Friedman [3] and
Ladyzhenskaja, Solonnikov, and Ural'ceva [9].

For an open set $Q$ in the layer $\mathbb{R}^n \times [0, T)$, the lateral boundary $\partial_\nu Q$ is the $x$-
boundary that is the closure of the set $\partial Q[(t = 0 \text{ or } t = T)]$. We say that $Q$ is
Lipschitz if its $x$-boundary is locally the graph of a function $x_j = \gamma(x_{1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n, t})$ that is Lipschitz.

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1. **Statement of results.** Let \( \Gamma_0 \) be \( \partial \Omega \cup B_0 \) for some ball \( B_0 \) centered at a point of \( \partial \Omega \).

We are interested in finding an open set \( Q_j \) and a function \( b_j \) entering the parabolic initial-boundary value problem

\[
(u_j)_t - \text{div}(a_j \nabla u_j) = 0 \quad \text{in} \ Q,
\]

\[
u_j = g \quad \text{on} \ S = \partial \Omega \times (0,T),
\]

\[
u = 0 \quad \text{on} \ \Omega \times \{0\},
\]

where

\[
(a_j = a_0 + \chi(Q_j)b_j > \epsilon > 0), \quad b_j \neq 0 \text{ on } \partial Q_j.
\]

It is well known that for any \( g \in C^{2,1}(\mathcal{S}) \), \( g = g_t = g_{tt} = 0 \) on \( \partial \Omega \times \{0\} \), there is a unique (generalized) solution \( u_j \) of this problem and \( u_j \in C^\lambda(\mathcal{Q}) \) for some \( \lambda \in (0,1) \), \( \nabla u_j \in L_2(Q_j) \), and \( u_j \in C(\overline{Q}_j) \). For this and for other results about the direct parabolic problem (1.1)–(1.4), we refer to Friedman [3] and Ladyzhenskaja, Solonnikov, and Ural’ceva [9, pp. 153, 204, and 227].

Our main result is the following theorem.

**Theorem 1.1.** Suppose \( Q_1 \) and \( Q_2 \) are open \( x \)-Lipschitz sets, \( Q_j \subset \Omega \times (-T, 2T) \), and

\[
\text{the sets } (Q \setminus \overline{Q}_j) \cap \{t = \tau \} \text{ are connected when } 0 < \tau < T.
\]

If solutions \( u_j \) to the initial-boundary value problems (1.1), (1.2), and (1.3) satisfy the equality

\[
\partial u_1 / \partial \nu = \partial u_2 / \partial \nu \quad \text{on } \Gamma_0 \times (0,T) \quad (\nu \text{ is a normal})
\]

for all \( g \in C^2(\partial \Omega \times [0,T]) \) with \( \text{supp} g \subset \Gamma_0 \times (0,T) \), then

\[
a_1 = a_2 \quad \text{on } Q.
\]

This result guarantees uniqueness of reconstruction of \( Q_j \) from all possible lateral measurements for an arbitrary \( T > 0 \).

The paper is organized as follows. In section 2, we will show that if equality (1.6) is valid for all Dirichlet boundary data, \( g \) implies certain integral relations which can be interpreted as orthogonality relations. To prove uniqueness in section 4, we will modify an approach from [5] (the use of singular solutions with the pole in those orthogonality relations) to obtain a contradiction when the pole converges to the boundary of one of the domains \( Q_j \). To show that some integrals in these relations are bounded while one of them is not, we will use estimates of integrals of singular solutions given in section 3, which is the most technically difficult part of the paper.

2. **Orthogonality relations.** In this section, we assume that the conditions of Theorem 1.1 are satisfied and obtain some auxiliary relations which will be used in its proof.

Denote by \( Q_{3j} \) the connected component of the open set \( \Omega \setminus (\overline{Q}_1 \cup \overline{Q}_2) \) whose boundary contains \( \Gamma_0 \). Here \( Q_{j\theta} \) is \( Q_j \cap \{t = \theta\}, \ j = 1, 2 \). Let \( Q_3 = \cup Q_{3j} \) over \( 0 < t < T \) and let \( Q_4 = Q \setminus \overline{Q}_3 \).
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LEMMA 2.1.

\[ (2.1) \quad \iint_{Q_1} b_1 \nabla v_1 \cdot \nabla u_2^j \, dx \, dt = \iint_{Q_2} b_2 \nabla v_1 \cdot \nabla u_2^j \, dx \, dt \]

for all solutions \( u_1 \) to equation (1.1) \((j = 1)\) near \( \bar{Q}_4 \) that are 0 when \( t < 0 \) and solutions \( u_2^j \) to the adjoint equation \((u_2^j)_t + \text{div}(a_2 \nabla u_2^j) = 0 \) near \( \bar{Q}_4 \) that are 0 when \( t > T \).

