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A STUDY ON HOLLOW-LIFTING MODULES

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This thesis was defended successfully on August 6, 2013

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I would love to start with the words of my divine source of inspiration, Almighty Allah.

Qol Kolon ya'mal 'ala shakelateh farabokom a'lam beman howa ahda sabeela

Also, I would like to express my utmost honest gratitude to my mentor, professor Mohammad Saleh for his endless generosity. At last, my sincere thanks to anybody thinks I owe him.

Declaration

I certify that this thesis, submitted for the degree of Master of Mathematics to the Department of Mathematics at Birzeit University, is of my own research except where otherwise acknowledged, and that this thesis (or any part of it) has not been submitted for a higher degree to any other university or institution.

Mahmoud Ghannam Signature …………
August 6, 2013

Abstract

This humble project aims mainly to reveal the characteristics of what so-called hollow-lifting modules. We start with providing some basics concerning modules. Afterwards, in part three, we study hollow-lifting modules and other relevant concepts. In part four, we move towards direct sums of such modules. Lastly, we will study hollow-lifting modules over commutative rings.

Keywords: hollow-lifting, f-hollow-lifting, X-hollow-lifting, amply supplemented, duo, UCC.

الملخص

الهدف الرئيسي من هذه الرسالة هو دراسة وكشف خصائص ما يسمى بالمقاسات المجوفة الرافعة. في بداية هذا المشروع نقوم بتقديم بعض الخصائص العامة للمقاسات، و بعد ذلك- في الفصل الثالث - نقوم بدراسة المقاسات المجوفة الرافعة وبعض المفاهيم المتعلقة بها

في الفصل الرابع، يتم دراسة المجموعات المباشرة للمقاسات التي تم دراستها في الفصول السابقة. وفي النهاية ، نقوم بإدخال الحلقات التبادلية ودراسة المقاسات المجوفة الرافعة تحت تأثير هذا النوع من الحلقات

كلمات البحث: مجوفة رافعة، مجوفة رافعة محدودة، تكملة ، مشاركة مغلقة وحيدة، ثنائية

Contents

1 Introduction

In the past few years, lifting modules were heavily studied as any researcher in this field can see. Several mathematicians have introduced different generalizations of lifting modules. Actually, the title of this thesis is just one of those generalizations.

It is commonly known and accepted that, in 2007, Orhan [15] is the one who has introduced hollow-lifting modules. He called a module M hollow-lifting if every submodule N of M with M/N hollow has a coessential submodule in M that is a direct summand of M.

After some research, it has turned out that such concept was provided in 1984 by K. Csiro [17]. Csiro called a submodule N of M *small liftable* if N has a coessential submodule in M that is a direct summand of M. According to this, he defined a module M to *satisfy the lifting property of hollow modules* if every submodule N of M with M/N hollow is small liftable.

It is obvious that the two definitions of Orhan and Csiro are the same but named differently.

In [3], hollow-lifting modules were represented in a different terminology. It has been said that a module M is hollow-lifting (or briefly h-lifting) if it is amply supplemented and every hollow submodule N of M has a coessential submodule in M that is a direct summand of M.

Through their paper, A. A. Hassan and B. H. Al-Bahraany [7] have introduced a generalization called *finitely hollow-lifting modules*.

In 2012, direct sums of hollow-lifting modules were studied by Wang and Wu [9]. Add on, they have provided a relative concept called *X-hollow-lifting modules*.

Only to be clear, most of this thesis work is not new. All what we are trying to do is to clarify what has been done so far and review it in our own way. Hopefully, we could add something with this promising future.

Notation

R	Ring with unity
M	Right Module
$End(M)$	Endomorphism ring of M
$Hom(M, M')$	Module homomorphisms from M into M'
$Rad(M)$	Jacobson radical of M
$Soc(M)$	Socle of M
$h(M)$	The hollow dimension of M
$M \cong M'$	M is isomorphic to M'
$A \leq M$	A is a submodule of M
$A \leq^{\oplus} M$	A is a direct summand of M
$A \ll M$	A is small in M
$A \trianglelefteq M$	A is essential in M
$A \leq_{ce} M$	A is coessential in M
$A \leq_{cc} M$	A is coclosed in M
$\bigoplus_{i \in I} M_i$	Direct sum of modules
$f.g.$	Finitely generated
$Ker(f)$	Kernel of a map f
$Im(f)$	Image of a map f

2 Basics & Preliminaries

2.1 General Basics

In this section we recall some basic concepts in ring and module theory. Notice that, throughout this thesis R is a ring with identity and every R-module M is a unitary right R-module, unless otherwise stated. The notion $A \leq M$ will mean A is a submodule of M.

The first two definitions summarize, in general, some types and classes of modules and submodules needed in this thesis. Later, we shall reference them.

Definition 2.1.1 (*Some Types Of Submodules*)**.** *Let M be a module with A, B, N, and L submodules of M. Then*

(a) A is called a minimal submodule of M if $A \neq 0$ and whenever $X \leq M$, $X \lneqq A$ implies $X = 0$.

(b) A is called a maximal submodule of M if $A \neq M$ and whenever $X \leq M$, $A \lneqq X$ implies $X = M$.

(c) A is called a direct summand of M (denoted by $A \leq^{\oplus} M$), if $\exists\ X \leq M$ such that $M = A \oplus X$.

(d) A is called a cyclic submodule of M if $\exists\ m \in M$ such that $A = mR$.

(e) A is called small (or superfluous) in M (denoted by $A \ll M$) if whenever $X \leq M$, $M = A + X$ implies $X = M$. In this case, M is called a small cover of M/A.

(f) A is called essential (or large) in M (denoted by $A \trianglelefteq M$) if whenever $X \leq M$, $0 = A \cap X$ implies $X = 0$.

(g) If $A \leq B \leq M$ and $A \trianglelefteq B$, then B is called an essential extension of A in M.

(h) If $A \leq B \leq M$, then A is called a coessential submodule of B (or the inclusion $A \subseteq B$ is called cosmall in M) (denoted by $A \leq_{ce} B$) if $B/A \ll M/A$. In that case, B is called a coessential extension of A in M.

(i) A is called closed in M, if A has no proper essential extension in M.

(j) *A is called coclosed in M (denoted by $A \leq_{cc} M$), if A has no proper coessential submodule in M. i.e., if $X \leq_{ce} A$ then $X = A$.*

(k) *A is called strongly coclosed in M (denoted by $A \leq_{scc} M$), if for any $X \leq M$ with $A \nleq X$, $X \nleq_{ce} (A + X)$.*

(l) *A is called a coclosure (or s-closure) of B in M, if A is a coessential submodule of B and A is coclosed in M.*

(m) *N is called a supplement of L in M, if N is minimal with respect to $M = N + L$. Equivalently, $M = N + L$ and $N \cap L \ll N$.*

(n) *N is called a supplement of L in M, if $M = N + L$ and $N \cap L \ll M$.*

(o) *N is called a strong supplement of L in M, if N is a supplement of L in M and $N \cap L \leq^{\oplus} L$.*

(p) *N is called a complement of L in M, if N is maximal with respect to $0 = N \cap L$. Equivalently, $0 = N \cap L$ and $(N + L)/N \trianglelefteq M/N$.*

(q) *A is called fully invariant if $f(A) \subseteq A \ \forall \ f \in End_R(M)$.*

<u>Definition</u> 2.1.2 (*Some Types Of Modules*)**.** *Let M be a module. Then*

(a) *M is called simple if $M \neq 0$ and has no nonzero proper submodules.*

(b) *M is called semisimple (or completely reducible) if every submodule of M is a direct summand. Clearly, any simple module is semisimple.*

(c) *M is called indecomposable or (directly indecomposable) if $M \neq 0$ and has only 0 and M as direct summands. Clearly, any simple module is indecomposable.*

(d) *M is called coatomic if every proper submodule of M is contained in a maximal submodule of M.*

(e) *M is called radical if $Rad(M) = M$.*

(f) *M is called hollow if $M \neq 0$ and every proper submodule is small in M. Equivalently, if $A, B \leq M$ with $A + B = M$, then either $A = M$ or $B = M$.*

(g) *M is called local if it is hollow and has a unique maximal submodule.*

(h) *M is called (weakly)supplemented if every submodule of M has a (weak) supplement in M.*

(i) *M is called amply supplemented if for every* $A, B \leq M$ *with* $M = A + B$, *A has a supplement in M contained in B.*

(j) *M is called lifting (or has* (D_1)*), if for every submodule N of M, there exists a direct summand K of M such that K is a coessential submodule of N in M.*

(k) *M is said to have* (D_2) *if whenever* $A \leq M$ *with* M/A *isomorphic to a direct summand in M, then* $A \leq^{\oplus} M$

(l) *M is said to have* (D_3)*, if for every direct summands A and B of M with* $M = A + B$, $A \cap B$ *is a direct summand of M.*

(m) *M is called quasi-discrete if it is lifting and has* (D_3).

(n) *M is called discrete if it has* (D_1) *and* (D_2).

(o) *M is called hollow-lifting if every submodule N of M with* M/N *hollow has a coessential submodule in M that is a direct summand of M.*

(p) *M is called a duo module if every submodule of M is fully invariant.*

(q) *M is said to have finite hollow dimension, if there exists an epimorphism* $f : M \longrightarrow \oplus_{i=1}^{k} H_i$ *with each* H_i *hollow and* $Ker f \ll M$. *In that case we say that a hollow dimension of M is k, denoted by* $h(M) = k$.

The following concepts are elementary and can be found in [1], [8], and [21].

Definition 2.1.3 (*Bases And Free Modules*)**.** *Let M be an R-module and $X \subseteq M$. Then*

(a) We define the submodule of M generated by X [denoted by $\mid X)$ or XR] as the following.

$$XR = \left\{ \begin{array}{ll} \left\{ \sum_{i=1}^{n} x_i r_i \quad \mid \quad x_i \in X, r_i \in R, \text{and } n \in \mathbb{N} \right\} & \text{if } X \neq \phi \\ 0 & \text{if } X = \phi \end{array} \right\}$$

(b) X is called a generating set of M if $XR = M$.

(c) X is called linearly independent if for any finite subset $\{x_1, \ldots, x_m\} \subseteq X$ with $x_i \neq x_j$, it follows from $\sum_{i=1}^{m} x_i r_i = 0$ with $r_i \in R$ that $r_i = 0$ $\forall\ i = 1, 2, \ldots, m$.

(d) X is called a basis of M if X is a linearly independent generating set of M.

(e) M is called free if it has a basis.

(f) M is called finitely generated (denoted by f.g.) if it has a finite generating set. By Zorn's Lemma, we can easily show that if M is a nonzero f.g. module, then every proper submodule is contained a maximal submodule of M.

Lemma 2.1.1 (Modular Law)**.** [8, *Lemma* 2.3.15] *Let M be a module and $A, B, C \leq M$ with $B \leq C$. Then $(A + B) \cap C = (A \cap C) + B$.*

Proof. Let $x \in (A+B) \cap C$. Then $x = c = a+b$ for some $a \in A$, $b \in B$, and $c \in C$. But $B \leq C$, therefore, $a = c-b \in (A \cap C)$ and hence $x \in (A \cap C)+B$. On the other hand, Let $y \in (A \cap C)+B$. Then $y = d+b$ for some $b \in B$ and $d \in (A \cap C)$. Again, since $B \leq C$, $y = d + b \in (A + B) \cap C$. This completes the proof. □

Lemma 2.1.2. [8, *Lemma* 3.1.5] *Let M, M', M'' be modules with $A \leq M$ and $B \leq M'$, and let $f : M \longrightarrow M'$, $g : M' \longrightarrow M''$ be homomorphisms. Then* *(a)* f *is a monomorphism* $\Leftrightarrow Ker(f) = 0$.

(b) $f^{-1}(f(A)) = A + Ker(f)$.

(c) $f(f^{-1}(B)) = B \cap Im(f)$.

(d) $Ker(gf) = f^{-1}(Ker(g))$.

(e) $Im(gf) = g(Im(f))$.

Proof. (a) If f is a monomorphism and $x \in Ker(f)$, then $f(x) = 0 = f(0)$ and hence $x = 0$. Conversely, if $Ker(f) = 0$ and $f(x) = f(y)$, then we get that $x - y \in Ker(f)$ and hence $x = y$.

(b) If $x \in f^{-1}(f(A))$ then $f(x) = f(a)$ for some $a \in A$ and hence $x - a \in Ker(f)$. Therefore, $x - a = k$ for some $k \in Ker(f)$ which implies $x = a+k \in A+Ker(f)$. On the other hand, if $x \in A+Ker(f)$ then $x = a+k$ for some $a \in A$ and $k \in Ker(f)$. Therefore, $f(x) = f(a+k) = f(a) \in f(A)$, hence $x \in f^{-1}(f(A))$.

(c) If $y \in f(f^{-1}(B))$ then $y = f(x)$ with $x \in f^{-1}(B)$ and hence $f(x) \in B$. Thus, $y \in B \cap Im(f)$. Conversely, if $y \in B \cap Im(f)$ then $y = b = f(x)$ with $b \in B$ and $x \in M$ which implies $x \in f^{-1}(B)$. Hence, $y \in f(f^{-1}(B))$.

(d) $Ker(gf) = (gf)^{-1}(0) = (f^{-1}g^{-1})(0) = f^{-1}(g^{-1}(0)) = f^{-1}(Ker(g))$.

(e) $Im(gf) = (gf)(M) = g(f(M)) = g(Im(f))$. □

Theorem 2.1.1 (*The Factor Theorem*)**.** [1, *Theorem* 3.6] *Let* M*,* N*, and* K *be modules with a homomorphism* $f : M \longrightarrow N$*. If* $g : M \longrightarrow K$ *is an epimorphism with* $Ker(g) \subseteq Ker(f)$*, then* $\exists!$ *homomorphism* $h : K \longrightarrow N$ *such that* $f = hg$*. Moreover,* $Im(h) = Im(f)$ *and* $Ker(h) = g(Ker(f))$*.Hence,* h *is an epimorphism iff* f *is an epimorphism and* h *is a monomorphism iff* $Ker(g) = Ker(f)$*.*

Proof. Since g is an epimorphism, for any $k \in K$ there is $m \in M$ with $g(m) = k$. Moreover, if $l \in M$ with $g(l) = k$, then $m - l \in Ker(g) \subseteq Ker(f)$ and hence $f(m) = f(l)$. Therefore, there exists a well-defined mapping $h : K \longrightarrow N$ and such h is indeed a module homomorphism. Now, for the uniqueness of h, if $\bar{h} : K \longrightarrow N$ is another homomorphism with $f = \bar{h}g$, then $hg = \bar{h}g$. Consequently, since g is an epimorphism, $h = \bar{h}$. The rest of this theorem is clear. □

Theorem 2.1.2 (*The Homomorphism Theorem*)**.** [8, *Corollary* 3.4.2] *Let* $f : M \longrightarrow M'$ *be a module homomorphism. Then* $M/Ker(f) \cong Im(f)$*.*

Proof. Consider the natural epimorphism $\pi : M \longrightarrow M/Ker(f)$. Then clearly $Ker(\pi) = Ker(f)$. Therefore, by The Factor Theorem, there exists a monomorphism $h : M/Ker(f) \longrightarrow M'$ with $f = h\pi$. Take the homomorphism $\bar{h} : M/Ker(f) \longrightarrow Im(f)$. Then $\bar{h}$ is an isomorphism. Thus, $M/Ker(f) \cong Im(f)$ □

Theorem 2.1.3 (*The First Isomorphism Theorem*). [8, *Theorem* 3.4.3] *Let A and B be submodules of M. Then $(A + B)/B \cong A/(A \cap B)$.*

Proof. Define the mapping $f : A \longrightarrow (A + B)/B$. Then easily we can show that f is an epimorphism and $Ker(f) = A \cap B$. Thus, the proof follows from The Homomorphism Theorem. □

Theorem 2.1.4 (*The Second Isomorphism Theorem*). [8, *Theorem* 3.4.6] *If $A \leq B \leq M$, then $M/B \cong (M/A)/(B/A)$.*

Proof. Define the mapping $f : M/A \longrightarrow M/B$. Then easily we can show that f is an epimorphism and $Ker(f) = B/A$. Thus, the proof follows from The Homomorphism Theorem. □

Corollary 2.1.1. [8, *Corollary* 3.4.4] *If $M = A \oplus B$, then $M/A \cong B$.*

Proof. $M/A = (A + B)/A \cong B/(A \cap B) = B/0 \cong B$. □

The next four definitions are basics in [1] & [8].

Definition 2.1.4 (*Small And Essential Homomorphisms*). *Let M, M' be modules and $f : M \longrightarrow M'$ be a homomorphism. Then f is called small (essential) if $Ker(f) \ll M$ ($Im(f) \trianglelefteq M'$).*

Definition 2.1.5 (*The Jacobson radical And Socle Of A Module*). *Let M be a module. Then*

(a) The Jacobson radical of M (denoted by $Rad(M)$) is the intersection of all maximal submodules of M. Equivalently, $Rad(M)$ is the sum of all small submodules of M. If there are no maximal submodules, we put $Rad(M) = M$.

(b) The socle of M (denoted by $Soc(M)$) is the sum of all minimal submodules of M. Equivalently, $Soc(M)$ is the intersection of all essential submodules of M. If there are no minimal submodules, we put $Soc(M) = 0$.

<u>Definition</u> 2.1.6 (*Exactness*)**.** *Let*

$$S = \cdots \xrightarrow{f_{i-2}} M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \cdots$$

be a sequence of homomorphisms f_i and modules M_i for $i \in I$. Then

(a) S is called exact if for any subsequence of the form

$$M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1}$$

it follows that $Im(f_{i-1}) = Ker(f_i)$.

(b) S is called split exact if it is exact and $Im(f_{i-1}) = Ker(f_i) \leq^{\oplus} M_i$ $\forall\ i \in I$.

(c) An exact sequence of the form

$$0 \longrightarrow M \xrightarrow{f} M' \xrightarrow{g} M'' \longrightarrow 0$$

is called a short exact sequence. Clearly, this short sequence is exact iff f is a monomorphism, g is an epimorphism, and $M'/Im(f) \cong M''$.

(d) If M is a module and $A \leq M$, then we get the following short exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} M \xrightarrow{\nu} M/A \longrightarrow 0$$

where ι is the inclusion map and ν is the natural epimorphism.

<u>Definition</u> 2.1.7 (*Split Homomorphisms*)**.** *Let M, M' be modules. Then*

(a) A monomorphism $f : M \longrightarrow M'$ is said to split if $Im(f) \leq^{\oplus} M'$. Equivalently, f is a split monomorphism iff there exists a homomorphism $f' : M' \longrightarrow M$ with $f'f = id_M$.

(b) An epimorphism $g : M \longrightarrow M'$ is said to split if $Ker(g) \leq^{\oplus} M$. Equivalently, g is a split epimorphism iff there exists a homomorphism $g' : M' \longrightarrow M$ with $gg' = id_{M'}$.

<u>Lemma</u> 2.1.3. [3, *Lemma* 1.24] *Let M be a module and $A, B, C \leq M$ with $A + B = M$, $(A \cap B) + C = M$. Then $A + (B \cap C) = B + (A \cap C) = M$.*

Proof.

$$\begin{aligned} A+(B\cap C) &= A+(A\cap B)+(B\cap C) \\ &= A+[(B+B\cap A)\cap(B\cap A+C)] \\ &= A+[B\cap(B\cap A+C)] \\ &= A+(B\cap M) \\ &= A+B \\ &= M \end{aligned}$$

A similar argument for $B+(A\cap C)=M$. □

Lemma 2.1.4. *If $M=A\oplus B$ and $X\leq A$ with $M=X+B$, then $A=X$.*

Proof. $A=A\cap M=A\cap(X+B)=(A\cap X)+(A\cap B)=X+0=X$. □

Lemma 2.1.5. [3, §2.2 & §2.3] *Let M, M', M'' be right R-modules. Then*

(a) $A\ll B\leq M$ *implies* $A\ll M$.

(b) $A_i\ll M$ $(i=1,\dots,n)\Longleftrightarrow \sum_{i=1}^{n} A_i\ll M$.

(c) If $A\leq B\leq M$, *then* $B\ll M\Leftrightarrow A\ll M$ *and* $B/A\ll M/A$.

(d) If $A\leq B\leq M$ *and* $B\leq^{\oplus} M$, *then* $A\ll M\Leftrightarrow A\ll B$.

(e) If $A\leq B\leq M$ *and* $B\leq_{cc} M$, *then* $A\ll M\Leftrightarrow A\ll B$.

(f) $A\ll M\Leftrightarrow$ *the natural epimorphism* $f:M\longrightarrow M/A$ *is a small epimorphism.*

(g) If $A\ll M$, *then* M *is f.g.* $\Leftrightarrow M/A$ *is f.g.*

(h) If $A,B,C\leq M$ *with* $B\leq C$, $M=A+B$, *and* $C/B\ll M/B$, *then* $(A\cap C)/(A\cap B)\ll M/(A\cap B)$.

(i) $A\ll M$ *and* $f\in Hom_R(M,M')$ *implies* $f(A)\ll M'$.

(j) Let $f:M\longrightarrow M'$ *be a small epimorphism. If* $C\ll M'$, *then* $f^{-1}(C)\ll M$.

(k) If $f:M\longrightarrow M'$ *and* $g:M'\longrightarrow M''$ *are small epimorphisms, then* $gf:M\longrightarrow M''$ *is also a small epimorphism.*

(l) If $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$, and $M = M_1 \oplus M_2$, then
$K_1 \oplus K_2 \ll M_1 \oplus M_2 \Leftrightarrow K_1 \ll M_1$ and $K_2 \ll M_2$.

Proof. (a) Let $X \leq M$ with $A + X = M$. Then $B + X = M$, therefore, $B + (X \cap C) = M \cap C = C$. But $B \ll C$, so $X \cap C = C$, hence $C \leq X$. This implies $X = A + X = M$. Thus $A \ll M$.

(b) Let $X \leq M$ with $\sum_{i=1}^{n} A_i + X = M$. Since $A_1 \ll M$, $\sum_{i=1}^{n-1} A_i + X = M$. Again, Since $A_2 \ll M$, $\sum_{i=1}^{n-2} A_i + X = M$. Proceed this process, you get $X = M$ and hence $\sum_{i=1}^{n} A_i \ll M$. The converse follows from (a).

(c) Suppose $A \leq B \ll M$. From (a), it follows that $A \ll M$. Moreover, if $B/A + X/A = M/A$ with $A \leq X \leq M$, then $B + X = M$. But $B \ll M$, so $X = M$ and hence $B/A \ll M/A$. Conversely, let $X \leq M$ with $X + B = M$. Then $M/A = (X + B)/A = (X + B + A)/A = B/A + (X + A)/A$. But $B/A \ll M/A$, so $(X + A)/A = M/A$ which implies $M = X + A$. Consequently, since $A \ll M$, $X = M$. Thus $B \ll M$.

