On the number of limit cycles of a generalized Abel equation*

Naeem Alkoumi and Pedro J. Torres

Departamento de Matemática Aplicada,
Universidad de Granada, 18071 Granada, Spain.

Abstract

New results are proved on the maximum number of isolated T-periodic (limit cycles) of a first order polynomial differential equation with periodic coefficients. The exponents of the polynomial may be negative. The results are compared with the available literature and applied to a class of polynomial systems on the cylinder.

AMS Subject Classification: 34C25

1 Introduction and main results.

This paper is motivated by some recent results on the number of isolated periodic solutions (limit cycles) of the first order differential equation with polynomial nonlinearity

$$u' = \sum_{i=0}^{n} a_i(t)u^i, \tag{1}$$

where the coefficients a_i are continuous and T-periodic functions for some $T > 0$. This is a classical problem. The first non-trivial situation is the Abel equation $n = 3$. If $a_3(t) > 0$, Pliss [12] proved that (1) has at most three limit cycles, but in the general case Lins Neto [10] gave examples with an arbitrary number of limit cycles. Such examples can be easily extended to higher-order polynomial equations, even with a constant leading coefficient a_n. Sufficient conditions for $n = 3$ to have at most three limit cycles were proved in [8, 2].

More recently, the equation with three terms

$$u' = a_{n_1}(t)u^{n_1} + a_{n_2}(t)u^{n_2} + a_{n_3}(t)u^{n_3}, \tag{2}$$

*Supported by projects MTM2008-02502, Ministerio de Educación y Ciencia, Spain, and FQM2216, Junta de Andalucía.

has been considered on some related works. From now on, a continuous function $f : [0, T] \to \mathbb{R}$ is said to have a definite sign if it is not null and either $f(t) \geq 0$ or $f(t) \leq 0$, and we write $a \succ 0$ in the first case and $a \prec 0$ in the second case. Gasull and Guillamon [7] proved that if $n_3 = 1$ and $a_{n_2}(t)$ or $a_{n_3}(t)$ have a definite sign, then (3) has at most two positive limit cycles. This gives a total maximum number of five limit cycles by the change $y = -u$, since $x = 0$ is always a solution. This remark leads us to focus the attention only in the positive limit cycles. In the same paper, if $n_3 > 1$ and only one of the coefficients has a definite sign, examples are given with an arbitrary number of limit cycles.

Therefore, for equations with 3 or more monomials, in order to obtain bounds on the number of limit cycles, it is natural in some sense to assume that two coefficients have a definite sign. A first result following this idea was proved by Alwash in [5], where it is proved that if $n \geq 3$ and $a_{n-3}(t) \leq 0$, the equation

$$u' = u^n + a_{n-1}(t)u^{n-2} + a_{n-3}(t)u^{n-3}, \tag{3}$$

has at most one positive limit cycle. This result has been generalized very recently in the following way.

Theorem 1 ([1]) *Consider the differential equation*

$$u' = a_{n_1}(t)u^{n_1} + a_{n_2}(t)u^{n_2} + a_{n_3}(t)u^{n_3} + a_m(t)u^m, \tag{4}$$

where $n_1 > n_2 > n_3 > m := 1$. Suppose that $a_{n_1}(t)$ and $a_{n_2}(t)$, or $a_{n_2}(t)$ and $a_{n_3}(t)$ have the same definite sign, or that $a_{n_1}(t)$ and $a_{n_3}(t)$ have opposite definite sign. Then, (4) *has at most two positive limit cycles. If moreover $a_m(t)$ has null integral over $[0, T]$, then* (4) *has at most one positive limit cycle.*

Our aim in this paper is to contribute to the literature by proving some related results which can be seen as a complement to the previous ones. Our main result is as follows.

Theorem 2 *Let us assume that a_{n_1} has a definite sign. Fix $n_1, n_2, n_3, m \in \mathbb{Z}$ entire numbers such that $n_1 > n_2 > n_3$ verify the condition*

$$n_1 - 2n_2 + n_3 = 0. \tag{5}$$

If

$$\Delta = a_{n_2}^2(m - n_2)^2 - 4a_{n_1}a_{n_3}(m - n_1)(m - n_3) \leq 0, \tag{6}$$

then (4) *has at most one positive limit cycle.*

Section 2 will be devoted to prove it and to state some clarifying consequences for the comparison between this result and those previously published. Here some comments are in order. The first original feature is that negative powers are possible. It is worthwhile to consider this case for applications to the study of the number of limit cycles in polynomial planar systems on the cylinder, as we will show in more detail in Section 4 with examples inspired by [3].

