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Devaney's Definition of Chaos and other Forms

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Dedication

Thankfully I dedicate this thesis to all those who contributed to its success. I dedicate it to my family: to soul of my father and to my mother, brothers and sisters -specially to Aisha- who through their encouragement, patience and support enabled me to continue my study and get this degree. I also dedicate it to my daughters Alaa, Alma and my wife for her consistent patience and support.

I am also very grateful to Professor Mohammad Saleh who through his patience, support and knowledge made this work succeeds. I would like also to express my thanks to my cousin Mohammad. There are too many important friends, they know who they are, for without their love and support I would not have been able to reach this point.

THANK YOU ALL

Abstract

We discuss in this thesis 17 notions of chaos which are commonly used in the mathematical literature and related definitions, namely those being introduced by Devaney, Turbulence, Liapounove, Robinson, Wiggins, Touhey, Experimantalists, Knudsen, P-chaos, Martelli, Block-Coppel, Li-Yorke, Entropy, Auslander, Smital, Kato,and S.Li respectively. We in particular show that for continuous mappings of a compact interval into itself the notions of chaos are equivalent (except the notion in sense of Li and Yorke) while each of these is sufficient but not necessary for chaos in the sense of Li & Yorke. We also give examples indicating that in the general context of continuous mappings between compact metric spaces the relation between these notions of chaos is more involved.

Declaration

I certify that this thesis, submitted for the degree of Master of Mathematics to the Department of Mathematics in Birzeit University, is of my own research except where otherwise acknowledged, and that this thesis has not been submitted for a higher degree to any other university or institution.

Bashir Abu Khalil Signature.....................

May 7, 2012

Contents

1 *Preliminaries*

1.1 Introduction

The term chaos has first been used in 1975 by Li & Yorke in their paper "Period three implies chaos ", but even before it has been observed that very simple functions may give rise to very complicated dynamics. One of the cornerstones in the development of chaotic dynamics in the 1964 paper "Coexistence of cycles of a continuous mapping of the line into itself " by Sarkovskii. During the seventies and eighties the interest in chaotic dynamics has been exploding and various attempts have been made to give the notion of chaos a mathematically precise meaning. Outstanding works in this context is the 1992 lecture notes "Dynamics in One Dimension "by Block & Coppel. While up to the end of the eighties the subject of chaotic dynamics was restricted mainly to research oriented publications, the 1986 book "An Introduction to chaotic Dynamical Systems " by Devaney marked the point where chaos became popular and began to enter university textbooks such as "A First Course in Discrete Dynamical Systems " by Holmgren 1994. The different definitions of chaos being around at the turn of the century have been designed to meet different purposes and they are based on very different backgrounds and levels of mathematical sophistication. Therefore it is not obvious how these notions universally accepted definition of chaos might evolve. We want to make a contribution to this question by picking 17 of the most popular definitions of chaos and investigating their mutual interconnections.

1.2 Basic Definitions

Here we will give some introductory definitions that will be used several times in this thises.

Definition 1.2.1: *A metric on a set* X *is a function* $d : X \times X \to R$ *that satisfies the following properties:*

a. $d(x, y) \geq 0$ *for all* $x, y \in X$.

b. $d(x, y) = 0$ *if and only if* $x = y$.

c. $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

d. $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Definition 1.2.2: *Consider the continuous function* $f : X \to X$.

The dynamical system defined by f *takes the form* $x_{n+1} = f(x_n)$ *and is written as* (X, f).

Such functions that describe dynamical system are called *maps*.

Definition 1.2.3: *Let* X *and* Y *be subsets of a metric space* Z*, such that* $X \subset Y$. We *say that* X *is dense in* Y *if* $\overline{X} = Y$*, i.e.* $\forall x \in Y, \forall \varepsilon > 0, N_\varepsilon(x)$ *contains a point in* X.

Definition 1.2.4: *Let f be a continuous map* $f : X \to X$. *A point* $x \in X$ *is said to be a fixed point for* f *if* $f(x) = x$. *The set of fixed points of* f *is denoted by:* $Fix(f) = \{x \in X : f(x) = x\}$.

Definition 1.2.5: *Let* f *be a function. The point* x *is a periodic point of* f *with period* k *if* $f^k(x) = x$.

In other words, a point is a periodic point of f with period k if it is a fixed point of f^k.