(d) Suppose $M = B \oplus C$ for some $C \leq M$ and $X \leq B$ with $A + X = B$. Then $A + X + C = B + C = M$. But $A \ll M$, so $X + C = M$. Now, $B = B \cap M = B \cap (X + C) = X + (B \cap C) = X + 0 = X$. Hence, $A \ll B$. The converse follows from (a).

(e) Let $X \leq B$ with $A + X = B$. The proof ends if we show that $B/X \ll M/X$. So let $X \leq C \leq M$ with $B/X + C/X = M/X$. Then $M = B + C = A + X + C$. But $A \ll M$, so $M = X + C = C$. Hence, $B/X \ll M/X$. The converse follows from (a).

(f) Since $Ker(f) = A$, the proof is trivial.

(g) If M has a finite generating set, say $\{m_i \mid i = 1, 2, \ldots, n\}$, then clearly the set $\{m_i + A \mid i = 1, 2, \ldots, n\}$ is a finite generating set of M/A. Conversely, let $\{m_i + A \mid i = 1, 2, \ldots, n\}$ is a finite generating set of M/A. If $m \in M$, then $m + A \in M/A$ and hence $m + A = \sum_{i=1}^{n} m_i r_i + A$. This means that $m_1 R + \cdots + m_n R + A = M$. But $A \ll M$, therefore $m_1 R + \cdots + m_n R = M$ and hence M is $f.g.$

(h) Let $M/(A\cap B) = (A\cap C)/(A\cap B) + X/(A\cap B)$ with $A\cap B \leq X \leq M$. Then $M = (A \cap C) + X$. But $B \leq C$, so $M = A + B \leq A + C$, hence $M = A + C$. Therefore, by lemma 2.1.3., $M = (A \cap X) + C$ and hence $M/B = C/B + ((A \cap X) + B)/B$, but $C/B \ll M/B$, hence $M = B + (A \cap X)$. Again, by lemma 2.1.3., $M = X + (A \cap B) = X$. Thus, $(A \cap C)/(A \cap B) \ll M/(A \cap B)$.

(i) Suppose that $f(A) + X = M'$ such that $X \leq M'$. Then we have $M = f^{-1}(M') = f^{-1}(f(A) + X) = A + f^{-1}(X)$. Since $A \ll M$, we get $f^{-1}(X) = M$, therefore $A \leq f^{-1}(X)$ which implies $f(A) \leq X$. Hence, $M' = X$. Thus $f(A) \ll M'$.

(j) Let $M = f^{-1}(C) + X$. Then $M' = f(f^{-1}(C) + X) = C + f(X)$. Since $C \ll M$, $M' = f(X)$. Hence, $M = f^{-1}(f(X)) = X + Ker(f)$. But $Ker(f) \ll M$, therefore $X = M$. Thus, $f^{-1}(C) \ll M$.

(k) Since f and g are epimorphisms, gf is also an epimorphism. Now, Let $X \leq M$ with $Ker(gf) + X = M$. Since $Ker(gf) = f^{-1}(Ker(g))$, $M' = f(M) = Ker(g) + f(X)$. But $Ker(g) \ll M'$, so $f(X) = M'$. Consequently, $M = f^{-1}(M') = f^{-1}(f(X)) = X + Ker(f)$. As $Ker(f) \ll M$, we have $X = M$. Thus, gf is a small epimorphism.

(l) Let $\pi_1 : M \longrightarrow M_1$, $\pi_2 : M \longrightarrow M_2$ be the epimorphism projections of M onto M_1 and M_2 respectively. Then $\pi_1(K_1) = K_1$ and $\pi_2(K_2) = K_2$. Since $x \in \pi_1(K_1)$, $x = \pi_1(k_1)$; $k_1 \in K_1$, therefore $x = \pi_1(k_1 + 0) = k_1 \in K_1$ and $k_1 \in K_1$. This implies that $k_1 = k_1 + 0 = \pi_1(k_1) \in \pi_1(K_1)$. So $\pi_1(K_1) = K_1$. Similarly, $\pi_2(K_2) = K_2$. Now, $\pi_1(K_1 \oplus K_2) = \pi_1(K_1) \oplus \pi_1(K_2)$ $= \pi_1(K_1) \oplus 0 = K_1$. Similarly, $\pi_2(K_1 \oplus K_2) = K_2$. But $K_1 \oplus K_2 \ll M$, hence $K_1 \ll M_1$ and $K_2 \ll M_2$. Conversely, if $K_1 \ll M_1 \leq M$ and $K_2 \ll M_2 \leq M$, then $K_1 \oplus K_2 = K_1 + K_2 \ll M = M_1 \oplus M_2$. □

Lemma 2.1.6. [3, §3.2] *Let M be a module. Then*

(a) The following are equivalent.

(i) $A \leq_{ce} B$ in M.

(ii) For any submodule $X \leq M$, $B + X = M$ implies $A + X = M$.

(b) $A \leq_{ce} B$ in M and $B \leq_{ce} C$ in M iff $A \leq_{ce} C$ in M.

(c) If $A \leq_{ce} B$ in M and $N \ll M$, then $A \leq_{ce} B + N$ in M.

(d) If $A \leq_{ce} B$ in M and $C \leq M$, then $A + C \leq_{ce} B + C$ in M. The converse is true if $C \ll M$.

(e) If $A \leq_{ce} B$ in M and $C \leq_{ce} D$ in M, then $A + C \leq_{ce} B + D$ in M.

(f) For $A \leq B \leq M$, $B \leq_{ce} C$ in M iff $B/A \leq_{ce} C/A$ in M/A.

(g) If $A \leq_{ce} B$ in M and $A + X = M$, then $A \cap X \leq_{ce} B \cap X$ in M.

(h) If $N \ll M$ and $A \leq M$, then $A \leq_{ce} A + N$ in M.

(i) If $M = A + B$, $A \leq C$, and $C \cap B \ll M$, then $A \leq_{ce} C$ in M.

(j) Let $f : M \longrightarrow M'$ be an epimorphism. If $A \leq_{ce} B$ in M, then $f(A) \leq_{ce} f(B)$ in M'.

(k) Let $f : M \longrightarrow M'$ be a small epimorphism. Then

(i) If $C \leq_{ce} D$ in M', then $f^{-1}(C) \leq_{ce} f^{-1}(D)$ in M.

(ii) If $M = M_1 \oplus M_2$, then $f(M_1) \cap f(M_2) \ll M'$.

Proof. (a) $(i) \implies (ii)$ Since $M = B + X = B + X + A$, we get that $M/A = B/A + (X + A)/A$. But $B/A \ll M/A$, so $X + A = M$.

$(ii) \implies (i)$ Let $A \leq X \leq M$ with $B/A+X/A = M/A$. Then $B+X = M$. By (ii), $M = A + X = X$. Hence $A \leq_{ce} B$ in M.

(b) $\boxed{\implies}$ Let $A \leq X \leq M$ with $C/A + X/A = M/A$. Then $C + X = M$ and hence $C/B + (X + B)/B = M/B$. But $C/B \ll M/B$, so $X + B = M$. Consequently, $X/A + B/A = M/A$. Since $B/A \ll M/A$, we have $X = M$. Thus, $A \leq_{ce} C$ in M.

$\boxed{\impliedby}$ Let $A \leq X \leq M$ with $B/A + X/A = M/A$. Then $B + X = M$ and hence $C + X = M$ since $B \leq C$. It follows that $C/A + X/A = M/A$. But $C/A \ll M/A$, therefore $X = M$. Thus, $A \leq_{ce} B$ in M. Now, let $B \leq X \leq M$ with $C/B + X/B = M/B$. Then $C + X = M$ and hence $C/A + X/A = M/A$. Since $C/A \ll M/A$, $X = M$. Therefore, $B \leq_{ce} C$ in M.

(c) Suppose $A \leq X \leq M$ with $(B + N)/A + X/A = M/A$. Then $B+N+X = M$. But $N \ll M$, so $B+X = M$ and hence $B/A+X/A = M/A$. Since $B/A \ll M/A$, $X = M$. Therefore, $A \leq_{ce} B + N$ in M.

(d) Let $A+C \leq X \leq M$ with $(B+C)/(A+X)+X/(A+C) = M/(A+C)$. Then $B+C+X = M$ and hence $(B+C+X)/A = M/A$. But $B/A \ll M/A$, hence $M = C + X = X$. Thus, $A + C \leq_{ce} B + C$ in M. For the converse, suppose $C \ll M$ and $A + C \leq_{ce} B + C$ in M. Let $A \leq X \leq M$ with $B/A + X/A = M/A$. Then we get $B + X = M$ and hence $(B + C)/(A + C)$ $= (X + C)/(A + C) = M/(A + C)$. Since $A + C \leq_{ce} B + C$ in M, we have $X + C = M$. But $C \ll M$, therefore $X = M$. Thus, $A \leq_{ce} B$ in M.

(e) From (d), $A + C \leq_{ce} B + C$ in M and $B + C \leq_{ce} B + D$ in M. Therefore, by (b), $A + C \leq_{ce} B + D$ in M.

(f) $\boxed{\Longrightarrow}$ Let $A \leq X \leq M$ and $B \leq X$ with $(C/A)/(B/A)+(X/A)/(B/A) = (M/A)/(B/A)$. Then $C + X = M$ and hence $C/B + X/B = M/B$. But $C/B \ll M/B$, therefore $X = M$. Thus, $B/A \leq_{ce} C/A$ in M/A.

$\boxed{\Longleftarrow}$ Let $B \leq X \leq M$ with $C/B + X/B = M/B$. This implies that $C/A+X/A = M/A$ and hence $(C/A)/(B/A)+(X/A)/(B/A) = (M/A)/(B/A)$. Since $B/A \leq_{ce} C/A$ in M/A, we have $X = M$. Therefore, $B \leq_{ce} C$ in M.

(g) Let $A\cap X \leq C \leq M$ with $(B\cap X)/(A\cap X)+C/(A\cap X) = M/(A\cap X)$. Then $(B \cap X) + C = M$. Moreover, $B + X = M$ since $A + X = M$ and $A \leq B$. Hence, by lemma 2.1.3, we have $M = B+(X\cap C) = A+(X\cap C) = (A \cap X) + C = C$. Thus, $A \cap X \leq_{ce} B \cap X$ in M.

(h) Let $X \leq M$ with $A+N+X = M$. Since $N \ll M$, we have $A+X = M$ and hence $A \leq_{ce} A + N$ (by part (a)).

(i) Let $X \leq M$ with $C + X = M$. Then $C = C \cap M = C \cap (A + B) = A + (C \cap B)$, hence $M = C + X = A + (C \cap B) + X$. Since $C \cap B \ll M$, $M = A + X$. Therefore, $A \leq_{ce} C$ in M.

(j) Let $M' = f(B) + Y$ for some $Y \leq M'$. Since f is an epimorphism, there is $X \leq M$ such that $f(X) = Y$. Therefore, $M = f^{-1}(M') = f^{-1}(f(B + X)) = B + X + Ker(f)$. But $A \leq_{ce} B$ in M, so ,by part (a), we have $M = A+X+Ker(f)$. Consequently, $M' = f(A)+f(X)+f(Ker(f)) = f(A) + Y$. Hence, $f(A) \leq_{ce} f(B)$ in M'.

(k) Let $M = f^{-1}(D) + X$ with $X \leq M$. Then $M' = f(f^{-1}(D) + X) = D + f(X)$ and hence $M' = C + f(X)$ since $C \leq_{ce} D$ in M'. This implies that $M = f^{-1}(C + f(X)) = f^{-1}(C) + X + Ker(f)$. But $Ker(f) \ll M$, so $M = f^{-1}(C) + X$. Thus, $f^{-1}(C) \leq_{ce} f^{-1}(D)$ in M.

(ii) Since $Ker(f) \ll M$, $M_1 \leq_{ce} M_1 + Ker(f)$ [*Lemma* 2.1.6(h)]. So, $[M_2 \cap (M_1 + Ker(f))]/(M_2 \cap M_1) \ll M/(M_2 \cap M_1)$ [*Lemma* 2.1.5(h)] and hence $M_2 \cap (M_1 + Ker(f)) \ll M$ since $M_1 \cap M_2 = 0$. Consequently, we get $f[M_2 \cap (M_1 + Ker(f))] \ll M'$. Since $M_1 + Ker(f) = f^{-1}(M_1)$, we have $f[M_2 \cap (M_1 + Ker(f))] = f(M_1) \cap f(M_2)$. Thus, $f(M_1) \cap f(M_2) \ll M'$. □

<u>Lemma</u> 2.1.7. [3, §3.7] *Let M be a module. Then*

(a) Any direct summand of M is coclosed in M.

(b) If $A \leq B \leq M$ and $B \leq_{cc} M$, then $B/A \leq_{cc} M/A$.

(c) If $A \leq_{cc} M$ and $N \ll M$, then $(A+N)/N \leq_{cc} M/N$.

(d) For $A \leq B \leq M$, if $A \leq_{cc} M$ then $A \leq_{cc} B$. The converse is true if $B \leq_{cc} M$.

(e) Let $f : M \longrightarrow M'$ be a small epimorphism. If $A \leq_{cc} M$, then $f(A) \leq_{cc} M'$.

Proof. (a) Let $A \leq^{\oplus} M$. Then $M = A \oplus B$ for some $B \leq M$. If $A' \leq_{ce} A$ in M, then $M/A' = (A+B)/A' = A/A' + (B+A')/A'$. But $A/A' \ll M/A'$, hence $B + A' = M$. Thus, by lemma 2.1.4, $A = A'$. Therefore, $A \leq_{cc} M$.

(b) Let $X/A \leq_{ce} B/A$ in M/A. Then, by the previous lemma, $X \leq_{ce} B$ in M. Since $B \leq_{cc} M$, $X = B$. Therefore, $B/A \leq_{cc} M/A$.

(c) Let $X/N \leq_{ce} (A+N)/N$ in M/N. Since $N \ll M$, it follows from the previous lemma that $X \leq_{ce} A+N$ in M. But $N+(A \cap X) = (N+A) \cap X = X$, hence $N + (A \cap X) \leq_{ce} A + N$. Therefore, we have $A \cap X \leq_{ce} A$ in M. Since $A \leq_{cc} M$, $A \cap X = A$ and so $A \leq X$. But $N \leq X$, so $X = A + N$. Thus, $(A+N)/N \leq_{cc} M/N$.

(d) Let $X \leq A$ with $A/X \ll B/X$. Then $A/X \ll M/X$. But $A \leq_{cc} M$, so $X = A$. Hence, $A \leq_{cc} B$. For the converse, suppose $A \leq_{cc} B$ and $B \leq_{cc} M$. Let $X \leq A \leq M$ with $A/X \ll M/X$. It follows from (b) that $B/X \leq_{cc} M/X$ and consequently $A/X \ll B/X$ which means that $X \leq_{ce} A$ in B. Since $A \leq_{cc} B$, $X = A$. Therefore, $A \leq_{cc} M$.

(e) Let $K' \leq_{ce} f(A)$ in M'. Since f is an epimorphism, there exists $K \leq A$ with $f(K) = K'$. By lemma 2.1.6(k), $f^{-1}(f(K)) \leq_{ce} f^{-1}(f(A))$ in M. Now, $f^{-1}(f(K)) = K + Ker(f)$, $f^{-1}(f(A)) = A + Ker(f)$, and $Ker(f) \ll M$. So $K \leq_{ce} K + Ker(f) \leq_{ce} A + Ker(f)$ [*Lemma* 2.1.6(h)] and hence $K \leq_{ce} A + Ker(f)$ [*Lemma* 2.1.6(b)]. Consequently, since $K \leq A$, we have $K \leq_{ce} A$ in M. But $A \leq_{cc} M$, so $K = A$ and hence $K' = f(K) = f(A)$. Therefore, $f(A) \leq_{cc} M'$. □

Theorem 2.1.5. [21, §21.6] *Let M be a module. Then*

(a) If $A \leq M$, then $Rad(A) \leq Rad(M)$.

(b) If $f : M \longrightarrow M'$ is a homomorphism, then $f(Rad(M)) \leq Rad(M')$.

(c) If $M = \bigoplus_{i \in I} M_i$, then $Rad(M) = \bigoplus_{i \in I} Rad(M_i)$.

(d) M is finitely generated $\Leftrightarrow$ $Rad(M) \ll M$ and $M/Rad(M)$ is finitely generated.

Proof. (a) If $X \ll A$, then $X \ll M$. Therefore, $Rad(A) \leq Rad(M)$.

(b) If $A \ll M$, then $f(A) \ll M'$. Hence, $f(Rad(M)) \leq Rad(M')$.

(c) Follows from (a) and (b).

(d) If M is finitely generated, then it has a maximal submodule. Therefore, $Rad(M) \neq M$. Now, if $Rad(M) + A = M$ and $A \neq M$, then A is contained is some maximal submodule, say B. This implies $M = Rad(M)+A = Rad(M) + B = B$, a contradiction. Hence $Rad(M) \ll M$. The converse follows from lemma 2.1.5,(g). □

Definition 2.1.8 (*Irredundant Sum Of Submodules*)**.** [21] *Let M be a module with $M = \sum_{i\in I} M_i$ as a sum of submodules $\{M_i \mid i \in I\}$. Then this sum is called irredundant sum if for every $i_0 \in I$, $\sum_{i\neq i_0} M_i \neq M$.*

Definition 2.1.9 (*Exchange Property*)**.** [15] *Let M be a module. Then M is said to have the (finite) exchange property if for any (finite) index set I, whenever $M\oplus N = \bigoplus_{i\in I} A_i$ for modules N and A_i, then $M\oplus N = M\oplus\left(\bigoplus_{i\in I} B_i\right)$ for submodules $B_i \leq A_i$. Notice that, a module M has the exchange property if $End(M)$ is local.*

Definition 2.1.10 (*Complement Direct Summands Decomposition*)**.** [8] *Let $M = \bigoplus_{i\in I} M_i$ be a decomposition of a module M as a direct sum of nonzero submodules $\{M_i \mid i \in I\}$. Then M is said to complement direct summands if for every $A \leq^{\oplus} M$, $\exists\ J \subseteq I$ such that $M = A \bigoplus \left(\bigoplus_{i\in J} M_i\right)$.*

The next theorem is a well known important one.

Theorem 2.1.6 (*Azumaya*)**.** [1, *Theorem* 12.6] *If a module M has a direct decomposition $M = \bigoplus_{i\in I} M_i$, where each endomorphism ring, $End(M_i)$, is local, then the following statements are true:*

(a) The decomposition $M = \bigoplus_{i\in I} M_i$ is indecomposable. That is, each submodule M_i is indecomposable.

(b) *Every nonzero direct summand of M has an indecomposable direct summand.*

(c) *The decomposition $M = \bigoplus_{i \in I}$ complements direct summands.*

Corollary 2.1.2. [1, *Corollary* 12.7] *If a module M has a finite direct decomposition $M = M_1 \oplus \cdots \oplus M_n$, with each $End(M_i)$ is local, then this decomposition complements direct summands.*

Definition 2.1.11 (*Noetherian Modules & Rings*)**.** *Let R be a ring and M a right R-module. Then*

1. *M is called a Noetherian module, if every nonempty set of submodules possesses a maximal element, with respect to inclusion.*
2. *R is called a right Noetherian ring, if R_R is Noetherian.*

Definition 2.1.12 (*ACC Condition*)**.** *A module M is said to satisfy the ascending chain condition (ACC), if every ascending chain of submodule of M terminates. That is, if $A_1 \leq A_2 \leq A_3 \leq \ldots$ with $A_i \leq M$, then there is $n \in \mathbb{Z}^+$ such that $A_n = A_i$ for all $i \geq n$.*

Lemma 2.1.8. [1, *Proposition* 10.1] *Let M be a right R-module. Then M is f.g. iff whenever $M = \sum_{i \in I} A_i$, then there exists a finite subset $I_0 \subseteq I$ such that $M = \sum_{i \in I_0} A_i$*

Proof. If M is $f.g.$ then $M = \sum_{j=1}^{n} m_j R$ with $m_j \in M$. Since $M = \sum_{i \in I} A_i$, each m_j can be written as a finite sum of elements from the $A_{i's}$. This means that there is a finite subset $I_0 \subseteq I$ such that $m_j \in \sum_{i \in I_0} A_i$. Therefore, we have $M = \sum_{i \in I_0} A_i$. Conversely, consider the set $\{mR \mid m \in M\}$ of submodule of M. By assumption, there exists $\{m_1, m_2, \ldots, m_n\} \subseteq M$ such that $M = \sum_{j=1}^{n} m_j R$. Hence, M is $f.g.$ □

Theorem 2.1.7 (Characterization Of Noetherian Modules). [1, *Proposition* 10.9] *Let M be a module and $A \leq M$. Then the following are equivalent.*

(a) *M is Noetherian.*

(b) *A and M/A are both Noetherian.*

(c) M satisfies the ACC condition.

(d) Every submodule of M is f.g..

(e) For every nonempty set $\{A_i \mid i \in I\}$ of submodules of M, $\exists$ a finite subset $I_0 \subseteq I$ such that $\sum\limits_{i\in I} A_i = \sum\limits_{i\in I_0} A_i$.

Proof. $(a) \Longrightarrow (b)$ Since any submodule of A is also a submodule of M, it follows that any nonempty set of submodules of A has a maximal element. Therefore, A is Noetherian. On the other hand, let $\pi : M \longrightarrow M/A$ be the natural epimorphism and $X = \{X_i \mid i \in I\}$ be a nonempty set of submodules of M/A. Then the set $Y = \{\pi^{-1}(X_i) \mid i \in I\}$ is a nonempty set of submodules of M. Since M is Noetherian, Y has a maximal element, say $\pi^{-1}(X_j)$. Now we claim that X_j is a maximal element of X. Suppose $X_j \leq X_i$. Then $\pi^{-1}(X_j) \leq \pi^{-1}(X_i)$. By maximality of $\pi^{-1}(X_j)$, we have $\pi^{-1}(X_i) = \pi^{-1}(X_j)$. Hence $X_j = \pi(\pi^{-1}(X_j)) = \pi(\pi^{-1}(X_i)) = X_i$. So our claim is true and therefore M/A is also Noetherian.

