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Uniqueness for a one-dimensional inverse parabolic problem

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Abstract. We prove the uniqueness of a discontinuous principal coefficient of a one-dimensional second-order parabolic equation of the form $a_0 + \chi(Q*)b$ with known smooth a_0 and unknown $b = b(x)$ from all possible lateral boundary measurements of solutions of this equation. In the proofs we make use of singular solutions of parabolic equations.

0. Introduction

We consider the problem of recovery of the coefficient a of the parabolic equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) = 0 \qquad \text{in} \quad Q = (0,1)\times(0,T) \tag{0.1}$$

with the initial and boundary conditions

$$u = 0 \qquad \text{on} \quad \Omega\times 0, \qquad \text{where} \quad \Omega = (0,1) \tag{0.2}$$

$$u = g \qquad \text{on} \quad S = 0, 1\times(0,T) \tag{0.3}$$

when $h = \partial u/\partial v$ is given for all (regular) g on $\bar{S}$ which satisfy the following compatibility conditions: $g(0) = g(1) = 0$. In this paper we prove the uniqueness of the recovery of discontinuous $a = a_0 + \chi(Q^*)b$, where $\chi(Q^*)$ is the indicator function of an open set $Q^* \subset Q$ with the Lipschitz lateral boundary $\partial_x Q^*$ changing with time, $a = a_0(x)$ and $b = b(x)$ are, respectively, given and unknown $C^2[0,1]$-functions. In addition, we prove uniqueness in the case of a piece-wise constant a when the Dirichlet-to-Neumann map is only given on one part of the boundary. Uniqueness for multidimensional parabolic equations ($n \geqslant 2$) has been proven in [3] where the connectedness of $Q\backslash Q^*$ is assumed for every fixed t, which we cannot assume for the one-dimensional problem. For elliptic equations nonuniqueness for the one-dimensional case has been shown in the paper of Isakov [6], but for higher dimensions ($n \geqslant 2$), uniqueness has been proven in the papers of Kohn and Vogelius [9] (piece-wise analytic a) and by Isakov [6] (Lipschitz Q^* and smooth b). Also, for elliptic equations, when one is making use of only one set of u, $\partial u/\partial v$ on $\partial\Omega$, some partial global uniqueness results for Q^* have been obtained by Friedman and Isakov [5]. In connection with parabolic equations we can also refer to the paper of Bellout [2] studying local stability in the inverse problem. This inverse parabolic problem is fundamental, in particular, in the search for ground water [11], and is important for many engineering applications.

We introduce some notation. For standard notation we refer to Friedman [4] and Ladyzenskaja *et al* [10]. For an open set Q in the layer $\mathbb{R}\times(0,T)$, the lateral boundary $\partial_x Q$, is the x-boundary, that is the closure of the set $\partial Q\backslash\{t = 0 \text{ or } T\}$. We say that Q is x-Lipschitz if its x-boundary is locally the graph of a function $x = \gamma(t)$ that is Lipschitz,

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and a is piece-wise Lipschitz constant on Q if there exists a cover $\{Q_j\}_{j=1}^n$ of Q such that $a = \sum \chi(Q_j)a_j$, Q_j are open x-Lipschitz sets, $Q_j \subset \Omega \times (-T, 2T)$, $a_j \neq a_{j+1}$ on ∂Q_j, a_j are constants, $j = 1, \ldots, n$.

1. Statement of results

We are interested in finding an open set Q_j and a function b_j entering the parabolic initial-boundary value problem

$$\frac{\partial u_j}{\partial t} - \frac{\partial}{\partial x}\left(a_j \frac{\partial}{\partial x} u_j\right) = 0 \qquad \text{in} \quad Q \tag{1.1}$$

$$u_j = g \qquad \text{on} \quad S \tag{1.2}$$

$$u_j = 0 \qquad \text{on} \quad \Omega \times 0 \tag{1.3}$$

where

$$a_j = a_0 + \chi(Q_j)b_j, \qquad b_j \neq 0 \qquad \text{on} \quad \partial Q_j. \tag{1.4}$$

We are also interested in finding an open set Q_j and a function a_j entering the parabolic initial-boundary value problem

$$\frac{\partial u_j}{\partial t} - \frac{\partial}{\partial x}\left(a_j \frac{\partial}{\partial x} u_j\right) = 0 \qquad \text{in} \quad Q \tag{1.1$'$}$$

$$u_j = g \qquad \text{on} \quad S' = 1 \times (0, T)$$

$$u_j = 0 \qquad \text{on} \quad S'' = 0 \times (0, T) \tag{1.2$'$}$$

$$u_j = 0 \qquad \text{on} \quad (0, 1) \times 0 \tag{1.3$'$}$$

where a is piece-wise constant.

It is well known that there is a unique (generalized) solution, u_j, of these problems when $g \in C^{2+\ell}(S)$, and $u_j \in C^\ell(Q)$ for some $\ell \in (0, 1)$, $\frac{\partial}{\partial x} u_j \in L_2(Q)$, and $\in C(Q \backslash Q_j)$. For this and for other results about the direct parabolic problem, (1.1)–(1.4), we refer the reader to the books of Friedman [4] and Ladyzenskaja *et al* [10, pp 153, 204, 227].

Theorem 1. *Suppose that* $a_j = a_0 + \chi(Q_j)b_j$, $j = 1, 2$, $b_j \neq 0$ *on* ∂Q_j, $a_0 = a_0(x)$ *and* $b = b(x)$ *are, respectively, given and unknown* $C^2[0, 1]$*-functions, and* Q_1, Q_2 *are open* x*-Lipschitz sets,* $Q_j \subset \Omega \times (-T, 2T)$.

If solutions u_j *to the initial-boundary value problems (1.1)–(1.3) satisfy the equality*

$$\partial u_1/\partial v = \partial u_2/\partial v \qquad \text{on} \quad S \ (v \text{ is a normal}) \tag{1.5}$$

for all $g \in C^2(\bar{S})$, *then*

$$a_1 = a_2 \qquad \text{on} \quad Q. \tag{1.6}$$

This result guarantees the uniqueness of the reconstruction of Q_j from all possible lateral measurements for an arbitrary $T > 0$.