The set of the periodic points of f with period k is denoted by:

$$Per_k(f) = \left\{x \in X : f^k(x) = x\right\}.$$

The set of all iterates of the point x is called *orbit* of x, and if x is a periodic point, then it and its iterates are called *a periodic orbit or a periodic cycle.*

Definition 1.2.6: *Let X be a metric space. We say X is a compact if every open cover of X has finite subcover, i.e. if $\{I_i\}_{i\in J}$ is a collection of open sets of X such that $X \subset \bigcup_{i\in J} I_i$ then there exists a finite subset A of J such that $X \subset \bigcup_{i\in A} I_i$.*

Definition 1.2.7: *A sequence $\{x_n\}_{n\in N} \in X$, where X is a metric space with metric d, is Cauchy if $\forall\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$, $\forall m, n > N$.*

X is said to be complete if every Cauchy sequence in X converges.

X is said to be separable if it has a countable dense subset.

Definition 1.2.8: *A homeomorphism $h : X \to Y$ is a continuous and bijective map with a continuous inverse.*

Definition 1.2.9: *Let X, Y be metric spaces and let $f : X \to X$ and $g : Y \to Y$ be continuous maps. The maps f and g are said to be topologically conjugate if there exists a homeomorphism $h : X \to Y$, such that $h \circ f(x) = g \circ h(x)$ $\forall x \in X$, i.e. the diagram:*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h \big\downarrow & & \big\downarrow h \\ Y & \xrightarrow[g]{} & Y \end{array}$$

commutes.

A homeomorphism satisfying this condition is called *topologically conjugacy.*

Definition 1.2.10: *Consider the continuous and differentiable map* $f : \mathbb{R} \to \mathbb{R}$*. Then the map* f *is said to be expanding if* $|f'(x)| > 1$, $\forall x \in \mathbb{R}$.

Definition 1.2.11: *Let* X *be a metric space and* f *be a continuous map* $f : X \to X$*. We say that* f *is topologically transitive if for every pair of nonempty open sets* U *and* V *in* X *there exists a positive integer* k *such that* $f^k(U) \cap V \neq \phi$.

There is another famous definition of topological transitivity which is the next definition.

Definition 1.2.12: *The map* $f : X \to X$ *is topologically transitive if* $\exists$ $x \in X$ *such that the orbit* $\{f^n(x) : n \geq 0\}$ *is dense in* X.

Remark 1.2.13: These two definitions of topological transitivity are not equivalent.

Example 1.2.14: Consider the continuous map $f : X \to X$ where $X = \{0\} \cup \left\{\frac{1}{n}, n \in N\right\}$ with metric $d = |x - y|$, $\forall x, y \in X$. [1]

The map f is defined by $f(0) = 0$ and $f(\frac{1}{n}) = \frac{1}{n+1}$ for $n = 1, 2, 3, ...$ Then if we choose $U = \left\{\frac{1}{2}\right\}$ and $V = \{1\}$ then f does not satisfy definition 1.2.11.

Now we can observe that the point $x = 1$ has *a dense orbit* in X so the definition 1.2.12 is satisfied and so it is not equivalent with definition 1.2.11.

Remark 1.2.15: Many authors ([1] , [2]) added assumptions on the phase space to make these two definitions equivalent such as compactness.

The next example shows that this compactness is not enough.

Example 1.2.16: [3] Let $X = \{a, b\}$ with discrete topology and let $f : X \to X$ be defined as the constant function $f(a) = f(b) = a$.

The orbit of b is dense. But f is not topologically transitive: if we choose $U = \{a\}, V = \{b\}$ then there is no k with $f^k(U) \cap V \neq \phi$.

Remark 1.2.17: In [2] , [4] another assumption was added in X. The assumption that X is a complete and separable space which is also not enough (see example two in [3]).

However, in [3] a sufficient condition was given under which the two notions are equivalent.

Proposition 1.2.18: *Let X be a thick (topological space is thick if no nonempty open subset U has a finite subset dense in U) and complete metric space with a countable base and $f : X \to X$ a continuous function. Then f is topologically transitive if and only if it possesses a dense orbit.*[3]

Proof: Let $(V_i)_{i\in I}$ be a countable base for X. For $i \in I$ the set $W = \cup_{n\geq 0} f^{-n}(V_i)$ is open by continuity of f. This set is also dense in X. Because of topological transitivity there exists a $k > 0$ with $f^k(U) \cap V \neq \phi$. This gives $f^{-k}(V_i) \cap U \neq \phi$ and $W_i \cap U \neq \phi$. Thus W_i is dense. By *Baire category theorem* the set $B = \cap_{i\in I} W$ is dense in X.