About condition (5), it is easy to realize that it is equivalent to impose that three of the terms of the equation have powers following an arithmetic sequence, that is, there exist $r \in \mathbb{N}, \beta \in \mathbb{Z}$, such that

$$n_1 = 2r + \beta, n_2 = r + \beta, n_3 = \beta.$$

If $r = 1$ we get consecutive powers. In spite of that, m is free so the result is quite flexible and give a whole family of new criteria.

The paper is divided into four Sections. In Section 2 we will prove Theorem 2 and discuss some consequences. The method of proof is based on the known result that a sign on derivative up to order three of the nonlinearity on a given region gives a bound on the number of limit cycles (see for instance [7, 8, 13]), but we exploit the fact that this sign is not invariant under changes of variables. In Section 3, we combine this technique with upper and lower solutions in order to get multiplicity results for the fourth-order differential equation. Finally, in the last section the main results are applied to some specific examples of polynomial planar systems in order to get information on the maximum number of limit cycles.

2 The equation with four monomials.

For the proof of Theorem 2, we will need the following result, which can be found in [7, 8, 13].

Proposition 1 *Let us consider a general first order equation*

$$x' = g(t, x), \tag{7}$$

with g continuous and T-periodic in t. Fix $k \in \{1, 2, 3\}$. Let J be an open interval and let us assume that $g(t, x)$ has continuous derivative $\frac{\partial^k}{\partial x^k} g(t, x)$ for all $(t, x) \in [0, T] \times J$. If $\frac{\partial^k}{\partial x^k} g(t, x) \geq 0$ for all $(t, x) \in [0, T] \times J$ (resp. $\frac{\partial^k}{\partial x^k} g(t, x) \leq 0$ for all $(t, x) \in [0, T] \times J$), then the equation (7) has at most k limit cycles with range contained in J.

Proof of Theorem 2. By means of the change in the independent variable $\tau = -t$, we can assume that $a_{n_1} \succ 0$ without loss of generality. Let us first consider the case $m = 1$. We write the equation as

$$u' = uF(t, u),$$

where

$$F(t, u) = a_{n_1} u^{n_1 - 1} + a_{n_2} u^{n_2 - 1} + a_{n_3} u^{n_3 - 1} + a_1.$$

By using the change of variable $u = e^x$, we get

$$x' = F(t, e^x) := g(t, x). \tag{8}$$

Now,

$$\begin{aligned} g_x(t,x) &= e^x F_x(t,e^x) \\ &= e^{(n_3-1)x}[(n_1-1)a_{n_1}e^{(n_1-n_3)x} + (n_2-1)a_{n_2}e^{(n_1-n_2)x} + (n_3-1)a_{n_3}]. \end{aligned}$$

If we call $S = e^{(n_1-n_2)x}$, then $S^2 = e^{(n_1-n_3)x}$ as a result of (5). Therefore, $g_x(t,x)$ can be written as

$$g_x(t,x) = e^{(n_3-1)x}[(n_1-1)a_{n_1}S^2 + (n_2-1)a_{n_2}S + (n_3-1)a_{n_3}]$$

The last factor is a quadratic polynomial with negative discriminant by hypothesis (6). Hence by Proposition 1 there exists at most one limit cycle of equation (8), which correspond to at most one positive limit cycle of (8).

For $m \geq 2$, the equation is written as

$$u' = u^m F(t,u).$$

Now the adequate change is $u = x^\alpha$, satisfying $(m-1)\alpha + 1 = 0$. This change is well defined for positive solutions and keeps the number of positive limit cycles. It leads to

$$x' = \frac{1}{\alpha} F(t, x^\alpha) := g(t,x).$$

The derivative is

$$\begin{aligned} g_x(t,x) &= x^{\alpha-1} F_x(t,x^\alpha) \\ &= \alpha x^{(n_3-m+1)\alpha-2}[a_{n_1}(n_1-m)S^2 + a_{n_2}(n_2-m)S + a_{n_3}(n_3-m)], \end{aligned}$$

where $S = x^{(n_1-n_2)\alpha}$. The conclusion is analogous. ■

After this proof, we will compare with the related literature through some corollaries. The first one generalizes the result by Alswalsh yet mentioned in the Introduction.