Proof. From well-known results about regularity of solutions to the parabolic initial-boundary value problem (1.1)-(1.3), it follows that \( u_j \) is in \( C^{2,1}(\bar{Q}_4) \) and in \( H^{2,1}(\bar{Q}_4) \), where \( \bar{Q}_4 = V \times (0, T) \) and \( V \) is a vicinity of \( \partial \Omega \) in \( \Omega \). Due to conditions (1.2) and (1.5), both \( u_1 \) and \( u_2 \) have the same Cauchy data on \( \Gamma_0 \times (0, T) \) and satisfy the same parabolic equation in \( \bar{Q}_4 \); thus from uniqueness of continuation for second-order parabolic equations (see, e.g., [7, Corollary 1.2.4]), we conclude that \( u_1 = u_2 \) on \( \bar{Q}_4 \). Letting \( u = u_2 - u_1 \), and subtracting the equations (1.1) with \( j = 1 \) from those with \( j = 2 \), we get

\[ (2.2) \quad \text{div}(a_0 + b_2 \chi(Q_2)) \nabla u - u_t = \text{div}(b_1 \chi(Q_2) - b_2 \chi(Q_2)) \nabla u_1 \quad \text{in} \ Q. \]

Now, using the definition of a weak solution to the parabolic equation under consideration, we obtain

\[ (2.3) \quad \iint_Q ((a_0 + b_2 \chi(Q_2)) \nabla u \cdot \nabla \psi + u_t \psi) = \iint_Q (b_1 \chi(Q) - b_2 \chi(Q_2)) \nabla u_1 \cdot \nabla \psi \]

for any function \( \psi \) from \( H^{1,1}_0(Q) \). Since \( u \) and \( \chi(Q_2) \) are zero outside \( \bar{Q}_4 \cap \{ t < T \} \), this relation remains valid for any function \( \psi \) from \( H^{1,1}(Q_2) \) (where \( Q_2 \) is an arbitrary vicinity of \( \bar{Q}_4 \) that is 0 when \( t > T \).

If \( \psi = u_2^j \) is an \( H^{1,1}(Q_2) \) solution to the adjoint equation from Lemma 2.1, then integrating the left side of (2.3) by parts with respect to \( t \) and using the definition of a weak solution to this adjoint equation with the test function \( u \) (which is zero outside \( Q_4 \cap \{ t < T \} \)), we conclude that the left side in (2.3) is zero. Thus we have relation (2.1) with \( u_1 \) instead of \( u_1 \).

Now by using the Runge property, we extend equality (2.1) onto all \( v_1 \) solving equation (1.1) with \( j = 1 \) near \( \bar{Q}_4 \) and satisfying the initial condition (1.3). Denote the space of such \( v_1 \) by \( X \). It is sufficient to prove that solutions \( u_1 \) to the initial-boundary value problem (1.1)-(1.3) with \( j = 1 \) (for various \( g \) supported in \( \Gamma_0 \times (0, T) \)) approximate in \( L_2(Q_4) \) any solution from \( X \). We denote the space of solutions to (1.1)-(1.3) (with various \( g \)) by \( X_1 \). Indeed, let \( u_1 \in X \). Then we can approximate it similarly by solutions from \( X \) in \( L_2(Q_7) \), where \( Q_7 \) is a Lipschitz domain containing \( Q_4 \) with \( \text{dist}(\partial Q_7, Q_4) > 0 \). From the well-known interior Schauder-type estimates for parabolic equations, it follows that these solutions from \( X_1 \) will approximate \( u_1 \) in \( H^{1,0}(Q_4) \).

To prove \( L_2 \) approximation in view of the Hahn–Banach theorem, it is sufficient to show that if \( f \) from the dual space \( L_2(Q_4) \) is orthogonal to \( X_1 \), then \( f \) is orthogonal to \( X \).

Let \( \Omega_0 \) be a bounded domain with \( C^2 \)-boundary such that \( \Omega \subset \Omega_0 \), \( \Omega \neq \Omega_0 \), and \( \partial \Omega \cap \Gamma_0 \) belong to \( \partial \Omega_0 \). Let \( K(x, t; y, s) \) be the Green function for the first initial value problem for the operator \( \partial_t + \text{div}(a_1 \nabla) \) in \( Q_0 \times (0, T) \). Let \( f \) be orthogonal to \( X_1 \).

