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Economic Theory

On the characterization of excess demand functions*

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Summary. In this paper, we give the necessary and sufficient conditions that characterize the individual excess demand function when it depends smoothly on prices and endowments. A given function is an excess demand function if and only if it satisfies, in addition to Walras' law and zero homogeneity in prices, a set of first order partial differential equations, its substitution matrix is symmetric and negative semidefinite. Moreover, we show that these conditions are equivalent to the symmetry and negative semidefiniteness of Slutsky matrix, Walras' law and zero homogeneity of Marshallian demand functions.

Keywords and Phrases: Direct utility function, Indirect utility function, Excess demand function, Slutsky matrix.

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1 Introduction

In consumer theory, the individual demand function is fully characterized by the well known conditions: (i) homogeneity of degree zero, (ii) Walras law and (iii) symmetry and negative semi-definiteness of its substitution matrix. This problem has been posed by Antonelli [2] and solved by Slutsky [9] about one century ago. The individual demand is the solution to the utility maximization problem under the budget constraint $p.x = y$, where p is the price vector and y is the individual income. The problem with several budget constraints was treated by Aloqeili [1].

We consider and exchange economy in which there are n commodities. Each consumer is endowed with certain quantity of each commodity. Our goal is to characterize the excess demand function of a typical consumer in this economy.

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To our knowledge, the problem of characterizing the individual excess demand function has never been discussed until a recent article by Chiappori and Ekeland [4]. In their article, Chiappori and Ekeland solved the problem of characterizing the individual excess demand when it is a smooth function of prices only, $z(p)$. They found that for each individual excess demand, there is a continuum of direct utility functions that rationalize it. Moreover, the "one to one" duality between the direct and the indirect utility functions is no longer valid.

In this article, we address the problem of characterizing the individual excess demand function when it depends smoothly on prices and initial endowments, $z(p, \omega)$. Such a setting is considered in Brown and Matzkin [3], and Chiappori et al. [5], (CEKP). The fact that z is defined as a function of prices and endowments has important implications on the structure of the aggregate excess demand function and the equilibrium manifold.

In their article (CEKP) discuss the testable implications of utility maximization on the equilibrium manifold. We know from the classical results of Debreu [6], Mantel [8] and Sonnenschein [10] (DMS) that no restrictions implied by utility maximization exist. In these articles, the individual excess demand is considered as a function of prices only. However, in the article of (CEKP), they consider an exchange economy where excess demand depends on both prices and initial endowments. While in the (DMS) perspective all the structure due to utility maximization is lost, in the (CEKP) setting, all the relevant structure is preserved in the sense that the initial economy can be recovered from the structure of the equilibrium manifold. Under regularity conditions, the price system can be solved in terms of initial endowments since price movements reflect fluctuations in the fundamentals of the economy.

We assume that the consumer is endowed with certain quantities of consumption commodities, his income is the market value of his endowment. We obtain a set of conditions that characterize the excess demand function when it depends smoothly on prices and endowments. More precisely, we get a symmetric negative semidefinite matrix, as in the Marshallian demand case, and a set of partial differential equations. The results of Chiappori and Ekeland regarding the integration problem and duality applies also to our setting.

The characterization problem will be introduced in the next section. Then, the main results are given. Proofs are grouped in an appendix.

2 Setting up the problem

Consider the problem

$$\max_x \; u(x)$$
$$p.x = p.\omega$$

Where $p \in \mathbb{R}^n_{++}$ is the price vector, $x \in \mathbb{R}^n_+$ is a consumption bundle and $\omega = (\omega^1, ..., \omega^n) \in \mathbb{R}^n_+$ is the vector of initial endowments. We suppose that u is of class C^k, $k \geq 2$, and $D^2_x u$ is negative definite on $\{D_x u\}^\perp$. Moreover, we assume that u is monotonic in the sense that $u(x) \geq u(y)$ whenever $x \neq y$ and $x \geq y$. Rewrite the above problem under the form

$$\max_z \ u(z+\omega)$$
$$p.z = 0$$

where $z = x - \omega$ is the individual excess demand function. We suppose here that the initial endowments are not fixed. Let $U(z,\omega) = u(z+\omega)$. Then, the above problem writes down

$$(\mathcal{P}) \begin{cases} \max_z \ U(z,\omega) \\ \quad p.z = 0 \end{cases}$$

