SIAM J. MATH. ANAL.
Vol. 28, No. 1, pp. 49–59, January 1997

ON UNIQUENESS OF RECOVERY OF THE DISCONTINUOUS CONDUCTIVITY COEFFICIENT OF A PARABOLIC EQUATION*

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Abstract. We prove uniqueness of a discontinuous principal coefficient of a second-order parabolic equation of the form $a_0 + \chi(Q^*)b$ with known smooth a_0 and unknown $b = b(x)$ from all possible lateral boundary measurements of solutions of this equation. In the proofs, we make use of singular solutions of parabolic equations.

Key words. partial differential equations, inverse problems

AMS subject classification. 35R30

PII. S0036141095286010

Introduction. We consider the problem of recovery of the coefficient a of the parabolic equation

$$u_t - \operatorname{div}(a\nabla u) = 0 \quad \text{in } Q = \Omega \times (0,T)$$

with the initial and boundary conditions

$$u = 0 \quad \text{on } \Omega \times \{0\}, \qquad u = g \quad \text{on } \partial\Omega \times [0,T]$$

when $\partial u/\partial \nu$ is given for all (regular) g. Here Ω is a bounded domain in $\mathbb{R}^n$, $2 \le n$, with the boundary $\partial\Omega \in C^2$. In this paper, we prove uniqueness of discontinuous $a = a_0 + \chi(Q^*)b$, where $\chi(Q^*)$ is the indicator function of an open set $Q^* \subset Q$ with the Lipschitz lateral boundary $\partial_x Q^*$ changing with time and $a_0 = a_0(x)$ and $b = b(x)$ are, respectively, given and unknown $C^2(\bar{\Omega})$-functions. For elliptic equations, uniqueness was proven by Kohn and Vogelius [8] (piecewise-analytic a) and Isakov [5] (Lipschitz Q^* and smooth b). Also for elliptic equations, when one is making use of only one set of u, $\partial u/\partial \nu$ on $\partial\Omega$, some partial global uniqueness results for Q^* were obtained by Friedman and Isakov [4]. Regarding parabolic equations, we can refer only to Bellout's study [2] of local stability in the inverse problem. This inverse parabolic problem is fundamental for groundwater search [12] in particular and important for many engineering applications.

We introduce some notation. For standard notation, we refer to Friedman [3] and Ladyzhenskaja, Solonnikov, and Ural'ceva [9].

For an open set Q in the layer $\mathbb{R}^n \times (0,T)$, the lateral boundary $\partial_x Q$ is the x-boundary that is the closure of the set $\partial Q | \{t = 0 \text{ or } t = T\}$. We say that Q is x-Lipschitz if its x-boundary is locally the graph of a function $x_j = \gamma(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n, t)$ that is Lipschitz.

* Received by the editors May 12, 1995; accepted for publication October 23, 1995.
http://www.siam.org/journals/sima/28-1/28601.html
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1. Statement of results. Let Γ_0 be $\partial\Omega \cup B_0$ for some ball B_0 centered at a point of $\partial\Omega$.

We are interested in finding an open set Q_j and a function b_j entering the parabolic initial-boundary value problem

$$(u_j)_t - \operatorname{div}(a_j \nabla u_j) = 0 \quad \text{in } Q, \tag{1.1}$$

$$u_j = g \quad \text{on } S = \partial\Omega \times (0,T), \tag{1.2}$$

$$u = 0 \quad \text{on } \Omega \times \{0\}, \tag{1.3}$$

where

$$a_j = a_0 + \chi(Q_j) b_j > \epsilon > 0, \quad b_j \neq 0 \text{ on } \partial Q_j. \tag{1.4}$$

It is well known that for any $g \in C^{2,1}(\bar{S})$, $g = g_t = g_{tt} = 0$ on $\partial\Omega \times \{0\}$, there is a unique (generalized) solution u_j of this problem and $u_j \in C^\lambda(\bar{Q})$ for some $\lambda \in (0,1)$, $\nabla_x u_j \in L_2(Q)$, and $\in C(\bar{Q}\backslash\bar{Q}_j)$. For this and for other results about the direct parabolic problem (1.1)–(1.4), we refer to Friedman [3] and Ladyzhenskaja, Solonnikov, and Ural'ceva [9, pp. 153, 204, and 227].

Our main result is the following theorem.

THEOREM 1.1. *Suppose Q_1 and Q_2 are open x-Lipschitz sets, $Q_j \subset \Omega \times (-T, 2T)$, and*

$$\text{the sets } (Q\backslash\overline{Q}_j) \cap \{t = \tau\} \text{ are connected} \quad \text{when } 0 < \tau < T. \tag{1.5}$$

If solutions u_j to the initial-boundary value problems (1.1), (1.2), *and* (1.3) *satisfy the equality*

$$\partial u_1/\partial\nu = \partial u_2/\partial\nu \quad \text{on } \Gamma_0 \times (0,T) \quad (\nu \text{ is a normal}) \tag{1.6}$$

for all $g \in C^2(\partial\Omega \times [0,T])$ with $\operatorname{supp} g \subset \Gamma_0 \times (0,T)$, *then*

$$a_1 = a_2 \quad \text{on } Q. \tag{1.7}$$

This result guarantees uniqueness of reconstruction of Q_j from all possible lateral measurements for an arbitrary $T > 0$.