The Green potential

\[ (2.4) \quad U(x, t; f) = \int_{Q_4} f K(x, t; ) \]

is equal to zero on $Q_1 \setminus Q_4$ because the function $u_1 = K(x, t; )$ belongs to $X_1$ if $(x, t) \in Q_1 \setminus Q_4$. Since $\operatorname{supp} f \subset Q_4$, this potential is a solution to the equation $\operatorname{div}(a_0 \nabla u) = u_1$ on $Q_1 \setminus Q_4$. The coefficient $a_0$ belongs to $C^1(Q_0)$, so this equation has the property of unique continuation. Therefore, $\mathcal{U}(\cdot; f) = 0$ on $Q_0 \setminus Q_4$. Now let $\nu \in X$; then $\nu$ is a solution to the homogeneous equation near $Q_0 \cup \partial \omega Q_5$, where $Q_5$ is an open set with $C^\infty$ lateral boundary and $\operatorname{dist}(\partial_2 Q_5, \partial_2 Q_4) > 0$. Using the representation of $\nu$ by a single layer potential, we obtain
\[
v(y, s) = \int_{\partial_\nu Q_5} gK(\cdot; y, s)d\Gamma
\]
for some $g \in C(\partial_\nu Q_5)$. By using this representation, (2.4), and Fubini's theorem, we obtain
\[
\int_{Q_4} f\nu = \int_{\partial_\nu Q_5} g\mathcal{U}(\cdot; f) = 0
\]
because $\mathcal{U}(\cdot; f) = 0$ on $\partial_\nu Q_5$. Accordingly, relation (2.1) is valid for any $\nu_1$ satisfying the conditions of Lemma 2.1.

The proof is complete.

Assume that
\[
Q_1 \neq Q_2.
\]
Then we may assume that $Q_1$ is not contained in $Q_2$. Hence, using condition (1.5) of Theorem 1.1 on $Q_1$, we conclude that there is a point $(x_0, t_0) \in \partial Q_1 \setminus \partial Q_2$ such that $(x_0, t_0) \in \partial \omega Q_3$. By considering $g = 0$ for $t < t_0$ and using the translations $t \rightarrow t - t_0$ and $x \rightarrow x - x_0$, we can reduce the general case to $t_0 = 0$ and $x_0 = 0$. We can choose a ball $B \subset \mathbb{R}^n$ centered at 0 and a cylinder $Z = B \times (0, 1)$ such that $B \subset \Omega$, $\overline{Z}$ does not intersect $\overline{Q}_2$, and $(\partial Q_1) \cap \partial \Omega$ is a Lipschitz surface. Due to well-known variants of the Whitney extension theorem, there is a $C^2(Q_1 \cup \partial \Omega)$-function $a_3$ that coincides with $a_1$ on $Q_1$. Extend $a_3$ onto $Q \cup (\overline{Q}_1 \cup \partial \Omega)$ as $a_0$.

**Lemma 2.2.** Under the conditions of Lemma 2.1,
\[
\int_{Q_1} b_1 \nabla u_3 \cdot \nabla u_3 = \int_{Q_2} b_2 \nabla u_3 \cdot \nabla u_3
\]
for any solution $u_3$ to the equation $\operatorname{div}(a_1 \nabla u) - (u_3)_t = 0$ near $\overline{Q}_4$ which is 0 when $t < 0$ and for any solution $u_3$ from Lemma 2.1.

Proof. Consider $u_3$ and let $Q_3$ be an open set with $C^\infty$-boundary $\partial_\nu Q_3$ and that contains $Q_4$ with $\operatorname{dist}(\partial_\nu Q_3, \partial_\nu Q_4) > 0$ such that $u_3$ is a solution to the equation $\operatorname{div}(a_3 \nabla u_3) - (u_3)_t = 0$ near $\overline{Q}_3$.

Introduce a sequence of open sets $Q_{4k}$ such that (i) $Q_{4k} \setminus Z = Q_4 \setminus Z$ and (ii) the (Hausdorff) distance from $\partial Q_{4k}$ to $\partial_\nu Q_4$ is less than $1/k$ and $\partial_\nu Q_{4k} \cap \partial_\nu Q_4$ does not intersect $\overline{Q}_4$.

Define a coefficient $a_{3k}$ as $a_3$ on $Q_3 \setminus (Q_{4k} \setminus Q_4)$ and as $a_0$ on $Q_{4k} \setminus Q_4$.

Since $\partial_\nu Q_4 \cap \partial_\Omega$ is a Lipschitz surface, we have
\[
\meas_n \{a_{3k} \neq a_3\} \to 0 \quad \text{as } k \to +\infty.
\]