Theorem 2. *Suppose that* a_1, a_2 *are two piece-wise Lipschitz constants, and suppose that solutions* u_j *to the initial-boundary value problems* (1.1′)–(1.3′) *satisfy the equality*

$$\partial u_1/\partial v = \partial u_2/\partial v \qquad \text{on} \quad S' \tag{1.6$'$}$$

for all $g \in C^2(\bar{S}')$, *then*

$$a_1 = a_2 \qquad \text{on} \quad Q. \tag{1.7$'$}$$

The paper is organized as follows: in section 2 we show that the validity of equality (1.6) for all Dirichlet boundary data g implies certain integral relations which can be interpreted as orthogonality relations. To prove uniqueness, in sections 4 and 5 we modify an approach from [6] (use of singular solutions with the pole in those orthogonality relations to obtain a contradiction when the pole converges to the boundary of one of the domains Q_j). To show that some integrals in these relations are bounded while one of them is not we use the estimates of integrals of singular solutions given in section 3. In the last section we prove theorem 2.

2. Orthogonality relations

In this section we assume that the conditions of theorem 1 are satisfied and obtain some auxiliary relations which will be used in its proof.

Denote by Q_{3t} the component of the open set $\Omega\backslash(\bar{Q}_{1t} \cup \bar{Q}_{2t})$ whose boundary meets S. Here $Q_{j\theta}$ is $Q_j \cap \{t = \theta\}$, $j = 1, 2$. Let $Q_3 = \cup Q_{3t}$ over $0 < t < T$ and let $Q_4 = Q\backslash\bar{Q}_3$.

Lemma 2.1.

$$\int_{Q_1} b_1 \frac{\partial v_1}{\partial x}\frac{\partial u_2^*}{\partial x}\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_1} b_2 \frac{\partial v_1}{\partial x}\frac{\partial u_2^*}{\partial x}\,\mathrm{d}x\,\mathrm{d}t \tag{2.1}$$

for all solutions v_1 to equation (1.1), with $j = 1$ near $\bar{Q}_4$, that are zero when $t < 0$, and solutions u_2^* to the adjoint equation

$$\frac{\partial u_2^*}{\partial t} + \frac{\partial}{\partial x}\left(a_2 \frac{\partial u_2^*}{\partial t}\right) = 0 \qquad \textit{near } \bar{Q}_4, \textit{ that are zero when } t > T.$$

Proof. From well known results about the regularity of solutions to the parabolic initial-boundary value problem (1.1)–(1.3) it follows that u_j is in $C^{2,1}(Q_3)$ and in $H^{2,1}(Q_5)$, where $Q_5 = V \times (0, T)$ and V is a vicinity of $\partial\Omega$ in Ω. Due to conditions (1.2), (1.5), both u_1 and u_2 have the same Cauchy data on S and satisfy the same parabolic equation in Q_3. Thus, from the uniqueness of continuation for second-order parabolic equations (see e.g. [7], corollary 1.2.4) we conclude that $u_1 = u_2$ on Q_3. Letting $u = u_2 - u_1$ and subtracting equations (1.1) with $j = 1$ from those with $j = 2$ we get

$$\frac{\partial}{\partial x}\left((a_0 + b_2\chi(Q_2))\frac{\partial u}{\partial x}\right) - \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left((b_1\chi(Q_1) - b_2\chi(Q_2))\frac{\partial u_1}{\partial x}\right) \qquad \text{in} \quad Q. \tag{2.2}$$

Now, using the definition of a weak solution to the parabolic equation under consideration we obtain

$$\int_Q \left((a_0 + b_2\chi(Q_2))\frac{\partial u}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial u}{\partial t}\psi\right) = \int_Q \left((b_1\chi(Q_1) - b_2\chi(Q_2))\frac{\partial u_1}{\partial x}\frac{\partial \psi}{\partial x}\right) \tag{2.3}$$

for any function ψ from $H_0^{1,1}(Q)$. Since u and $\chi(Q_j)$ are zero outside $\bar{Q}_4 \cap \{t < T\}$ this relation remains valid for any function ψ from $H^{1,1}(Q_6)$ (where Q_6 is an arbitrary vicinity of Q_4) that is zero when $t > T$.

If $\psi = u_2^*$ is an $H^{1,1}(Q_6)$ solution to the adjoint equation from lemma 2.1, then integrating the left side of (2.3) by parts with respect to t and using the definition of a weak solution to this adjoint equation with the test function u which is zero outside $\bar{Q}_4 \cap \{t < T\}$, we conclude that the left-hand side in (2.3) is zero. So we have the relation (2.1) with u_1 instead of v_1.

Now, by using the Runge property we extend equality (2.1) onto all v_1 solving equation (1.1) with $j = 1$ near $\bar{Q}_4$, and satisfying the initial condition (1.3). Denote the space of such v_1 by X. It is sufficient to prove that solutions u_1 to the initial-boundary value problem (1.1)–(1.3) with $j = 1$ (for various g supported in S) approximate in $L_2(Q_7)$ any solution from X. We denote the space of solutions to (1.1)–(1.3) by X_1. Indeed, let $v_1 \in X$. Then we can similarly approximate it by solutions from X in $L_2(Q_7)$, where Q_7 is a Lipschitz domain containing Q_4 with $\operatorname{dist}(\partial_x Q_7, Q_4) > 0$. From well known interior Schauder-type estimates for parabolic equations it follows that these solutions from X_1 will approximate v_1 in $H^{1,0}(Q_4)$.

To prove L_2-approximation, in view of the Hahn–Banach theorem, it is sufficient to show that if f from the dual space $L_2(Q_4)$ is orthogonal to X_1, then f is orthogonal to X.