This is a parametrized maximization problem with parameters p and ω. The implicit function theorem implies that the solution to this problem, called the individual excess demand function, is a C^{k-1} function of p and ω. Define the indirect utility function as follows

$$V(p,\omega) = \max\{U(z,\omega) \mid p.z = 0\} \tag{1}$$

Let $z(p,\omega)$ be the solution to this problem and $\lambda(p,\omega)$ be the corresponding Lagrange multiplier. Then, we have

$$V(p,\omega) = U(z(p,\omega),\omega) - \lambda(p,\omega)p.z(p,\omega) \tag{2}$$

The solution $z(p,\omega)$ is homogeneous of zero in p. This property and Walras' law imply that $(D_p z)p = 0,\ p'(D_p z) + z' = 0$ and $p'(D_\omega z) = 0$. The function V is quasiconvex and homogeneous of degree zero in p. We show later that $D^2_{pp}V$ is negative definite on $\{z\}^{\perp}\backslash\ span\{p\}$ provided that u satisfies the above assumptions. It is also true that $\frac{\partial V}{\partial \omega^i} > 0$ and $\frac{\partial V}{\partial p_i} z^i \leq 0$. The last inequality means that the function V is decreasing in p if $z^i \geq 0$, that is, if the consumer is net demander of good i. Similarly, V is increasing in p if $z^i \leq 0$, that is, if the consumer is net supplier of good i.

The next theorem gives relations between the indirect utility function and λ, z, p. These relations will be used to derive the conditions that characterize the individual excess demand function $z(p,\omega)$.

Theorem 1 *Let V be the indirect utility function of problem $(\mathcal{P})$, $z(p,\omega)$ be the solution of this problem and $\lambda(p,\omega)$ be the corresponding Lagrange multiplier . Then, the following relations are satisfied.*

$$\frac{\partial V}{\partial p_i}(p,\omega) = -\lambda(p,\omega)z^i(p,\omega) \tag{3}$$

$$\frac{\partial V}{\partial \omega^i}(p,\omega) = \lambda(p,\omega)p_i \tag{4}$$

Relations (3) and (4) give a sort of Roy's Identity in the excess demand case. Namely, we have

$$z^i = -\frac{p'\omega\ \partial V/\partial p_i}{\sum_i(\partial V/\partial \omega^i)\omega^i} \tag{5}$$

or equivalently

$$z^i = -\frac{\partial V/\partial p_i}{\partial V/\partial \omega^i} p_i$$

Notice that, the above (Roy's) identity implies that for each indirect utility function, there exists a unique excess demand function $z(p, \omega)$ in contrast to the case where z is a function of p only. In that case, for each V, there exist a continuum of excess demand functions. Just pick any positive, homogeneous of degree -1 function $\lambda(p)$ then $z(p) = \frac{1}{\lambda(p)} D_p V$ is an excess demand function, see Chiappori and Ekeland [4]. In our setting, λ cannot be chosen arbitrarily, from the above relations we must have $\lambda(p, \omega) = \frac{1}{p'\omega}(D_\omega V)'\omega$.

In the next section, we give the necessary and sufficient conditions for the existence of V and λ satisfying the above relations.

3 The main results

In the first part of this section we derive the necessary conditions satisfied by the excess demand function $z(p, \omega)$. Then, we see that these conditions are indeed sufficient. That is, any function satisfying these relations can be rationalized by some utility function, in fact, as we shall see, by a continuum of (direct) utility functions. Using relations (3) and (4), the differential of V takes the form

$$dV = -\lambda \left(\sum_i z^i(p, \omega) dp_i - \sum_i p_i d\omega^i \right) \tag{6}$$

Define the differential 1-form α as follows

$$\alpha = \sum_i z^i(p, \omega) dp_i - \sum_i p_i d\omega^i \tag{7}$$

It follows that

$$-\frac{1}{\lambda} dV = \alpha \tag{8}$$

The following theorem, which follows directly from Frobenius' theorem, gives the necessary and sufficient conditions for the existence of V and λ satisfying (8). In other words, we solve the *mathematical integration* problem.

Theorem 2 *There exist two functions λ and V such that $-\frac{1}{\lambda} dV = \alpha$ if and only if $\alpha \wedge d\alpha = 0$.*

The above theorem gives the necessary and sufficient conditions for mathematical integration. We can write these conditions more explicitly. This is established in the following theorem

Theorem 3 *Let α be the linear form defined by (7). Then, $\alpha \wedge d\alpha = 0$ if and only if*

$$\frac{\partial z^i}{\partial p_k}+\frac{1}{p.\omega}\left(\sum_j \frac{\partial z^i}{\partial \omega^j}\omega^j+\omega^i\right) z^k = \frac{\partial z^k}{\partial p_i}+\frac{1}{p.\omega}\left(\sum_j \frac{\partial z^k}{\partial \omega^j}\omega^j+\omega^k\right) z^i \quad (9)$$

$$\frac{\partial z^i}{\partial \omega^k} + \delta^i_k - \frac{1}{p.\omega}\left(\sum_j \frac{\partial z^i}{\partial \omega^j}\omega^j + \omega^i\right) p_k = 0 \quad (10)$$

where δ^i_k is the Kronecker symbol.