Let $u_{3k}$ be solutions to the initial-boundary value problems
\[
\operatorname{div}(a_{3k} \nabla u_{3k}) - (u_{3k})_t = 0 \quad \text{in } Q_3, \quad u_{3k} = u_3 \quad \text{on } \partial_\nu Q_3, \quad u_{3k} = 0 \quad \text{on } Q_3 \cap \{t = 0\}.
\]
Since \( u_{3k} = a_0 + \chi(Q_1)b_1 \) near \( \overline{Q}_4 \), relation (2.1) is valid for any \( u_1 = u_{3k} \). The difference \( u_k = u_{3k} - u_3 \) satisfies the equation
\[
\text{div}(a_{3k} \nabla u_k) - (u_k)_t = \text{div}((a_3 - a_{3k}) \nabla u_3) \quad \text{in } Q_8,
\]
and \( u_k = 0 \) on \( \partial Q_8 \cap \{ t < T \} \) because \( u_{3k} \) and \( u_3 \) coincide on the lateral boundary of \( Q_8 \) and when \( t = 0 \). From the definition of a weak solution to this initial-boundary value problem with the test function \( u_k \), we have
\[
\int_{Q_8} a_{3k} \nabla u_k \cdot \nabla u_k + \int_{Q_8 \cap \{ t = T \}} \frac{u_k^2}{2} = \int_{Q_8} (a_3 - a_{3k}) \nabla u_3 \cdot \nabla u_k.
\]
According to the assumptions, \( \varepsilon < a_{3k} \) for certain positive \( \varepsilon \). Using this inequality, dropping the second integral in the left side, and bounding the right side by the inequality \( x \cdot y \leq c^{-1}/2|x|^2 + \varepsilon/2|y|^2 \), we obtain
\[
\int_{Q_8} \varepsilon |\nabla u_k|^2 \leq C(c) \int_{Q_8} |a_3 - a_{3k}|^2 |\nabla u_3|^2 + \frac{\varepsilon}{2} \int_{Q_8} |\nabla u_k|^2.
\]
Since \( \nabla u_3 \) belongs to \( L^2(Q_8) \), we conclude from (2.6) that the first integral in the right side tends to 0. Therefore, \( \nabla u_k \) converges to 0 in \( L^2(Q_8) \). Putting \( u_1 = u_{3k} = u_3 + u_k \) into relation (2.1) and letting \( k \to \infty \), we complete the proof of Lemma 2.2.

3. Estimates of integrals of singular solutions. We will make use of solutions \( u_3 \) and \( u_3^* \) with singularities outside \( Q_4 \). Solutions of elliptic equations of second order with arbitrary power singularities were constructed by Alessandrini [1]; we do not know of similar results for parabolic equations. To simplify obtaining bounds on the integrals of such solutions, we introduce new variables. We can assume that the direction \( e_1 \) of the \( x_n \)-axis coincides with the interior unit normal to \( \partial_s Q_1 \cap \{ t = 0 \} \). According to our assumptions, \( \partial_s Q_1 \) near the origin is the graph of a Lipschitz function \( x_n = q_t(x_1, \ldots, x_{n-1}, t) \) which can be assumed to be defined and Lipschitz on the whole \( \mathbb{R}^n \). The substitution
\[
x_k = x_k^*, \quad k = 1, \ldots, n-1, \quad x_n = x_n^* + q_t(x_1^*, \ldots, x_{n-1}^*, t), \quad t = t^*
\]
transforms the equations (1.1) into similar equations with additional first-order differentiation with respect to \( x_n^* \) multiplied by a Lipschitz function of \( t \). The domains \( Q_j \) are transformed onto domains with similar properties and with the additional property that the points \( (0, t) \), \( 0 < t < T \), belong to \( \partial_s Q_1 \). Since the (hyper)plane \( \{ x_n^* = 0 \} \) is tangent to this surface at the origin, we can find a cone \( C = \{ |x^*|/|x^*| - e_1| < \theta, |x^*| < \varepsilon \} \) such that the cylinder \( C \times (0, T) \) is inside \( Q_1 \). Henceforth, we drop the sign *.

Let \( K^+ \) be the fundamental solution of the Cauchy problem for the forward parabolic equation \( \text{div}(a_3 \nabla u_3) - (u_3)_t = 0 \) in \( s \)-coordinates. Let \( K^- \) be the fundamental solution of the backward Cauchy problem for the backward parabolic equation \( \text{div}(a_3 \nabla u_3) + u_3_t = 0 \) in these coordinates. It is known that
\[
(3.1) \quad K^+ = K^+_1 + K^+_0, \quad K^- = K^-_1 + K^-_0,
\]
where \( K^+_1 \) and \( K^-_1 \) are the principal parts of \( K^+ \) and \( K^- \) (parametrices) and \( K^+_0 \) and \( K^-_0 \) are.
and $K_0^-$ are the remainders. The principal parts are

$$K_1^+(x, t; y, \tau) = \frac{C}{(a_2(y)(t - \tau))^{n/2}} \exp\left(-\frac{|x - y|^2}{4a_2(y)(t - \tau)}\right),$$

$$K_1^-(x, t; y, \tau) = \frac{C}{(a_0(y)(\tau - t))^{n/2}} \exp\left(-\frac{|x - y|^2}{4a_0(y)(\tau - t)}\right).$$

(3.2)