$(b) \Longrightarrow (c)$ Let $A_1 \leq A_2 \leq A_3 \leq \ldots$ be an ascending chain of submodules of M and $\pi : M \longrightarrow M/A$ be the canonical epimorphism. Consider the following sets: $\Gamma := \{A_i \mid i \in \mathbb{Z}^+\}$, $\pi(\Gamma) := \{\pi(A_i) \mid i \in \mathbb{Z}^+\}$, and the set $\Gamma_A := \{A_i \cap A \mid i \in \mathbb{Z}^+\}$. Clearly, both $\pi(\Gamma)$ and Γ_A are nonempty since Γ is nonempty. Therefore, by assumption, both $\pi(\Gamma)$ and Γ_A have maximal elements, say $\pi(A_j)$ and $A_k \cap A$, respectively. Let $m = \max\{j, k\}$. This implies that $\pi(A_m) = \pi(A_i)$ and $A_m \cap A = A_i \cap A$ for all $i \geq m$. Our claim is $A_m = A_i$ for all $i \geq m$. From $\pi(A_m) = \pi(A_i)\ \forall\ i \geq m$ it follows that

$$A_m + A = \pi(\pi^{-1}(A_m)) = \pi(\pi^{-1}(A_i)) = A_i + A \qquad \forall\ i \geq m.$$

Moreover, we have

$$\begin{aligned} A_i &= (A_i + A) \cap A_i \quad \text{(By Modular Law)} \\ &= (A_m + A) \cap A_i \\ &= A_m + (A \cap A_i) \quad (\text{Since } A_m \leq A_i\ \forall\ i \geq m) \\ &= A_m + (A \cap A_m) \\ &= A_m \end{aligned}$$

Hence our claim is true. This ends the proof.

$(c) \Longrightarrow (a)$ For sake of a contradiction, assume that there is a nonempty set Γ with no maximal elements. Then for every submodule $X \in \Gamma$, $\exists\ X' \in \Gamma$

such that $X \lneqq X'$. Now, for any $X \in \Gamma$ choose and fix such X'. This means that if Y is any submodule in Γ, then the chain: $Y \lneqq Y' \lneqq Y'' \lneqq \ldots$ is an ascending chain which never terminates, a contradiction.

$(d) \Longleftrightarrow (e)$ Follows directly from the previous lemma [you just take $M = \sum_{i \in I} A_i$ in (e)].

$(a) \Longleftrightarrow (a)$ Suppose M is Noetherian and $\Gamma = \{A_i \mid i \in I\}$. Let Λ be the set of all finite sums of elements of Γ. Since M is Noetherian, Λ must have a maximal element, say $D = \sum_{i \in I_0} A_i$, where $I_0 \subseteq I$ and I_0 is finite. From maximality of D, it follows that if $i \in I$ then $D + A_i = D$, hence $A_i \leq D$ for all $i \in I$. Therefore, $\sum_{i \in I} A_i = \sum_{i \in I_0} A_i$.

$(e) \Longleftrightarrow (c)$ Let $A_1 \leq A_2 \leq \ldots$ be an ascending chain of submodules of M. Consider the set $\{A_i \mid i \in \mathbb{Z}^+\}$. By assumption, it follows that $\exists\, n \in \mathbb{Z}^+$ such that $\sum_{i=1}^{\infty} A_i = \sum_{i=1}^{n} A_i$. Consequently, $A_n = A_i \; \forall\, i \geq n$.

□

Corollary 2.1.3. *Let R be a ring and M a right R-module. Then*

(1) If M is a finite sum of Noetherian submodules, then M is Noetherian.

(2) If R is a right Noetherian ring and M is f.g., then M is Noetherian.

Proof. (1) Suppose $M = \sum_{i=1}^{n}$ with $A_i \leq M$. The proof is by induction on n. If $n = 1$, then the assertion is trivial. So suppose $n > 1$ and that the assumption holds for $n-1$. That is, $L = \sum_{i=1}^{n-1}$ is Noetherian. By the isomorphism theorems, we have $M/A_n = (L + A_n)/A_n \cong L/(L \cap A_n)$. From the previous theorem we know that $L/(L \cap A_n)$ is Noetherian, hence M/A_n is also Noetherian. Consequently, M is Noetherian since A_n is Noetherian.

(2) Suppose $M = \sum_{i=1}^{n} m_i R$. For each $i = 1, 2, \ldots, n$, consider the homomorphism $\phi_i : R_R \longrightarrow M$ with $r \longmapsto m_i r$. By the Homomorphism Theorem, $R/Ker(\phi_i) \cong Im(\phi_i) = m_i R$. Since R_R is Noetherian, it follows that $m_i R$ is also Noetherian for each $i = 1, 2, \ldots, n$. Hence, From (1), M is Noetherian.

□

Definition 2.1.13 (*Artinian Modules*)**.** *Let R be a ring and M a right R-module. Then M is called a Artinian module, if every nonempty set of submodules possesses a minimal element, with respect to inclusion. Equivalently, M satisfies the descending chain condition (DCC).*

It has been proved in [3, §2.16 & §5.4] that Any nonzero Artinian module has a finite hollow dimension. Besides, Any nonzero module with a finite hollow dimension has a hollow factor module.

Now, we are going to introduce an important example of $\mathbb{Z}$-modules called the Prüfer p-group.

Definition 2.1.14. [21, §15] *Let M be a $\mathbb{Z}$-module and p a prime number. Then the p-component of M is defined as*

$$p(M) = \{m \in M \quad | \quad p^k m = 0 \ \textit{for some} \ k \in \mathbb{N}\}$$

If we let $M = \mathbb{Q}/\mathbb{Z}$, then the p-component of M is called the Prüfer p-group and denoted by $\mathbb{Z}_{p^\infty}$.

The following proposition provides some properties of the Prüfer p-group as a $\mathbb{Z}$-module.

Proposition 2.1.1. [21, §15.10, §15.11, §17.11, §17.13, & §41.23] *Let p be a prime number. Then*

(a) $\mathbb{Q}/\mathbb{Z} = \bigoplus_{p:prime} \mathbb{Z}_{p^\infty}$.

(b) $\mathbb{Z}_{p^\infty} = \{q + \mathbb{Z} \quad | \quad q \in \mathbb{Q}, \ p^k q \in \mathbb{Z} \ \textit{for some} \ k \in \mathbb{N}\}$.

(c) Every nonzero proper submodule of $\mathbb{Z}_{p^\infty}$ is of the form $\left(\frac{1}{p^k} + \mathbb{Z}\right)\mathbb{Z}$.

(d) The submodules of $\mathbb{Z}_{p^\infty}$ are totally ordered by inclusion.

(e) $\mathbb{Z}_{p^\infty}$ is Artinian but not Noetherian module. Moreover, it has no maximal submodules.

(f) Every submodule of $\mathbb{Z}_{p^\infty}$ is fully invariant. Thus, $\mathbb{Z}_{p^\infty}$ is a duo module.

(g) Every proper submodule of $\mathbb{Z}_{p^\infty}$ is small. (Therefore, $\mathbb{Z}_{p^\infty}$ is hollow but not local).

2.2 Hollow Modules

Recall that a nonzero module M is said to be *hollow* if every proper submodule of M is small in M [3, §2.12].

Theorem 2.2.1 (Characterization Of Hollow Modules). [3, §2.12]
Let M be a module. Then the following are equivalent.

(a) M is hollow.

(b) Every factor module of M is indecomposable.

(c) For any nonzero modules K and N and any module homomorphisms $K \xrightarrow{f} M \xrightarrow{g} N$, gf is an epimorphism implies that both f and g are epimorphisms.

Proof. $(a) \Longrightarrow (b)$ Since M is hollow, any factor module of M is hollow and hence indecomposable.

$(b) \Longrightarrow (a)$ Suppose M is not hollow. Then there exist proper submodules U and V of M with $M = U + V$. Therefore,

$$\begin{aligned} M/(U \cap V) &= (U + V)/(U \cap V) \\ &= U/(U \cap V) + V/(U \cap V) \\ &= U/(U \cap V) \oplus V/(U \cap V) \\ &\cong (U + V)/V \oplus (U + V)/U \\ &= M/V \oplus M/U \end{aligned}$$

This means that $M/(U \cap V)$ is a decomposable factor module of M, a contradiction.

$(a) \Longrightarrow (c)$ Let $K \xrightarrow{f} M \xrightarrow{g} N$ be module homomorphisms with K, N nonzero modules, M hollow, and gf epimorphism. Then g is an epimorphism. Now, gf is an epimorphism $\Longrightarrow gf(K) = N \Longrightarrow g^{-1}(gf(K)) = g^{-1}(N) \Longrightarrow Im(f) + Ker(g) = M$. But $Ker(g) \neq M$ (since if $Ker(g) = M$ then $0 = M/Kerg \cong N \neq 0$, impossible), hence $Ker(g) \ll M$, therefore $Im(f) = M$. Thus, f is an epimorphism.
$(c) \Longrightarrow (a)$ Let $K \lneqq M$ and $X \leq M$ with $M = X + K$. Consider the homomorphisms $X \xrightarrow{i} M \xrightarrow{\pi} M/K$, where i is the inclusion monomorphism and π the natural epimorphism. Since $M/K \cong X/(X \cap K)$, πi is an epimorphism. By assumption, i is an epimorphism. Consequently, $X = i(X) = M$, so $K \ll M$. Thus, M is hollow. □

<u>Lemma</u> 2.2.1. [3, §2.15] *Let M be a module. Then the following hold*

(a) If M is hollow, then every factor module is hollow.

(b) If $A \ll M$ and M/A is hollow, then M is hollow.

(c) Let H and M be modules such that H is hollow and $H \subseteq M$. Then either $H \ll M$ or $H \leq_{cc} M$.

(d) If $h(M) = 1$, then M is hollow.

Proof. (a) let A, B, and X be submodules of M with $A \leq B \lneqq M$, $A \leq X \leq M$, and $B/A + X/A = M/A$. Then $B + X = M$. But $B \ll M$, therefore $X = M$. Hence, $B/A \ll M/A$.

(b) Let B be a proper submodule of M with $B + X = M$ for some $X \leq M$. Then $(B+X)/A = M/A$, so $(B+A)/A + (X+A)/A = M/A$. But M/A is hollow, hence either $B + A = M$ or $X + A = M$. Since $A \ll M$, $B + A = M$ implies $B = M$, a contradiction. Thus, $X + A = M$. Again, since $A \ll M$, we have $X = M$. Therefore, $B \ll M$ which implies that M is hollow.

(c) Suppose that H is not coclosed in M. Then there exists a proper submodule A of H such that $H/A \ll M/A$. Since H is hollow, $A \ll H$ and hence $A \ll M$. But $H/A \ll M/A$ and $A \ll M$ is equivalent to $H \ll M$. This ends the proof.

(d) If $h(M) = 1$, then there exists a hollow module H and a small epimorphism $f : M \longrightarrow H$. But $M/Ker(f) \cong H$, hence $M/Ker(f)$ is also hollow. Since $Ker(f) \ll M$, it follows that M is hollow. □

Remember that a module M is called *local* if it is hollow and has a unique maximal submodule (namely $Rad(M)$) [3, §2.12]. Also recall that if M is nonzero and $f.g.$, then every proper submodule is contained in a maximal submodule of M.

<u>Theorem</u> 2.2.2. [3, §2.15] *Let M be an R-module. Then the following are equivalent.*

(a) M is local.

(b) M is hollow and $Rad(M) \neq M$.

(c) M is hollow and cyclic (or $f.g.$).

Proof. $(a) \Longrightarrow (b)$ Suppose that M is local. Then, by definition, M is hollow and has a unique maximal submodule which is $Rad(M)$, hence $Rad(M) \neq M$.

$(b) \Longrightarrow (c)$ $Rad(M) \neq M$ and M is hollow implies that $Rad(M) \ll M$ and it is the unique maximal submodule of M. Take any $x \notin Rad(M)$. Then $xR + Rad(M) = M$, consequently, $xR = M$. Thus, M is cyclic.

$(c) \Longrightarrow (a)$ Since M is $f.g.$, every proper (hence small) submodule of M is contained in a maximal submodule. Now, if $A \lneqq M$ and N is maximal submodule with $A \nleq N$, then $A + N = M$ and consequently $N = M$, a contradiction. Therefore, all proper submodules of M must be contained in a unique maximal submodule which is surely $Rad(M)$. Hence, M is local. □

Example 2.1. *We consider the following examples:*

1. *Any simple* $\mathbb{Z}$*-module is local* [3, §2.13].
2. *For any prime* p*, the* $\mathbb{Z}$*-module* $\mathbb{Z}_{p^k}$ $(k \in \mathbb{N})$ *is local and hence hollow, whereas the module* $\mathbb{Z}_{p^\infty}$ *is hollow but not local* [3, §2.13].
3. *Neither* $\mathbb{Q}$ *nor* $\mathbb{Q}/\mathbb{Z}$ *is hollow as modules over* $\mathbb{Z}$ [3, §2.15].
4. *The* $\mathbb{Z}$*-modules* $\mathbb{Q} \oplus \mathbb{Q}/p\mathbb{Z}$ *and* $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/p\mathbb{Z}$ *show that a module might be non local but still has a unique maximal submodule* [3, §2.15].

2.3 Supplemented Modules

Recall that, From [3] & [21], if A and B are submodules of a module M, then

1. A is called a *weak supplement* of B in M if $A+B=M$ and $A\cap B \ll M$.
2. A is called a *supplement* of B in M if $A+B=M$ and $A\cap B \ll A$.
3. A is called a *strong supplement* of B in M if A is a supplement of B in M and $A\cap B \leq^{\oplus} B$.
4. The module M is called *weakly supplemented* if every submodule of M has a weak supplement in M.
5. The module M is called *supplemented* if every submodule of M has a supplement in M.
6. The module M is called *amply supplemented* if for every $A, B \leq M, M = A+B$ implies A has a supplement in M contained in B.

Corollary 2.3.1. *The following implications are true.*
Amply supplemented $\Longrightarrow$ *supplemented* $\Longrightarrow$ *weakly supplemented.*

Lemma 2.3.1. [3, §17.9] *Let M be a module with $A \leq B \leq M$. Then*

(a) If A and B have the same weak supplement in M, then $A \leq_{ce} B$ in M.

(b) If $A \leq_{ce} B$ in M and B has a weak supplement in M, then $B = A+N$ for some $N \ll M$.

Proof. (a) Let C be the weak supplement for both A and B. Then we have $M = A+C = B+C$, $A\cap C \ll M$, and $B\cap C \ll M$. Since $A \leq B$, we have $B = B\cap M = B\cap(A+C) = A+(B\cap C)$. Let $X \leq M$ with $M = B+X$. Then $M = A+(B\cap C)+X$. But $B\cap C \ll M$, hence $M = A+X$. Therefore, by [*Lemma* 2.1.6, (a)], $A \leq_{ce} B$ in M.

(b) Let C be a weak supplement of B in M. Then $M = B+C$ and $B\cap C \ll M$. Again, by [*Lemma* 2.1.6, (a)], we get $M = A+C$. Therefore, $B = B\cap M = B\cap(A+C) = A+(B\cap C)$ and $B\cap C \ll M$. □

Lemma 2.3.2. [21, §41.1] *Let M be a module with submodules A and B. Assume that A is a supplement of B in M. Then*

(a) If $A+C=M$ for some $C \leq B$, then A is a supplement of C in M.

(b) If $C \ll M$, then A is a supplement of $B + C$ in M.

(c) If $C \leq B$, then $(A + C)/C$ is a supplement of B/C in M/C.

Proof. (a) Since $C \leq B$ and A is a supplement of B in M, we have $A \cap C \leq A \cap B \ll A$ and hence $A \cap C \ll A$. Thus, A is a supplement of C in M.

(b) Clearly $A + B + C = M + C = M$. Moreover, if $X \leq A$ with $B + C + X = M$, then $B + X = M$ since $C \ll M$. Consequently, $X = A$. Therefore, A is a supplement of $B + C$ in M.

(c) Clearly $(A + C)/C + B/C = M/C$. Now, $(A + C)/C \cap B/C = ((A + C) \cap B)/C = ((A \cap B) + C)/C$. Consider the natural epimorphism $\pi : A \longrightarrow (A + C)/C$. Since $A \cap B \ll A$, we have $\pi(A \cap B) = ((A \cap B) + C)/C \ll (A+C)/C$. Hence, $(A+C)/C$ is a supplement of B/C in M/C. □

Lemma 2.3.3. [13, *Proposition* 1.2.1] *Let M be a module and $N \leq M$. Consider the following conditions.*

(a) N is a supplement submodule of M.

(b) $N \leq_{cc} M$.

(c) $\forall\ X \leq N$, $X \ll M$ implies $X \ll N$.

Then $(a) \Longrightarrow (b) \Longrightarrow (c)$ hold. If M is a weakly supplemented module then $(c) \Longrightarrow (a)$ holds.

Proof. $(a) \Longrightarrow (b)$ Let N be a supplement of a submodule $L \leq M$. Then $M = N + L$ and N is minimal to this property. Now, let $K \leq N$ with $N/K \ll M/K$. This implies that $N + L + K = M + K = M$, hence $N/K + (L + K)/K = M/K$. But $N/K \ll M/K$, so $L + K = M$. By the minimality of N, $N = K$. This means $N \leq_{cc} M$.

$(b) \Longrightarrow (c)$ Let $X \leq N$ with $X \ll M$. Assume $Y \leq N$ such that $N = X + Y$. Since $N \leq_{cc} M$, it suffices to show that $N/Y \ll M/Y$. So let $M/Y = N/Y + H/Y$ with $Y \leq H$. Then $M = N + H = X + Y + H = X + H$. But $X \ll M$, hence $H = M$. Therefore $N/Y \ll M/Y$. Consequently, since $N \leq_{cc} M$, $N = Y$. Thus, $X \ll N$.

$(c) \Longrightarrow (a)$ Suppose M is a weakly supplemented module and $N \leq M$ that satisfies the assumption in (c). Since M is weakly supplemented, there is $L \leq M$ such that $M = N+L$ and $N \cap L \ll M$. By assumption, $N \cap L \ll N$. Thus, N is a supplement of L in M. This completes the proof. □

Corollary 2.3.2. *If N is a supplement submodule of M and $N' \leq N$, then $N' \leq_{cc} N$ iff $N' \leq_{cc} M$.*

Proof. Follows immediately from [*Lemma* 2.3.3] and [*Lemma* 2.1.7, (d)]. □

Lemma 2.3.4. [11, *Lemma* 1.4] *Let M be a module with $A \leq B \leq M$. Then*

(a) If B is a supplement in M, then B/A is a supplement in M/A.

(b) If A is a supplement in M and B/A is a supplement in M/A, then B a supplement in M.

(c) If M is a weakly supplemented module such that $B/A \leq_{cc} M/A$ and $A \leq_{cc} M$, then $B \leq_{cc} M$.

Proof. (a) Let $N \leq M$ with $M = B+N$ and $B \cap N \ll B$. We claim that B/A is a supplement of $(N+A)/A$ in M/A. Obviously, $B/A+(N+A)/A = M/A$. Also, $B/A \cap (N+A)/A = (B \cap (N+A))/A = ((B \cap N)+A)/A$. Consider the epimorphism $\pi : B \longrightarrow B/A$. Then $\pi(B \cap N) = ((B \cap N) + A)/A \ll B/A$ since $B \cap N \ll B$. Hence, our claim is verified.

(b) Let A be a supplement of A' in M and B/A a supplement of B'/A in M/A. Then $M = A + A'$, $A \cap A' \ll A$, $M/A = B/A + B'/A$, and $B/A \cap B'/A \ll B/A$. Also, $A \cap A' \ll B$ since $A \leq B$. From $M = (B \cap B')+A'$ and $M = B+B'$ it follows that $M = B+(A' \cap B')$ [*Lemma* 2.1.3]. Moreover, $B = B \cap (A + A') = A + (B \cap A')$ and $(B \cap B')/A \ll B/A$ implies that $(B \cap B' \cap A')/(A \cap A') \ll B/(A \cap A')$ [*Lemma* 2.1.5, (h)]. Since $A \cap A' \ll B$, we have $(B \cap B' \cap A') \ll B$. Thus, B is a supplement of $B' \cap A'$ in M.

(c) Follows directly from (b) & [*Lemma* 2.3.3]. □

Proposition 2.3.1. [11, *Proposition* 1.5] *If M is an amply supplemented module, then every submodule of M has a coclosure in M.*

Proof. Let $A \leq M$. Since M is amply supplemented, $\exists\ B \leq M$ such that B is a supplement of A in M. i.e., $M = A + B$ and B is minimal with this property. Again since M is amply supplemented and $M = A + B$, $\exists\ C \leq A$ such that C is a supplement of B in M. i.e., $M = B+C$ and $B \cap C \ll C$. We claim that C is a coclosure of A in M. To verify this, let $C \leq X \leq M$ with

$A/C+X/C = M/C$. Then $M = A+X$. Now, $X = X\cap M = X\cap(B+C) = (X\cap B)+C$, hence $M = X+A = (X\cap B)+C+A = (X\cap B)+A$. By the minimality of B, $X\cap B = B$, so $B \le X$. This implies $M = B+C \le X$ and so $M = X$. Therefore, $A/C \ll M/C$, that is, $C \le_{ce} A$ in M. Moreover, C is coclosed since it a supplemented submodule in M. Thus, C is a coclosure of A in M. □

Proposition 2.3.2. *If M is an amply supplemented module, then any supplement submodule , direct summand, or factor module of M is amply supplemented.*

Proof. First, let A be a supplement of B in M. Then $M = A+B$ and $A\cap B \ll A$. Let $X, Y \le A$ such that $A = X+Y$. This implies that $M = B+X+Y$. But M is amply supplemented, so there exists $Y' \le Y$ such that Y' is a supplement of $B+X$ in M which means $Y'+B+X = M$ and $Y'\cap(B+X) \ll Y'$. We will show that Y' is a supplement of X in A. Now, $A = A\cap M = A\cap Y'+B+X = (A\cap B)+(X+Y')$ and so $A = X+Y'$ since $A\cap B \ll A$. Moreover, From $X\cap Y' \le (X+B)\cap Y' \ll Y'$, it follows that $X\cap Y' \ll Y'$. Hence, A is amply supplemented. On the other hand, Any direct summand of M is a supplement submodule and hence amply supplemented by the previous argument. Finally, the assertion for factor modules follows from [*Lemma* 2.3.2].

□

Theorem 2.3.1. [11, *Lemma* 1.7] *A module M is amply supplemented iff M is weakly supplemented and every submodule has a coclosure in M.*

Proof. $\boxed{\Longrightarrow}$ Follows directly from definitions and [*Proposition* 2.3.1].