The paper is organized as follows. In section 2, we will show that if equality (1.6) is valid for all Dirichlet boundary data, g implies certain integral relations which can be interpreted as orthogonality relations. To prove uniqueness in section 4, we will modify an approach from [5] (the use of singular solutions with the pole in those orthogonality relations) to obtain a contradiction when the pole converges to the boundary of one of the domains Q_j. To show that some integrals in these relations are bounded while one of them is not, we will use estimates of integrals of singular solutions given in section 3, which is the most technically difficult part of the paper.

2. Orthogonality relations. In this section, we assume that the conditions of Theorem 1.1 are satisfied and obtain some auxiliary relations which will be used in its proof.

Denote by Q_{3t} the connected component of the open set $\Omega\backslash(\overline{Q}_{1t} \cup \overline{Q}_{2t})$ whose boundary contains Γ_0. Here $Q_{j\theta}$ is $Q_j \cap \{t = \theta\}$, $j = 1,2$. Let $Q_3 = \cup\, Q_{3t}$ over $0 < t < T$ and let $Q_4 = Q\backslash\overline{Q}_3$.

LEMMA 2.1.

$$\int_{Q_1} b_1 \nabla v_1 \cdot \nabla u_2^* \, dx\, dt = \int_{Q_2} b_2 \nabla v_1 \cdot \nabla u_2^* \, dx\, dt \tag{2.1}$$

for all solutions v_1 to equation (1.1) ($j = 1$) near $\bar{Q}_4$ that are 0 when $t < 0$ and solutions u_2^* to the adjoint equation $(u_2^*)_t +$ $\operatorname{div}(a_2 \nabla u_2^*) = 0$ near $\bar{Q}_4$ that are 0 when $t > T$.

Proof. From well-known results about regularity of solutions to the parabolic initial-boundary value problem (1.1)–(1.3), it follows that u_j is in $C^{2,1}(Q_3)$ and in $H^{2,1}(Q_5)$, where $Q_5 = V \times (0,T)$ and V is a vicinity of $\partial\Omega$ in Ω. Due to conditions (1.2) and (1.5), both u_1 and u_2 have the same Cauchy data on $\Gamma_0 \times (0,T)$ and satisfy the same parabolic equation in Q_3; thus from uniqueness of continuation for second-order parabolic equations (see, e.g., [7, Corollary 1.2.4]), we conclude that $u_1 = u_2$ on Q_3. Letting $u = u_2 - u_1$ and subtracting the equations (1.1) with $j = 1$ from those with $j = 2$, we get

$$\operatorname{div}((a_0 + b_2\chi(Q_2))\nabla u) - u_t = \operatorname{div}((b_1\chi(Q_1) - b_2\chi(Q_2))\nabla u_1) \quad \text{in } Q. \tag{2.2}$$

Now using the definition of a weak solution to the parabolic equation under consideration, we obtain

$$\int_Q ((a_0 + b_2\chi(Q_2))\nabla u \cdot \nabla\psi + u_t\psi) = \int_Q (b_1\chi(Q_1) - b_2\chi(Q_2))\nabla u_1 \cdot \nabla\psi \tag{2.3}$$

for any function ψ from $H_0^{1,1}(Q)$. Since u and $\chi(Q_j)$ are zero outside $\overline{Q}_4 \cap \{t < T\}$, this relation remains valid for any function ψ from $H^{1,1}(Q_6)$ (where Q_6 is an arbitrary vicinity of Q_4) that is 0 when $t > T$.

If $\psi = u_2^*$ is an $H^{1,1}(Q_6)$ solution to the adjoint equation from Lemma 2.1, then integrating the left side of (2.3) by parts with respect to t and using the definition of a weak solution to this adjoint equation with the test function u (which is zero outside $Q_4 \cap \{t < T\}$), we conclude that the left side in (2.3) is zero. Thus we have relation (2.1) with u_1 instead of v_1.

Now by using the Runge property, we extend equality (2.1) onto all v_1 solving equation (1.1) with $j = 1$ near $\overline{Q}_4$ and satisfying the initial condition (1.3). Denote the space of such v_1 by X. It is sufficient to prove that solutions u_1 to the initial-boundary value problem (1.1)–(1.3) with $j = 1$ (for various g supported in $\Gamma_0 \times (0,T)$) approximate in $L_2(Q_4)$ any solution from X. We denote the space of solutions to (1.1)–(1.3) (with various g) by X_1. Indeed, let $v_1 \in X$. Then we can approximate it similarly by solutions from X in $L_2(Q_7)$, where Q_7 is a Lipschitz domain containing Q_4 with $\operatorname{dist}(\partial_x Q_7, Q_4) > 0$. From the well-known interior Schauder-type estimates for parabolic equations, it follows that these solutions from X_1 will approximate v_1 in $H^{1,0}(Q_4)$.