From the known bounds of fundamental solutions of parabolic equations [9, p. 377], we have

$$|\nabla_x K_1^+(x, t; y, \tau)| \leq C(t - \tau)^{-n/2} \exp\left(-\frac{|x - y|^2}{C(t - \tau)}\right),$$

$$|\nabla_x K_1^-(x, t; y, \tau)| \leq C(\tau - t)^{-n/2} \exp\left(-\frac{|x - y|^2}{C(\tau - t)}\right).$$

(3.3)

When $(y, 0)$ and $(y, \tau)$ are outside $Q_1$, the functions $K_1^+(; y, 0)$ and $K_1^-(; y, \tau)$ are $(x, t)$-solutions to the homogeneous parabolic equations with bounded measurable coefficients satisfying zero initial and final conditions. Using Lemma 2.2 with $u_3 = K_1^+(; y, 0)$ and $u_2 = K_1^-; y, \tau$, we get

$$\int_{Q_1 \cap Z} b_1 \nabla_x K_1^+(; y, 0) \cdot \nabla_x K_1^-; y, \tau$$

$$= -\int_{Q_1 \cap Z} b_1 \nabla_x K_1^+(; y, 0) \cdot \nabla_x K_1^-; y, \tau$$

$$+ \int_{Q_2} b_2 \nabla_x K_1^+(; y, 0) \cdot \nabla_x K_1^-; y, \tau.$$

(3.4)

From the estimates in (3.3) and similar estimates for $\nabla_x K_1^+$ and $\nabla_x K_1^-$, we conclude that the integrands are bounded by an integrable function uniformly with respect to $y$ outside $Q_1$. By the Lebesgue dominated-convergence theorem, we may let $y \to 0$ and replace $y$ in (3.4) by 0. Using representation (3.1), we obtain from (3.4) that

$$|I_1| \leq |I_2| + |I_3|,$$

(3.5)

where

$$I_1 = \int_{Q_1 \cap Z} b_1 \nabla_x K_1^+(; 0, 0) \cdot \nabla_x K_1^-; 0, \tau$$

is formed from the principal parts of $K$ and the remainders are collected in

$$I_2 = -\int_{Q_1 \cap Z} b_1 \nabla_x K_1^+(; 0, 0) \cdot \nabla K^-; 0, \tau + \int_{Q_2} b_2 \nabla_x K_1^+(; 0, 0) \cdot \nabla_x K^-; 0, \tau$$

and

$$I_3 = \int_{Q_1 \cap Z} b_1 (\nabla_x K_0^+(; 0, 0) \cdot \nabla K_0^-; 0, \tau + \nabla_x K_0^-(; 0, 0) \cdot \nabla_x K_0^-; 0, \tau)$$

$$+ \nabla_x K_0^-(; 0, 0) \cdot \nabla K_0^-; y, \tau).$$
In the following three lemmas, $I_1$ is bounded from below and $I_2$ and $I_3$ is bounded from above.

**Lemma 3.1.**

$$|I_1| \geq C^{-1}r^{-n} \int_0^r \rho^{n-1}e^{-4q^2/(mr)}d\rho,$$

where $m = \inf(a_3,a_0)$ over $Q$.

**Proof.** Using the fact that $b_1(0) \neq 0$ and choosing $\epsilon$ in the definition of $C$ to be sufficiently small, we obtain

$$|I_1| \geq C^{-1} \int_{C_Q \times (0,\tau)} \nabla_x K_1^+(x,t;0,0) \cdot \nabla_x K_1^-(x,t;0,\tau)$$

$$= C^{-1} \int_0^\tau \int_C t^{-n/2-1} \exp \left(-\frac{|x|^2}{a_3(x)}\right) x \cdot (r-t)^{-n/2-1} \exp \left(-\frac{|x|^2}{a_0(x)(r-t)}\right) x \, dx \, dt$$

$$\geq C^{-1} \int_0^r \int_C |x|^2((r-t)t)^{-n/2-1} \exp \left(-\frac{|x|^2}{mt(r-t)}\right) dt \, dx.$$  

Using the inequality

(3.6) \quad \frac{1}{tr} \leq \frac{1}{t(r-t)} \leq \frac{2}{tr} \quad \text{when } 0 < t < \frac{r}{2},

we bound from below the integral shown above by

$$C^{-1} \int_0^r \int_C |x|^2 \frac{1}{(tr)^{n/2-1}} \exp \left(-\frac{2|x|^2}{mt}\right) dt \, dx$$

$$= \frac{1}{C^r r^{n/2+1}} \int_C \int_0^\infty \int_0^w w^{n/2-1}e^{-w}dw \, dx,$$

where we substituted $w = 2|x|^2/mt$.

The function $w^{n/2-1}$ is increasing, so replacing it by its minimal value at $w = 4|x|^2/(mr)$, we bound the last integral from below by

$$\int_C x^{-n/2} \left(\int_{4|x|^2/(mr)}^\infty e^{-w}dw\right) \, dx = C^{-1}r^{1-n/2} \int_{(0,R)} \rho^{n-1}e^{-4q^2/(mr)}d\rho.$$

The proof is complete.