Corollary 1 *If $n_1 > n_2 > n_3$ holds the condition* (5) *and a_{n_1}, a_{n_3} have opposite definite signs, then the equation* (3) *has at most two nontrivial limit cycles, at most one positive and at most one negative.*

Proof. For the existence of at most one positive limit cycle, just take $m = n_2$ and apply Theorem 2. For the negative one, make the change $y = -x$. ■

For comparison with Theorem 1, note that it does not cover the case of a_{n_1} and $a_{n_3}(t)$ with the same definite sign. In fact, in [1] the authors provides examples under this assumption with at least three limit cycles. Now we get the following complementary result.

Corollary 2 *Fix $n_1 > n_2 > n_3 > m := 1$ verifying (5) and assume that a_{n_1} and a_{n_3} have the same definite sign. If*

$$a_{n_1}(t)a_{n_3}(t) \geq \frac{(n_2-1)^2}{4(n_1-1)(n_3-1)} a_{n_2}(t)^2$$

for all t, then (4) *has at most one positive limit cycle.*

The proof is direct. Other variant is the following one.

Corollary 3 *Fix $n_1 > n_2 > n_3$ verifying* (5) *and assume that a_{n_1} and a_{n_3} have the same definite sign. If*

$$4a_{n_1}(t)a_{n_3}(t) > a_{n_2}^2(t),$$

for all t, there exists $m_0 > 0$ such that if $|m| > m_0$ then (4) *has at most one positive limit cycle.*

The number m_0 is explicitly computable, for the proof follows easily from a pass to the limit in condition (6).

We finish the section by pointing out that Theorem 2 and its corollaries can be complemented with stability and exact multiplicity information by using the explicit behavior near the origin, as it is done for instance in [2, 7].

3 The complete fourth-order equation.

The aim of this section is to provide some sufficient conditions for limiting the number of limit cycles of the (4,3,2,1,0)-polynomial equation

$$u' = a_4(t)u^4 + a_3(t)u^3 + a_2(t)u^2 + a_1(t)u + a_0(t). \tag{9}$$

In [7, Theorem 5], it is proved that (9) with $a_4(t) \equiv 1$ may have an arbitrary number of T-periodic solutions. On the other hand, when $a_0 \equiv 0$, the main result of [1] implies that (9) has at most two positive T-periodic solutions if $a_4, a_3 \succ 0$, or $a_3, a_2 \succ 0$, or $a_4 \succ 0 \succ a_2$. Our results can be seen as a partial counterpart.

We will need some basic facts about the concept of upper and lower solutions. See for instance [11] for more details.

Definition 1 *A T-periodic function ϕ is called a strict lower (resp. upper) solution of equation* (4) *if*

$$\phi'(t) < g(t, \phi(t)) \quad (\textit{resp. } \phi'(t) > g(t, \phi(t))),$$

for all t.

Lemma 1 *A T-periodic solution does not intersect any eventual strict upper or lower solution.*

Our first result is very similar to some results in [4] for the fifth-order homogeneous equation.

Theorem 3 *If $a_2, a_4 \succ 0$ and $a_3^2 - \frac{8}{3}a_4a_2 \leq 0$, equation* (9) *has at most two limit cycles.*

Proof. The second derivative of the right-hand side of equation (9) is

$$12a_4(t)u^2 + 6a_3(t)u + 2a_2(t).$$

Looking this as a second-order polynomial, the discriminant is $36a_3^2 - 96a_4a_2$. By hypothesis, this is negative, then by Proposition (1) there exist at most two limit cycles. ■

On the other hand, next results are of a different nature.

Theorem 4 *Let us assume that* $a_0(t)a_4(t) > 0$ *for all* t*. If* $4\sqrt[4]{a_0a_4^3} + a_3 \geq 0$ *, equation* (9) *has at most two positive limit cycles.*

Proof. We can assume without loss of generality that a_0, a_4 are both strictly positive functions. After the change $x = \frac{1}{u}$, the equation is

$$x' = -xF(t, \frac{1}{x}),$$

where

$$F(t,x) = a_4(t)x^3 + a_3(t)x^2 + a_2(t)x + a_1(t) + \frac{a_0(t)}{x}.$$

By defining $g(t,x) := -xF(t, \frac{1}{x})$, the second derivative is

$$g_{xx}(t,x) = \frac{-1}{x^3}F_{xx}(t, \frac{1}{x}).$$

Therefore, the proof is reduced to show that $F_{xx}(t,x)$ is positive for $x > 0$. It turns out that