Let $\Omega = (0, 1)$, and Ω_0 be a bounded interval such that $\Omega \subset \Omega_0, \Omega \neq \Omega_0$. Let $K(x, t; y, s)$ be the Green function to the first initial value problem for the operator $\frac{\partial}{\partial t} + \frac{\partial}{\partial x}(a\frac{\partial}{\partial x})$ in $Q_0 = \Omega_0 \times (0, T)$. Let f be orthogonal to X_1. The Green potential

$$U(x, t; f) = \int_{Q_4} f K(x, t; \,) \tag{2.4}$$

is equal to zero on $Q_0\backslash\bar{Q}_4$ because the function $u_1 = K(x, t; \,)$ belongs to X_1 if $(x, t) \in Q_0\backslash\bar{Q}_4$. Since supp $f \subset \bar{Q}_4$, this potential is a solution to the equation

$$-\frac{\partial}{\partial t}(a_0 \nabla u) = \frac{\partial u}{\partial t} \qquad \text{on} \quad Q_0\backslash\bar{Q}_4.$$

The coefficient a_0 belongs to $C^1(\bar{Q}_0)$, so this equation has the property of the unique continuation. Therefore, $U(; f) = 0$ on $Q_0\backslash\bar{Q}_4$. Now, let $v \in X_1$, then v is a solution to the homogeneous equation near $Q_5 \cup \partial_x Q_5$, where Q_5 is an open set with a C^∞ lateral boundary and $\operatorname{dist}(\partial_x Q_5, \partial_x Q_4) > 0$. Using the representation of v by a single-layer potential, we obtain

$$v(y, s) = \int_{\partial_x Q_5} g K(; y, s)\, \mathrm{d}\Gamma$$

for some $g \in C(\partial_x Q_5)$. By using this representation (2.4) and the Fubini theorem we yield

$$\int_{Q_4} f v = \int_{\partial_x Q_5} g U(; f) = 0$$

because $U(; f) = 0$ on $\partial_x Q_5$. Accordingly, relation (2.1) is valid for any v_1 satisfying the conditions of lemma 2.1. Hence the proof is complete. □

Assuming that

$$Q_1 \neq Q_2 \tag{2.5}$$

then we may assume that Q_1 is not contained in Q_2; hence, we conclude that there is a point $(x_0, t_0) \in \partial Q_1\backslash\bar{Q}_2$ such that $(x_0, t_0) \in \partial_x Q_3$. By considering $g = 0$ for $t < t_0$ and using the translations $t \to t - t_0$, $x \to x - x_0$, we can reduce the general case to $t_0 = 0$, $x_0 = 0$. We can choose a ball $B = (0, \varepsilon)$ and a cylinder $Z = B(0, \tau)$ such that $\bar{B} \subset \Omega$, $\bar{Z}$ does not intersect $\bar{Q}$, and $(\partial_x Q_1) \cap \bar{Z}$ is a Lipschitz curve. Due to the well known variants of the Whitney extension theorem there is a $C^2(\bar{Q}_1 \cup \bar{Z})$-function a_3 that coincides with a_1 on Q_1. Extend a_3 onto $Q\backslash(\bar{Q}_1 \cup \bar{Z})$ as a_0.

Lemma 2.2. *Under the conditions of lemma 2.1,*

$$\int_{Q_1} b_1 \frac{\partial u_3}{\partial x}\frac{\partial u_2^*}{\partial x} = \int_{Q_2} b_2 \frac{\partial u_3}{\partial x}\frac{\partial u_2^*}{\partial x}$$

for any solution u_3 to the equation $\frac{\partial}{\partial x}(a_3\frac{\partial u_3}{\partial x}) - \frac{\partial}{\partial t}(u_3) = 0$ near $\bar{Q}_4$ which is zero when $t < 0$ and for any solution u_2^* from lemma 2.1.

Proof. Consider u_3 and let Q_8 be an open set with a C^∞-boundary $\partial_x Q_8$ and containing Q_4 with $\mathrm{dist}(\partial_x Q_8, Q_4) > 0$ such that u_3 is a solution to the equation

$$\frac{\partial}{\partial x}\left(a_3\frac{\partial u_3}{\partial x}\right) - \frac{\partial}{\partial t}(u_3) = 0 \qquad \text{near } \bar{Q}_8.$$

Introduce a sequence of open sets Q_{4k} such that (i) $Q_{4k}\backslash Z = Q_4\backslash Z$, (ii) the (Hausdorff) distance from $\partial Q_{4k} \cap Z$ is less than $1/k$ and $\partial_x Q_{4k} \cap Z$ does not intersect $\bar{Q}_4$. Define a coefficient a_{3k} as a_3 on $Q_8\backslash(Q_{4k}\backslash Q_4)$ and as a_0 on $Q_{4k}\backslash Q_4$. Since $\partial Q_4 \cap Z$ is a Lipschitz surface we have

$$\mathrm{meas}_n\{a_{3k} \neq a_3\} \to 0 \qquad \text{as} \quad k \to +\infty. \tag{2.6}$$

Let u_{3k} be solutions to the initial-boundary value problems:

$$\frac{\partial}{\partial x}\left(a_{3k}\frac{\partial u_{3k}}{\partial x}\right) - \frac{\partial}{\partial t}(u_k) = 0 \quad \text{in} \quad Q_8, \qquad u_{3k} = u_3 \quad \text{on} \quad \partial_x Q_8,$$

$$u_{3k} = 0 \qquad \text{on} \quad Q_8 \cap \{t = 0\}.$$

Since $a_{3k} = a_0 + \chi(Q_1)b_1$ near $\bar{Q}_1$ relation (2.1) is valid for any $u_1 = u_{3k}$. The difference $u_k = u_{3k} - u_3$ satisfies the equation

$$\frac{\partial}{\partial x}\left(a_{3k}\frac{\partial u_{3k}}{\partial x}\right) - \frac{\partial}{\partial t}(u_k) = \frac{\partial}{\partial x}\left((a_3 - a_{3k})\frac{\partial u_{3k}}{\partial x}\right) \qquad \text{in} \quad Q_8$$

and $u_k = 0$ on $\partial Q_8 \cap \{t < T\}$ because u_{3k} and u_3 coincide on the lateral boundary of Q_8 and when $t = 0$. From the definition of a weak solution to this initial-boundary value problem with the test function u_k we have

$$\int_{Q_8} a_{3k}\frac{\partial u_k}{\partial x}\frac{\partial u_k}{\partial x} + \int_{Q_8\cap\{t=T\}} u_k^2/2 = \int_{Q_8}(a_3 - a_{3k})\frac{\partial u_3}{\partial x}\frac{\partial u_k}{\partial x}.$$