**Lemma 3.2.**

$$|I_2| \leq C\tau^{-n/2+1}e^{-2e^{-2\epsilon^2/(Ms)}},$$

where $M$ depends only on $\sup(a_3,a_0)$ over $Q$.

**Proof.** $I_2$ consists of two integrals. The first one is bounded by

$$C \int_{<|x|<R,0<t<\tau} |\nabla_x K^+(x,t;0,0) \cdot \nabla_x K^-(x,t;0,\tau)|$$

$$\leq C \int_{<|x|<R} \int_0^{t/2} \frac{1}{((r-t)t)^{n/2+1/2}} \exp \left(-\frac{|x|^2}{Mt(r-t)}\right) dt \, dx.$$
The bound on $|\nabla_x K^+ \cdot \nabla_x K^-|$ follows from the direct differentiation of (3.2), the inequality

$$|\eta| (t - \tau) - n/2 - 1 \exp \left( -\frac{|\xi|^2}{4(t - \tau)} \right) \leq C (t - \tau)^{-n/2 - 1/2} \exp \left( -\frac{|\xi|^2}{8(t - \tau)} \right),$$

and the bounds in (3.3).

Applying inequality (3.6) as above, we bound the last integral by

$$\frac{C}{\tau^{n/2+1/2}} \int_{0 < |\xi| < R} \int_{0}^{\tau/2} \int_{|\eta| < \tau} \int_{0}^{\infty} \frac{1}{\tau^{n/2+1/2}} \exp \left( -\frac{|\xi|^2}{Mt} \right) \exp \left( -\frac{|\eta|^2}{M\tau} \right) dt \, d\eta \, dx$$

$$\leq \frac{C}{\tau^{n/2+1/2}} \int_{0 < |\xi| < R} |\xi|^{-n} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} w^{n/2-1} w^{-1/2} e^{-w} dw \, dx \, d\rho \, d\sigma \, d\tau \, d\rho$$

when we use the substitution $w = |\xi|^2/(Mt)$. The function $w^{-1/2}$ is decreasing. Replacing it by its value at $2|\xi|^2/(M\tau)$, we increase the integral, and we also use the inequality $w^{n/2-1} e^{-w} \leq C e^{-w/2}$ and calculate the resulting integral with respect to $w$. Then the last integral will be less than

$$C_{\tau}^{-n/2} \int_{0 < |\xi| < R} |\xi|^{n} \exp \left( -|\xi|^2/(M\tau) \right) \, dx$$

$$\leq C_{\tau}^{-n/2} \int_{0 < |\xi| < R} |\xi|^{n} \exp \left( -\rho^2/(M\tau) \right) \, d\rho$$

when we use the polar coordinates in $\mathbb{R}^n$. Replacing $\rho^{-2}$ by its maximal value at $\epsilon$ and calculating the remaining integral with respect to $\rho$, we complete the bounding of the integral over $Q_0 \setminus Z$.

A similar argument works for the integral over $Q_2$.

The proof is complete.

**Lemma 3.3.**

$$|I_3| \leq C_{\tau}^{-n/2},$$

where $M$ depends only on the upper bounds of $|a_0|$, $|a_0|$.

**Proof.** We bound the integral of the first of the three functions, forming $I_3$ as defined after (3.5).

As follows from (3.2), (3.3), and the argument in Lemma 3.2, replacing $|\xi|$ by some power of $(t - \tau)$, the absolute value of this integral is less than

$$C \int_{|\xi| < \epsilon} \int_{(0, \tau/2)} (t - \tau) - n/2 - 1/2 \exp \left( -|\xi|^2 / (Mt (t - \tau)) \right) dt \, dx$$

$$\leq C \int_{|\xi| < \epsilon} \int_{(0, \tau/2)} (t - \tau) - n/2 - 1/2 \exp \left( -|\xi|^2 / (Mt) \right) dt \, dx,$$

where we used inequality (3.6). Substituting $w = |\xi|^2/(Mt)$ in the inner integral yields

$$C_{\tau}^{-n/2} \int_{|\xi| < \epsilon} |\xi|^{-n} \int_{|\xi|^2/(M\tau) - \epsilon}^{\infty} w^{n/2 - 1/2} e^{-w} dw \, dx \leq C_{\tau}^{-n/2} \int_{|\xi| < \epsilon} |\xi|^{-n} \, dx.$$

Using the polar coordinates, we bound the last integral by $C_{\epsilon}$.

The other terms can be bounded in a similar way. The proof is complete.
4. Proof of Theorem 1.1. Now we will complete the proof of Theorem 1.1. Let

\[ \varepsilon^2 = \alpha \tau, \]

where (large) \( \alpha \) will be chosen later.