$\boxed{\Longleftarrow}$ Let $A, B \le M$ with $A+B = M$. Since M is weakly supplemented and $A\cap B \le M$, there exists $C \le M$ such that $(A\cap B)+C = M$ and $(A\cap B)\cap C \ll M$. Therefore, $M = A+(B\cap C)$ [*Lemma* 2.1.3]. Now, let D be a coclosure of $B\cap C$ in M. Then $(B\cap C)/D \ll M/D$ and $D \le_{cc} M$. But $M/D = (A+(B\cap C))/D = (A+D)/D+(B\cap C)/D$, hence $M = A+D$. Furthermore, $A\cap D \le A\cap B\cap C \ll M$ implies that $A\cap D \ll M$. Moreover, it follows from $A\cap D \le D \le_{cc} M$ and $A\cap D \ll M$ that $A\cap D \ll D$ [*Lemma* 2.3.3]. Thus, $D \le B$ and D is a supplement of A in M. Hence, M is amply supplemented.

□

Example 2.2. *Consider the following examples.*

1. *The* $\mathbb{Z}$*-module* $\mathbb{Z}$ *in not supplemented* [8, §5.2].

2. *Artinian modules and semisimple modules are supplemented* [8, §5.2].

3. *Every lifting module is amply supplemented* [*Proposition* 2.4.1].

4. $\mathbb{Q}/\mathbb{Z}$ *is a weakly supplemented* $\mathbb{Z}$*-module* [3, *Example* 17.10].

5. $\mathbb{Q}$ *is a weakly supplemented* $\mathbb{Z}$*-module* [3, *Example* 17.15].

6. $\mathbb{Q}$ *is not a supplemented* $\mathbb{Z}$*-module* [3, *Example* 20.12].

2.4 Lifting Modules

From [3], Recall that a module M is called *lifting (or has* (D_1)*)* if for every submodule $N \leq M$, there exists $K \leq^{\oplus} M$ such that $K \leq N$ and $K \leq_{ce} N$ in M.

Now we start with a lemma that will be so valuable in characterizing lifting and hollow-lifting modules.

<u>Lemma</u> 2.4.1. [21, *Lemma* 41.11] *Let M be a module and $A \leq M$. Then the following are equivalent.*

(a) There is $X \leq^{\oplus} M$ with $X \leq_{ce} A$ in M.

(b) There is $X \leq^{\oplus} M$ and $Y \ll M$ with $A = X \oplus Y$.

(c) There is a decomposition $M = X \oplus X'$ with $X \leq A$ and $A \cap X' \ll X'$.

(d) A has a strong supplement in M.

Proof. $(a) \Longrightarrow (b)$ Let $M = X \oplus X'$ and $X \leq_{ce} A$ in M. Then $A = A \cap M = A \cap (X \oplus X') = X \oplus (A \cap X')$. It remains to show that $A \cap X' \ll X'$. Consider the homomorphism $f : M/X \longrightarrow X'$. We can consider f as an isomorphism since $M/X = (X \oplus X')/X \cong X'$. Hence, since $A/X \ll M/X$, we have $f(A/X) = A \cap X' \ll X'$ and so we are done.

$(b) \Longrightarrow (c)$ Let $M = X \oplus X'$ and $A = X \oplus Y$ with $Y \ll M$. Then clearly X' is a supplement of X in M. But $Y \ll M$, therefore X' is a supplement of $X + Y = A$ in M [*Lemma* 2.3.2, (b)]. Consequently, $A \cap X' \ll X'$.

$(c) \Longrightarrow (d)$ Suppose that $M = X \oplus X'$ with $X \leq A$ and $A \cap X' \ll X'$. We claim that X' is a strong supplement of A in M. Since $X \leq A$, $M = A + X'$ and hence X' is a supplement of A in M. Moreover, $A = A \cap (X \oplus X') = X \oplus (A \cap X')$ which means that $A \cap X' \leq^{\oplus} A$. Thus, our claim is true.

$(d) \Longrightarrow (a)$ Let B be a strong supplement of A in M. Then $M = A + B$, $A \cap B \ll B$, and $(A \cap B) \oplus C = A$ for some $C \leq A$. Therefore, we have $M = (A \cap B) + C + B$ and $0 = (B \cap A) \cap C = B \cap C$. This means $M = B \oplus C$ and so $C \leq^{\oplus} M$. Now, we will show that $C \leq_{ce} A$ in M. Let $C \leq X \leq M$ with $A/C + X/C = M/C$. Then $M = A + C = (A \cap B) + C + X = (A \cap B) + X$. But $A \cap B \ll B$ implies $A \cap B \ll M$, so $X = M$. Thus, $C \leq_{ce} A$ in M. $\square$

Proposition 2.4.1. *Let M be a lifting module. Then the following are true.*

(1) Any coclosed submodule of M is a direct summand.

(2) M is amply supplemented.

(3) M is hollow iff it is indecomposable.

Proof. (1) Let $A \leq_{cc} M$. Since M is lifting, $\exists\ K \leq^{\oplus} M$ such that $K \leq_{ce} A$ in M. But A has no proper coessential submodules, hence $A = K$.

(2) Let $A,\ B \leq M$ with $M = A + B$. We will show that B contains a supplement of A in M. From the previous lemma, $B = X \oplus Y$, where $X \ll M$ and $Y \leq^{\oplus} M$. Therefore, $M = A + Y$. Again by the same lemma, $A \cap Y = N \oplus S$ with $S \ll M$ and $N \leq^{\oplus} M$ and hence $S \ll Y$ and $N \leq^{\oplus} Y$. Let $Y = N \oplus N'$ for some $N' \leq Y$. Clearly, N' is a supplement of N in Y. But $S \ll Y$, therefore N' is a supplement of $N + S$ in Y [*Lemma* 2.3.2, (b)] which means $Y = N' + N + S$ and $N' \cap (N + S) \ll N'$ implying that $(A \cap Y) + N' = N + S + N' = Y$. Consequently, $M = A + N + S + N' = A + N'$. Moreover, $A \cap N' = (A \cap Y) \cap N' = (N + S) \cap N' \ll N'$. Hence, N' is a supplement of A in M with $Y' \leq B$.

(3) If M is hollow and $M = A \oplus B$, then either $A = M$ or $B = M$, hence M is indecomposable. Conversely, suppose M is an indecomposable lifting module and let A be a proper submodule of M. Since M is lifting, there exists a direct summand K of M with $A/K \ll M/K$. But M is indecomposable, hence $K = 0$ and so $A \ll M$. Thus, M is hollow. □

Theorem 2.4.1 (Characterization Of Lifting Modules). [3, §22.3] *Let M be a module. Then the following are equivalent.*

(a) M is lifting.

(b) For every $A \leq M$, there is a decomposition $M = X \oplus X'$ with $X \leq A$ and $A \cap X' \ll M$.

(c) Every $A \leq M$ can be written as $A = X \oplus Y$ with $X \leq^{\oplus} M$ and $Y \ll M$.

(d) M is amply supplemented and every coclosed submodule of M is a direct summand.

(e) M is amply supplemented and every supplement submodule of M is a direct summand.

Proof. $(a) \Longleftrightarrow (b) \Longleftrightarrow (c)$ Follow from [*Lemma* 2.4.1].

$(a) \Longrightarrow (d)$ Follows from [*Proposition* 2.4.1].

$(d) \Longleftrightarrow (e)$ Follows from [*Lemma* 2.3.3].

$(e) \Longleftrightarrow (a)$ Follows from [*Proposition* 2.3.1]. □

Corollary 2.4.1. *Any hollow module is lifting.*

Proof. Let $A \leq M$. If $A = M$ then $A = M \oplus 0$ with $0 \ll M$ and $M \leq^{\oplus} M$. And if $A \neq M$, then $A = A \oplus 0$ with $A \ll M$ and $0 \leq^{\oplus} M$. Hence, M is lifting. □

Corollary 2.4.2. *Any coclosed submodule (hence any direct summand) of a lifting module M is also lifting.*

Proof. Let M be a lifting module and $A \leq_{cc} M$. From [*Proposition* 2.4.1], $A \leq^{\oplus} M$. Consequently, A is amply supplemented [*Proposition* 2.3.2]. Now, let $A' \leq_{cc} A$. Since A is a supplement in M, $A' \leq_{cc} M$ [*Corollary* 2.3.2]. Hence, $A' \leq^{\oplus} M$ which implies $A' \leq^{\oplus} A$. Therefore, by the previous theorem, A is lifting. □

Example 2.3. *Consider the following:*

1. *Since any hollow module is lifting, it follows that the $\mathbb{Z}$-modules $\mathbb{Z}_{p^k}$ and $\mathbb{Z}_{p^\infty}$ are lifting, where p is a prime integer.*

2. *Consider the Z-modules $A = \mathbb{Z}/8\mathbb{Z}$ and $B = \mathbb{Z}/2\mathbb{Z}$. Since A and B are hollow, they are lifting. But the module $M = A \oplus B$ is not lifting since if U is the submodule generating by $(2 + 8\mathbb{Z}, 1 + 2\mathbb{Z})$, then U is not small in M and U does not contain a nonzero direct summand of M* [3, *Example* 22.5].

3. *By following* [14, *Proposition A*.7], *the $\mathbb{Z}$-module $\mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ is lifting, whereas the module $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ is not lifting.*

In order to proceed, we must recall the following about a module M [14].

(1) M is said to have (D_2) , if whenever $A \leq M$ with M/A isomorphic to a direct summand of M, then $A \leq^{\oplus} M$.

(2) M is said to have (D_3) , if whenever $M = M_1 + M_2$ such that M_1 and M_2 are direct summands of M, then $M_1 \cap M_2 \leq^{\oplus} M$.

(3) M is called discrete, if it is lifting and has (D_2).

(4) M is called quasi-discrete, if it is lifting and has (D_3).

We must point out that any hollow module is quasi-discrete and any direct summand of a module with (D_2) or (D_3) also has (D_2) or (D_3) respectively [14, *Lemma* 4.7].

Lemma 2.4.2. [14, *Lemma* 4.6] *Let M be a module with (D_2). Then every epimorphism $f : M_1 \longrightarrow M_2$ splits, provided that M_1 and M_2 are direct summands of M.*

Proof. Let $M = M_1 \oplus N_1$ for some $N_1 \leq M$, and $f : M_1 \longrightarrow M_2$ be an epimorphism. By The Homomorphism Theorem, $M/(Ker(f) \oplus N_1) = (M_1 \oplus N_1)/(Ker(f) \oplus N_1) \cong M_1/Ker(f) \cong M_2$. But M has (D_2), therefore $Ker(f) \oplus N_1 \leq^{\oplus} M$ and hence $Ker(f) \leq^{\oplus} M$. Since $Ker(f) \leq M_1$, we have also $Ker(f) \leq^{\oplus} M_1$. Thus, f splits. □

Corollary 2.4.3. *Any Module satisfies (D_2) also satisfies (D_3)*

Proof. Let $M = M_1 + M_2 = M_1 \oplus N_1$, where $M_1, M_2 \leq^{\oplus} M$ and $N_1 \leq M$. Then $N_1 \cong N_1/0 = N_1/(M_1 \cap N_1) \cong (M_1 \oplus N_1)/M_1 = M/M_1 = (M_1 + M_2)/M_1 \cong M_2/(M_1 \cap M_2)$. Therefore, by the previous lemma, $M_1 \cap M_2$ is a direct summand of M, hence M satisfies (D_3). □

Now we can see that any discrete module is quasi-discrete, and direct summands of discrete (quasi-discrete) modules are also discrete (quasi-discrete).

2.5 Duo Modules

Definition 2.5.1. [18] *Let M be a module and $N \leq M$. Then N is called fully invariant if $f(N) \leq N$ for all $f \in End(M)$. Moreover, we call M a duo module if every submodule of M is fully invariant.*

Clearly, the trivial submodules of M (which are 0 and M) are fully invariant. Besides, Any simple module is a duo module.

Lemma 2.5.1. [18, *Lemma* 1.1] *Let M be a right R-module. Then M is a duo module iff for each $f \in End(M)$ and each $m \in M$, there exists $r \in R$ such that $f(m) = mr$.*

Proof. $\boxed{\Longrightarrow}$ Let $f \in End(M)$ and $m \in M$. Then $mR \leq M$. But M is duo, therefore $f(mR) \leq mR$ and hence $f(m) = f(m.1) \in mR$. This means that $\exists\, r \in R$ with $f(m) = mr$.

$\boxed{\Longleftarrow}$ Let $N \leq M$ and $f \in End(M)$. Now, if $n \in N$ then, by hypothesis, there exists $r \in R$ with $f(n) = nr \in nR \leq N$, hence $f(N) \leq N$. Therefore, M is a duo module. □

Proposition 2.5.1. [18, *Proposition* 1.3] *Any direct summand of a duo module is also duo.*

Proof. Let M be a duo module and $K \leq^{\oplus} M$. Then $M = K \oplus L$ for some $L \leq M$. Now, let $f \in End(M)$ and $K' \leq K$. Consider the projection map $\pi : M \longrightarrow K$ and the inclusion map $i : K \longrightarrow M$. Let $g = if\pi$, then $g \in End(M)$. Moreover, $g(K') = if\pi(K') = if(K') = f(K')$. Since M is duo, $g(K') \leq K'$ and hence $f(K') \leq K'$. Thus, K is a duo module. □

Using [18, *Lemma* 2.1], we get the following result.

Lemma 2.5.2. *Let $M = M_1 \oplus M_2$ be a module and A a fully invariant submodule of M. Then $A = (A \cap M_1) \oplus (A \cap M_2)$. Moreover, $M/A = (A + M_1)/A \oplus (A + M_2)/A$.*

Proof. Let π_1 and π_2 be the projection epimorphisms onto M_1 and M_2 respectively and let $a \in A$. Then $\pi_1(a) + \pi_2(a) = a$. Since $\pi_1(A) \leq A$ and $\pi_2(A) \leq A$, it follows that $\pi_1(a) \in A \cap M_1$ and $\pi_2(a) \in A \cap M_2$ and hence $A \leq (A \cap M_1) \oplus (A \cap M_2)$. Consequently, $A = (A \cap M_1) \oplus (A \cap M_2)$. Moreover, this implies that $(A + M_1) \cap (A + M_2) \leq [(M_1 + M_2 + A) \cap A] + [M_1 + A + A) \cap M_2] = A + [M_1 + (A \cap M_1) \oplus (A \cap M_2)] \cap M_2 = A$. Thus, $M/A = (A + M_1)/A \oplus (A + M_2)/A$. □

2.6 UCC Modules

The concept of UCC modules was introduced in 2002 by the Indian mathematician N. Vanaja [5]. This notion was a dual to UC modules which were studied by Patrick. F. Smith. In this section, we will review some results of Vanaja about UCC modules which we need in our thesis.

Definition 2.6.1. *A module M is called a unique coclosure module (denoted by UCC), if every submodule of M has a unique coclosure in M.*

Example 2.4. *Hollow modules and semisimple modules are UCC modules.*

Notice that, an amply supplemented module need not be UCC. For example, consider the $\mathbb{Z}$-module $M = \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ [5, *Proposition* 4.4]. However, using theorem 2.3.1, we have the following result.

Corollary 2.6.1. *A weakly supplemented UCC module is amply supplemented.*

According to Vanaja, we can characterize UCC modules through the next two theorems.

Theorem 2.6.1. [5, *Theorem* 3.3 & *Proposition* 3.9] *Let M be a module. Then the following are equivalent.*

(a) M is a UCC module.

(b) Every factor module of M is a UCC module.

(c) Every coclosed submodule of M is a UCC module

(d) Given a submodule $N \leq M$, there exists a coclosure K of N in M such that if $L \leq_{ce} N$ in M, then $K \leq L$.

(e) Every submodule has a coclosure in M, and if $N, X, Y \leq M$ such that $X \leq_{ce} N$ in M and $Y \leq_{ce} N$ in M, then $X \cap Y \leq_{ce} N$ in M

(f) If $\{X_i \mid i \in I\}$ is any family of submodules of M and $N \leq M$ with $X_i \leq_{ce} N$ in M $\forall i$, then $\bigcap_{i \in I} X_i \leq_{ce} N$ in M.

Theorem 2.6.2. [5, *Theorem* 3.12] *Let M be an amply supplemented module. Then the following are equivalent.*

(a) M is a UCC module.

(b) *If $N \ll M$ and $K, L \leq M$, then $K \cap L \leq_{ce} [(N+K) \cap (N+L)]$ in M.*

(c) *There exists $N_0 \ll M$ such that M/N_0 is a UCC module , and for all $K, L \leq M$, $K \cap L \leq_{ce} [(N_0+K) \cap (N_0+L)]$ in M.*

Now we continue this section by introducing a new type of submodules which gives a different look at UCC modules.

Definition 2.6.2. [5] *Let M be a module and $K \leq M$. Then K is called strongly coclosed submodule (Denoted by $K \leq_{scc} M$), if whenever $X \leq M$ with $K \nleq X$, then $X \nleq_{ce} K+X$ in M.*

The next proposition provide us some properties of strongly coclosed submodules.

Proposition 2.6.1. [5, *Proposition* 3.14] *Let M be a module and $K \leq M$. Then*

(i) *If $K \leq_{scc} M$, then $K \leq_{cc} M$.*

(ii) *If $K \leq_{scc} M$ and $K \leq H \leq M$, then $K \leq_{scc} H$*

(iii) *If $K \leq_{scc} M$ and $N \leq M$, then $(K+N)/N \leq_{scc} M/N$.*

(iv) *If $K_i \leq_{scc} M \ \forall\ i \in I$, then $\sum_{i \in I} K_i \leq_{scc} M$.*

Finally, the next theorem defines UCC module with respect to their coclosed submodules.

Theorem 2.6.3. [5, *Theorem* 3.16] *Let M be an amply supplemented module. Then the following are equivalent.*

(a) *M is a UCC module.*

(b) *Every coclosed submodule of M is strongly coclosed.*

(c) *The sum of any family of coclosed submodules of M is coclosed.*

(d) *The sum of two coclosed submodules of M is coclosed.*

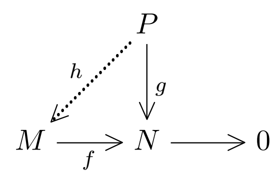
(e) *For any epimorphism $f : M \longrightarrow M'$. If $K \leq_{cc} M$ then $f(K) \leq_{cc} M'$.*

(f) *For all nonempty index sets I and all submodules $K_i \leq L_i \leq M$, if $L_i/K_i \leq_{cc} M/K_i \ \forall\ i \in I$, then $\sum_{i \in I} L_i / \sum_{i \in I} K_i \leq_{cc} M / \sum_{i \in I} K_i$.*

2.7 Projective Modules

The notion of projective modules is a generalization of the concept of free modules. It was first introduced in 1956 by Henri Cartan and Samuel Eilenberg.

Definition 2.7.1. [14, *Definition* 4.29] *Let P, M be right R-modules. Then P is called M-projective if for every epimorphism $f : M \longrightarrow N$ and every homomorphism $g : P \longrightarrow N$, $\exists$ a homomorphism $h : P \longrightarrow M$ such that $fh = g$. i.e., the following diagram commutes.*



The module P is called *projective* if P is *M-projective* $\forall$ M, and it is called *quasi-projective (or self-projective)* if it is *P-projective*. Moreover, if P is M-projective and M is P-projective, then we say that P and M are *relatively projective modules.*

Theorem 2.7.1. *Let P and M be modules. Then P is M-projective iff whenever $K \leq M$ and $f : P \longrightarrow M/K$ is a module homomorphism, then f can be lifted, through the natural epimorphism $\pi : M \longrightarrow M/K$, by a homomorphism $h : P \longrightarrow M$.*

Proof. $\boxed{\Longrightarrow}$ By definition.
$\boxed{\Longleftarrow}$ By The Factor Theorem of module homomorphism. □

Proposition 2.7.1. [14, *Proposition* 4.31] *If M is N-projective and $A \leq N$, then M is A-projective and N/A-projective.*

Definition 2.7.2. *Let M be a module. A small epimorphism $f : P \longrightarrow M$ with a projective module P, is called a projective cover of M.*

Now we will recall briefly some well-known results and properties concerned projective modules. Therefore, most of the next results will not be proved or shortly proved.

Proposition 2.7.2. [14, *Proposition* 4.32] *A module M is $(\bigoplus_{i=1}^{n} A_i)$-projective iff M is A_i-projective for all $i = 1, 2, \ldots, n$.*

Corollary 2.7.1. *A finite direct sum* $(\bigoplus_{i=1}^{n} M_i)$ *is quasi-projective iff* M_i *is* M_j*-projective for all* $i, j = 1, 2, \ldots, n$.

Lemma 2.7.1. [14, *Proposition* 4.30] *If* M *is* N*-projective, then any epimorphism* $f : N \longrightarrow M$ *splits. Furthermore, if* N *is indecomposable then* f *is an isomorphism.*

Proof. Consider the following diagram

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow{\scriptstyle id_M} & & \\ N & \xrightarrow[f]{} & M & \longrightarrow & 0 \end{array}$$

Since M is N-projective, there exists a homomorphism $\bar{f} : M \longrightarrow N$ such that $f\bar{f} = id_M$. Hence, f splits. Now suppose that N is indecomposable. Since f splits, $Ker(f) \leq^{\oplus} N$. Consequently, $Ker(f) = 0$. Therefore, f is an isomorphism. □

Theorem 2.7.2. [1, *Proposition* 17.2] *The following are equivalent for a module* M.

(a) M *is a projective module.*

(b) M *is a direct summand of a free module.*

(c) Every short exact sequence $0 \longrightarrow K \xrightarrow{f} N \xrightarrow{g} M \longrightarrow 0$ *splits.*

The following lemma can be found in [3].

Lemma 2.7.2. [21, §41.14] *Let* M *be a module and* U, V *be submodules of* M. *Consider the following conditions.*

(a) If $M = U + V$ *and* $U \leq^{\oplus} M$, *then* $\exists\, V' \leq V$ *such that* $M = U \oplus V'$.

(b) If $M = U \oplus V$, *then* V *is* U*-projective (and* U *is* V*-projective).*

(c) If $M = U \oplus V$ *and* $U \cong V$, *then* M *is self- projective.*

(d) If $M = U + V$ *and* $U, V \leq^{\oplus} M$, *then* $U \cap V \leq^{\oplus} M$.

Then $(a) \Leftrightarrow (b)$, $(b) \Rightarrow (c)$, *and* $(a) \Rightarrow (d)$.

Proof. $(a) \Rightarrow (b)$ Suppose $M = U \oplus V$ and $A \leq U$. Let $p : U \longrightarrow U/A$ be the natural epimorphism and $f : V \longrightarrow U/A$ be any homomorphism. Define the submodule $N = \{u - v \mid u \in U, v \in V, \text{and } p(u) = f(v)\}$. Since p is an epimorphism, $M = U + N$. So, by (a), $\exists\, N' \leq N$ with $M = U \oplus N'$. Now, let $\pi : M = U \oplus N' \longrightarrow U$ be the projection of M onto U. This gives the homomorphisms $V \longrightarrow M \longrightarrow U$. Now, $(1 - \pi)(V) \leq N' \leq N$. Therefore, if $v \in V$ then $v - \pi(v) \in N$ and hence $f(v) = \pi p(v)$. i.e., $f = \pi p$. Thus, V is U-projective.