According to the assumptions, $\varepsilon < a_{3k}$ for certain positive ε. Using this inequality, dropping the second integral in the left-hand side, and majorating the right-hand side by the inequality $x \cdot y \leqslant \varepsilon^{-1}/2|x|^2 + \varepsilon/2|y|^2$ we obtain

$$\int_{Q_8} \varepsilon\left|\frac{\partial u_k}{\partial x}\right|^2 \leqslant C(\varepsilon)\int_{Q_8}|a_3 - a_{3k}|^2\left|\frac{\partial u_3}{\partial x}\right|^2 + \varepsilon/2\int_{Q_8}\varepsilon\left|\frac{\partial u_k}{\partial x}\right|^2.$$

Since $\frac{\partial}{\partial x}u_3$ belongs to $L_2(Q_8)$ we conclude from (2.6) that the first integral in the right-hand side tends to zero. Therefore, $\frac{\partial}{\partial x}u_k$ converges to zero in $L_2(Q_8)$. Putting $u_1 = u_{3k} = u_3 + u_k$ into the relation (2.1) and letting $k \to \infty$, we complete the proof of lemma 2.2. □

3. Estimates of integrals of singular solutions

We make use of the solutions u_3, u_2^* with singularities outside Q_4. Solutions of elliptic equations of second order with arbitrary power singularities were constructed by Alessandrini [1]. To simplify obtaining bounds on the integrals of such solutions we introduce new variables. According to our assumptions, $\partial_x Q_1$ near the origin is the graph of a Lipschitz function $x = q_1(t)$ which can be assumed to be defined and Lipschitz on $\mathbb{R}$. The substitution

$$x = x^* + q_1(t), \qquad t = t^*$$

transforms equations (1.1) into similar equations with additional first-order differentiation with respect to x^* multiplied by a Lipschitz function of t. The domains Q_j are transformed onto domains with similar properties and with the additional property that the points $(0, t)$, $0 < t < T$ belong to $\partial_x Q_1$. Further on we drop the superscript *.

Let K^+ be the fundamental solution of the Cauchy problem for the forward parabolic equation $\frac{\partial}{\partial x}(a_3\frac{\partial u_3}{\partial x}) - \frac{\partial}{\partial t}(u_3) = 0$ in *-coordinates. Let K^- be the fundamental solution of the backward Cauchy problem for the backward parabolic equation $\frac{\partial}{\partial x}(a_2\frac{\partial u_2}{\partial x}) + \frac{\partial}{\partial t}(u_2) = 0$ in these coordinates. It is known that

$$K^+ = K_1^+ + K_0^+ \qquad K^- = K_1^- + K_0^- \tag{3.1}$$

where K_1^+, K_1^- are the principal parts of K^+, K^- (parametrices) and K_0^+, K_0^- are the remainders. The principal parts are

$$\begin{aligned} K_1^+(x,t;y,\tau) &= \frac{C}{(t-\tau)^{1/2}}\exp\left(-\frac{|x-y|^2}{a_3(x)(t-\tau)}\right) \\ K_1^-(x,t;y,\tau) &= \frac{C}{(\tau-t)^{1/2}}\exp\left(-\frac{|x-y|^2}{a_0(x)(\tau-t)}\right). \end{aligned} \tag{3.2}$$

From the known bounds of fundamental solutions of parabolic equations ([10], p 377) we have

$$\begin{aligned} \left|\frac{\partial}{\partial x}K_0^+(x,t;y,\tau)\right| &\leqslant \frac{C}{(t-\tau)^{1/2}}\exp\left(-|x-y|^2/C(t-\tau)\right) \\ \left|\frac{\partial}{\partial x}K_0^-(x,t;y,\tau)\right| &\leqslant \frac{C}{(\tau-t)^{1/2}}\exp\left(-|x-y|^2/C(\tau-t)\right). \end{aligned} \tag{3.3}$$

When $(y,0)$, (y,τ) are outside Q_1 the functions $K^+(;y,0)$, $K^-(;y,\tau)$ are (x,t)-solutions to the homogeneous parabolic equations with bounded measurable coefficients satisfying zero initial and final conditions. Using lemma 2.2 with $u_3 = K^+(;y,0)$, $u_2^* = K^-(;y,\tau)$ we get

$$\begin{aligned} \int_{Q_1\cap Z} b_1\frac{\partial}{\partial x}K^+(;y,0)\frac{\partial}{\partial x}K^-(;y,\tau) &= -\int_{Q_1\setminus Z} b_1\frac{\partial}{\partial x}K^+(;y,0)\frac{\partial}{\partial x}K^-(;y,\tau) \\ &\quad + \int_{Q_2} b_2\frac{\partial}{\partial x}K^+(;y,0)\frac{\partial}{\partial x}K^-(;y,\tau). \end{aligned} \tag{3.4}$$

From the estimates (3.3) and similar estimates for $\frac{\partial}{\partial x}K_1^+$, $\frac{\partial}{\partial x}K_1^-$ we conclude that the integrands are majorated by an integrable function uniformly with respect to y outside Q_1, by the Lebesgue dominated convergence theorem we may let $y \to 0$, and replace y in (3.4) by zero.

Using representation (3.1) we obtain from (3.4) that

$$|I_1| \leqslant |I_2| + |I_3| \tag{3.5}$$

where

$$I_1 = \int_{Q_1\cap Z} b_1\frac{\partial}{\partial x}K_1^+(;0,0)\frac{\partial}{\partial x}K_1^-(;0,\tau)$$

is formed from the principal parts of K and the remainders are collected in

$$\begin{aligned} I_2 = \int_{Q_1\cap Z} b_1 & \frac{\partial}{\partial x}K_1^+(;0,0)\frac{\partial}{\partial x}K_0^-(;0,\tau) + \frac{\partial}{\partial x}K_0^+(;0,0)\frac{\partial}{\partial x}K_1^-(;0,\tau) \\ & + \frac{\partial}{\partial x}K_0^+(;0,0)\frac{\partial}{\partial x}K_0^-(;0,\tau) \end{aligned}$$

and

$$I_3 = \int_{Q_1\setminus Z} b_1\frac{\partial}{\partial x}K^+(;0,0)\frac{\partial}{\partial x}K^-(;0,\tau) + \int_{Q_2} b_2\frac{\partial}{\partial x}K^+(;0,0)\frac{\partial}{\partial x}K^-(;0,\tau).$$

In the following three lemmas we bound I_1 from below and I_2, I_3 from above.