First, we bound \( I_1 \) from below. From Lemma 3.1, substituting \( w = 4y^2/(m\alpha) \) in the integral and using condition (4.1), we obtain

\[ |I_1| \geq C^{-1} \tau^{-n/2} \int_{(0,4E/m)} w^{n/2-1} e^{-w} dw \geq C^{-1} \tau^{-n/2} \]

provided \( \alpha > m \).

From (3.5), Lemmas 3.1–3.3, (4.1), and (4.2), it follows that

\[ C^{-1} \tau^{-n/2} \leq C(\tau^{-n/2} + \varepsilon^2 \exp(-E/M) + \tau^{-n/2} \varepsilon). \]

Using (4.1) again and multiplying both sides by \( C\tau^{n/2} \), we obtain

\[ 1 \leq CE^{-1} \exp(-E/M) + C\varepsilon \leq CE^{-1} + C\varepsilon. \]

Let \( \tau < 1 \). Choose \( \alpha \) so large that \( E^{-1} < 1/(4C) \) and \( \varepsilon < 1/(4C) \); then the right side is smaller than \( 1/2 \). We have a contradiction.

This contradiction shows that \( Q_1 = Q_2 \).

The next step of the proof is to show that

\[ b_1 = b_2 \quad \text{on} \quad \partial_2 Q_1. \]

As in the proof for \( Q_j \), we assume the opposite. Then we can assume that the origin \( 0 \in \partial_2 Q_1 \) and \( b_1(0) < b_2(0) \). By continuity, \( b_1(0) - b_2(0) > C^{-1} \) for some \( C \) on a certain ball \( B \) centered at the origin. Let \( Z = B \times (0,T) \). Extend \( u_2 \) from \( Q_2 \) onto \( \mathbb{R}^n \) as a \( C^2 \)-function \( a_4 > 0 \). By repeating the proof of Lemma 3.2, we obtain the following orthogonality relation:

\[ \int_{Q_1} (b_1 - b_2) \nabla u_3 \cdot \nabla u_2 = 0. \]

for all solutions \( u_3 \) to the equation \( \text{div}(a_3 \nabla u_3) - u_3 = 0 \) near \( Q_4 \) which are zero when \( t < 0 \) and for all solutions \( u_2^* \) to the adjoint equation \( \text{div}(a_3 \nabla u_4) + u_4 = 0 \) near \( Q_4 \) which are zero when \( t > T \). Let \( K^+ \) be a fundamental solution to the forward Cauchy problem for the first equation and \( K^- \) be the fundamental solution to the backward Cauchy problem for the adjoint equation with the coefficient \( a_4 \). Using the representation (3.1) of these fundamental solutions and splitting \( Q_1 \) into \( Q_1 \cap Z \) and its complement, as in section 3, we obtain from (4.4) the inequality

\[ |I_4| \leq |I_5| + |I_6|, \]

where

\[ I_4 = \int_{Q_1 \cap Z} (b_1 - b_2) \nabla \cdot \nabla_2 K^+_1 \cdot \nabla_2 K^-_1. \]
is related to the supposedly singular part and

\[ I_5 = \int_{Q_1 \setminus \mathcal{Z}} (b_1 - b_2) \nabla K^+ \cdot \nabla K^- , \]

\[ I_6 = \int_{Q_1} (b_1 - b_2) \nabla K_0^+ \cdot \nabla K_0^- . \]

It is easy to see that Lemmas 3.1, 3.2, and 3.3 are valid for \( I_4, I_5, \) and \( I_6, \) respectively. Therefore, as in the proof above, we arrive at the contradiction that \( Q_1 = Q_2. \)

This shows that the assumption about \( b_1 \) and \( b_2 \) is wrong and that \( b_1 = b_2 \) on \( \partial \Omega Q_1. \)

Let \( \Omega_0 \) be the intersection of all \( Q_{1\theta} \) over \( 0 < \theta < T. \) Since \( b_1 \) and \( b_2 \) do not depend on \( t \) and are equal on \( \partial \Omega Q_1, \) they coincide on \( Q_{1\theta} \setminus \Omega_0. \) Letting \( Q_0 = \Omega_0 \times (0, T), \) we obtain from (4.4) the relation

\[ \int_{Q_0} (b_1 - b_2) \nabla u_3 \cdot \nabla u_3^* = 0 \]

for all \( u_3 \) and \( u_3^* \) in (4.4). As in the proof of Lemma 3.2, this implies that

\[ \int_{Q_0} (b_1 - b_2) \nabla u_6 \cdot \nabla u_6^* = 0 \]

for solutions \( u_5 \) to the equation \( \text{div}((a_0 + b_1 \chi(Q_0)) \nabla u_5) - u_{5\theta} = 0 \) near \( Q_0, \) which are zero when \( t < 0 \) and for solutions to the adjoint equation \( \text{div}((a_0 + b_2 \chi(Q_0)) \nabla u_5^*) - u_{5\theta}^* = 0 \) near \( Q_0, \) which are zero when \( t < T. \)