To prove L_2 approximation in view of the Hahn–Banach theorem, it is sufficient to show that if f from the dual space $L_2(Q_4)$ is orthogonal to X_1, then f is orthogonal to X.

Let Ω_0 be a bounded domain with C^2-boundary such that $\Omega \subset \Omega_0$, $\Omega \neq \Omega_0$, and $\partial\Omega \backslash \Gamma_0$ belong to $\partial\Omega_0$. Let $K(x,t;y,s)$ be the Green function to the first initial value problem for the operator $\partial_t + \operatorname{div}(a_1\nabla)$ in $Q_0 \times (0,T)$. Let f be orthogonal to X_1. The Green potential

$$U(x,t;f) = \int_{Q_4} fK(x,t;\ ) \tag{2.4}$$

is equal to zero on $Q_0\backslash\overline{Q}_4$ because the function $u_1 = K(x,t;\ )$ belongs to X_1 if $(x,t) \in Q_0\backslash\overline{Q}_4$. Since supp $f \subset \overline{Q}_4$, this potential is a solution to the equation $-\operatorname{div}(a_0\nabla u) = u_t$ on $Q_0\backslash\overline{Q}_4$. The coefficient a_0 belongs to $C^1(\overline{Q}_0)$, so this equation has the property of unique continuation. Therefore, $U(\ ;f) = 0$ on $Q_0\backslash\overline{Q}_4$. Now let $v \in X$; then v is a solution to the homogeneous equation near $Q_5 \cup \partial_x Q_5$, where Q_5 is an open set with C^∞ lateral boundary and $\operatorname{dist}(\partial_x Q_5, \partial_x Q_4) > 0$. Using the representation of v by a single layer potential, we obtain

$$v(y,s) = \int_{\partial_x Q_5} gK(\ ;y,s)d\Gamma$$

for some $g \in C(\partial_x Q_5)$. By using this representation, (2.4), and Fubini's theorem, we obtain

$$\int_{Q_4} fv = \int_{\partial_x Q_5} gU(\ ;f) = 0$$

because $U(\ ;f) = 0$ on $\partial_x Q_5$. Accordingly, relation (2.1) is valid for any v_1 satisfying the conditions of Lemma 2.1.

The proof is complete.

Assume that

$$Q_1 \neq Q_2. \tag{2.5}$$

Then we may assume that Q_1 is not contained in Q_2. Hence, using condition (1.5) of Theorem 1.1 on Q_j, we conclude that there is a point $(x_0,t_0) \in \partial Q_1\backslash\overline{Q}_2$ such that $(x_0,t_0) \in \partial_x Q_3$. By considering $g = 0$ for $t < t_0$ and using the translations $t \to t - t_0$ and $x \to x - x_0$, we can reduce the general case to $t_0 = 0$ and $x_0 = 0$. We can choose a ball $B \subset \mathbb{R}^n$ centered at 0 and a cylinder $Z = B \times (0,\tau)$ such that $\overline{B} \subset \Omega$, $\overline{Z}$ does not intersect $\overline{Q}_2$, and $(\partial_x Q_1) \cap \overline{Z}$ is a Lipschitz surface. Due to well-known variants of the Whitney extension theorem, there is a $C^2(\overline{Q}_1 \cup \overline{Z})$-function a_3 that coincides with a_1 on Q_1. Extend a_3 onto $Q\backslash(\overline{Q}_1 \cup \overline{Z})$ as a_0.

LEMMA 2.2. *Under the conditions of Lemma* 2.1,

$$\int_{Q_1} b_1 \nabla u_3 \cdot \nabla u_2^* = \int_{Q_2} b_2 \nabla u_3 \cdot \nabla u_2^*$$

for any solution u_3 to the equation $\operatorname{div}(a_3\nabla u_3) - (u_3)_t = 0$ *near* $\overline{Q}_4$ *which is* 0 *when* $t < 0$ *and for any solution* u_2^* *from Lemma* 2.1.

Proof. Consider u_3 and let Q_8 be an open set with C^∞-boundary $\partial_x Q_8$ and that contains Q_4 with $\operatorname{dist}(\partial_x Q_8, Q_4) > 0$ such that u_3 is a solution to the equation $\operatorname{div}(a_3\nabla u_3) - (u_3)_t = 0$ near $\overline{Q}_8$.