Let E be the $n \times n$ matrix defined by

$$E_{ik} = \frac{\partial z^i}{\partial p_k} + \frac{1}{p.\omega}\left(\sum_j \frac{\partial z^i}{\partial \omega^j}\omega^j + \omega^i\right) z^k \quad (11)$$

Using matrix notation, the matrix E takes the form

$$E = D_p z + \frac{1}{p.\omega}(D_\omega z + I_n)\omega z' \quad (12)$$

Conditions (9) and (10) write down

(a) $E = E'$.
(b) $D_\omega z + I_n - \frac{1}{p.\omega}(D_\omega z + I_n)\omega p' = 0$

One may wonder about the relation between the matrix E and the Slutsky matrix. In fact, the matrix E is the exact equivalent of Slutsky matrix. To see this, recall that $z(p,\omega) = x(p, p'\omega) - \omega$ where $x(p, y)$ is the Marshallian demand function and the Slutsky matrix $S = D_p x + (D_y x)x'$, where y is the individual income. Differentiating with respect to p we get, $D_p z = D_p x + (D_y x)\omega'$, $y = p'\omega$ is the individual income in our setting. Similarly, $D_\omega z = (D_y x)p' - I_n$. It follows that, $D_y x = \frac{1}{p'\omega}(D_\omega z + I_n)\omega$. Then we have

$$\begin{aligned} S = D_p x + (D_y x)x' &= D_p z - (D_y x)\omega' + (D_y x)(z' + \omega') \\ &= D_p z + \frac{1}{p'\omega}(D_\omega z + I_n)\omega z' \\ &= E \end{aligned}$$

It follows from this remark that the matrix E has the same definiteness property as S. So we have the following result

Theorem 4 *The matrix E is symmetric and negative semi-definite.*

Moreover, we have the following

Lemma 1 *If $D^2_{zz}U$ is negative definite on $\{D_z U\}^\perp$ then $D_p z$ and $D^2_{pp}V$ are both negative definite on $\{z\}^\perp \backslash span\{p\}$.*

Let us work out a simple example

Example 1 In this example, we find the excess demand function and the corresponding matrix E of the famous Cobb-Douglas utility function $U(x^1, x^2, ..., x^n) = k(x^1)^{a_1}(x^2)^{a_2}...(x^n)^{a_n}$, where $k > 0$, and $a_1 + \cdots + a_n = 1,\ a_i > 0$. Notice that the Cobb-Douglas utility function doesn't satisfy the assumptions we imposed on the direct utility function. However, we choose it for simplicity. We need to solve the problem

$$\max k(z^1 + \omega^1)^a (z^2 + \omega^2)^{a_2} ....(z^n + \omega^n)^{a_n}$$

$$p_1 z^1 + p_2 z^2 + \cdots + p_n z^n = 0$$

The solution to this problem is the excess demand function $z = (z^1, z^2, ..., z^n)$ given by

$$z^i(p, \omega) = \frac{a_i}{p_i} p'\omega - \omega^i \tag{13}$$

After simple calculations, we find that

$$E_{ij} = -\frac{a_i(\delta^i_j - a_j)}{p_i p_j} p'\omega \tag{14}$$

Its clear that $E_{ij} = E_{ji}$ and that all diagonal entries are negative. Take, for example, $n = 2$ and $a_1 = a$, $a_2 = 1 - a$. The (rank one) matrix E corresponding to the excess demand function in this case is given by

$$E = a(1-a) \begin{bmatrix} -\dfrac{p.\omega}{(p_1)^2} & \frac{p.\omega}{p_1 p_2} \\ \dfrac{p.\omega}{p_1 p_2} & -\frac{p.\omega}{(p_2)^2} \end{bmatrix}$$

Note that $|E| = 0$ and the upper left and lower right entries are negative. It follows that this matrix is negative semidefinite. □

Two remarks are in order

- The matrix E satisfies the conditions $Ep = 0$.
- Conditions (b) imply that $D_\omega z = -I_n$ on $\{p\}^\perp$. It follows that $D_\omega z$ is negative definite on $\{p\}^\perp$ and that the matrix $D_\omega z - \frac{1}{p'\omega}(D_\omega z + I_n)\omega p'$ is symmetric and negative definite on this space.