$(b) \Rightarrow (a)$ Let $M = U + V$ and $M = U \oplus X$, where $X \leq M$. By (b), we can assume that X is U-projective. Consider the following diagram.

$$\begin{array}{ccccccc} U & \longrightarrow & M & \longrightarrow & M/V & \longrightarrow & 0 \\ & & & & \uparrow & & \\ & & & & X & & \end{array}$$

Since X is U-projective, this diagram can be lifted commutatively by a homomorphism $f : X \longrightarrow U$. Therefore, if $x \in X$ then $x + V = f(x) + V$ and hence $(1 - f)(X) \leq V$. So we have $M = U + X \leq U + (1 - f)(X)$ which implies that $M = U + (1 - f)(X)$. Let $V' = U + (1 - f)(X)$. Then $V' \leq V$ and $M = U + V'$. Now, if $y \in U \cap V'$ then $y = u = (1 - f)(x)$ for some $u \in U$ and $x \in X$. Consequently, $x = u + f(x) \in U \cap X = 0$ and hence $y = 0$, that is $U \cap V' = 0$. Thus, $M = U \oplus V'$.

$(b) \Rightarrow (c)$ Clear.

$(a) \Rightarrow (d)$ Let $U, V \leq^{\oplus} M$ with $M = U + V$. By (a), we can find the submodules $U' \leq U$ and $V' \leq V$ such that $M = U \oplus V'$ and $M = U' \oplus V$. Therefore, $M = (U \cap V) + (U' + V')$ and $0 = (U \cap V) \cap (U' + V')$. Hence, $M = (U \cap V) \oplus (U' + V')$. That is, $U \cap V \leq^{\oplus} M$. $\square$

3 Hollow-Lifting Modules

In this section we will try to uncover the structure of hollow-lifting modules as possible as we can. After that, we introduce what so-called completely hollow-lifting modules by adding a new condition. Eventually, we represent a generalization called finitely hollow-lifting modules.

Just to know, it is not worthless to remind again that throughout R denotes a ring with identity and every R-module M is a unitary right R-module, unless otherwise stated.

3.1 Definition & Characterization

Definition 3.1.1. *A module M is called hollow-lifting if every submodule N of M with M/N hollow has a coessential submodule in M that is a direct summand of M.*

Example 3.1. *It is clear that lifting modules(hence hollow modules) and semisimple modules are hollow-lifting. Moreover, any module having no hollow factor modules is trivially hollow-lifting.*

The following proposition provides a simple characterization of indecomposable hollow-lifting modules

Proposition 3.1.1. [15, *Proposition* 2.7] *Let M be an indecomposable module. Then M is hollow-lifting iff M is hollow, or else M has no hollow factor modules.*

Proof. The sufficiency is clear. Now, Suppose that M has a proper submodule N such that M/N is hollow. Then, since M is hollow-lifting, $\exists\, K \leq^{\oplus} M$ with $K \leq_{ce} N$ in M. But M is indecomposable, therefore $K = 0$. Consequently, $N/K = N/0 \cong N$ and $M/K = M/0 \cong M$, hence $N \ll M$. Thus, M is hollow. □

Now, we can link hollow-lifting and amply supplemented modules through the following proposition.

Proposition 3.1.2. *If M is hollow-lifting, then every coclosed submodule N of M with M/N hollow is a direct summand of M. The converse is true if M is amply supplemented.*

Proof. Let M be a hollow-lifting module and $N \leq_{cc} M$ with M/N hollow. Then $\exists\, L \leq^{\oplus} M$ such that $N/L \ll M/L$. But $N \leq_{cc} M$, so $N = L$. Hence, N is a direct summand of M. The converse follows directly from [*Proposition* 2.3.1]. □

Lemma 3.1.1. [1, *Proposition* 5.5] *Let $M = H_1 \oplus H_2$ be a module and N be a submodule of M. Let $\pi_1 : M \longrightarrow H_1$ be the projection epimorphism of M onto H_1. Then $M = N \oplus H_2 \Leftrightarrow \pi_1|_N : N \longrightarrow H_1$ is an isomorphism.*

Proof. First, $Ker(\pi_1|_N) = N \cap Ker(\pi_1) = N \cap H_2$. So $\pi_1|_N$ is monomorphism $\Leftrightarrow N \cap H_2 = 0$. Moreover, since $\pi_1|_{H_1} = id_{H_1}$ and $Ker(\pi_1) = H_2$,

$$\begin{aligned}\pi_1(N) &= \pi_1(N + H_2)\\ &= \pi_1((N + H_2) \cap (H_1 + H_2))\\ &= \pi_1[((N + H_2) \cap H_1) + H_2]\\ &= \pi_1((N + H_2) \cap H_1)\\ &= (N + H_2) \cap H_1.\end{aligned}$$

So $\pi_1|_N$ is an epimorphism $\Leftrightarrow (N + H_2) \cap H_1 = H_1 \Leftrightarrow H_1 \leq N + H_2$ $\Leftrightarrow M = N + H_2$. This completes the proof. □

The following result provides another example of hollow-lifting modules.

Proposition 3.1.3. [15, *Proposition* 2.1] *Let H_1 and H_2 be hollow modules. Then the following are equivalent for the module $M = H_1 \oplus H_2$.*

(a) M is hollow-lifting.

(b) M is lifting.

Proof. $(b) \Longrightarrow (a)$ Trivial.
$(a) \Longrightarrow (b)$ Let $N \leq M$. Consider the natural projections $\pi_1 : M \longrightarrow H_1$ and $\pi_2 : M \longrightarrow H_2$. If $\pi_1(N) \neq H_1$ and $\pi_2(N) \neq H_2$. Then, since H_1 and H_2 are hollow, $\pi_1(N) \ll H_1$ and $\pi_2(N) \ll H_2$ and hence we get that $\pi_1(N) \oplus \pi_2(N) \ll H_1 \oplus H_2$. Now claim that $N \subseteq \pi_1(N) \oplus \pi_2(N)$. To verify that, let $n \in N$ then $n = (h_1, h_2)$, where $h_1 \in H_1$, $h_2 \in H_2$. Therefore, $\pi_1(n) = h_1$ and $\pi_2(n) = h_2$, that is, $n = (\pi_1(n), \pi_2(n))$. So our claim is verified. Consequently, $N \ll M$. Thus, M is lifting. Now, assume that $\pi_1(N) = H_1$. Then $M = N + H_2$. By the second isomorphism theorem, $M/N = (N + H_2)/N \cong H_2/(N \cap H_2)$. But H_2 is hollow, so $H_2/(N \cap H_2)$ is hollow, therefore M/N is also hollow. Hence, since M is hollow-lifting, there exists $K \leq^{\oplus} M$ with $K \leq_{ce} N$ in M. Thus, M is again lifting. □

Example 3.2. *Using* [*Example* 2.2] *and the previous proposition, we get that the $\mathbb{Z}$-module $\mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ is hollow-lifting but $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ is not.*

To introduce a useful characterization of hollow-lifting modules, we start with the following lemma.

Lemma 3.1.2. [15, *Proposition* 2.3] *Let M be a module and $U \leq M$. Then the following are equivalent.*

(a) U has a strong supplement in M.

(b) U has a coessential submodule that is a direct summand of M.

Proof. $(a) \implies (b)$ Let V be a strong supplement of U in M. Then we have $M = V + U$, $V \cap U \ll V$, and $U = (V \cap U) \oplus W$ for some $W \leq U$. Consequently, we have $M = V + U = V + (V \cap U) + W = V + W$ and also $0 = W \cap (V \cap U) = W \cap V$. This means that $M = V \oplus W$, hence $W \leq^{\oplus} M$. Now, if $W \leq X \leq M$ and $U/W + X/W \ll M/W$, then $U + X = M$ and so $M = (U \cap V) + W + X = (U \cap V) + X$. But $V \cap U \ll V$, so $V \cap U \ll M$, hence $X = M$. Therefore, $U/W \ll M/W$. Thus, W is the submodule we seek.

$(b) \implies (a)$ Let $A \leq_{ce} U$ with $A \leq^{\oplus} M$. Then $U/A \ll M/A$ and $A \oplus B = M$ for some $B \leq M$. We claim that B is a strong supplement of U in M. To verify this claim, we must show that $B + U = M$, $B \cap U \ll B$, and $B \cap U \leq^{\oplus} U$. Now, $A + B = M$ and $A \leq U$ implies $B + U = M$. Also, $(B \cap U) \cap A = 0 \cap U = 0$. Moreover, $(B \cap U) + A = (B + A) \cap U = M \cap U = U$. Hence, $(B \cap U) \oplus A = U$. This means $(B \cap U) \leq^{\oplus} U$. Finally, if $X \leq B$ with $(B \cap U) + X = B$, then $M = A + B = A + (B \cap U) + X = U + X$. Hence, $M/A = U/A + (X + A)/A$. But $U/A \ll M/A$, so $X + A = M$. Therefore, $B = B \cap M = B \cap (X + A) = X + 0 = X$. Thus, $B \cap U \ll B$. This completes the proof. □

Corollary 3.1.1. *Let M be a module. Then the following are equivalent.*

(a) M is hollow-lifting

(b) Every submodule $N \leq M$ with M/N hollow has a strong supplement in M.

Proposition 3.1.4. [15, *Proposition* 2.5] *Let M be a module. Then the following are equivalent.*

(a) M is hollow-lifting.

(b) Every submodule $N \leq M$ with M/N hollow can be written as $N = K \oplus L$, where $K \leq^{\oplus} M$ and $L \ll M$.

Proof. $(a) \Longrightarrow (b)$ Let $N \leq M$ with M/N hollow. Then, since M is hollow-lifting, $\exists\ K \leq^{\oplus} M$ such that $K \leq N$ and $N/K \ll M/K$. Let $F \leq M$ with $M = K \oplus F$. We claim that $L = F \cap N$ is the one we seek. To verify that, we have to show that $N = K \oplus L$ and $L \ll M$. First of all, we have $K + L = K + (F \cap N) = (K + F) \cap N = M \cap N = N$. Also, $K \cap L = K \cap (F \cap N) = (K \cap F) \cap N = 0 \cap N = 0$. Hence, $N = K \oplus L$. To show $L \ll M$, it suffices to prove that $L \ll F$. So let $X \leq F$ with $L+X = F$. Then $F = (F \cap N) + X = F \cap (N + X)$ which implies $F \leq N + X$ and hence $M = F + K \leq N + X + K = N + X$. This means $M = N + X$. Now, $(N + X + K)/K = N/K + (X + K)/K = M/K$, but $N/K \ll M/K$, therefore $X + K = M$. Consequently, $X = F$. This completes this direction.

$(b) \Longrightarrow (a)$ Let $N \leq M$ with M/N hollow. Then, by the assumption, $N = K \oplus L$ with $K \leq^{\oplus} M$ and $L \ll M$. Now, let $X \leq M$ with $K \leq X$ and $N/K + X/K = M/K$. Then $N + X = M$, so $K + L + X = M$. But $L \ll M$, therefore $X + K = X = M$. This implies that $N/K \ll M/K$. Thus, M is hollow-lifting. □

Now, we are ready to introduce the following characterization.

<u>**Theorem**</u> **3.1.1 (Characterization Of Hollow-lifting Modules).** *Let M be a module. Then the following are equivalent.*

(a) M is hollow-lifting.

(b) Every submodule $N \leq M$ with M/N hollow can be written as $N = K \oplus L$, where $K \leq^{\oplus} M$ and $L \ll M$.

(c) Every submodule $N \leq M$ with M/N hollow has a strong supplement in M.

(d) For every submodule $N \leq M$ with M/N hollow, there exists a decomposition $M = X \oplus X'$ with $X \leq N$ and $N \cap X' \ll M$.

Proof. The theorem is just a combination of [*Lemma* 2.4.1], [*Corollary* 3.1.1], and [*Proposition* 3.1.4], □

<u>*Remark*</u>. An example of a hollow-lifting module which is not lifting was introduced by C. Lomp [12, *Proposition* 2.1]. Actually, he provided a full paper for this purpose. Unfortunately, it has turned out that this example has some mistakes as Lomp himself declared. However, Orhan [15, *Remark* 2.10] has pointed out how to construct such example. The next two propositions clarify our point.

Proposition 3.1.5. *Any indecomposable module with no hollow factor modules is hollow-lifting but not lifting.*

Proof. Trivially, M is hollow-lifting. To show that M is not lifting, we first claim that M has a nonzero proper submodule. If not, then M is a simple module and hence hollow, a contradiction. So let A be a nonzero proper submodule of M. Now, if M is lifting then A can be written as $A = X \oplus Y$ with $X \leq^{\oplus} M$ and $Y \ll M$ [*Theorem* 2.4.1]. But M is indecomposable and $A \lneqq M$, therefore $X = 0$ and hence $Y = A$, a contradiction. Thus, M is not lifting. □

Proposition 3.1.6. *Let K be a semisimple module and N an indecomposable module with no hollow factor modules. If $M = N \oplus K$, then M is hollow-lifting but not lifting.*

Proof. Let L be a submodule of M such that M/L is hollow . First of all, $M/L = (N+K)/L = (N+L)/L+(K+L)/L$. But M/L is hollow, therefore either $N + L = M$ or $K + L = M$. If $N + L = M$ then $M/L \cong N/(N \cap L)$ which is impossible because M/L is hollow and N has no hollow factor modules. So, $M = K + L$. But K is semisimple, hence $(K \cap L) \leq^{\oplus} K$. It follows that there exists $E \leq K$ such that $K = (K \cap L) \oplus E$. Therefore, $M = K + L = E \oplus (K \cap L) + L = E \oplus L$. Hence, $L \leq^{\oplus} M$ and so M is hollow-lifting. Now, if M is lifting, then N is also lifting [*Corollary* 2.4.2]. Since M is indecomposable, it follows that N is hollow [*Proposition* 2.4.1], a contradiction. Thus, M is not lifting. □

3.2 More Properties Of Hollow-lifting Modules

We begin this part by the following quick result.

Proposition 3.2.1. [15, *Proposition* 2.9] *Let $M_1, M_2, \dots, M_n$ be modules with no hollow factor modules. Then $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is hollow-lifting.*

Proof. We claim that M has no hollow factor modules. Otherwise, if $N \leq M$ with M/N hollow, then $(M_1+N)/N + \cdots + (M_n+N)/N = M/N$ is hollow. Therefore, there exists $i \in \{1, 2, \dots, n\}$ with $(M/N) = (M_i + N)/N$ which means M_i has a hollow factor module, a contradiction. So our claim is true. Consequently, M is hollow-lifting. □

The following lemma [15, *Lemma* 2.11] is useful in supporting some results.

Lemma 3.2.1. *Let M be a module and $K \leq^{\oplus} M$ such that K has the finite exchange property. If $K \leq N \leq M$ and N has a strong supplement in M, then N/K has a strong supplement in M/K.*

Proposition 3.2.2. [15, *Proposition* 2.12] *Let M be a hollow-lifting module and $K \leq^{\oplus} M$ such that K has the finite exchange property. Then M/K is hollow-lifting.*

Proof. Let $N \leq M$ with $K \leq N$ and $(M/K)/(N/K)$ is hollow. Then $M/N \cong (M/K)/(N/K)$ and hence M/N is hollow. But M is hollow-lifting, so N has a strong supplement in M. Consequently, by the above lemma, N/K has a strong supplement in M/K. Thus, M/K is hollow-lifting. □

Proposition 3.2.3. [15, *Proposition* 2.13] *Let M be a hollow-lifting module having a nonsmall hollow submodule. Then M has a hollow direct summand.*

Proof. Let H be a nonsmall hollow submodule of M. Then $\exists\ N \lneqq M$ with $M = H + N$. Therefore, $M/N = (H+N)/N \cong H/(H \cap N)$. But H is hollow, so $H/(H \cap N)$ is also hollow, hence M/N is hollow. Since M is hollow-lifting, $\exists\ L \leq^{\oplus} M$ such that $N/L \ll M/L$. Moreover, we have $M/L = (H+N)/L = (H+N+L)/L = (H+L)/L + N/L$. But $N/L \ll M/L$, so $M/L = (H+L)/L \cong H/(H \cap L)$. This means that M/L is hollow. Now, let $K \leq M$ with $M = L \oplus K$. Then $M/L = (L+K)/L \cong K/(K \cap L) = K/0 \cong K$, hence K is hollow. Thus, K is a hollow direct summand of M. □

<u>Lemma</u> 3.2.2. *Let M be a module with submodules A and B. If A is a strong supplement of B in M, then $A \leq^{\oplus} M$. Moreover, if B is a maximal submodule and A is a strong supplement of B, then A is a local direct summand of M.*

Proof. Suppose A is a strong supplement of B in M. Then $M = A + B$, $A \cap B \ll A$, and $(A \cap B) \oplus C = B$ for some $C \leq M$. Claim that $M = A \oplus C$. Firstly, $A \cap C = A \cap C \cap B = 0$. Moreover, $B = C + (A \cap B) = (C + A) \cap B$, therefore $B \leq C + A$, hence $M = A + B = A + C + A = A + C$. So our claim is verified and so $A \leq^{\oplus} M$.

Now, suppose B is a maximal submodule and with a strong supplement A. Then M/B is simple module and hence hollow. Also, we have that $M/B = (A + B)/B \cong A/(A \cap B)$ which implies $A/(A \cap B)$ is simple, hence hollow. But $A \cap B \ll A$, so A is also hollow. Now, $A \cap B$ is maximal submodule of A, moreover it is unique since if $X \leq A$ is any other maximal submodule of A, then $X + (A \cap B) = A$. But $A \cap B \ll A$, so $X = A$, contradicting the maximality of X. Hence, A is local. This ends the proof. □

As a result of the previous lemma, we provide the following proposition.

<u>Proposition</u> 3.2.4. [15, *Lemma* 2.14] *Let M be a hollow-lifting module having a maximal submodule N. Then M has a local direct summand.*

Proof. Since N is maximal, M/N is a simple module and so it is hollow. But M is hollow-lifting, so N has a strong supplement in M, say K. By the previous lemma, K is a local direct summand of M. □

In order to continue reviewing our properties, we give the following terminology.

<u>Definition</u> 3.2.1. [15] *A module M is called coatomic if every proper submodule of M is contained in a maximal submodule.*

<u>Lemma</u> 3.2.3. *If M is a coatomic module, then $Rad(M) \ll M$.*

Proof. For the sake of a contradiction, assume $Rad(M)$ is not small in M. Then $\exists$ a proper submodule $K \leq M$ such that $M = Rad(M) + K$. But M is coatomic, so K is contained in a maximal submodule $N \leq M$. But $Rad(M)$ is the intersection of all maximal submodules of M, therefore we get that $M = Rad(M) + K \leq N$ and hence $M = N$, a contradiction. Thus, $Rad(M) \ll M$. □

The next result gives a good structure of coatomic hollow-lifting modules.

Proposition 3.2.5. [15, *Proposition* 2.16] *Let M be a coatomic hollow-lifting module. Then M can be written as an irredundant sum of local direct summands of M.*

Proof. Since M is coatomic, $Rad(M) \ll M$ and hence a proper submodule, therefore M has a maximal submodule containing $Rad(M)$. But M is hollow-lifting, so it has a local direct summand. Let N be the sum of all local direct summands of M. If N is a proper submodule of M, then there is a maximal submodule $L \leq M$ with $N \leq L$. Now, M/L is simple and hence hollow, but M is hollow-lifting, so L has a strong supplement P in M. Moreover, P is a local direct summand of M since L is a maximal submodule, therefore $P \leq N$. This implies that $M = L + P \leq L + N = L$, contradicting the maximality of L. Hence, $M = N$. So let $M = \sum\limits_{i \in I} A_i$, where each A_i is a local direct summand of M. Then

$$\frac{M}{Rad(M)} = \frac{\sum\limits_{i \in I} A_i}{Rad(M)} = \sum_{i \in I} \frac{A_i + Rad(M)}{Rad(M)}$$

But each $\dfrac{A_i + Rad(M)}{Rad(M)} \cong \dfrac{A_i}{Rad(M) \cap A_i}$ which is simple [3, §20.4]

Hence, by [1, *Proposition* 9.3], $\exists\ K \subseteq I$ such that

$$\frac{M}{Rad(M)} = \bigoplus_{k \in K} \left[\frac{A_k + Rad(M)}{Rad(M)}\right]$$

Thus, $M = \sum\limits_{k \in K} A_k$ since $Rad(M) \ll M$. This ends the proof. □

Proposition 3.2.6. [15, *Corollary* 2.17] *Let M be a coatomic module with $Rad(M) = 0$. Then the following are equivalent.*

(a) M is hollow-lifting.

(b) M is supplemented.

(c) M is semisimple.

Proof. $(a) \implies (c)$ Since M is a coatomic hollow-lifting module, M can be written as a sum of local direct summands. Let H be any local direct summand of M. Then $Rad(H) \leq Rad(M) = 0$, so $Rad(H) = 0$, hence the zero submodule is the only small submodule of H. But H is local, hence hollow, therefore H contains no proper submodules other than 0. Thus, H is simple. This means that M is a sum of simple submodules. So M is semisimple [1, *Proposition* 9.3].

$(c) \implies (b)$ Suppose M is semisimple module and let $L \leq M$. Then $L \leq^{\oplus} M$, therefore $M = L \oplus K$ for some $K \leq M$. This means K is a supplement of L in M. Hence, M is supplemented.