Introduce a sequence of open sets Q_{4k} such that (i) $Q_{4k}\backslash Z = Q_4\backslash Z$ and (ii) the (Hausdorff) distance from ∂Q_{4k} to $\partial_x Q_4$ is less than $1/k$ and $\partial_x Q_{4k} \cap Z$ does not intersect $\overline{Q}_4$. Define a coefficient a_{3k} as a_3 on $Q_8\backslash(Q_{4k}\backslash Q_4)$ and as a_0 on $Q_{4k}\backslash Q_4$. Since $\partial Q_4 \cap Z$ is a Lipschitz surface, we have

$$\operatorname{meas}_n\{a_{3k} \neq a_3\} \to 0 \quad \text{as } k \to +\infty. \tag{2.6}$$

Let u_{3k} be solutions to the initial-boundary value problems

$$\operatorname{div}(a_{3k}\nabla u_{3k}) - (u_{3k})_t = 0 \quad \text{in } Q_8, \qquad u_{3k} = u_3 \quad \text{on } \partial_x Q_8, \qquad u_{3k} = 0 \quad \text{on } Q_8 \cap \{t = 0\}.$$

Since $u_{3k} = a_0 + \chi(Q_1)b_1$ near $\overline{Q}_1$, relation (2.1) is valid for any $u_1 = u_{3k}$. The difference $u_k = u_{3k} - u_3$ satisfies the equation

$$\operatorname{div}(a_{3k}\nabla u_k) - (u_k)_t = \operatorname{div}((a_3 - a_{3k})\nabla u_3) \quad \text{in } Q_8,$$

and $u_k = 0$ on $\partial Q_8 \cap \{t < T\}$ because u_{3k} and u_3 coincide on the lateral boundary of Q_8 and when $t = 0$. From the definition of a weak solution to this initial-boundary value problem with the test function u_k, we have

$$\int_{Q_8} a_{3k}\nabla u_k \cdot \nabla u_k + \int_{Q_8 \cap \{t=T\}} \frac{u_k^2}{2} = \int_{Q_8} (a_3 - a_{3k})\nabla u_3 \cdot \nabla u_k.$$

According to the assumptions, $\epsilon < a_{3k}$ for certain positive ϵ. Using this inequality, dropping the second integral in the left side, and bounding the right side by the inequality $x \cdot y \leq \epsilon^{-1}/2|x|^2 + \epsilon/2|y|^2$, we obtain

$$\int_{Q_8} \epsilon|\nabla u_k|^2 \leq C(\epsilon)\int_{Q_8} |a_3 - a_{3k}|^2|\nabla u_3|^2 + \frac{\epsilon}{2}\int_{Q_8} |\nabla u_k|^2.$$

Since ∇u_3 belongs to $L_2(Q_8)$, we conclude from (2.6) that the first integral in the right side tends to 0. Therefore, ∇u_k converges to 0 in $L_2(Q_8)$. Putting $u_1 = u_{3k} = u_3 + u_k$ into relation (2.1) and letting $k \to \infty$, we complete the proof of Lemma 2.2.

3. Estimates of integrals of singular solutions. We will make use of solutions u_3 and u_2^* with singularities outside Q_4. Solutions of elliptic equations of second order with arbitrary power singularities were constructed by Alessandrini [1]; we do not know of similar results for parabolic equations. To simplify obtaining bounds on the integrals of such solutions, we introduce new variables. We can assume that the direction e_n of the x_n-axis coincides with the interior unit normal to $\partial_x Q_1 \cap \{t = 0\}$. According to our assumptions, $\partial_x Q_1$ near the origin is the graph of a Lipschitz function $x_n = q_1(x_1, \ldots, x_{n-1}, t)$ which can be assumed to be defined and Lipschitz on the whole $\mathbb{R}^n$. The substitution

$$x_k = x_k^*, \quad k = 1, \ldots, n-1, \qquad x_n = x_n^* + q_1(x_1^*, \ldots, x_{n-1}^*, t), \quad t = t^*$$

transforms the equations (1.1) into similar equations with additional first-order differentiation with respect to x_n^* multiplied by a Lipschitz function of t. The domains Q_j are transformed onto domains with similar properties and with the additional property that the points $(0, t), 0 < t < T$, belong to $\partial_x Q_1$. Since the (hyper)plane $\{x_n^* = 0\}$ is tangent to this surface at the origin, we can find a cone $\mathcal{C} = \{|x^*/|x^*| - e_n| < \theta, |x^*| < \epsilon\}$ such that the cylinder $\mathcal{C} \times (0, T)$ is inside Q_1. Henceforth, we drop the sign $*$.