Now the orbit of any point $x \in B$ is dense in X. Because, given any nonempty open $U \subset X$, there is an $i \in I$ with $V_i \subset U$ and $k > 0$ with $x \in f^{-k}(V_i)$. This means $f^k(x) \in V_i \subset U$. Thus the orbit of x enters any U.

Now let $x \in X$ have a dense orbit and let U and $V \subset X$ be nonempty open subsets. By denseness, the orbit of x will enter both U and V. Let m, n be the least integers such that $f^m(x) \in U$ and $f^n(x) \in V$. Assume first $m < n$ set $k = n - m$. Then obviously $f^k(U) \cap V \neq \phi$.

Now let $m \geq n$. The orbit of x can enter V several times before it enters U. Call these points: $f^{l_1}(x), f^{l_2}(x), ..., f^{l_k}(x)$, $(n \leq l_i \leq m, i = 1, 2, ..., p)$.

As the set of these points is not dense in V, there is an open subset $\phi \neq V' \subset V$ such that V' does not contain any of these points. But the orbit of x is dense in X, so there is $q > 0$ with $f^q(x) \in V'$. This q is greater than m and for $t = q - m$ we get $f^t(U) \cap V \neq \phi$.■

Definition 1.2.19 : *Consider the metric space X with the metric d and the continuous map $f : X \to X$. We say that the map f exhibits sensitive dependence on initial conditions if $\exists \delta > 0$ such that for any $x \in X$ and any open neighborhood $N_\varepsilon(x)$ of x for some $\varepsilon > 0$ there exists a point $y \in N_\varepsilon$ and $n \geq 0$ such that*:

$$d(f^n(x), f^n(y)) \geq \delta.$$

2 Definitions of chaos

2.1 Devaney's Definition of Chaos

The Devaney's definition of chaos is the most popular and widely used definition.

Namely, a self map on a metric space X is chaotic on X if it has three essential ingredient: the periodic points of the map must form a dense subset, the map must have sensitive dependence on initial conditions, and the map must be topologically transitive.[5]

Definition 2.1.1: *The function $f : X \to X$ is chaotic if it satisfies the following three properties:*

$i)$ f is transitive,

$ii)$ the set of periodic points of f is dense in X,

$iii)$ f is sensitive dependence on initial conditions.

We will denote this chaos by *D-Chaos*.

Remark 2.1.2: In [6] Banks et al prove that [i] and [ii] imply [iii] in devaney's definition in any metric space.

Theorem 2.1.3 [6]**:** *Let $f : X \to X$ be a continuous map where X is a metric space. Then if f is topologically transitive and has dense periodic point then f exhibits sensitive dependence on initial conditions.*

Proof: First we observe that we can choose two periodic points p_0 and q_0 such that $O(p_0) \cap O(q_0) = \phi$. ($O = orbit$). If $O(p_0) = \{p_0, p_1, p_2, ..., p_{n-1}\}$, and $O(q_0) = \{q_0, q_1, q_2, ..., q_{m-1}\}$, we let $\varepsilon_0 = \frac{1}{2} \min\{d(p_i, q_j)\} : i = 0, 1, 2, ...n - 1; j = 0, 1, 2, ..., m - 1$.

Let $x \in X$ be an arbitrary point. Then by *triangle inequality* we have for any $r, s \in N$:

$$2\varepsilon_0 \leq d(f^r(p_0), f^s(q_o)) \leq d(f^r(p_0), x) + d(x, f^s(q_0)).$$

So if $d(f^r(p_0), x) \leq \varepsilon_0$, then $d(x, f^s(q_0)) \geq \varepsilon_0$ for all $s \in N$ and if $d(f^s(q_0), x) \leq \varepsilon_0$, then

$$d(x, f^r(p_0)) \geq \varepsilon_0 \ \forall r \in N.$$

Now we let our final $\varepsilon = \frac{1}{4}\varepsilon_0$ where ε_0 as in above observation.