Lemma 3.1.

$$|I_1| \geqslant \frac{1}{C\tau^{1/2}}$$

where C is independent of τ.

Proof. Using that $b_1(0) \neq 0$ and choosing ε sufficiently small, we obtain

$$|I_1| = C^{-1}\int_0^\tau\int_0^\varepsilon \frac{\partial}{\partial x}K_1^+(;0,0)\frac{\partial}{\partial x}K_1^-(x,t;0,\tau)$$

$$= C^{-1}\int_0^\tau\int_0^\varepsilon \frac{x}{t^{3/2}}\exp\left(-\frac{x^2}{a_3(x)t}\right)\frac{x}{(\tau-t)^{3/2}}\exp\left(-\frac{x^2}{a_0(x)(\tau-t)^{3/2}}\right)\mathrm{d}x\,\mathrm{d}t$$

$$\geqslant C^{-1}\int_0^\varepsilon\int_0^{\tau/2} x^2\frac{1}{(t(\tau-t))^{3/2}}\exp\left(-\frac{x^2\tau}{mt(\tau-t)}\right)\mathrm{d}t\,\mathrm{d}x,$$

where $m = \inf(a_3, a_0)$ over Q.

Using the inequality

$$\frac{1}{t\tau} \leqslant \frac{1}{t(\tau-t)} \leqslant \frac{2}{t\tau} \qquad \text{when} \quad 0 < t < \tau/2 \tag{3.6}$$

from below we bound the above integral by

$$C^{-1}\int_0^\varepsilon\int_0^{\tau/2}\frac{x^2}{(t\tau)^{3/2}}\exp\left(-\frac{2x^2}{mt}\right)\mathrm{d}t\,\mathrm{d}x = \frac{1}{C\tau^{3/2}}\int_0^\varepsilon\int_0^{\tau/2}\frac{x^2}{t^{3/2}}\exp\left(-\frac{2x^2}{mt}\right)\mathrm{d}t\,\mathrm{d}x$$

$$= \frac{1}{C\tau^{3/2}}\int_0^\varepsilon\int_{\frac{4x^2}{m\tau}}^\infty \frac{x}{w^{1/2}}\mathrm{e}^{-w}\,\mathrm{d}w\,\mathrm{d}x, \qquad \text{where we substituted } w = 2x^2/mt$$

$$= \frac{1}{C\tau^{3/2}}\int_0^\infty\left(\int_0^{\min(\varepsilon,(m\tau w)^{1/2}/2)} x\,\mathrm{d}x\right)w^{-1/2}\mathrm{e}^{-w}\,\mathrm{d}w$$

$$\geqslant \frac{1}{C\tau^{3/2}}\int_0^{2\varepsilon^2/m\tau}\left(\int_0^{(m\tau w)^{1/2}/2} x\,\mathrm{d}x\right)w^{-1/2}\mathrm{e}^{-w}\,\mathrm{d}w$$

$$\geqslant \frac{1}{C\tau^{3/2}}\int_0^{2\varepsilon^2/m\tau} w^{-1/2}\mathrm{e}^{-w}\,\mathrm{d}w \geqslant 1/C\tau^{1/2}.$$

The proof is complete. □

Lemma 3.2.

$$|I_2| \leqslant C$$

where C is independent of τ.

Proof. We bound the integral of the first of the three functions forming I_2 as defined after (3.5):

$$|I_2| \leqslant C^{-1}\int_0^\tau\int_0^\varepsilon \frac{\partial}{\partial x}K_1^+(x,t;0,0)\cdot\frac{\partial}{\partial x}K_1^-(x,t;0,\tau)$$

$$= C^{-1}\int_0^\tau\int_0^\varepsilon \frac{x}{t^{3/2}}\exp\left(-\frac{x^2}{a_3(x)t}\right)\frac{1}{(\tau-t)^{1/2}}\exp\left(-\frac{x^2}{a_0(x)(\tau-t)}\right)\mathrm{d}t\,\mathrm{d}x.$$

Using the inequality

$$\frac{|x|}{t^{3/2}}\exp(-|x|^2/4t) \leqslant \frac{C}{t}\exp(-|x|^2/8t)$$

we get

$$\leqslant C^{-1}\int_0^\tau\int_0^\varepsilon \frac{1}{t}\exp\left(-\frac{x^2}{a_3(x)t}\right)\frac{1}{(\tau-t)^{1/2}}\exp\left(-\frac{x^2}{a_0(x)(\tau-t)}\right)\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant C^{-1}\int_0^\varepsilon\int_0^{\tau/2}\frac{1}{t^{1/2}(t(\tau-t))^{1/2}}\exp\left(-\frac{x^2\tau}{Mt(\tau-t)}\right)\mathrm{d}t\,\mathrm{d}x,$$

where $M = \sup(a_3, a_0)$ over Q

where we used the symmetry of t and that the maximum of $\exp(-\frac{x^2\tau}{Mt(\tau-t)})$ was attained at $t = \tau/2$. Using inequality (3.6), we bound the integral from above by

$$C^{-1}\int_0^{\varepsilon}\int_0^{\tau/2}\frac{1}{t^{1/2}(t\tau)^{1/2}}\exp\left(-\frac{x^2}{Mt}\right)\mathrm{d}t\,\mathrm{d}x = \frac{1}{C\tau^{1/2}}\int_0^{\varepsilon}\int_0^{\tau/2}\frac{1}{t}\exp\left(-\frac{x^2}{Mt}\right)\mathrm{d}t\,\mathrm{d}x$$