$$F_{xx}(t,x) = 6a_4(t)x + 2a_3(t) + \frac{2a_0(t)}{x^3}.$$

Since, a_0, a_4 are strictly positive, the function $6a_4(t)x + \frac{2a_0(t)}{x^3}$ attains its global minimum at $a_4(t)^{1/4}a_0(t)^{-1/4}$. Hence, for any $x > 0$

$$F_{xx}(t,x) \geq 8a_0(t)^{1/4}a_4(t)^{3/4} + 2a_3(t) \geq 0$$

and the proof is done. ■

Theorem 5 *Let us assume that* $a_4(t) > 0$ *for all* t*. Then, equation* (9) *has at most three limit cycles bigger (resp. smaller) than* $\frac{-a_3(t)}{4a_4(t)}$*.*

Proof. The third derivative of the right-hand side of equation (9) is

$$g_{uuu}(t,u) = 24a_4(t)u + 6a_3(t).$$

Then $g_{uuu}(t,u) > 0$ if $u > \frac{-a_3(t)}{4a_4(t)}$. By Proposition (1) there exist at most three limit cycles bigger than $\frac{-a_3(t)}{4a_4(t)}$. In the same way, it is proved that there are at most three limit cycles smaller than $\frac{-a_3(t)}{4a_4(t)}$. ■

Of course, in this latter result additional T-periodic solutions crossing $\frac{-a_3(t)}{4a_4(t)}$ may appear. This possibility is excluded with an additional assumption.

Corollary 4 *Let us assume that* $\frac{-a_3(t)}{4a_4(t)}$ *is an upper (resp. lower) solution of eq.* (9)*. Then, there are at most* 6 *limit cycles.*

Proof. If $\frac{-a_3(t)}{4a_4(t)}$ is an upper (or lower) solution, by Lemma 1 a T-periodic solution can not cross it, so there are at most 3 of them above and at most 3 below. ■

4 Applications to polynomial systems in the cylinder.

In this section we study the maximum number of limit cycles of some polynomial vector fields in $\mathbb{R}^2$, the so-called Hilbert number. The first example is known in the literature as a *rigid system* (see for instance [9, 7]).

The planar system

$$x' = -y + xP(x,y) \quad , \quad y' = x + yP(x,y) \tag{10}$$

where $P(x,y)$ is a polynomial, it is known in the related literature as a *rigid system* (see for instance [7, 8, 9] and their references). In polar coordinates, the system is rewritten as

$$r' = rP(rcos\theta, rsin\theta), \quad \theta' = 1.$$

If r is considered as a function of θ, we get the first order differential equation

$$\frac{dr}{d\theta} = rP(rcos\theta, rsin\theta), \tag{11}$$

and now it is easy to give applications of the results of Section 2 for suitable choices of the polynomial P.

In the recent paper [3], the authors study the number of non-contractible limit cycles of a family of systems in the cylinder $\mathbb{R} \times \mathbb{R}/[0, 2\pi]$ of the form

$$\left\{ \begin{array}{lcl} \frac{d\rho}{dt} & = & \tilde{\alpha}(\theta)\rho + \tilde{\beta}(\theta)\rho^{k+1} + \tilde{\gamma}(\theta)\rho^{2k+1}, \\ \frac{d\theta}{dt} & = & b(\theta) + c(\theta)\rho^k, \end{array} \right. \tag{12}$$

where $k \in \mathbb{Z}^+$ and all the above functions in θ are continuous and 2π-periodic. A contractible limit cycle is an isolated periodic orbit which can be deformed continuously to a point, on the contrary it is called non-contractible. This type of systems arises as the polar expression of several types of planar polynomial systems. Of course, when $b(\theta) \equiv 1$ and $c(\theta) \equiv 0$ we have a rigid system. In general, if $b(\theta)$ does not vanishes, a widely used change of variables due to Cherkas [6] transforms the system into a common Abel equation. We will consider the reciprocal case $b(\theta) \equiv 0$, $c(\theta) \equiv 1$. Let us consider the system

$$\left\{ \begin{array}{rcl} \frac{d\rho}{dt} & = & \tilde{\alpha}(\theta)\rho + \tilde{\beta}(\theta)\rho^{N_3} + \tilde{\gamma}(\theta)\rho^{N_2} + \tilde{\delta}(\theta)\rho^{N_1}, \\ \frac{d\theta}{dt} & = & \rho^k, \end{array} \right. \tag{13}$$

where $N_1 > N_2 > N_3 > 0$ and $k > 0$. A limit cycle of this system is always non-contractible and as a function of θ it is a limit cycle or the first order equation

$$r' = \tilde{\beta}(\theta)r^{n_1} + \tilde{\gamma}(\theta)r^{n_2} + \tilde{\delta}(\theta)r^{n_3} + \tilde{\alpha}(\theta)r^m,$$

where $n_i = N_i - k$ for $i = 1, 2, 3$ and $m = 1 - k$. Now, a direct application of Theorem 2 gives the following result.