After investigating the connection between the matrix E and the Slutsky matrix S, what can we say about conditions (b)?

Notice that if we multiply $D_p z = D_p x + (D_y x)\omega'$ on the right by p, we get the homogeneity condition on x. In fact, conditions (b) is a variant of the zero homogeneity of x. To see this, we know that $D_p z = D_p x + (D_y x)\omega'$. Substituting $D_y x = \frac{1}{p'\omega}(D_\omega z + I_n)$ and multiplying (on the right) by p we get $(D_p x)p + (D_\omega z + I_n)\omega = 0$. By homogeneity of x, replace $(D_p x)p$ by $-(D_y x)y$ and divide both sides by $-y = -p'\omega$ to get $D_y x - \frac{1}{p'\omega}(D_\omega z + I_n)\omega = 0$. Now, right multiplying the last equality by p' and using the fact that $(D_y x)p' = D_\omega z + I_n$ imply that $D_\omega z + I_n - \frac{1}{p'\omega}(D_\omega z + I_n)\omega p' = 0$ which is (b).

We have proved that we can get the last relations using only the zero homogeneity of x. One can check that, in the standard individual problem, if we go through the same calculation as we did above, we will obtain the Slutsky matrix and the zero homogeneity of x with respect to prices and income.

We can write the matrix E under a different form as follows:

Lemma 2 *Let E be the matrix defined above. Then, $E = -(D_\omega z)(D_p z)$.*

Now, we give the necessary and sufficient conditions for the existence of an indirect utility function V and a positive function λ. In other words, we solve the *economic integration* problem.

Theorem 5 *Let $z(p, \omega) \in \mathbb{R}^n$ be of class C^1 and homogeneous of degree zero in p. Then, there exist a quasi-convex function $V(p, \omega)$ in p and a positive function $\lambda(p, \omega)$ such that $dV = -\lambda\alpha$ if conditions (9) and (10) are satisfied and if the matrix E is negative semi-definite.*

The above theorem holds locally. That is, the functions V, λ and z are defined locally in a neighborhood $\mathcal{U}$ of some point $(\bar{p}, \bar{\omega})$ of $\mathbb{R}^n_{++} \times \mathbb{R}^n_+$.

Now, given regular functions $V(p, \omega)$ and $z(p, \omega)$, satisfying the conditions of Theorem (5), can we find some utility function $U(p, \omega)$ such that $z(p, \omega)$ solves problem $(\mathcal{P})$. The process of going back from z to U is the standard integration problem in our setting where the excess demand function depends smoothly on prices and endowments.

Notice that, as in [4], while U is a function of $2n$ variables (in fact, n of them are parameters), V and z are functions of $2n-1$ variables because of homogeneity in p. Therefore, we have a non-uniqueness result, again as in [4], of direct utility function U that rationalizes $z(p, \omega)$. Assume, without loss of generality, that $\|\bar{p}\| = 1$.

To construct the function U, we mimic the steps carried out in [4] with minor modifications. In order to overcome the problem of dimension difference between V, z and U, define the map

$$\phi(p, \omega, t) = t(z(p, \omega), \omega)$$

from $S^{n-1} \times \mathbb{R}^n_+ \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n_+$. Then, the matrix $D_{p,\omega,t}\phi$ is of rank $2n$. It follows that ϕ is a diffeomorphism from some neighborhood $\mathcal{U} \times [1-\epsilon, 1+\epsilon]$ of $(\bar{p}, \bar{\omega}, 1)$ in $S^{n-1} \times \mathbb{R}^n \times \mathbb{R}$ onto a neighborhood of $(z(\bar{p}, \bar{\omega}), \bar{\omega})$ in $\mathbb{R}^n \times \mathbb{R}^n_+$. Therefore, the set

$$\mathcal{V} = \{(z, \omega) \mid (p, \omega) \in \mathcal{U}\} = \{\phi(p, \omega, 1) \mid (p, \omega) \in \mathcal{U}\}$$

is diffeomorphic to $\mathcal{U}$. Define the function $\tilde{U}$ as follows

$$\tilde{U}_A(p, \omega, t) = V(p, \omega) + (t-1)A(p, \omega, t)$$

where $A : \mathbb{R}^{2n}_{++} \times \mathbb{R} \to \mathbb{R}$ is any C^{k-1} function such that $A(p, \omega, 1) = \frac{\partial \tilde{U}_A}{\partial t}$. The proof in [4] applies here to show that the function $U = \tilde{U}_A$ on $\mathcal{V}$ is indeed strictly quasi-concave.