$(b) \implies (a)$ Let $L \leq M$ with M/L hollow. Since M is supplemented, $\exists\ H \leq M$ such that H is a supplement of L in M. i.e., $M = L + H$ and $L \cap H \ll H$, hence $L \cap H \ll M$. But $Rad(M) = 0$, therefore $L \cap H = 0$. Thus, $M = L \oplus H$. This means that H is a strong supplement of L in M. Hence, M is hollow-lifting. □

Lemma 3.2.4. *If M is an amply supplemented module with finite hollow dimension, then M has a coclosed submodule K with M/K hollow.*

Proof. Since M has a finite hollow dimension, $\exists\ N \leq M$ such that M/N is hollow [13, § 3.1.4]. But M is amply supplemented, so by [*Proposition* 2.3.1] N has a coclosure in M, say K. That means that K is coclosed in M and $N/K \ll M/K$. Now, $M/N \cong (M/K)/(N/K)$, hence $(M/K)/(N/K)$ is hollow which implies, since $N/K \ll M/K$, that M/K is hollow. □

Proposition 3.2.7. [15, *Lemma* 2.18] *Let M be an amply supplemented hollow-lifting module, and $K \leq_{cc} M$ such that M/K has a finite hollow dimension. Then $K \leq^{\oplus} M$.*

Proof. The proof is by induction on $h(M/K)$. If $h(M/K) = 1$, then M/K is hollow [*Lemma* 2.2.1]. But M is hollow-lifting and $K \leq_{cc} M$, so $K \leq^{\oplus} M$ [*Proposition* 3.1.2]. Now, assume that $h(M/K) = n$ and for any $T \leq_{cc} M$ with $h(M/T) < n$, $T \leq^{\oplus} M$. Since M is amply supplemented, M/K is amply supplemented [*Proposition* 2.3.2]. But M/K has a finite hollow dimension, so from the previous lemma $\exists\ H/K \leq_{cc} M/K$ with $(M/K)/(H/K)$ hollow, hence $H \leq_{cc} M$ [*Lemma* 2.3.4]. But M is hollow-lifting and $M/H \cong (M/K)/(H/K)$ is hollow, so $H \leq^{\oplus} M$, that is, $M = H \oplus H'$ for some $H' \leq M$. Consequently, $H \cap (K \oplus H') = (H \cap K) \oplus (H \cap H') = K \oplus 0 = K$, and $(H/K) \oplus ((K \oplus H')/K) = (H \oplus K \oplus H')/K = (M \oplus K)/K = M/K$, Therefore, $(K \oplus H')/K \leq_{cc} M/K$, hence $K \oplus H' \leq_{cc} M$ [*Lemma* 2.3.4]. Thus, by induction, $K \oplus H' \leq^{\oplus} M$. Hence, $K \leq^{\oplus} M$. □

Proposition 3.2.8. [15, *Proposition* 2.19] *Let M be an amply supplemented module with a finite hollow dimension. Then the following are equivalent.*

(a) M is lifting.

(b) M is hollow-lifting.

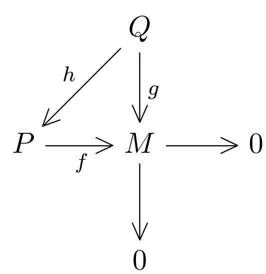
Proof. $(a) \Longrightarrow (b)$ Clear

$(b) \Longrightarrow (a)$ Let $N \leq M$. Since M is amply supplemented, N has a coclosure in M. i.e., $\exists\ K \leq M$ with $K \leq_{cc} M$ and $K \leq_{ce} N$ in M. Now, since M has a finite hollow dimension, M/K has also a finite hollow dimension, hence $K \leq^{\oplus} M$ by the previous proposition. Thus, M is lifting. □

Now, consider the following lemma.

Lemma 3.2.5. [1, *Lemma* 17.17] *Suppose that a module M has a projective cover $f : P \longrightarrow M$. If Q is a projective module and $g : Q \longrightarrow M$ is an epimorphism, then Q has the decomposition $Q = P_1 \oplus P_2$ such that $P_1 \cong P$, $P_2 \leq Ker(g)$, and the restriction $g|_{P_1} : P_1 \longrightarrow M$ is a projective cover for M.*

Proof. Consider the following diagram



Since Q is projective, the above diagram is commutative with exact row and column. Moreover, since f is a small epimorphism and $fh = g$, h is also an epimorphism. Now, h splits because P is projective [*Lemma* 2.7.1]. Therefore, there exists a monomorphism $\bar{h} : P \longrightarrow Q$ such that $h\bar{h} = id_P$ and hence $Q = Im(\bar{h}) \oplus Ker(h)$. Let $P_1 = Im(\bar{h})$ and $P_2 = Ker(h)$. Then $Q = P_1 \oplus P_2$, $P_1 \cong P$ since $\bar{h}$ is a monomorphism and $P_2 \leq Ker(g)$ because $fh = g$. Now, $M = g(Q) = g(P_1)$ and so the sequence

$$P_1 \xrightarrow{g|_{P_1}} M \longrightarrow 0$$

is exact. Moreover, $g\bar{h} = fh\bar{h} = f$, therefore we have

$$Ker(g|_{P_1}) = \bar{h}(Ker(f)) \ll \bar{h}(P) = P_1.$$

Thus, $g|_{P_1}$ is a projective cover of M. This completes the proof. □

Using the last lemma, we can deduce the next result.

Proposition 3.2.9. *Let M be a projective module. Then M is hollow-lifting iff for every submodule N of M with M/N hollow, M/N has a projective cover.*

Proof. $\boxed{\Longrightarrow}$ Since M is hollow-lifting, it follows that M has the decomposition $M = X \oplus Y$ with $X \leq N$ and $N \cap Y \ll M$ [*Theorem* 3.1.1] and hence $N \cap Y \ll Y$. Consider the following short exact sequence

$$0 \longrightarrow N \cap Y \xrightarrow{\iota} Y \xrightarrow{\pi} Y/(N \cap Y) \longrightarrow 0$$

Now, Y is projective since M is projective and $Y \leq^{\oplus} M$. Moreover, $Ker(\pi) = N \cap Y \ll Y$. Hence, $\pi : Y \longrightarrow Y/(N \cap Y)$ is a small epimorphism with Y projective. That is, $Y/N \cap Y$ has a projective cover. But $M/N = (Y + N)/N \cong Y/(N \cap Y)$. Thus, M/N has a projective cover. □

$\boxed{\Longleftarrow}$ Let $N \leq M$ with M/N hollow. Then, by assumption, M/N has a projective cover, say $f : P \longrightarrow M/N$. Now, consider the natural epimorphism $\pi : M \longrightarrow M/N$. Then, by the last lemma, M has the decomposition $M = P_1 \oplus P_2$ such that $P_1 \cong P$, $P_2 \leq Ker(\pi) = N$, and the restriction $\pi|_{P_1} : P_1 \longrightarrow M/N$ is projective cover for M/N. Therefore, $Ker(\pi|_{P_1}) = N \cap P_1 \ll M$. Hence, M is hollow-lifting.

3.3 Completely Hollow-lifting Modules

The idea of completely hollow-lifting modules has come out by Orhan [15] as a result of not being able to find a hollow-lifting module with a direct summand that is not hollow-lifting .

Definition 3.3.1. *A module M is called completely hollow-lifting if every direct summand of M is hollow-lifting.*

Clearly, any completely hollow-lifting module is hollow-lifting. But the converse is not obvious yet as we pointed before.

Let us now start with the following result.

Proposition 3.3.1. [15, *Proposition* 5.1] *Let M be a weakly supplemented module UCC module. If M is hollow-lifting, then it is completely hollow-lifting.*

Proof. Let $N \leq^{\oplus} M$ and $A \leq N$ with N/A hollow. Then $M = N \oplus N'$ for some $N' \leq M$. Moreover, M must amply supplemented [*Theorem* 2.3.1] and hence N is also amply supplemented by [*Proposition* 2.3.2]. Therefore, there is $A' \leq_{cc} N$ with $A' \leq A$ and $A/A' \ll N/A'$. Now, since $N \leq_{cc} M$ [*Lemma* 2.3.3], it follows that $A' \leq_{cc} M$ from [*Lemma* 2.1.7, (d)]. Hence, $A' \oplus N' \leq_{cc} M$ by [*Theorem* 2.6.3]. Now, $(N/A')/(A/A') \cong N/A$ which is hollow. But $A/A' \ll N/A'$, therefore N/A' is hollow [*Lemma* 2.2.1, (b)] and so $M/(N' \oplus A')$ is hollow. Since M is hollow-lifting, $N' \oplus A' \leq^{\oplus} M$ implying that $A' \leq^{\oplus} N$. Thus, N is hollow-lifting. This ends the proof. □

The next proposition provides a different condition so that a hollow-lifting module becomes completely hollow-lifting.

Proposition 3.3.2. [15, *Proposition* 5.2] *If M is a hollow-lifting module and satisfies (D_3), then it is completely hollow-lifting.*

Proof. Let $M = N \oplus N'$ and $A \leq N$ with N/A hollow. We will show that N is hollow-lifting. Clearly, $M/(N' \oplus A)$ is hollow because $M/A = N/A \oplus (N' \oplus A)/A$. But M is hollow-lifting, hence there exists $K \leq^{\oplus} M$ such that $K \leq N' \oplus A$ and $(N' \oplus A)/K \ll M/K$. Therefore, $M = K + N$ and so $[N \cap (N' \oplus A)]/(K \cap N) \ll M/(K \cap N)$ [*Lemma* 2.1.5, (h)]. But M has (D_3), therefore $K \cap N \leq^{\oplus} M$ and hence $K \cap N \leq^{\oplus} N$. Since $N/(K \cap N) \leq^{\oplus} M/(K \cap N)$, it follows that $A/(K \cap N) \ll N/(K \cap N)$. Thus, N is hollow-lifting. □

Now by recalling what a duo module means, we have the following two results.

Proposition 3.3.3. [15, *Lemma* 5.5] *If M is a hollow-lifting module and A is a fully invariant submodule of M, then M/A is hollow-lifting.*

Proof. Let $U/A \leq M/A$ with $(M/A)/(U/A)$ hollow. Then we have that $(M/A)/(U/A) \cong M/U$. But M is hollow-lifting, therefore there is $B \leq^{\oplus} M$ with $B \leq U$ and $U/B \ll M/B$. Suppose $M = B \oplus B'$. It follows that $(A+U)/(A+B) = U/(A+B) \ll M/(A+B)$ [*Lemma* 2.1.5, (i)]. We claim that $(B+A)/A \leq^{\oplus} M/A$. By [*Lemma* 2.5.2] and since $M = B \oplus B'$, we have $M/A = (A+B)/A \leq^{\oplus} (A \oplus B')/A$. So our claim is true. Thus, M/A is hollow-lifting. □

Corollary 3.3.1. *If M is a duo hollow-lifting module, then it is completely hollow-lifting.*

Proof. Let $N \leq^{\oplus} M$. Then $M = N \oplus N'$ for some $N' \leq M$. By assumption, M/N' is hollow-lifting. Now, $M/N' \cong N$. Therefore, N is hollow-lifting. This completes the proof. □

The last corollary helps us to introduce the next result.

Theorem 3.3.1. [15, *Theorem* 6.3] *If $M = M_1 \oplus M_2$ is a duo module, then M is hollow-lifting iff both M_1 and M_2 are hollow-lifting.*

Proof. The forward direction is an immediate result of [*Corollary* 3.3.1]. Now, let $A \leq M$ with M/A hollow. Then A is a fully invariant submodule of M, hence from [*Lemma* 2.5.2] we have $A = (A \cap M_1) \oplus (A \cap M_2)$ and so $M/A = (A + M_1)/A \oplus (A + M_2)/A$. But M/A is hollow, therefore we get that $(A + M_1)/A = M/A$ and so $M_2 \leq A$. Since M_1 is hollow-lifting and $(A+M_1)/A \cong M_1/(A \cap M_1)$, there exists $B_1 \leq^{\oplus} M_1$ such that $B_1 \leq A \cap M_1$ and $(A \cap M_1)/B_1 \ll M_1/B_1$. Therefore, $A/(B_1 \oplus M_2) \ll M/(B_1 \oplus M_2)$. Since $B_1 \oplus M_2 \leq^{\oplus} M$, it follows that M is hollow-lifting. □

Corollary 3.3.2. *If $M = M_1 \oplus \cdots \oplus M_n$ is a duo module, then M is hollow-lifting iff each M_i $(i = 1, \ldots, n)$ is hollow-lifting.*

Proof. The proof is by induction using [*Proposition* 2.5.1]. □

The following two lemmas can be found in [14]. They will be helpful to end this part.

Lemma 3.3.1. [14, *Lemma* 3.19] *Let M be a module having the exchange property and suppose $A = M \oplus N \oplus L = (\bigoplus_{i \in I} A_i) \oplus L$. Then there exist submodules $B_i \leq A_i$ such that $A = M \oplus (\bigoplus_{i \in I} B_i) \oplus L$.*

Proof. Let π be the projection epimorphism of M onto $\bigoplus_{i \in I} A_i$. Then we get $Ker(\pi) = L$. Moreover, the restriction of π to $M \oplus N$ is an isomorphism. Now, $\pi(M) \oplus \pi(N) = \bigoplus_{i \in I} A_i$. Since M has the exchange property and $\pi(M) \cong M$, we have $\pi(M) \oplus \pi(N) = \pi(M) \oplus (\bigoplus_{i \in I} B_i)$ with $B_i \leq A_i$. Therefore, $A = M \oplus N \oplus L = \pi^{-1}[\pi(M) \oplus (\bigoplus_{i \in I} B_i)] = M \oplus (\bigoplus_{i \in I} B_i) \oplus L$. □

Lemma 3.3.2. [14, *Lemma* 3.20] *Let $M = X \oplus Y$ be a module. If X and Y have the exchange property, then so is M.*

Proof. Let $A = M \oplus N = \bigoplus_{i \in I} A_i$. Then $A = X \oplus Y \oplus N = Y \oplus (\bigoplus_{i \in I} B_i)$ with $B_i \leq A_i$. Then, from the previous lemma, we have $A = X \oplus Y \oplus (\bigoplus_{i \in I} C_i)$ with $C_i \leq B_i$. Hence, M has the exchange property. □

The following two results end this part.

Proposition 3.3.4. [15, *Proposition* 5.7] *Let $M = \bigoplus_{i \in I} M_i$ be a module such that the decomposition $\bigoplus_{i \in I} M_i$ complements direct summands and $End(M_i)$ is local $\forall\ i \in I$. If M is hollow-lifting, then it is completely hollow-lifting.*

Proof. Let A be a direct summand of M. Then $M = A \oplus B$ for some $B \leq M$. From [14, *Theorem* 2.25] we get that M has the finite exchange property. Therefore, from the previous lemma, B must has the finite exchange property. But M is hollow-lifting, so M/B is also hollow-lifting by [*Proposition* 3.2.2]. Since $M/B \cong A$, it follows that A is hollow-lifting. Hence, M is completely hollow-lifting. □

Corollary 3.3.3. *Let $M = M_1 \oplus \cdots \oplus M_n$ be a module such that each $End(M_i)$ is local. If M is hollow-lifting, then it is completely hollow-lifting.*

Proof. Follows from [*Corollary* 2.1.2]. □

3.4 Finitely Hollow-lifting Modules

The concept of finitely hollow-lifting modules (Shortened: f-hollow-lifting) was introduced in [7] as a generalization of hollow-lifting modules.

Definition 3.4.1. *A module M is called f-hollow-lifting if every f.g. submodule N of M with M/N hollow has a coessential submodule in M that is a direct summand of M.*

Example 3.3. *Consider the following examples.* [7, *Example* 2.1]

1. *Any hollow-lifting module is f-hollow-lifting.*
2. *The $\mathbb{Z}$-module $\mathbb{Z}/4\mathbb{Z}$ is f-hollow lifting.*
3. *The $\mathbb{Z}$-module $\mathbb{Q}$ is f-hollow-lifting but not hollow-lifting.*
4. *$\mathbb{Z}$, as a submodule of $\mathbb{Q}_{\mathbb{Z}}$, is not f-hollow-lifting.*

Remark. The last two examples above tell us that a f-hollow-lifting module need not be hollow-lifting. Besides, a submodules of a f-hollow-lifting module also need not be f-hollow-lifting. However, if a module M is Noetherian, then hollow-lifting and f-hollow-lifting modules are equivalent [*Theorem* 2.1.7].

Let us now start representing some properties about f-hollow-lifting modules. Before that, we must point out that most results will appear as imitations of those hollow-lifting modules possess.

Proposition 3.4.1. [7, *Proposition* 2.4] *Let M be a module and N a f.g. fully invariant submodule of M. Then M/N is f-hollow-lifting.*

Proof. Similar to the proof of [*Proposition* 3.3.3]. □

Example 3.4. [7, *Remark* 2.5] *Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ and let $A = 2\mathbb{Z}/4\mathbb{Z} \oplus 0$ be a submodule of M. Then M is hollow-lifting* [*Example* 3.2] *and hence it is f-hollow-lifting. But M/A is not f-hollow-lifting since $M/A = [\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}]/[2\mathbb{Z}/4\mathbb{Z} \oplus 0] \cong [(\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z})] \oplus 0 \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})$ which is not f-hollow-lifting.*

The process of characterizing f-hollow-lifting modules has turned out to be easy because it is very similar to that of hollow-lifting modules. The next five results support our thought.

Lemma 3.4.1. *Let M be a module and U a f.g. submodule of M. Then U has a strong supplement in M iff U has a coessential submodule that is a direct summand in M.*

Proof. This is an immediate result of [*Lemma* 3.1.2]. □

Corollary 3.4.1. *A module M is f-hollow-lifting iff any f.g. submodule U of M with M/U hollow has a strong supplement in M.*

Proposition 3.4.2. [7, *Theorem* 3.4] *Let M be a module. Then M is f-hollow-lifting iff every f.g. submodule U of M with M/U hollow can be written as $U = K \oplus L$ with $K \leq^{\oplus} M$ and $L \ll M$.*

Proof. Just imitate the proof of [*Proposition* 3.1.4]. □

Proposition 3.4.3. [7, *Theorem* 3.1] *Let M be a module. Then M is f-hollow-lifting iff for every f.g. submodule U of M with M/U hollow, there is a decomposition $M = X \oplus X'$ with $X \leq U$ and $U \cap X' \ll M$.*

Proof. Follows from [*Lemma* 2.4.1]. □

Proposition 3.4.4. *Let M be a f-hollow-lifting module. Then every f.g. coclosed submodule U of M with M/U hollow is a direct summand of M. The converse is true if M is amply supplemented.*

Proof. The same proof of [*Proposition* 3.1.2]. □

Now we can introduce the next characterization.

Corollary 3.4.2 (Characterization Of F-hollow-lifting Modules). *Let M be a module. Then the following are equivalent.*

(a) M is f-hollow-lifting.

(b) Every f.g. submodule $N \leq M$ with M/N hollow can be written as $N = K \oplus L$, where $K \leq^{\oplus} M$ and $L \ll M$.

(c) Every f.g. submodule $N \leq M$ with M/N hollow has a strong supplement in M.

(d) For every f.g. submodule $N \leq M$ with M/N hollow, there is a decomposition $M = X \oplus X'$ with $X \leq N$ and $N \cap X' \ll M$.

We proceed by providing more properties of f-hollow-lifting modules.

Proposition 3.4.5. *Let M be an indecomposable module. The following are equivalent.*

(i) *M is f-hollow-lifting.*

(ii) *M is hollow, or else M has no $f.g$ submodule N such that M/N is hollow*

Proof. $(ii) \Longrightarrow (i)$ Trivial.

$(i) \Longrightarrow (ii)$ Suppose M has a $f.g.$ submodule N with M/N is hollow. Then there exists $K \leq^{\oplus} M$ with $K \leq_{ce} N$ in M. But M is indecomposable, hence $K = 0$. Therefore, $N \ll M$ and hence M is hollow. □

Proposition 3.4.6. [7, *Proposition* 2.7] *Let M be a $f.g.$ duo f-hollow-lifting module. Then every direct summand of M is f-hollow-lifting.*

Proof. Let $M = N \oplus N'$. We will show that N is f-hollow-lifting. Now, $M/N' = (N + N')/N \cong N$. By [*Proposition* 3.4.1] we get that M/N' is f-hollow-lifting. Hence, N is f-hollow-lifting. □

Proposition 3.4.7. [7, *Proposition* 2.8] *Let M be a $f.g.$ f-hollow-lifting module. If M has (D_3) then every direct summand of M is f-hollow-lifting.*

Proof. Let $N \leq^{\oplus} M$ and K a $f.g.$ submodule of N with N/K hollow. Suppose $M = N \oplus N'$. Then $M/(N' \oplus K)$ is hollow. Since M is $f.g.$, so is $N' \oplus K$. But M is f-hollow-lifting, therefore there exists $A \leq^{\oplus} M$ with $(N' \oplus K)/A \ll M/A$. Now $M/A = (N + A)/A + (N' + A)/A$. Moreover, if $M = N' + A$ then $M = N' + K$, a contradiction. Therefore, since M/A is hollow, $M = N + A$. This implies that $[N \cap (N' \oplus K)]/(N \cap A) \ll M/(N \cap A)$ and so $N \cap A \leq_{ce} K$ in M. But M has (D_3), hence $N \cap A \leq^{\oplus} M$ implying $N \cap A \leq^{\oplus} N$. Therefore, $N \cap A \leq_{ce} K$ in N [*Lemma* 2.1.5]. This completes the proof. □

Proposition 3.4.8. [7, *Proposition* 3.6] *Suppose that M is a module with $Rad(M) = 0$. Then M is f-hollow-lifting iff every f.g. submodule N of M with M/N hollow is a direct summand of M.*

Proof. Suppose that M is f-hollow-lifting and N a $f.g.$ submodule with M/N hollow. Then $N = K \oplus L$ with $L \ll M$ and $K \leq^{\oplus} M$ [Corollary 3.4.2]. But $Rad(M) = 0$, hence $L = 0$. Thus $N \leq^{\oplus} M$. The converse is clear. □

Now we give the following concept which was introduced in [3].

Definition 3.4.2. *A module M is called f-lifting if for any $f.g.$ submodule N of M, there exists $K \leq^{\oplus} M$ such that $K \leq_{ce} N$ in M.*

Using the last concept, we can provide the next proposition which gives another example of f-hollow-lifting modules.

Proposition 3.4.9. [7, *Proposition* 2.10] *Let H_1 and H_2 be two hollow modules and $M = H_1 \oplus H_2$. Then M is f-hollow-lifting iff it is f-lifting.*

Proof. The same proof of [*Proposition* 3.1.3]. □

The following result is just an imitation of [*Proposition* 3.2.9]

Proposition 3.4.10. *Let M be a projective module. Then M is f-hollow-lifting iff for every $f.g.$ submodule N of M with M/N hollow, M/N has a projective cover.*

Proof. The same proof of [*Proposition* 3.2.9] □

3.5 X-hollow-lifting Modules

In this part, we discuss the relative properties of hollow-lifting modules. Let M and X be modules. Harmanic, in [6], defined the following set.

$$\mathbb{B}(M,X) = \{A \leq M; \exists Y \leq X, \exists f \in Hom(M, X/Y), ker(f)/A \ll M/A\}$$

Now, we provide the following properties.

(*) $\mathbb{B}(M,X)$-lifting : For any $N \in \mathbb{B}(M,X)$, there exists $K \leq^{\oplus} M$ with $N/K \ll M/K$.

(*) $\mathbb{B}(M,X)$-hollow-lifting : For any $N \in \mathbb{B}(M,X)$ with M/N hollow, there exists $K \leq^{\oplus} M$ with $N/K \ll M/K$.

(*) $\mathbb{B}(M,X)$-(D_3) : For any $A \in \mathbb{B}(M,X)$ and $B \leq M$, if A and B are direct summands of M with $M = A + B$, then $A \cap B \leq^{\oplus} M$.