Let K^+ be the fundamental solution of the Cauchy problem for the forward parabolic equation $\operatorname{div}(a_3\nabla u_3) - (u_3)_t = 0$ in $*$-coordinates. Let K^- be the fundamental solution of the backward Cauchy problem for the backward parabolic equation $\operatorname{div}(a_2\nabla u_2) + u_{2t} = 0$ in these coordinates. It is known that

$$K^+ = K_1^+ + K_0^+, \qquad K^- = K_1^- + K_0^-, \tag{3.1}$$

where K_1^+ and K_1^- are the principal parts of K^+ and K^- (parametrices) and K_0^+

and K_0^- are the remainders. The principal parts are

$$K_1^+(x,t;y,\tau) = \frac{C}{(a_3(y)(t-\tau))^{n/2}} \exp\left(-\frac{|x-y|^2}{4a_3(y)(t-\tau)}\right),$$
$$K_1^-(x,t;y,\tau) = \frac{C}{(a_0(y)(\tau-t))^{n/2}} \exp\left(-\frac{|x-y|^2}{4a_0(y)(\tau-t)}\right). \tag{3.2}$$

From the known bounds of fundamental solutions of parabolic equations [9, p. 377], we have

$$|\nabla_x K_0^+(x,t;y,\tau)| \le C(t-\tau)^{-n/2} \exp\left(-\frac{|x-y|^2}{(C(t-\tau))}\right),$$
$$|\nabla_x K_0^-(x,t;y,\tau)| \le C(\tau-t)^{-n/2} \exp\left(-\frac{|x-y|^2}{(C(\tau-t))}\right). \tag{3.3}$$

When $(y,0)$ and (y,τ) are outside $\overline{Q}_1$, the functions $K^+(\ ;y,0)$ and $K^-(\ ;y,\tau)$ are (x,t)-solutions to the homogeneous parabolic equations with bounded measurable coefficients satisfying zero initial and final conditions. Using Lemma 2.2 with $u_3 = K^+(\ ;y,0)$ and $u_2^* = K^-(\ ;y,\tau)$, we get

$$\begin{aligned}\int_{Q_1\cap Z} & b_1 \nabla_x K^+(\ ;y,0)\cdot\nabla_x K^-(\ ;y,\tau)\\ &= -\int_{Q_1\setminus Z} b_1 \nabla_x K^+(\ ;y,0)\cdot\nabla_x K^-(\ ;y,\tau)\\ &\quad + \int_{Q_2} b_2 \nabla_x K^+(\ ;y,0)\cdot\nabla_x K^-(\ ;y,\tau).\end{aligned} \tag{3.4}$$

From the estimates in (3.3) and similar estimates for $\nabla_x K_1^+$ and $\nabla_x K_1^-$, we conclude that the integrands are bounded by an integrable function uniformly with respect to y outside Q_1. By the Lebesgue dominated-convergence theorem, we may let $y \to 0$ and replace y in (3.4) by 0. Using representation (3.1), we obtain from (3.4) that

$$|I_1| \le |I_2| + |I_3|, \tag{3.5}$$

where

$$I_1 = \int_{Q_1\cap Z} b_1 \nabla_x K_1^+(\ ;0,0)\cdot\nabla_x K_1^-(\ ;0,\tau)$$

is formed from the principal parts of K and the remainders are collected in

$$I_2 = -\int_{Q_1\setminus Z} b_1 \nabla_x K^+(\ ;0,0)\cdot\nabla K^-(\ ;0,\tau) + \int_{Q_2} b_2 \nabla_x K^+(\ ;0,0)\cdot\nabla_x K^-(\ ;0,\tau)$$

and

$$\begin{aligned} I_3 = \int_{Q_1\cap Z} b_1 (\nabla_x K_1^+(\ ;0,0)&\cdot\nabla K_0^-(\ ;0,\tau) + \nabla_x K_0^+(\ ;0,0)\cdot\nabla_x K_1^-(\ ;0,\tau)\\ &+ \nabla_x K_0^+(\ ;0,0)\cdot\nabla K_0^-(\ ;y,\tau)).\end{aligned}$$

In the following three lemmas, I_1 is bounded from below and I_2 and I_3 is bounded from above.

LEMMA 3.1.

$$|I_1| \geq C^{-1}\tau^{-n}\int_0^\epsilon \rho^{n-1}e^{-4p^2/(m\tau)}d\rho,$$

where $m = \inf(a_3, a_0)$ *over* Q.

Proof. Using the fact that $b_1(0) \neq 0$ and choosing ϵ in the definition of $\mathcal{C}$ to be sufficiently small, we obtain

$$\begin{aligned}
|I_1| \geq\ & C^{-1}\int_{\mathcal{C}\times(0,\tau)} \nabla_x K_1^+(x,t;0,0)\cdot\nabla_x K_1^-(x,t;0,\tau)\\
&= C^{-1}\int_0^\tau\int_{\mathcal{C}} t^{-n/2-1}\exp\left(-\frac{|x|^2}{a_3(x)t}\right)x\cdot(\tau-t)^{-n/2-1}\exp\left(\frac{|x|^2}{a_0(x)(\tau-t)}\right)x\,dx\,dt\\
&\geq\ C^{-1}\int_{\mathcal{C}}\int_0^{\tau/2}|x|^2((\tau-t)t)^{-n/2-1}\exp\left(-\frac{|x|^2\tau}{mt(\tau-t)}\right)dt\,dx.
\end{aligned}$$