Appendix: Proofs of theorems

Proof of Theorem 1. Relations (3) follow from the envelope theorem. To derive (4), just differentiate $V(P,\omega) = u(z(p,\omega) + \omega)$ to get

$$\frac{\partial V}{\partial \omega^i} = \sum_k \frac{\partial u}{\partial x^k} \frac{\partial x^k}{\partial \omega^i} \tag{15}$$

where $x^k = z^k + \omega^k$. We have that

$$\frac{\partial x^k}{\partial \omega^i} = \frac{\partial z^k}{\partial \omega^i} + \delta^i_k$$

We get, using the first order conditions,

$$\begin{aligned}\frac{\partial V}{\partial \omega^i} &= \sum_k \lambda p_k (\frac{\partial z^k}{\partial \omega^i} + \delta^i_k) \\ &= \lambda(p,\omega) p_i\end{aligned}$$

This completes the proof. □

Proof of Theorem 3. We have, $\alpha \wedge d\alpha = 0$ if and only if there exists 1-form β such that $d\alpha = \beta \wedge \alpha$. Define the vector field ξ as follows

$$\xi = \sum_i p_i \frac{\partial}{\partial p_i} + \sum_i \omega^i \frac{\partial}{\partial \omega^i} \tag{16}$$

It follows, using Walras' law, that $< \alpha, \xi >= -p.\omega$. Applying the 2-form $d\alpha = \beta \wedge \alpha$ to the vector field ξ, we get $< d\alpha, (\xi, .) >=< \beta, \xi > \alpha - < \alpha, \xi > \beta$. This gives a formula for β given by $\beta = \frac{1}{p.\omega} < \beta, \xi > \alpha - \frac{1}{p.\omega} < d\alpha, (\xi, .) >$. It follows that $d\alpha = -\frac{1}{p.\omega} < d\alpha, (\xi, .) > \wedge \alpha$. By simple calculations, we can show that

$$< d\alpha, (\xi, .) >= \alpha + \sum_{i,k} \left(\frac{\partial z^i}{\partial \omega^k} \omega^k + \omega^i \right) dp_i \tag{17}$$

and

$$d\alpha = \sum_{i,k} \frac{\partial z^i}{\partial p_k} dp_k \wedge dp_i + \sum_{i,k} \frac{\partial z^i}{\partial \omega^k} d\omega^k \wedge dp_i - \sum_{i,k} \delta^i_k dp_i \wedge d\omega^k \tag{18}$$

The result follows from the two last equation. □

Proof of Lemma 1. Derive $V(p,\omega) = U(z(p,\omega),\omega)$ twice with respect to p we get

$$D^2_{pp} V = (D_p z)' D^2_{zz} U (D_p z) + \sum_k \frac{\partial U}{\partial z^k} D^2_{pp} z^k \tag{19}$$

To deal with the second term, just derive Walras' law twice with respect to p to see that

$$\sum_k \frac{\partial U}{\partial z^k} D^2_{pp} z^k = -\lambda(D_p z + (D_p z)') \tag{20}$$

Substituting from (20) into (19) and using the fact that $D^2_{pp}V = -\lambda(D_p z)' - z(D_p\lambda)'$, we get

$$D_p z = \frac{1}{\lambda}[(D_p z)' D^2_{zz} U(D_p z) + z(D_p\lambda)']$$

Let $\xi \in \{z\}^\perp$ then $(D_p z)\xi \in \{p\}^\perp$ since $p'(D_p Z)\xi = -z'\xi = 0$. It follows that

$$\xi'(D_p z)\xi = \frac{1}{\lambda}\eta'(D^2_{zz}U)\eta < 0$$

where $\eta = (D_p z)\xi$ (notice that if $\xi = p$ then $\eta = 0$) and the last inequality follows from the assumption that $D^2_{zz}U$ is negative definite on $\{D_z U\}^\perp$. □

Proof of Lemma 2. Differentiating $p.z(p,\omega) = 0$ with respect to p, we get $p'(D_p z) + z' = 0$. Substituting for z' in E, we get $E = D_p z + \frac{1}{p.\omega}(D_\omega z + I_n)\omega(-p' D_p z)$. Using conditions (b), E writes down $E = D_p z - (D_\omega z + I_n)(D_p z)$. It follows that $E = -(D_\omega z)(D_p z)$. □

Proof of Theorem 5. We apply the Convex Darboux theorem [7] . Conditions (9) and (10) guarantee the existence of λ and V such that $dV = -\lambda\alpha$ and the negativity condition on E implies that V is quasi-convex in p and that λ is positive using the convex Darboux theorem. This completes the proof. □

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