$$= \frac{1}{C\tau^{1/2}}\int_0^{\varepsilon}\int_{\frac{2x^2}{m\tau}}^{\infty} w^{-1}\mathrm{e}^{-w}\,\mathrm{d}w\,\mathrm{d}z, \qquad \text{where we substitued } w = 2x^2/mt$$

$$= \frac{1}{C\tau^{1/2}}\int_0^{\infty}\left(\int_0^{\min(\varepsilon,(M\tau w)^{1/2}/2)} 1\,\mathrm{d}x\right)w^{-1}\mathrm{e}^{-w}\,\mathrm{d}w$$

$$\leqslant \frac{1}{C\tau^{1/2}}\int_0^{\infty}\left(\int_0^{C(\tau w)^{1/2}} 1\,\mathrm{d}x\right)w^{-1}\mathrm{e}^{-w}\,\mathrm{d}w$$

$$= C\int_0^{\infty} w^{-1/2}\mathrm{e}^{-w}\,\mathrm{d}w = C.$$

The proof is complete. □

Lemma 3.3.

$$|I_3| \leqslant C.$$

Proof. I_3 consists of two integrals. The first one is bounded by

$$C\int_{\varepsilon<x<R}\left|\frac{\partial}{\partial x}K^+(;0,0)\frac{\partial}{\partial x}K^-(;0,\tau)\right|$$

$$\leqslant C\int_{\varepsilon<x<R}\int_0^{\tau/2}\frac{1}{((\tau-t)t)^1}\exp\left(-\frac{x^2\tau}{Mt(\tau-t)}\right)\mathrm{d}t\,\mathrm{d}x.$$

The bound on $|\frac{\partial}{\partial x}K^+\frac{\partial}{\partial x}K^-|$ follows from the direct differentiation of (3.2), the inequality

$$|x|(t-\tau)^{-3/2}\exp(-|x|^2/4(t-\tau)) \leqslant C(t-\tau)^{-1}\exp(-|x|^2/8(t-\tau))$$

and the bounds (3.3).

Applying, as above, inequality (3.6) we bound the last integral by

$$\frac{C}{\tau}\int_{\varepsilon}^{R}\int_0^{\tau/2}\frac{1}{t}\exp\left(-\frac{x^2}{Mt}\right)\mathrm{d}t\,\mathrm{d}x = \frac{C}{\tau}\int_{\varepsilon}^{R}\int_{\frac{2x^2}{M\tau}}^{\infty} w^{-1}\mathrm{e}^{-w}\,\mathrm{d}w\,\mathrm{d}z$$

when we use the substitution $w = x^2/(Mt)$. The function $w^{-1/2}$ is decreasing. Replacing it by its value at $2x^2/(M\tau)$ we increase the integral, and it becomes

$$\leqslant \frac{C}{\tau}\int_{\varepsilon}^{R}\int_{\frac{2x^2}{M\tau}}^{\infty}\left(\frac{2x^2}{M\tau}\right)^{-1}\mathrm{e}^{-w}\,\mathrm{d}w\,\mathrm{d}z$$

$$\leqslant C\int_{\varepsilon}^{R} x^{-2}\mathrm{e}^{-\frac{2x^2}{M\tau}}\,\mathrm{d}w\,\mathrm{d}z = C.$$

□

4. Proof of theorem 1

Now we will finish the proof of the theorem. Using lemmas 1–3 it follows that:

$$\frac{1}{\tau^{1/2}} \leqslant C.$$

Choosing τ small enough we get a contradiction. This contradiction shows that $Q_1 = Q_2$.

The next step of the proof is to show that

$$b_1 = b_2 \qquad \text{on} \quad \partial_x Q_1. \tag{4.1}$$

As in the proof for Q_j we assume the opposite. Then we can assume that the origin $0 \in \partial_x Q_1$ and $b_1(0) > b_2(0)$. By continuity, $b_1(0) - b_2(0) > C^{-1}$ for some C in a certain ball B centred at the origin, let $Z = B(0, T)$. Extend a_2 from Q_2 onto $\mathbb{R}^n$ as a C^2-function $a_4 > 0$. By repeating the proof of lemma 3.2 we obtain the following orthogonality relation:

$$\int_{Q_1} (b_1 - b_2)\frac{\partial}{\partial x}u_3 \frac{\partial}{\partial x}u_4^* = 0 \tag{4.2}$$

for all solutions u_3 to the equation $\frac{\partial}{\partial x}(a_3 \frac{\partial}{\partial x} u_3) - \frac{\partial}{\partial x} u_3 = 0$ near Q_4 which are zero when $t < 0$ and for all solutions u_4^* to the adjoint equation near Q_4 which are zero when $t > T$. Let K^+ be a fundamental solution to the forward Cauchy problem for the first equation and K^- be the fundamental solution to the backward Cauchy problem for the adjoint equation with the coefficient a_4. Using the representation (3.1) of these fundamental solutions, splitting Q_1 into $Q_1 \cap Z$ and its complement, as in section 3, we obtain from (4.2) the inequality

$$|I_4| \leqslant |I_5| + |I_6| \tag{4.3}$$

where

$$I_4 = \int_{Q_1 \cap Z} (b_1 - b_2)\frac{\partial}{\partial x}K_1^+ \frac{\partial}{\partial x}K_1^-$$

is related to the supposedly singular part and

$$I_5 = \int_{Q_1 \backslash Z} (b_1 - b_2)\frac{\partial}{\partial x}K^+ \frac{\partial}{\partial x}K^-$$

$$I_6 = \int_{Q_1} (b_1 - b_2)\frac{\partial}{\partial x}K_0^+ \frac{\partial}{\partial x}K_0^-.$$

It is easy to see that lemmas 3.1–3.3 are valid for I_4, I_5 and I_6, respectively. So we arrive at a contradiction, exactly as in the above proof, i.e. that $Q_1 = Q_2$.

This shows that the assumption about b_1, b_2 is wrong and that $b_1 = b_2$ on $\partial_x Q_1$.