Using the inequality

$$\frac{1}{t\tau}\leq\frac{1}{t(\tau-t)}\leq\frac{2}{t\tau}\quad\text{when } 0<t<\frac{\tau}{2}, \tag{3.6}$$

we bound from below the integral shown above by

$$\begin{aligned}
&C^{-1}\int_{\mathcal{C}}\int_0^{\tau/2}|x|^2\frac{1}{(t\tau)^{n/2+1}}\exp\left(-\frac{2|x|^2}{mt}\right)dt\,dx\\
&\qquad=\frac{1}{C\tau^{n/2+1}}\int_{\mathcal{C}}|x|^{2-n}\int_{\frac{4|x|^2}{m\tau}}^\infty w^{n/2-1}e^{-w}dw\,dx,
\end{aligned}$$

where we substituted $w = 2|x|^2/mt$.

The function $w^{n/2-1}$ is increasing, so replacing it by its minimal value at $w = 4|x|^2/(m\tau)$, we bound the last integral from below by

$$\int_{\mathcal{C}}\tau^{1-n/2}\left(\int_{(4|x|^2/(m\tau),\infty)}e^{-w}dw\right)dx = C^{-1}\tau^{1-n/2}\int_{(0,\epsilon)}\rho^{n-1}e^{-4\rho^2/(m\tau)}d\rho.$$

The proof is complete.

LEMMA 3.2.

$$|I_2| \leq C\tau^{-n/2+1}\epsilon^{-2}e^{-\epsilon^2/(M\tau)},$$

where M *depends only on* $\sup(a_3, a_0)$ *over* Q.

Proof. I_2 consists of two integrals. The first one is bounded by

$$\begin{aligned}
&C\int_{\epsilon<|x|<R,0<t<\tau}|\nabla_x K^+(\ ;0,0)\cdot\nabla_x K^-(\ ;0,\tau)|\\
&\quad\leq\ C\int_{\epsilon<|x|<R}\int_0^{\tau/2}\frac{1}{((\tau-t)t)^{n/2+1/2}}\exp\left(-\frac{|x|^2\tau}{Mt(\tau-t)}\right)dt\,dx.
\end{aligned}$$

The bound on $|\nabla_x K^+ \cdot \nabla_x K^-|$ follows from the direct differentiation of (3.2), the inequality

$$|x|(t-\tau)^{-n/2-1}\exp\left(-\frac{|x|^2}{4(t-\tau)}\right) \le C(t-\tau)^{-n/2-1/2}\exp\left(-\frac{|x|^2}{8(t-\tau)}\right),$$

and the bounds in (3.3).

Applying inequality (3.6) as above, we bound the last integral by

$$\frac{C}{\tau^{n/2+1/2}}\int_{\epsilon<|x|<R}\int_0^{\tau/2}\frac{1}{t^{n/2+1/2}}\exp\left(-\frac{|x|^2}{Mt}\right)dt\,dx$$
$$\le \frac{C}{\tau^{n/2+1/2}}\int_{\epsilon<|x|<R}|x|^{1-n}\int_{\frac{2|x|^2}{M\tau}}^{\infty} w^{n/2-1}w^{-1/2}e^{-w}dw\,dx$$

when we use the substitution $w=|x|^2/(Mt)$. The function $w^{-1/2}$ is decreasing. Replacing it by its value at $2|x|^2/(M\tau)$, we increase the integral, and we also use the inequality $w^{n/2-1}e^{-w}\le Ce^{-w/2}$ and calculate the resulting integral with respect to w. Then the last integral will be less than

$$C\tau^{-n/2}\int_{\epsilon<|x|<R}|x|^{-n}\exp(-|x|^2/(M\tau))dx$$
$$\le C\tau^{-n/2}\int_{(\epsilon,\infty)}\rho^{-2}\rho\exp(-\rho^2/(M\tau))d\rho$$

when we use the polar coordinates in $\mathbb{R}^n$. Replacing ρ^{-2} by its maximal value at ϵ and calculating the remaining integral with respect to ρ, we complete the bounding of the integral over $Q_1\backslash Z$.

A similar argument works for the integral over Q_2.

The proof is complete.

LEMMA 3.3.

$$|I_3|\le C\epsilon\tau^{-n/2},$$

where M depends only on the upper bounds of $|a_3|$, $|a_0|$.

Proof. We bound the integral of the first of the three functions, forming I_3 as defined after (3.5).