Let $x \in X$ and $\delta > 0$ be given with $\delta < \varepsilon$. By the density of the set of periodic points, there exists $a-k$ periodic point $y \in X$ such that $d(x, y) < \delta$. By above observation there exists a periodic point p such that:

$$d(x, f^n(p)) \geq \varepsilon_0 = 4\varepsilon, \forall n \in N.$$

Let

$$U = \bigcap_{i=0}^{k-1} f^{-i}(B_\varepsilon(f^i(p)).$$

Then U is open and nonempty since $p \in U$. Since f is transitive there exists $z \in U$ and $m \in N$ such that $f^m(z) \in U$.

Let $r \in N$ such that

$$\frac{m}{k} < r < \frac{m}{k} + 1 or 0 < kr - m < k.$$

Now :

$$d(x, f^{kr-m}(p)) \leq d(x, y) + d(y, f^{kr}(z)) + d(f^{kr}(z), f^{kr-m}(p)).$$

Since

$$f^{kr}(z) = f^{kr-m}(q) \in B_\varepsilon(f^{kr-m}(p))$$

for some $q \in f^m(z)$ or $z = f^{-m}(q)$, it follows that

$$d(f^{kr}(z), f^{kr-m}(p)) < \varepsilon.$$

Hence we get:

$$4\varepsilon \leq d(x, f^{kr-m}, (p)) < 2\varepsilon + d(y, f^{kr}(z)).$$

This implies that: $d(y, f^{kr}(z)) > 2\varepsilon$ and since y is of periodic k, $d(f^{kr}(y), f^{kr}(z)) > 2\varepsilon$, since y is k-periodic. Therefore by *triangle inequality,*

$$2\varepsilon < d(f^{kr}(y), f^{kr}(z)) \leq d(f^{kr}(y), f^{kr}(x)) + d(f^{kr}(x), f^{kr}(z)).$$

Therefore either

$$d(f^{kr}(y), f^{kr}(x)) > \varepsilon \text{ or } d(f^{kr}(x), f^{kr}(z)) > \varepsilon.\blacksquare$$

Remark 2.1.4: In [7] David Assaf shows $[i]$ and $[iii]$ do not imply $[ii]$ in Devaney's definition and $[ii]$, $[iii]$ do not imply $[i]$, the next two examples showing this.

Example 2.1.5: Consider the continuous map $f : X \to X$ defined by $f(e^{i\theta}) = e^{2i\theta}$ and $X = S^1 \setminus \{e^{\frac{2\pi pi}{q}} : p, q \in Z, q \neq 0\}$ is a metric space equipped with the arc length metric.

Now every non-empty subset of X is eventually expanded under iteration to cover X, so f is transitive.

Also by defining in this way the set X we let out all the periodic points of f (we removed the $2^n - 1$ roots of unity for all n.[5]) So f has no (dense) periodic points.

Finally for any given two points in X say $e^{i\theta}$, and $e^{i\varphi}$ such that $0 < |\theta - \varphi| < \pi$ we can choose n that satisfies $2^n < |\theta - \varphi| \leq \pi < 2^{n+1} |\theta - \varphi| \Rightarrow$

f is sensitive with sensitivity constant $\frac{\pi}{2}$ since $d(f^n(e^{i\theta}), f^n(e^{i\varphi})) > \frac{\pi}{2}$.

So the map f is not *D-chaotic*.

Example 2.1.6: [7] Consider the continuous map $f : X \to X$ where $X = S^1 \times [0,1]$ is a metric space equipped with taxicab metric;

$$d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| - |y_2 - y_1|$$

for every pair $(x_1, y_1), (x_2, y_2) \in X$.

We define f by

$$f(e^{i\theta}, t) = (e^{2i\theta}, t).$$

Clearly a point $z = (e^{i\theta}, t)$ will be a periodic point for f when $e^{i\theta}$ is the root of unity of order $2^n - 1$ for some n. So the periodic points of f are dense in X.

On other hand if we take two sets A and B where $A = S^1 \times [0, \frac{1}{2})$ and $B = S^1 \times (\frac{1}{2}, 1]$,

then $\forall n \in N$, we have that $f^n(A) \cap B = A \cap B = \phi \Rightarrow$ the map is not transitive.