Let Ω_0 be the intersection of all $Q_{1\theta}$ over $0 < \theta < T$. Since b_1, b_2 do not depend on t and are equal on $\partial_x Q_1$, they concide on $Q_{1\theta} \backslash \Omega_0$. Letting $Q_0 = \Omega_0 \times (0, T)$ from (4.2) we obtain the relation

$$\int_{Q_0} (b_1 - b_2)\frac{\partial}{\partial x}u_3 \frac{\partial}{\partial x}u_4^* = 0$$

for all u_3, u_4^* in (4.4). As in the proof of lemma 3.2 it implies that

$$\int_{Q_0} (b_1 - b_2)\frac{\partial}{\partial x}u_5 \frac{\partial}{\partial x}u_6^* = 0 \tag{4.4}$$

for all solutions u_5 to the equation $\frac{\partial}{\partial x}((a_0 + b_1\chi(Q_0))\frac{\partial}{\partial x}u_5) - \frac{\partial}{\partial t}u_5 = 0$ near Q_0 which are zero when $t < 0$ and for all solutions to the adjoint equation $\frac{\partial}{\partial x}((a_0 + b_2\chi(Q_0)\frac{\partial}{\partial x}u_6^*) + \frac{\partial}{\partial t}u_6^* = 0$ near Q_0 which are zero when $t > T$.

Now we will show that the equations for u_5, u_6 have the same lateral Dirichlet-to-Neumann maps. Let u_5, u_6 be the solutions to these equations with zero initial conditions and the same lateral Dirichlet data. By subtracting these equations and letting $u = u_6 - u_5$ we obtain

$$\frac{\partial}{\partial x}\left((a_0 + b_2\chi(Q_0))\frac{\partial}{\partial x}u\right) = \frac{\partial}{\partial x}\left((b_1 - b_2)\chi(Q_0)\frac{\partial}{\partial x}u_5\right) \qquad \text{in} \quad Q.$$

From the definition of a weak solution to this equation, we have

$$\int_0^T a_0(u_\nu\psi(1,t) - u_\nu\psi(0,t)) - \int_Q (a_0 + b_2\chi(Q_0))\frac{\partial}{\partial x}u \cdot \frac{\partial}{\partial x}\psi - \int_Q \frac{\partial u}{\partial t}\cdot\psi$$
$$= -\int_{Q_0}(b_1 - b_2)\frac{\partial}{\partial x}u_5 \cdot \frac{\partial}{\partial x}\psi$$

for any function $\psi \in H^{1,1}(Q)$. Using $\psi = u_6^*$, integrating by parts in the third integral of the left-hand side and using again the definition of a weak solution to the equation $\frac{\partial}{\partial x}((a_0 + b_2\chi(Q_0))\frac{\partial}{\partial x}u_6^*) + \frac{\partial}{\partial t}u_6^* = 0$ with the test function u which is zero on $\partial Q \cap \{t < T\}$ we conclude that the sum of the second and third integrals in the left-hand side is zero. The right-hand side is zero due to (4.4). So the first integral in the left-hand side is zero. Since the lateral Dirichlet data $\psi = u_6^*$ can be any function in $C_0^\infty(\partial\Omega \times (0,T))$ we get $u_\nu = 0$ on $\partial\Omega \times (0,T)$. So, $u_{5\nu} = u_{6\nu}$ on the lateral boundary, which means that we have the same lateral Dirichlet-to-Neumann maps.

Take as the Dirichlet data g, a function which does not depend on t when $t > \tau$. Since the coefficients of the equations $\frac{\partial}{\partial x}((a_0 + b_j\chi(Q_0))\frac{\partial}{\partial x}u_j) - \frac{\partial}{\partial t}u_j = 0$ are time independent, the solutions $u_j(x,t)$ of the initial boundary value problems on $\Omega \times (0,\infty)$ will be analytic with respect to $t > \tau$. They have the same Cauchy data on $\partial\Omega \times (0,\infty)$, therefore, as above by the uniqueness in the lateral Cauchy problem, $u_5 = u_6$ on $(\Omega\backslash\Omega_0) \times (0,T)$. By uniqueness of the analytic continuation, they are also equal on $(\Omega\backslash\Omega_0) \times (0,\infty)$. Then we use theorem 9.2.1 in [8] to get $b_1 = b_2$ on Ω_0, and so on Q. The proof is complete. □

5. Proof of theorem 2

To prove theorem 2 we need the following lemma.

Lemma 5.1.

$$\int_Q (a_1 - a_2)\frac{\partial}{\partial x}u_1\frac{\partial}{\partial x}u_2^* = 0 \tag{5.1}$$

for all solutions u_1 to the equation $\frac{\partial}{\partial x}(a_1\frac{\partial}{\partial x}u_1) - \frac{\partial u_1}{\partial t} = 0$ in Q which are zero when $t < 0$ and satisfy (1.2′) and for all solutions u_2^* to the adjoint equation $\frac{\partial}{\partial x}(a_2\frac{\partial}{\partial x}u_2^*) + \frac{\partial}{\partial t}u_2^* = 0$ near Q which are zero when $t > T$ and $u_2^* = 0$ on S'' and $u_2^* = \psi$ on S' where $\psi \in C_0^{2+\lambda}(S')$.

Proof of the lemma. Denote by Λ the Dirichlet-to-Neumann operator which maps the Dirchlet data $u|_{\partial\Omega\times(0,T)}$ into the Neumann data $(\frac{\partial}{\partial x}u)|_{\partial\Omega\times(0,T)}$.

Using [10], a generalized solution of equation (0.1) is a function $u \in H^{1,1}(Q)$ which satisfies the identity

$$\int_Q a\frac{\partial}{\partial x}u\frac{\partial}{\partial x}\phi - \int_Q u\phi_t = \int_S (\Lambda u)\phi \tag{5.2}$$

for every $\phi \in H^{1,1}(Q)$ that is zero for $t = T$, and also zero in S''.