Following these properties, we get the next concepts.

(1) M is called X-lifting if it satisfies the $\mathbb{B}(M,X)$-lifting property.

(2) M is called X-hollow-lifting if it satisfies the $\mathbb{B}(M,X)$-hollow-lifting property.

(3) M is called X-quasi-discrete if it satisfies both the $\mathbb{B}(M,X)$-lifting and $\mathbb{B}(M,X)$-(D_3) properties.

(4) M is called X-amply supplemented if for and submodules A and B of M with $A \in \mathbb{B}(M,X)$ and $M = A + B$, there exists a supplement P of A in M with $P \leq B$.

Lemma 3.5.1. [6, *Lemma* 2.2] *Let M, N, and X be modules. Then*

(a) If $A \in \mathbb{B}(M,X)$ and $B \leq_{ce} A$ in M, then $B \in \mathbb{B}(M,X)$.

(b) If $B \leq A \leq M$, then $A \in \mathbb{B}(M,X)$ iff $A/B \in \mathbb{B}(M/B,X)$.

(c) If $h : N \longrightarrow M$ is an epimorphism and $A \in \mathbb{B}(M,X)$, then $h^{-1}(A) \in \mathbb{B}(N,X)$

(d) If $h : M \longrightarrow N$ is an epimorphism and $A \in \mathbb{B}(M,X)$ with $Ker(h) \leq A$, then $h(A) \in \mathbb{B}(N,X)$. The converse is true if $Ker(h) \leq A$.

Let us start with the properties of X-hollow-lifting modules.

Proposition 3.5.1. [20, *Proposition* 4.3] *If M is an X-hollow-lifting module, then every coclosed submodule $N \in \mathbb{B}(M,X)$ with M/N hollow is a direct summand of M. The converse is true if M is X-amply supplemented and $\mathbb{B}(M,X)$ is closed under supplement submodules.*

Proof. The forward direction is obvious. Now, let $N \in \mathbb{B}(M,X)$ with M/N hollow. Since M is X-amply supplemented, N has a supplement L in M. But $\mathbb{B}(M,X)$ is closed under supplement submodules, hence $L \in \mathbb{B}(M,X)$. Moreover, N contains a supplement Y of L in M and so $Y \in \mathbb{B}(M,X)$. Therefore, since $N \cap L \ll L$, it follows that $(N \cap L)/(Y \cap L) \ll L/(Y \cap L)$. But $(N \cap L)/(Y \cap L) \cong N/Y$ and $L/(Y \cap L) \cong M/Y$, hence $N/Y \ll M/Y$. We claim that $Y \leq^{\oplus} M$. To verify this, let $f : M/Y \longrightarrow M/N$ be an epimorphism with $Ker(f) = N/Y$ and let K/Y be a proper submodule of N/Y. Then either $(K+N)/N \ll M/N$ or $K+N = M$. If $K+N = M$ then $K = M$, a contradiction. This implies that $K/Y \ll M/Y$ and hence M/Y is hollow. Thus, our claim is true. This completes the proof. □

Proposition 3.5.2. [20, *Proposition* 4.6] *Let M be an X-hollow-lifting module. Then M/N is X-hollow-lifting for every fully invariant submodule N of M.*

Proof. Let $A/N \in \mathbb{B}(M/N,X)$ with $(M/N)/(A/N)$ hollow. Then M/A is hollow and $A \in \mathbb{B}(M,X)$ [*Lemma* 3.5.1]. But M is X-hollow-lifting, therefore $\exists\ B \leq^{\oplus} M$ with $B \leq A$ and $A/B \ll M/B$. Assume $M = B \oplus B'$. Then $A/(B+N) \ll M/(B+N)$ and $M/N = (B+N)/N \oplus (B'+N)/N$ since N is fully invariant. Thus, M/N is X-hollow-lifting. □

Corollary 3.5.1. *If M is a duo X-hollow-lifting module, then every direct summand of M is X-hollow-lifting.*

The next two result give us different conditions for a direct summand of an X-hollow-lifting module to be so.

Proposition 3.5.3. [20, *Proposition* 4.5] *If M is an X-hollow-lifting module having $\mathbb{B}(M,X)$-(D_3), then every direct summand of M is X-hollow-lifting.*

Proof. Let $N \leq^{\oplus} M$ and $K \in \mathbb{B}(N,X)$ with N/K hollow. Suppose that $M = N \oplus N'$. Since $M/K = N/K \oplus (N' \oplus K)/K$, $M/(N' \oplus K)$ is hollow. Let $\pi : M \longrightarrow N$ be the projection epimorphism of M onto N. Then $\pi(N' \oplus K) = K$ and $Ker(\pi) = N'$. So $N' \oplus K \in \mathbb{B}(M,X)$ [*Lemma* 3.5.1, (d)].

But M is X-hollow-lifting, so there exists $A \leq^{\oplus} M$ such that $A \leq N' \oplus K$ and $(N' \oplus K)/A \ll M/A$. Hence, $A \in \mathbb{B}(M,X)$ and $M = A + N$. So $K/(A \cap N) \ll M/(A \cap N)$ [*Lemma* 2.1.5, (h)]. Since M has $\mathbb{B}(M,X)$-(D_3) property, $A \cap N \leq^{\oplus} M$ and hence $N/(A \cap N) \leq^{\oplus} M/(A \cap N)$. Therefore, N is X-hollow-lifting. □

Lemma 3.5.2. *Every epimorphic image of an X-amply supplemented module M is X-amply supplemented.*

Proof. Let $h : M \longrightarrow N$ be an epimorphism. Let A and B be submodules of N with $N = A+B$ and $A \in \mathbb{B}(N,X)$. Then $M = h^{-1}(A)+h^{-1}(B)$ and hence $h^{-1}(A) \in \mathbb{B}(M,X)$ [*Lemma* 3.5.1, (c)]. Since M is X-amply supplemented, there exists $H \leq M$ with $M = H + h^{-1}(A)$ and $H \cap h^{-1}(A) \ll H$. Now, since $N = A + h(H)$, $h[h^{-1}(A) \cap H] = A \cap h(H) \ll h(H)$. Thus, N is X-amply supplemented. □

Proposition 3.5.4. [20, *Proposition* 4.4] *Let M be a UCC X-amply supplemented module such that $\mathbb{B}(M,X)$ is closed under supplement submodules. If M is X-hollow-lifting, then every direct summand of M is X-hollow-lifting.*

Proof. Let $M = N \oplus N'$ and $A \in \mathbb{B}(N,X)$ with N/A hollow. By the previous lemma, N is X-amply supplemented. Hence, There exists a coclosed submodule A' of N with $A' \leq A$ and $A/A' \ll N/A'$ and so $A' \in \mathbb{B}(N,X)$. Moreover, $A' \oplus N' \leq_{cc} M$. Since N/A' is hollow, it follows that $M/(N' \oplus A')$ is also hollow. Now, consider the projection epimorphism $\pi : M \longrightarrow N$. Then $\pi(N' \oplus A') = A'$ and $Ker(\pi) = N'$, hence $N' \oplus A' \in \mathbb{B}(M,X)$. But M is X-hollow-lifting, therefore $N' \oplus A' \leq^{\oplus} M$ and so $A \leq^{\oplus} N$. Hence, N is X-hollow-lifting.

□

Lemma 3.5.3. [20, *Proposition* 4.15] *Let M be a module. If the sequence $0 \to X' \to X \to X'' \to 0$ is exact, then $\mathbb{B}(M,X') \cup \mathbb{B}(M,X'') \subseteq \mathbb{B}(M,X)$.*

Proof. Since the given sequence is exact, we may assume that $X' \leq X$ and $X'' = X/X'$. Clearly, $\mathbb{B}(M,X') \subseteq \mathbb{B}(M,X)$. Now, let $A \in \mathbb{B}(M,X'')$. Then there exists $K \leq X''$ and $f \in Hom(M, X''/K)$ such that $Ker(f)/A \ll M/A$. Since $X'' = X/X'$, $K = K'/X'$ for some $K' \leq X$. Therefore, we have $f : M \longrightarrow X/K'$. Hence, $A \in \mathbb{B}(M,X)$. This completes the proof. □

The following result is an immediate consequence of the last lemma.

Corollary 3.5.2. *If* $0 \to X' \to X \to X''$ *is an exact sequence and* M *is an* X*-hollow-lifting module, then* M *is both* X'*-hollow-lifting and* X''*-hollow-lifting.*

The next result is similar to Proposition 3.1.3

Proposition 3.5.5. [20, *Proposition* 4.2] *Let* H_1 *and* H_2 *be two hollow modules and* $M = H_1 \oplus H_2$. *Then* M *is* X*-hollow-lifting iff it is* X*-lifting.*

Proof. An analogy to the proof of [*Proposition* 3.1.3]. □

4 Direct Sum Of Hollow-lifting Modules

4.1 Direct Sum Of Two Hollow-lifting Modules

At the beginning, let us point out that a (finite) direct sum of hollow-lifting modules need not be so.

Example 4.1. *Consider the* $\mathbb{Z}$*-modules* $A = \mathbb{Z}/2\mathbb{Z}$ *and* $B = \mathbb{Z}/8\mathbb{Z}$*. Then both* A *and* B *are hollow-lifting because they are hollow. But* $M = A \oplus B$ *is not hollow-lifting* [*Example* 3.2]

The main question in this part is : *When the sum of two hollow-lifting modules is so? The theorem below answers that.*

Theorem 4.1.1. *Let* M_1 *and* M_2 *be two hollow-lifting modules such that* $M = M_1 \oplus M_2$*. If any one of the following conditions holds, then* M *is hollow-lifting.*

(a) M *is a duo module.*

(b) M_1 *and* M_2 *are relatively projective.*

(c) M_1 *is radical,* M_2 *is coatomic, and* M_1 *and* M_2 *are relatively h-small projective.*

(d) M *is amply supplemented and any of the following holds.*

- *(i)* M_1 *and* M_2 *are relatively *cojective.*
- *(ii)* M_1 *is h-small* M_2*-projective and every coclosed submodule* N *of* M *with* M/K *hollow and* $M = K + M_1$ *is a direct summand of* M*.*
- *(iii)* M_1 *and* M_2 *are relatively h-small projective and every coclosed submodule* N *of* M *with* M/K *hollow and* $M = K + M_1 = K + M_2$ *is a direct summand of* M*.*
- *(iv)* M_2 *is* M_1*-projective and* M_1 *is h-small* M_2*-projective.*
- *(v)* M_1 *is semisimple and h-small* M_2*-projective.*

Before we prove this theorem, we must define some concepts and provide some results.

Following [22] *we get the next definition.*

<u>Definition</u> 4.1.1. *Let M_1 and M_2 be two modules. Then M_2 is said to be M_1-*cojective if for any supplement M_1' of M_1 in $M_1 \oplus M_2$, we have a decomposition $M_1 \oplus M_2 = M_1' \oplus M_1'' \oplus M_2'$ with $M_1'' \leq M_1$ and $M_2' \leq M_2$. We say that M_1 and M_2 are relatively *cojective if M_2 is M_1-*cojective and M_1 is M_2-*cojective.*

Y. Wang, in [20]*, has introduced the following result.*

<u>Proposition</u> 4.1.1. [20, *Corollary* 3.5] *Let $M = M_1 \oplus M_2$ be an amply supplemented module with M_1, M_2 hollow-lifting. If M_1 and M_2 are relatively *cojective, then M is hollow-lifting.*

The following definitions can be found in [2] *and* [9].

<u>Definition</u> 4.1.2. *Let M_1, M_2 be modules. Then*

(a) M_1 is said to be small (nearly) M_2-projective, if every homomorphism $f : M_1 \longrightarrow M_2/A$; $A \leq M_2$ and $Im(f) \ll M_2/A$ ($Im(f) \neq M_2/A$), can be lifted to a homomorphism $\phi : M_1 \longrightarrow M_2$. That is, the following diagram commutes ($f = \pi\phi$).

$$\begin{array}{ccccc} M_2 & \xrightarrow{\pi} & M_2/A & \longrightarrow & 0 \\ & \nwarrow^{\phi} & \uparrow f & & \\ & & M_1 & & \end{array}$$

(b) M_1 is h-small M_2-projective if any homomorphism $f : M_1 \longrightarrow M_2/A$, where $A \leq M_2$, M_2/A hollow, and $Im(f) \ll M_2/A$, can be lifted to a homomorphism $\phi : M_1 \longrightarrow M_2$.

(c) M_1 and M_2 are called relatively small (nearly or h-small) projective if M_1 is small (nearly or h-small) M_2-projective and M_2 is small (nearly or h-small) M_1-projective.

<u>Remark</u>. clearly, nearly projectivity $\Rightarrow$ small projectivity $\Rightarrow$ h-small projectivity. However, If M_2 is hollow, then h-small M_2-projectivity, small M_2-projectivity, and nearly M_2-projectivity coincide. Moreover, the properties of small (nearly or h-small) projectivity are inherited by direct summands.

The next result is a direct consequence of [*Lemma* 2.7.2].

Corollary 4.1.1. *Let M_1 and M_2 be modules and let $M = M_1 \oplus M_2$. Then M_1 is M_2-projective iff for every submodule N of M with $M = N + M_2$, there exists $N' \leq N$ such that $M = N' \oplus M_2$.*

Proposition 4.1.2. [15, *Proposition* 6.2] *Let M_1 and M_2 be two hollow-lifting modules and $M = M_1 \oplus M_2$. If M_1 and M_2 are relatively projective, then M is hollow-lifting.*

Proof. Let L be a submodule of M such that M/L is hollow. Then we have $M/L = (M_1 + L)/L + (M_2 + L)/L$. Since M/L is hollow, it follows that either $M = M_1 + L$ or $M = M_2 + L$. Without loss of generality, assume $M = M_1 + L$. Then $M/L \cong M_1/(M_1 \cap L)$ and hence $M_1/(M_1 \cap L)$ is hollow. Since M_2 is M_1-projective, there exists $A \leq^{\oplus} L$ with $M = M_1 \oplus A$ [*Corollary* 4.1.1]. Therefore, $L = L \cap M = (L \cap M_1) \oplus A$. Now, because M_1 is hollow-lifting and $M_1/(M_1 \cap L)$ is hollow, there is $X \leq^{\oplus} M_1$ with $X \leq M_1 \cap L$ and $(M_1 \cap L)/X \ll M_1/X$. Hence, $M = M_1 \oplus A = X' \oplus X \oplus A$ for some $X' \leq M_1$ which implies that $X \oplus A \leq^{\oplus} M$ and $X \oplus A \leq L$. We claim that $X \oplus A \leq_{ce} L$ in M. Let $B \leq M$ with $X \oplus A \leq B$ and $B/(X \oplus A) + L/(X \oplus A) = M/(X \oplus A)$. Then $M = L + B = (L \cap M_1) + B$. Since $(M_1 \cap L)/X \ll M_1/X$, we have $B = M$. So our claim is verified. Thus, M is hollow-lifting. □

The following three lemmas characterize the types of projectivity which were mentioned in [*Definition* 4.1.2].

Lemma 4.1.1. [9, *Lemma* 2.3] *Let M_1 and M_2 be two modules such that $M = M_1 \oplus M_2$. Then M_1 is nearly M_2-projective iff for every submodule N of M with $M = N + M_2$ and $M \neq N + M_1$, there exists $N' \leq N$ such that $M = N' \oplus M_2$.*

Proof. $\boxed{\Longrightarrow}$ Consider the homomorphism $g : M_1 \longrightarrow M/N$ such that $m_1 \longmapsto m_1 + N$ and the epimorphism $f : M_2 \longrightarrow M/N$; $f(m_2) = m_2 + N$. Then $Im(g) = (N + M_1)/N \neq M/N$. Therefore, there exists a homomorphism $\phi : M_1 \longrightarrow M_2$ with $f\phi = g$. Define $N' = \{m_1 - \phi(m_1) | \quad m_1 \in M_1\}$. Then $N' \leq N$ and $M = N' \oplus M_2$.

$\boxed{\Longleftarrow}$ Let $A \leq M_2$, $f : M_1 \longrightarrow M_2/A$ with $Im(f) \neq M_2/A$ and let $\pi : M_2 \longrightarrow M_2/A$ be the natural epimorphism. Now, define the submodule $N = \{m_1 + m_2 \in M_1 \oplus M_2 \quad | \quad f(m_1) = -\pi(m_2)\}$. Then $M = N + M_2$. Since $Im(f) \neq M_2/A$, $M \neq N + M_1$. Therefore, there exists $N' \leq N$ with $M = N' \oplus M_2$. Let $h : N' \oplus M_2 \longrightarrow M_2$ be the projection epimorphism. Then $f = \pi h|_{M_1}$.Thus, M_1 is nearly M_2-projective. □

Lemma 4.1.2. [11, *Lemma* 2.4] *Let M_1 and M_2 be two modules such that $M = M_1 \oplus M_2$. Then M_1 is small M_2-projective iff for every submodule N of M with $(M_1 + N)/N \ll M/N$, there exists $N' \leq N$ such that $M = N' \oplus M_2$.*

Proof. $\boxed{\Longrightarrow}$ Let $N \leq M$ with $(N + M_1)/N \ll M/N$. Then $M = N + M_2$. Consider the homomorphism $g : M_1 \longrightarrow M/N$, where $g(m_1) = m_1 + N$, and the epimorphism $f : M_2 \longrightarrow M/N$ with $f(m_2) = m_2 + N$. Then $Im(g) = (N + M_1)/N \ll M/N$. But M_1 is small M_2-projective, therefore there exists a homomorphism $\phi : M_1 \longrightarrow M_2$ such that $f\phi = g$. Define the submodule $N' = \{m_1 - \phi(m_1) \mid m_1 \in M_1\}$. Then $N' \leq N$ and $M = N' \oplus M_2$.

$\boxed{\Longleftarrow}$ Let $A \leq M_2$, $f : M_1 \longrightarrow M_2/A$ with $Im(f) \ll M_2/A$, and let $\pi : M_2 \longrightarrow M_2/A$ be the natural epimorphism. Now, define the submodule $N = \{m_1 + m_2 \in M_1 \oplus M_2 \mid f(m_1) = -\pi(m_2)\}$. Then $N \leq M$, $A \leq N$ and $M = N + M_2$. Let $Im(f) = X/A$ with $X \leq M_2$ and consider the homomorphism $h : M_2/A \longrightarrow M/N$, where $h(m_2 + A) = m_2 + N$. Since $Im(f) \ll M_2/A$, $h(X/A) = (X + N)/N \ll M/N$. Moreover, we have $(N + M_1)/N \ll M/N$ because $(N + M_1)/N \leq (X + N)/N$. Hence, by assumption, there exists $N' \leq N$ with $M = N' \oplus M_2$. Now, consider the projection epimorphism $\alpha : N' \oplus M_2 \longrightarrow M_2$. Then f can be lifted by the restriction $\alpha|_{M_1} : M_1 \longrightarrow M_2$. Thus, M_1 is small M_2-projective. $\square$

Lemma 4.1.3. [15, *Proposition* 6.6] *Let M_1 and M_2 be two modules and $M = M_1 \oplus M_2$. Then M_1 is h-small M_2-projective iff for every submodule N of M with M/N hollow and $M \neq M_1 + N$, there exists $N' \leq N$ such that $M = N' \oplus M_2$.*

Proof. $\boxed{\Longrightarrow}$ Let $N \leq M$ with M/N hollow and $M \neq M_1 + N$. Now, $M/N = (M_1 + M_2)/N = (M_1 + N)/N + (M_2 + N)/N$. But M/N is hollow and $M \neq M_1 + N$, therefore $M = M_2 + N$. The proof proceeds exactly as the previous lemma.

$\boxed{\Longleftarrow}$ Let $A \leq M_2$, $f : M_1 \longrightarrow M_2/A$ with $Im(f) \ll M_2/A$ and M_2/A hollow. Let $\pi : M_2 \longrightarrow M_2/A$ be the natural epimorphism. Define the submodule $N = \{m_1 + m_2 \in M_1 \oplus M_2 \mid f(m_1) = -\pi(m_2)\}$. Again, follow the steps of the previous lemma's proof and then the proof will complete $\square$

The next result helps us connect between projectivity and small (and nearly) projectivity.

Lemma 4.1.4. *Let M_1 and M_2 be two modules and $M = M_1 \oplus M_2$. Assume that for any submodule N of M, $M = N + M_2 \Longrightarrow M \neq N + M_1$. Then if M_1 is small M_2-projective, then M_1 is M_2-projective.*

Proof. Let $N \leq M$ with $M = N + M_2$. Then $(M_1 + N)/N \ll M/N$. But M_1 is small M_2-projective, hence there exists $N' \leq N$ such that $M = N' \oplus M_2$. Thus, M_1 is M_2-projective [*Corollary* 4.1.1]. □

The last lemma provides the following result.

Corollary 4.1.2. *Let M_1 and M_2 be two modules and $M = M_1 \oplus M_2$. Assume that for any submodule N of M, if $M = N + M_2$, then $M \neq N + M_1$. Then the following are equivalent.*

1. *M_1 is M_2-projective.*
2. *M_1 is small M_2-projective.*
3. *M_1 is nearly M_2-projective.*

Lemma 4.1.5. [15, *Proposition* 6.11] *Let M_1 and M_2 be two modules with M_2 hollow-lifting and $M = M_1 \oplus M_2$. If M_1 is h-small M_2-projective, then every coclosed submodule K of M with M/K hollow and $M \neq K + M_1$ is a direct summand of M.*

Proof. Let K be a coclosed submodule of M such that M/K is hollow and $M \neq K + M_1$. Then there is $K' \leq K$ with $M = K' \oplus M_2$ [*Lemma* 4.1.3]. Therefore, $M/K' \cong M_2$ and hence M/K' is hollow-lifting. Moreover, we have $K/K' \leq_{cc} M/K'$ [*Lemma* 2.1.7, (b)]. Also, $K/K' \leq^{\oplus} M/K'$ because $(M/K')/(K/K') \cong M/K$ which is hollow. Thus, $K \leq^{\oplus} M$. □

Recall that a module M is called coatomic if every proper submodule of M is contained in a maximal submodule. Moreover, it is called radical if $Rad(M) = M$. [15]

Proposition 4.1.3. [15, *Proposition* 6.16] *Let M_1 and M_2 be two hollow-lifting modules and $M = M_1 \oplus M_2$. Assume that M_1 is radical, M_2 is coatomic, and M_1 and M_2 are relatively h-small projective. Then M is hollow-lifting.*

Proof. Let $N \leq M$ with M/N hollow. Then $M = M_1 + N$ or $M = M_2 + N$. We claim that the last two equalities cannot happen simultaneously. Now, if $M = M_1 + N = M_2 + N$, then $M/N \cong M_1/(M_1 \cap N) \cong M_2/(M_2 \cap N)$. Therefore, $M_2/(M_2 \cap N)$ is both coatomic and radical. This implies that $M_2/(M_2 \cap N) = 0$ and $N = 0$, impossible. So our claim is true. Now, since M_1 and M_2 are relatively h-small projective, the proof completes by [*Proposition* 4.1.2] and [*Lemma* 4.1.3]. □

Now, we back to the proof of theorem 4.1.1

Proof of theorem 4.1.1

(a) *Follows from* [*Theorem* 3.3.1].