Observe that by choosing \( T \) small, we can guarantee that \( \Omega_0 \) is a Lipschitz domain. Indeed, for any point of \( \partial \Omega Q_1 \cap \{ t = 0 \}, \) there is a neighborhood where \( Q_1 \) is the subgraph of the Lipschitz function \( x_j < \alpha_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n, t). \) We can cover the compact set \( \partial \Omega Q_1 \cap \{ t = 0 \} \) by a finite number of such neighborhoods. Then there is \( T_1 \) such that \( \partial \Omega Q_1 \cap \{ t < T_1 \} \) is contained in the union of these neighborhoods. Let \( T = T_1; \) then \( \Omega_0 \) is Lipschitz because locally (in the corresponding neighborhood) its boundary is given by the equation \( x_j = \inf \alpha_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n, t) \) over \( t \in (0, T), \) and the inf of a family of uniformly Lipschitz functions is a Lipschitz function.

Now we will show that the equations for \( u_5 \) and \( u_6 \) have the same lateral Dirichletto-Neumann maps. Let \( u_5 \) and \( u_6 \) be a solution to these equations with zero initial conditions and the same lateral Dirichlet data. By subtracting these equations and letting \( u = u_5 - u_6, \) we obtain

\[ \text{div}((a_0 + b_1 \chi(Q_0)) \nabla u) = \text{div}((b_1 - b_2) \chi(Q_0) \nabla u_5) \quad \text{in} \quad Q. \]

From the definition of a weak solution of this equation, we have

\[ \int_{\partial \Omega \times (0, T)} a_0 u \cdot \psi - \int_Q ((a_0 + b_2) \chi(Q_0)) \nabla u \cdot \nabla \psi - \int_Q u_5 \psi = - \int_{Q_0} (b_1 - b_2) \nabla u_5 \cdot \nabla \psi \]

for any function \( \psi \in H^{1,1}(Q). \) Using \( \psi = u_5^* \), integrating by parts in the third integral of the left side, and again using the definition of a weak solution to the equation \( \text{div}((a_0 + b_2 \chi(Q_0)) \nabla u_5^*) + u_{5\theta}^* = 0 \) with the test function \( u \) which is zero on
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We conclude that the sum of the second and third integrals in the left side is zero. The right side is zero due to (4.6). Thus the first integral in the left side is zero. Since the lateral Dirichlet data \( \psi = u^*_\alpha \) can be any function in \( C^0(\partial \Omega \times (0, T)) \), we get \( u_\alpha = 0 \) on \( \partial \Omega \times (0, T) \). Therefore, \( u_{u\alpha} = u_{\alpha u} \) on the lateral boundary, which means that we have the same lateral Dirichlet-to-Neumann maps.

Take as the Dirichlet data \( g \) which does not depend on \( t \) when \( t > \tau \). Since the coefficients of the equations \( \text{div}(a_0 + b_0 \chi(Q_0) \nabla u_j) - a_j t = 0 \) are time independent, the solution \( u_j(x, t) \) of the initial-boundary value problems on \( \Omega \times (0, \infty) \) will be analytic with respect to \( t > \tau \). They have the same Cauchy data on \( \partial \Omega \times (0, T) \); therefore, as above, by uniqueness in the lateral Cauchy problem, \( u_5 = u_6 \) on \( \Omega \setminus \Omega_0 \times (0, T) \). By uniqueness of the analytic continuation, they are equal also on \( \Omega \setminus \Omega_0 \times (0, \infty) \). Now we modify the argument of [6] and consider the Laplace transforms

\[
U_j(x, s) = \int_{(0, \infty)} e^{-st} u_j(x, t) \, dt.
\]

They solve the following Dirichlet problems:

\[
(4.7) \quad \text{div}(a_0 + b_0 \chi(Q_0) \nabla U_j) - sU_j = 0 \quad \text{in} \; \Omega, \quad U_j = G \quad \text{on} \; \partial \Omega,
\]

and \( U_5 = U_6 \) on \( \Omega \setminus \Omega_0 \). Letting \( \tau \to 0 \) we obtain \( G(x, s) = g_0(x) s^{-1} \), where \( g_0(x) = g(x, t) \) when \( t > \tau \). Applying the results of [5] and [11] on identification of elliptic equations, we conclude that \( b_1 = b_2 \) on \( \Omega_0 \). In fact, this result is obtained in [5] when \( n \geq 3 \), but the recent global uniqueness theorem of Nachman [10] extends it to \( n = 2 \).

The proof is complete.

REFERENCES