Letting $\phi = u^*$ in (5.2), where u^* is a solution to the adjoint equation

$$\frac{\partial}{\partial t}(u^*) + \frac{\partial}{\partial x}\left(a_j \frac{\partial}{\partial x} u^*\right) = 0 \qquad \text{in} \quad Q \tag{5.3}$$

which is zero on S'', we have

$$\int_Q a \frac{\partial}{\partial x} u \frac{\partial}{\partial x} u^* - \int_Q u u_t^* = \int_S \Lambda u \cdot u^*. \tag{5.4}$$

Integrating the first integral by parts, we get

$$\int_S \Lambda_* u^* \cdot u = \int_S \Lambda u \cdot u^* \tag{5.5}$$

where we used that u^* satisfies the adjoint equation (5.3).

Recalling that $(\,,\,)(S)$ denotes the scalar product in the Hilbert space of real-valued L_2 functions and using (5.5) we get that

$$(\Lambda_* u^*, u)(S) = (u^*, \Lambda u)(S) \tag{5.6}$$

for any solution u to the first equation and any solution u^* to the adjoint equation, which implies that

$$\Lambda_*^* = \Lambda \tag{5.7}$$

where Λ_*^* denotes that adjoint of the Dirchlet-to-Neumann operator of the adjoint equation. Now, using the definition of a weak solution to the adjoint equation (5.3) with $u^* = 0$ on T, and $u^* = \psi$ on S' and zero on S'', we have that u^* satisfies

$$\int_Q a \frac{\partial}{\partial x} u^* \frac{\partial}{\partial x} \phi + \int_Q u^* \frac{\partial}{\partial t} \phi = \int_S (\Lambda_* u^*) \phi \tag{5.8}$$

for every $\phi \in H^{1,1}(Q)$ that is zero for $t = 0$, and also zero in S''.

Now using (5.4) with $a = a_1$, $u = u_1$, $u^* = u_2^*$, we get

$$\int_Q a_1 \frac{\partial}{\partial x} u_1 \frac{\partial}{\partial x} u_2^* - \int_Q u_1 \frac{\partial}{\partial t} u_2^* = \int_S (\Lambda u_1) u_2^* \tag{5.9}$$

and using (5.8) with $a = a_2$, $u^* = u_2^*$, $\phi = u_1$, we get

$$\int_Q a_2 \frac{\partial}{\partial x} u_2^* \frac{\partial}{\partial x} u_1 + \int_Q u_2^* \frac{\partial}{\partial t} u_1 = \int_S (\Lambda_* u_2^*) u_1. \tag{5.10}$$

Now subtracting (5.10) from (5.9) and using (5.7) we get the result of lemma 5.1. □

Proof of theorem 2. Let $\{Q_j^1\}_{j=1}^n$ and $\{Q_k^2\}_{k=1}^m$ be two covers to which a_1, a_2 are piece-wise constant, respectively.

Let $Q_i' = Q_j^1 \cap Q_k^2$.

Note that the collection of all Q_i' constitute a cover of Q relative to which a_1, a_2 are piece-wise constant. We collect Q_i' into layers M_k in the following way.

Let $Q_0 = Q$.

$M_1 = \cup\{Q_k' \subset Q : \partial Q_k' \cap S'$ has nonempty interior in $S'\}$.

On M_1, we have subdomains Q_i' which have a boundary part in common with S' where both u_1, u_2 have the same Cauchy data and since both functions satisfy the same parabolic equation in M_1, thus from the uniqueness of continuation of the second-order parabolic equation, we get $u_1 = u_2$ in M_1. Using lemma (5.1), if $Q_1 = Q \backslash M_1$ we have

$$\int_{Q_1} (a_1 - a_2) \frac{\partial}{\partial x} u_1 \frac{\partial}{\partial x} u_2^* = 0 \tag{5.11}$$

for all solutions u_1 to the equation $\frac{\partial}{\partial x}(a_1 \frac{\partial}{\partial x} u_1) - \frac{\partial}{\partial t} u_1 = 0$ in Q which are zero when $t < 0$ and satisfy (1.2′) and for all solutions u_2^* to the adjoint equation $\frac{\partial}{\partial x}(a_2 \frac{\partial}{\partial x} u_2^*) + \frac{\partial}{\partial t} u_2^* = 0$ in Q which are zero when $t > T$ and $u_2^* = 0$ on S'' and $u_2^* = \psi$ on S' where $\psi \in C_0^{2+\lambda}(S')$.

Using the Runge property, (5.11) still holds for all solutions u_2 to the equation

$$\frac{\partial}{\partial x}\left(a_1 \frac{\partial}{\partial x} u_1\right) - \frac{\partial}{\partial t} u_1 = 0 \qquad \text{near} \quad Q_1$$

which are zero when $t < 0$ and satisfies (1.2′) and for all solutions u_2^* to the adjoint equation

$$\frac{\partial}{\partial x}\left(a_2 \frac{\partial}{\partial x} u_2^*\right) + \frac{\partial}{\partial t} u_2^* = 0 \qquad \text{near} \quad Q_1$$

which are zero when $t > T$ and $u_2^* = 0$ on S''.

Set

$$M_k = \cup\{Q'_\ell \subset Q_{k-1} : \partial Q'_\ell \cap \partial M_{k-1} \text{ has nonempty interior in } \partial M_{k-1}\}, \qquad k = 2, 3, \ldots$$
$$Q_k = Q_{k-1} \backslash M_k, \qquad k = 2, 3, \ldots.$$

We show that $a_1 = a_2$ on ∂M_k.

Suppose the contrary. Then there exists a point (x, t), a ball B, and $\varepsilon > 0$ such that $a_1(x, t) - a_2(x, t) > \varepsilon$ on B, using (5.11), we have

$$0 = \int_{Q_k} (a_1 - a_2) \frac{\partial}{\partial x} u_1 \frac{\partial}{\partial x} u_2^* = \int_{Q_k \cap B} (a_1 - a_2) \frac{\partial}{\partial x} u_1 \frac{\partial}{\partial x} u_2^* + \int_{Q_k \backslash B} (a_1 - a_2) \frac{\partial}{\partial x} u_1 \frac{\partial}{\partial x} u_2^*$$

using the same arguments as in the proof of theorem 1 and using Green's function as in [3], we get the contradiction. □

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