(b) *Follows from* [*Proposition* 4.1.2].

(c) *Follows from* [*Proposition* 4.1.3].

(d) *(i)* *Follows from* [*Proposition* 4.1.1].

(ii) *Follows from* [*Proposition* 2.3.1] *and* [*Lemma* 4.1.5].

(iii) *Follows from* [*Proposition* 2.3.1] *and* [*Lemma* 4.1.5].

(iv) *Follows from* (*ii*).

(v) *Follows from* (*iv*).

4.2 Direct Sum Of Hollow Modules

In this section, we will provide some conditions so that a direct sum of hollow modules would be hollow-lifting. Just to be clear, we will use directly some former results, especially from [14].

First of all, let us begin with the following definition which was introduced in [14].

Definition 4.2.1. *M_1 is called almost M_2-projective if for every epimorphism $f : M_2 \longrightarrow K$ and every homomorphism $g : M_1 \longrightarrow K$, either there exists $h : M_1 \longrightarrow M_2$ with $fh = g$ or there exists a nonzero direct summand N of M_2 and $\bar{h} : N \longrightarrow M_1$ with $g\bar{h} = f|_N$.*

The next two lemmas will help us provide some results

Lemma 4.2.1. *Let M_1 be a hollow module and M_2 be an indecomposable module. Assume that there is no epimorphism from M_2 to M_1.Then M_1 is almost M_2-projective if and only if M_1 is M_2-projective.*

Proof. Suppose that M_1 is almost M_2-projective. Let A be a submodule of M_2 with $f : M_1 \longrightarrow M_2/A$ any homomorphism and consider the natural epimorphism $\pi : M_2 \longrightarrow M_2/A$. If there exists a nonzero $N \leq^{\oplus} M_2$ and a homomorphism $h : N \longrightarrow M_1$ with $fh = \pi|_N$, then $N = M_2$ because M_2 is indecomposable. Consequently, $\pi|_N = \pi$ and hence fh is an epimorphism. But M_1 is hollow, so $h : M_1 \longrightarrow M_2$ is epimorphism, a contradiction. Therefore, there exists a homomorphism $g : M_1 \longrightarrow M_2$ with $\pi g = f$. Hence, M_1 is M_2-projective. The converse is always true. □

Lemma 4.2.2. *Let $M = M_1 \oplus M_2$ be a module. Assume that for every proper submodule $N \leq M$, if $M = N + M_2$ then $M \neq N + M_1$. Then there is no epimorphism between M_1 to M_2.*

Proof. Suppose that there exists an epimorphism $f : M_1 \longrightarrow M_2$. Let $N = \{m_1 + f(m_1) \mid m_1 \in M_1\}$. Then $N \leq M$, $N \cong M_1$, and $M = N \oplus M_2$. But f is an epimorphism, so $M = N + M_1$, a contradiction. □

The next characterization is well known and has a vital importance in our thesis.

Theorem 4.2.1. [2, *Theorem* 1] *Let* $M = \bigoplus_{i=1}^{n} M_i$ *be a direct sum of hollow modules such that each* $End(M_i)$ *is local. Then the following are equivalent.*

(a) M *is lifting.*

(b) M_i *is* M_j*-projective for all* $i \neq j$.

(c) For any subset J *of* $I = \{1, 2, \ldots, n\}$, $\bigoplus_{j \in J} M_j$ *is almost* $\bigoplus_{i \in I-J} M_i$*-projective.*

Theorem 4.2.2. [15, *Proposition* 4.5] *Let* M_1 *and* M_2 *be hollow modules with local endomorphism rings. Assume that there is no epimorphism between* M_1 *and* M_2. *Let* $M = M_1 \oplus M_2$. *Then the following are equivalent.*

(a) M *is hollow-lifting.*

(b) M *is lifting.*

(c) M *is quasi-discrete.*

(d) M_1 *and* M_2 *are relatively projective.*

(e) M_1 *and* M_2 *are relatively almost projective.*

Proof. $(a) \Longleftrightarrow (b)$ Follows from [*Proposition* 3.1.3].
$(d) \Longleftrightarrow (e)$ Follows from [*Lemma* 4.2.2].
$(e) \Longleftrightarrow (b)$ Follows from [*Theorem* 4.2.1].
$(c) \Longleftrightarrow (d)$ Follows from [14, *Corollary* 4.50]. □

Corollary 4.2.1. *Let* M_1 *and* M_2 *be hollow modules with local endomorphism rings. Assume that* $Rad(M_1) = M_1$, M_2 *is local, and* $M = M_1 \oplus M_2$. *Then the previous theorem is true for* M.

Proof. We only need to prove that there is no epimorphism between M_1 and M_2. So suppose that $f : M_1 \longrightarrow M_2$ is an epimorphism. Then $M_2 = f(M_1) = f(Rad(M_1)) \leq Rad(M_2)$. Therefore, $M_2 = Rad(M_2)$, a contradiction since M_2 is local. On the other hand, let $g : M_2 \longrightarrow M_1$ be an epimorphism. Since M_2 is local, $Rad(M_2) \neq M_2$ and hence $Rad(M_2) \ll M_2$. This implies that $g(Rad(M_2)) \ll g(M_2) = M_1$, hence $Rad(M_1) \neq M_1$, a contradiction. This completes the proof. □

Now, we can use [Corollary 4.1.2] to get the next result below.

Corollary 4.2.2. [15, *Theorem* 4.8] *Let $M = M_1 \oplus M_2$ where M_1 and M_2 are hollow modules with local endomorphism rings. Assume that for every proper submodule $N \leq M$, if $M = N + M_i$ then $M \neq N + M_j$ $\forall$ $j \neq i$. Then the following are equivalent.*

(a) M is hollow-lifting.

(b) M is lifting.

(c) M_1 and M_2 are relatively projective.

(d) M_1 and M_2 are relatively small projective.

(e) M_1 and M_2 are relatively nearly projective.

(f) M_1 and M_2 are relatively almost projective.

(g) M is quasi-discrete.

The next theorem provides some sufficient condition for nearly projectivity in direct sums of hollow modules.

Theorem 4.2.3. [15, *Theorem* 4.1] *Let $M = \bigoplus_{i \in I} M_i$, where all M_i are hollow and $\bigoplus_{i \in I} M_i$ complements direct summands. If M is hollow-lifting, then $\bigoplus_{i \neq j} M_i$ is nearly M_j-projective.*

Proof. Let $A \leq M_j$ and $f : \bigoplus_{i \neq j} M_i \longrightarrow M_j/A$ be a homomorphism with $Im(f) \neq M_j/A$. Consider the natural epimorphism $\pi : M_j \longrightarrow M_j/A$. Define the submodule $B = \{x + y \mid x \in \bigoplus_{i \neq j} M_i, y \in M_j, \text{and } f(x) = -\pi(y)\}$. Then $M = B + M_j$ and $A \leq B$ and $M/B = (B + M_j)/B \cong M_j/(B \cap M_j)$, hence M/B is hollow. But M is hollow-lifting, so there exists $D \leq^{\oplus} M$ with $B/D \ll M/D$. Since $M/B \cong (M/D)/(B/D)$, M/D is also hollow. Now, $\bigoplus_{i \in I} M_i$ complements direct summands and $D \leq^{\oplus} M$ implies that $\exists$ $k \in I$ with $M = D \oplus M_k$. Besides, $M/D = (B + M_j)/D = B/D + (D + M_j)/D$, but $B/D \ll M/D$, therefore $M = D + M_j$. Now, if $k \neq j$, then f will be an epimorphism, a contradiction. So $k = j$ and hence $M = D \oplus M_j$. Let $\alpha : M = D \oplus M_j \longrightarrow M_j$ be the projection map, and take the homomorphism $\beta = \alpha|_{\bigoplus_{i \neq j} M_i} : \bigoplus_{i \neq j} M_i \longrightarrow M_j$. Then $\pi\beta = f$. Thus, the direct sum $\bigoplus_{i \neq j} M_i$ is nearly M_j-projective. $\square$

The previous theorem grants the next two results.

Corollary 4.2.3. *Let $M = \bigoplus_{i \in I} M_i$, where all M_i are hollow and $\bigoplus_{i \in I} M_i$ complements direct summands. If M is hollow-lifting, then for all $i \neq j$, M_i is nearly (small) M_j-projective.*

Proof. Suppose $i \neq j$. then $\bigoplus_{i \neq j} M_i$ is nearly (small)M_j-projective. But $M_i \leq^{\oplus} \bigoplus_{i \neq j} M_i$, hence M_i is nearly (small) M_j-projective. □

Corollary 4.2.4. *Let M_1 and M_2 be hollow modules with local endomorphism rings. If $M = M_1 \oplus M_2$ is hollow lifting, then M_1 and M_2 are relatively nearly (small) projective.*

Proof. Follows immediately from [$Theorem$ 4.2.3] and [$Theorem$ 2.1.6]. □

Proposition 4.2.1. [15, $Proposition$ 4.9] *Let $M = \bigoplus_{i \in I} H_i$ be a module with all H_i hollow. If M is hollow-lifting and has (D_3), then for every $i \in I$, $\bigoplus_{j \neq i} H_j$ is H_i-projective.*

Proof. By [$Lemma$ 2.2.1, (c)], If $H_i \ll M$ then $\bigoplus_{j \neq i} H_j = M$ and hence $H_i = 0$ which implies that $\bigoplus_{j \neq i} H_j = M$ is H_i-projective. So assume H_i is not small in M and let N be a proper submodule of M with $M = N + H_i$. Now, $M/N = (N + H_i)/N \cong H_i/(N \cap H_i)$, hence M/N is hollow. But M is hollow-lifting, so $\exists\ K \leq^{\oplus} M$ with $K \leq N$ and $N/K \ll M/K$. Therefore, from $M/K = (N + H_i + K)/K$ we get $M = K + H_i$. Since M has (D_3), $(K \cap H_i) \leq^{\oplus} M$, but this cannot happen unless $K \cap H_i = 0$. Therefore, $M = K \oplus H_i$. This means that we have a submodule $K \leq \bigoplus_{j \neq i} H_j$ with $M = K \oplus H_i$. Thus, $\bigoplus_{j \neq i} H_j$ is H_i-projective [$Lemma$ 2.7.2]. □

Theorem 4.2.4. *Let $M = \bigoplus_{i=1}^{n} H_i$ be a finite direct sum of hollow modules H_i and suppose M has (D_3). Then the following are equivalent.*

(a) M is hollow-lifting.

(b) M is lifting.

(c) M is quasi-discrete.

(d) H_i is H_j-projective for all $i \neq j$.

Proof. $(c) \Leftrightarrow (d)$ Follows from [14, *Corollary* 4.50].
$(a) \Rightarrow (d)$ Follows from the previous proposition.
$(c) \Rightarrow (b) \Rightarrow (a)$ Trivial. □

Using the previous result, we can prove the following theorem.

Theorem 4.2.5. *Let M be a hollow-lifting module satisfying (D_3). If M has a finite hollow dimension n, then M is lifting.*

Proof. It suffices to show that M is a finite direct sum of hollow modules. The proof is by induction on $h(M) = n$. If $n = 1$, we are done. So assume $n > 1$ and for any hollow-lifting module N having (D_3) with $h(N) < n$, N is a finite direct sum of hollow modules. We claim that M is not indecomposable. To verify our claim, assume not. Then, Since M has a finite hollow dimension, there exists A proper submodule $A \leq M$ with M/A hollow. But M is hollow-lifting, therefore there exists $B \leq^{\oplus} M$ such that $B \leq A$ and $A/B \ll M/B$. Hence, M is hollow, a contradiction. So Our claim is true. This implies that M has a decomposition $M = M_1 \oplus M_2$ such that M_1 and M_2 are nonzero. Since $h(M) = h(M_1) + h(M_2)$, it follows that $h(M_1) < n$ and $h(M_2) < n$. Moreover, M_1 and M_2 are hollow-lifting [*Proposition* 3.3.2]. Hence, by the induction hypothesis, Both M_1 and M_2 are finite direct sums of hollow modules. Thus, M is a finite direct sum of hollow modules and hence it is lifting.

□

Now, we are able to give the next characterization

Theorem 4.2.6. *Let $M = \bigoplus_{i \in I} H_i$ be a direct sum of local modules H_i such that $Rad(M) \ll M$ and there is no epimorphism between H_i and H_j $\forall\, i \neq j$. Then the following are equivalent.*

(a) M is quasi-discrete.

(b) M is hollow-lifting and the decomposition $\bigoplus_{i \in I} H_i$ complements direct summands.

Proof. Follows from [14, *Theorem* 4.48]. □

5 Hollow-lifting Modules Over Commutative Rings

Finally, in this section, we will have a brief overview on hollow-lifting modules over commutative rings.

We start with the following result.

Proposition 5.0.2. *Let R be a Noetherian ring and M a $f.g.$ hollow-lifting R-module. Then M is a finite direct sum of local modules.*

Proof. Since M is $f.g.$, it has a maximal submodule. Therefore, M has a local direct summand N [$Proposition$ 3.2.4]. Moreover, M is Noetherian [$Corollary$ 2.1.3] and hence N is $f.g.$ [$Theorem$ 2.1.7, (d)]. This implies that $End(N)$ is local, so N has the finite exchange property [21, $Theorem$ 4.2]. It follows that M/N is hollow-lifting [$Proposition$ 3.2.2]. Hence, by induction, M is a direct sum of local modules. □

The following theorem characterizes hollow-lifting modules over commutative Noetherian rings.

Theorem 5.0.7. *Let M be a nonzero indecomposable module over a commutative Noetherian ring. Then the following are equivalent.*

(a) M is hollow-lifting.

(b) M is lifting.

(c) M is hollow.

Proof. $(c) \Longrightarrow (b) \Longrightarrow (a)$ are clear.

$(a) \Longrightarrow (c)$ From [19, $Proposition$ 2.24 & $Theorem$ 4.30], M has an Artinian factor module. But every Artinian module has a finite hollow dimension and hence M has a hollow factor module [3, §5.4]. Thus, M is hollow [$Proposition$ 3.1.1]. □

The next lemma helps us provide a property of hollow-lifting modules over local commutative rings.

Lemma 5.0.3. [23, *Folgerung* 3.3] *Let R be a commutative local ring and M a f.g. R-module. Then the following are equivalent.*

(a) M is lifting.

(b) Every submodule N of M with M/N cyclic has a strong supplement in M.

Proposition 5.0.3. [15, *Proposition* 3.3] *Let R be a commutative local ring and M a f.g. R-module. If M is hollow-lifting, then it is lifting.*

Proof. Let $N \leq M$ such that M/N is cyclic. Then M/N is a local R-module since R is local. Hence, M/N is hollow. But M is hollow-lifting, therefore N has a strong supplement in M [Corollary 3.1.1]. Thus, M is lifting by the previous lemma. □

Remark. Actually, it has turned out by Orhan [15] that the condition: " R is a local ring " is not necessary in the previous lemma. Therefore, we have the following result.

Proposition 5.0.4. [15, *Corollary* 3.4] *If M is a f.g. module over a commutative ring R, then M is lifting iff it is hollow-lifting.*

Remark. Wang and Yu, in [20], has used the previous result to prove that a factor module of a hollow-lifting module need not be a hollow-lifting module [20, Example 2.2 & Example 2.3].

Now, we shall provide some results about f-hollow-lifting modules over commutative rings. Before doing that, we have to recall some concepts. If M is a right R-module and $x \in M$, then the Annihilator of x is a right ideal of R defined by:

$$Ann(x) = \{r \in R \quad | \quad xr = o\}$$

Also, if R is an integral domain and M is an R-module, then M is called torsion-free if $Ann(x) = 0$ for all nonzero x in M [7].

Proposition 5.0.5. [7, *Proposition* 3.7] *Let R be an indecomposable integral domain and M a torsion-free R-module. If M is f-hollow-lifting module and has a hollow factor module M/xR for some $x \in M$, then either M is hollow or $xR \leq^{\oplus} M$.*

Proof. Since M is f-hollow-lifting module, it follows that $xR = K \oplus L$, where $K \leq^{\oplus} M$ and $L \ll M$ [*Proposition* 3.4.2]. Now consider the following short exact sequence

$$0 \longrightarrow Ann(x) \xrightarrow{\iota} R \xrightarrow{f} xR \longrightarrow 0$$

Where ι is the inclusion monomorphism and $f(r) = xr \ \forall \ r \in R$. Since M is torsion-free, we have $Ann(x) = 0$ and hence $R \cong xR$. But R is indecomposable, hence xR is indecomposable as an R-module. Therefore, either $xR = K$ or $xR = L$. Now, If $xR = K$ then $xR \leq^{\oplus} M$. And if $xR = L$ then $xR \ll M$ and so M is hollow [*Lemma* 2.2.1]. □

The next two results are concerned with a commutative f-hollow-lifting ring R. Just recall that an element $e \in R$ is called an idempotent if $e^2 = e$.

Proposition 5.0.6. [7, *Proposition* 3.9] *Let R be a commutative f-hollow-lifting ring. Then for each $x \in R$ with R/xR hollow, there exists an idempotent $e \in xR$ such that $x(1-e) \in Rad(R)$.*

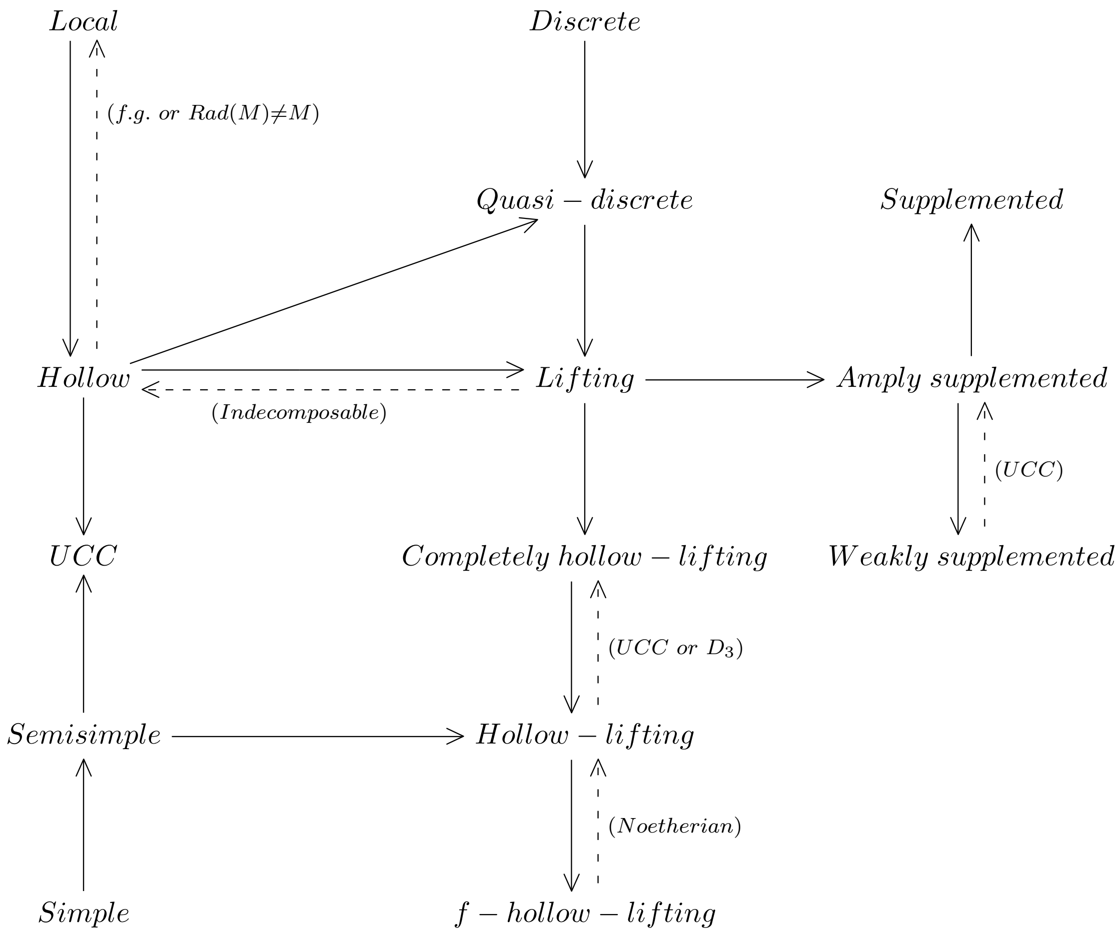
Proof. Let $x \in R$ with R/xR hollow. Since R is f-hollow-lifting, there is a decomposition $R = K \oplus N$ such that $K \leq xR$ and $xR \cap N \ll R$ [*Proposition* 3.4.3]. Therefore, there exists an idempotent $e \in R$ such that $K = eR$ and $N = (1-e)R$. Now, we claim that $xR \cap N = x(1-e)R$. Let $y \in xR \cap N$, then $y = xr_1 = (1-e)r_2$ for some r_1, r_2 in R. Therefore, $y = (1-e)(1-e)r_2 = (1-e)xr_2 = x(1-e)r_2 \in x(1-e)R$. Conversely, let $z \in x(1-e)R$. Then $z = x(1-e)r$ for some $r \in R$. Clearly, $z \in xR$. Moreover, $z = xr - xer = xr(1-e) \in R(1-e) = N$. Hence, our claim is verified. But $xR \cap N \ll R$, so $x(1-e)R \subseteq Rad(R)$. It follows that $x(1-e) \in Rad(R)$. □

Corollary 5.0.5. *Let R be a commutative f-hollow-lifting ring. Then for each $x \in R$ with R/xR hollow, there exists $y \in R$ such that $(xy)^2 = xy$ and $x - xyx \in Rad(R)$.*

Proof. From the previous proposition, there is an idempotent $e \in xR$ such that $x(1-e) \in Rad(R)$. Therefore, $e = xy$ for some $y \in R$ and hence $(xy)^2 = e^2 = e = xy$. Moreover, $x(1-e) = x(1-xy) = x - xyx \in Rad(R)$. □

Remark. Since any hollow-lifting ring is f-hollow-lifting, the last two results hold for a commutative hollow-lifting ring.

The following diagram summarizes some classes of modules and the relations between them.



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