Corollary 5 *Take N_1, N_2, N_3 such that $N_1 - 2N_2 + N_3 = 0$ and assume*

$$\tilde{\gamma}(\theta)^2(N_2 - 1)^2 - 4\tilde{\beta}(\theta)\tilde{\delta}(\theta)(N_1 - 1)(N_3 - 1) \le 0.$$

Then, system (13) *has at most one limit cycle in the semiplane* $\{\rho > 0\}$.

In particular, the result holds if $\tilde{\gamma}(\theta) \equiv 0$ and $\tilde{\beta}(\theta), \tilde{\delta}(\theta)$ have opposite definite signs.

Similarly, the results contained in Section 3 can be applied to rigid systems when the polynomial $P(x, y)$ is a sum of homogeneous polynomials up to fourth degree or to a suitable system in the cylinder. We omit further details.

As a last remark, let us comment that the study non-contractible limit cycles on a general system on the cylinder

$$\left\{ \begin{array}{rcl} \frac{d\rho}{dt} & = & P(\theta, \rho), \\ \frac{d\theta}{dt} & = & Q(\theta, \rho) \end{array} \right.$$

where components of the field (P, Q) are periodic in θ and polynomial in ρ, leads to the study of the existence and multiplicity of periodic solutions of a first order equation with a rational (quotient of two polynomials) nonlinearity. This is a difficult problem which deserves further developments.

Acknowledgement: We would like to thank to prof. J.L. Bravo for a first critical reading of a first version of this work.

References

[1] A. Alvarez, J.L. Bravo, M. Fernández, The number of limit cycles for generalized Abel equations with periodic coefficients of definite sign, Communications on Pure and Applied Analysis, Vol. 8, n. 5 (2009), 1493-1501.

[2] M.J. Alvarez, A. Gasull, H. Giacomini, A new uniqueness criterion for the number of periodic orbits of Abel equations, J. Differential Equations 234 (2007) 161-176.

[3] M.J. Alvarez, A. Gasull, R. Prohens, On the number of limit cycles of some systems on the cylinder, Bull. Sci. Math., Vol. 131 (2007), 620-637.

[4] M.A.M. Alwash, Periodic solutions of polynomial non-autonomous differential equations, Electronic Journal of Differential Equations, Vol. 2005 (2005), No. 84, pp. 1û8.

[5] M.A.M. Alwash, Periodic solutions of Abel differential equations, J. Math. Anal. Appl. 329, (2007) 1161-1169.

[6] L.A. Cherkas, Number of limit cycles of an autonomous second-order system, Differential euqations 5 (1976), 666-668.

[7] A.Gasull, A.Guillamon, Limit cycles for generalized Abel equations, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 16 (2006), no. 12, 3737–3745.

[8] A. Gasull, J. Llibre, Limit cycles for a class of Abel equations, SIAM J. Math. Anal. 21 (5) (1990) 1235-1244.

[9] A. Gasull, J. Torregrosa, Exact number of limit cycles for a family of rigid systems, Proc. Amer. Math. Soc., 133 no. 3 (2005), 751û758.

[10] A. Lins-Neto, On the number of solutions of the equation $\frac{dx}{dt} = \sum_{j=0}^{4} a_j(t)x^j, 0 \leq t \leq 1$, for which $x(0) = x(1)$, Invent. Math. 59 (1980), 69-76.

[11] M.N. Nkashama, A generalized upper and lower solutions method and multiplicity results for nonlinear first-order ordinary differential equations, J. Math. Anal. Appl. 140 (1989), 381-395.

[12] V.A. Pliss, Non-local problems of the theory of oscillations, Academic Press, New York, 1966.

[13] A. Sandqvist, K.M. Andersen, On the Number of Closed Solutions to an Equation $x' = f(t, x)$, where $f_{x^n}(t, x) \geq 0 (n = 1, 2, or 3)$. J. Math. Anal. Appl. 159 (1991), 127-146.