As follows from (3.2), (3.3), and the argument in Lemma 3.2, replacing $|x|$ by some power of $(t-\tau)$, the absolute value of this integral is less than

$$C\int_{|x|<\epsilon}\int_{(0,\tau/2)}((\tau-t)t)^{-n/2}t^{-1/2}\exp(-|x|^2\tau/(Mt(\tau-t))dt\,dx$$
$$\le C\int_{|x|<\epsilon}\int_{(0,\tau/2)}(\tau t)^{-n/2}t^{-1/2}\exp(-|x|^2/(Mt))dt\,dx,$$

where we used inequality (3.6). Substituting $w=|x|^2/(Mt)$ in the inner integral yields

$$C\tau^{-n/2}\int_{|x|<\epsilon}|x|^{1-n}\int_{(2|x|^2/(M\tau),\infty)}w^{n/2-3/2}e^{-w}dw\,dx\le C\tau^{-n/2}\int_{|x|<\epsilon}|x|^{1-n}dx.$$

Using the polar coordinates, we bound the last integral by $C\epsilon$.

The other terms can be bounded in a similar way. The proof is complete.

4. Proof of Theorem 1.1. Now we will complete the proof of Theorem 1.1. Let

$$\epsilon^2 = E\tau, \tag{4.1}$$

where (large) E will be chosen later.

First, we bound I_1 from below. From Lemma 3.1, substituting $w = 4\rho^2/(m\tau)$ in the integral and using condition (4.1), we obtain

$$|I_1| \geq C^{-1}\tau^{-n/2} \int_{(0,4E/m)} w^{n/2-1}e^{-w}dw \geq C^{-1}\tau^{-n/2} \tag{4.2}$$

provided $E > m$.

From (3.5), Lemmas 3.1–3.3, (4.1), and (4.2), it follows that

$$C^{-1}\tau^{-n/2} \leq C(\tau^{-n/2+1}\epsilon^{-2}\exp(-E/M) + \tau^{-n/2}\epsilon).$$

Using (4.1) again and multiplying both sides by $C\tau^{n/2}$, we obtain

$$1 \leq CE^{-1}\exp(-E/M) + C\epsilon \leq CE^{-1} + C\epsilon.$$

Let $\tau < 1$. Choose E so large that $E^{-1} < 1/(4C)$ and $\epsilon < 1/(4C)$; then the right side is smaller than $1/2$. We have a contradiction.

This contradiction shows that $Q_1 = Q_2$.

The next step of the proof is to show that

$$b_1 = b_2 \quad \text{on } \partial_x Q_1. \tag{4.3}$$

As in the proof for Q_j, we assume the opposite. Then we can assume that the origin $0 \in \partial_x Q_1$ and $b_1(0) < b_2(0)$. By continuity, $b_1(0) - b_2(0) > C^{-1}$ for some C on a certain ball B centered at the origin. Let $Z = B \times (0,T)$. Extend a_2 from Q_2 onto $\mathbb{R}^n$ as a C^2-function $a_4 > 0$. By repeating the proof of Lemma 3.2, we obtain the following orthogonality relation:

$$\int_{Q_1} (b_1 - b_2)\nabla u_3 \cdot \nabla u_4^* = 0 \tag{4.4}$$

for all solutions u_3 to the equation $\operatorname{div}(a_3\nabla u_3) - u_{3t} = 0$ near Q_4 which are zero when $t < 0$ and for all solutions u_4^* to the adjoint equation $\operatorname{div}(a_4\nabla u_4) + u_{4t} = 0$ near Q_4 which are zero when $t > T$. Let K^+ be a fundamental solution to the forward Cauchy problem for the first equation and K^- be the fundamental solution to the backward Cauchy problem for the adjoint equation with the coefficient a_4. Using the representation (3.1) of these fundamental solutions and splitting Q_1 into $Q_1 \cap Z$ and its complement, as in section 3, we obtain from (4.4) the inequality

$$|I_4| \leq |I_5| + |I_6|, \tag{4.5}$$

where

$$I_4 = \int_{Q_1 \cap Z} (b_1 - b_2)\nabla_x K_1^+ \cdot \nabla_x K_1^-$$

is related to the supposedly singular part and

$$I_5 = \int_{Q_1\setminus Z} (b_1 - b_2)\nabla K^+ \cdot \nabla K^-,$$

$$I_6 = \int_{Q_1} (b_1 - b_2)\nabla K_0^+ \cdot \nabla K_0^{-2}.$$

It is easy to see that Lemmas 3.1, 3.2, and 3.3 are valid for I_4, I_5, and I_6, respectively. Therefore, as in the proof above, we arrive at the contradiction that $Q_1 = Q_2$.

This shows that the assumption about b_1 and b_2 is wrong and that $b_1 = b_2$ on $\partial_x Q_1$.

Let Ω_0 be the intersection of all $Q_{1\theta}$ over $0 < \theta < T$. Since b_1 and b_2 do not depend on t and are equal on $\partial_x Q_1$, they coincide on $Q_{1\theta}\setminus\Omega_0$. Letting $Q_0 = \Omega\times(0,T)$, we obtain from (4.4) the relation

$$\int_{Q_0} (b_1 - b_2)\nabla u_3 \cdot \nabla u_4^* = 0$$

for all u_3 and u_4^* in (4.4). As in the proof of Lemma 3.2, this implies that

$$\int_{Q_0} (b_1 - b_2)\nabla u_6 \cdot \nabla u_6^* = 0 \tag{4.6}$$

for solutions u_5 to the equation $\operatorname{div}((a_0 + b_1\chi(Q_0))\nabla u_5) - u_{5t} = 0$ near Q_0 which are zero when $t < 0$ and for solutions to the adjoint equation $\operatorname{div}((a_0 + b_2\chi(Q_0))\nabla u_6^*) - u_{6t}^* = 0$ near Q_0 which are zero when $t < T$.

Observe that by choosing T small, we can guarantee that Ω_0 is a Lipschitz domain. Indeed, for any point of $\partial_x Q_1 \cap \{t = 0\}$, there is a neighborhood where Q_1 is the subgraph of the Lipschitz function $x_j < q_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n, t)$. We can cover the compact set $\partial_x Q_1 \cap \{t = 0\}$ by a finite number of such neighborhoods. Then there is T_1 such that $\partial_x Q_1 \cap \{t < T_1\}$ is contained in the union of these neighborhoods. Let $T = T_1$; then Ω_0 is Lipschitz because locally (in the corresponding neighborhood) its boundary is given by the equation $x_j = \inf q_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n, t)$ over $t \in (0,T)$, and the inf of a family of uniformly Lipschitz functions is a Lipschitz function.

Now we will show that the equations for u_5 and u_6 have the same lateral Dirichlet-to-Neumann maps. Let u_5 and u_6 be a solution to these equations with zero initial conditions and the same lateral Dirichlet data. By subtracting these equations and letting $u = u_0 - u_5$, we obtain

$$\operatorname{div}((a_0 + b_2\chi(Q_0))\nabla u) = \operatorname{div}((b_1 - b_2)\chi(Q_0)\nabla u_5) \quad \text{in } Q.$$

From the definition of a weak solution of this equation, we have

$$\int_{\partial\Omega\times(0,T)} a_0 u_\nu \psi - \int_Q ((a_0 + b_2\chi(Q_0))\nabla u \cdot \nabla\psi - \int_Q u_t\psi = -\int_{Q_0} (b_1 - b_2)\nabla u_5 \cdot \nabla\psi$$

for any function $\psi \in H^{1,1}(Q)$. Using $\psi = u_6^*$, integrating by parts in the third integral of the left side, and again using the definition of a weak solution to the equation $\operatorname{div}((a_0 + b_2\chi(Q_0))\nabla u_6^*) + u_{6t}^* = 0$ with the test function u which is zero on

$\partial Q \cap \{t < T\}$, we conclude that the sum of the second and third integrals in the left side is zero. The right side is zero due to (4.6). Thus the first integral in the left side is zero. Since the lateral Dirichlet data $\psi = u_6^*$ can be any function in $C_0^\infty(\partial\Omega \times (0,T))$, we get $u_\nu = 0$ on $\partial\Omega \times (0,T)$. Therefore, $u_{5\nu} = u_{6\nu}$ on the lateral boundary, which means that we have the same lateral Dirichlet-to-Neumann maps.

Take as the Dirichlet data g a function which does not depend on t when $t > \tau$. Since the coefficients of the equations $\operatorname{div}((a_0 + b_j\chi(Q_0)\nabla u_j) - u_{jt} = 0$ are time independent, the solution $u_j(x,t)$ of the initial-boundary value problems on $\Omega \times (0,\infty)$ will be analytic with respect to $t > \tau$. They have the same Cauchy data on $\partial\Omega \times (0,T)$; therefore, as above, by uniqueness in the lateral Cauchy problem, $u_5 = u_6$ on $(\Omega\backslash\Omega_0) \times (0,T)$. By uniqueness of the analytic continuation, they are equal also on $(\Omega\backslash\Omega_0) \times (0,\infty)$. Now we modify the argument of [6] and consider the Laplace transforms

$$U_j(x,s) = \int_{(0,\infty)} e^{-st} u_j(x,t)\, dt.$$

They solve the following Dirichlet problems:

$$\text{(4.7)} \qquad \operatorname{div}((a_0 + b_j\chi(\Omega_0))\nabla U_j) - sU_j = 0 \quad \text{in } \Omega, \qquad U_j = G \quad \text{on } \partial\Omega,$$

and $U_5 = U_6$ on $\Omega\backslash\Omega_0$. Letting $\tau \to 0$ we obtain $G(x,s) = g_0(x)s^{-1}$, where $g_0(x) = g(x,t)$ when $t > \tau$. Applying the results of [5] and [11] on identification of elliptic equations, we conclude that $b_1 = b_2$ on Ω_0. In fact, this result is obtained in [5] when $n \geq 3$, but the recent global uniqueness theorem of Nachman [10] extends it to $n = 2$.

The proof is complete.

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