Finally, if we work in the same way as in the above example, it is easy to conclude that the map f is sensitive.

Remark 2.1.7: [**10**] Vellekoop and Berglund showed that for continuous maps on an interval in $\mathbb{R}$, transitivity implies that the set of periodic points is dense. It follows from *Theorem* 2.1.3 that transitivity implies chaos.

The proof of this result will be facilitated by first establishing the following lemma.

Lemma 2.1.8: [**10**] *Let $f : J \to J$ be a continuous map on an interval J in R.*

Suppose that there exists a subinterval I of J such that I contains on periodic points of f . If $x, f^m(x)$, and $f^n(x)$ are all in I, with $0 < m < n$, then either $x < f^m(m) < f^n(x)$ or $x > f^m(x) > f^n(x)$.

Proof: Let m, n be integers such that $0 < m < n$, and let $I \subset J$ be an interval with no periodic points of f.

Suppose that for some $x \in I$ we have $x < f^m(x)$, $f^m(x) > f^n(x)$ and $f^m(x)$, $f^n(x) \in I$. Define a new function $g = f^m$. Then we have $s < g(x)$. Claim that for all $K > 1$,

$$x < g(x) \leq g^k(x).$$

Now, if $g^2(x) < g(x)$, then the function $h(z) = g(z) - z$ is positive at $z = x$ and negative at $z = g(x)$.

By intermediate value theorem, this implies that $h(y) = g(y) - y = 0$ for some y between x and $g(x)$.

This mean that $g(y) = f^m(y) = y$ and y is thus a periodic point of f. But $y \in I$, so we have a contradiction.

This proves that $x < g(x) < g^2(x)$. By mathematical induction, we may complete the proof of the claim.

Now, $x < g^k(x)$ for all $k \in Z^+$ and in particular for $k = n - m$ we have

$$x < g^{n-m}(x) = f^{(n-m)m}(x).$$

By letting $h = f^{n-m}$, we have $x < h^m(x)$.

Now

$$f^{n-m}(f^m(x)) = f^n(x) < f^m(x)$$

or

$$h(f^m(x)) < f^m(x).$$

By an argument similar to that used for g we have:

$$h^m(f^m(x)) < f^m(x).$$

It easy to see that the function $p(y) = h^m(y) - y$ is positive at $y = x$ and negative at $y = f^m(x)$. Thus by intermediate value theorem, there exists $z \in$

I between x and $f^m(x)$ Such that $h^m(z) = f^{(n-m)m}(z) = z$, a contradiction.

This prove that $x < f^m(x) < f^n(x)$.

The other case be proven analogously.■

Remark 2.1.9: The above lemma can be found in [4] (chapter IV, corollary 10) in a more general form:

"If J is a subinterval of I which contains no periodic point of f then, for any $x \in I$, the point of trajectory $O(x)$ which lie in J form a strictly monotonic (finite or infinite sequence)".

Theorem 2.1.10 [10]**:** *Let I be an interval (not necessarily finite) and f:I $\rightarrow$I a continuous and topologically transitive map. Then 1- The periodic points of f are dense in I and 2- f has sensitive dependence on initial conditions, that is f is chaotic in the sense of devaney.*

Proof: Suppose that f is continuous and topologically transitive. Because of the result of theorem 2.1.3 we any need to prove that the periodic points are dense in I. Suppose that this is not the case, then there exists an interval $J \subset I$ containing no periodic points. Take an $x \in J$ which is not an endpoint of J, an open neighborhood $N \subsetneq J$ of x and an open interval $E \subset J \setminus N$. Since f is topologically transitive on I there exists a natural number $m > 0$ with $f^m(N) \cap E \neq \phi$ and thus a $y \in J$ with $f^m(y) \in E \subset J$. Since J contains no periodic points we know that $y \neq f^m(y)$ and since f is continuous this implies that we can find a neighborhood U of y with $f^m(U) \cap U = \phi$. Since U is open set we can use the topologically transitive of f again and find an $n > m$ and a point $z \in U$ with $f^n(z) \in U$. But then we have $0 < m < n$ and $z, f^n(z) \in U$ while $f^m(z) \notin U$ and this violates our earlier lemma (lemma 2.1.8).■

Remark 2.1.11: The first result (1) in the above theorem can be found in [4] chapter V1.5, lemma 41 but the proof uses a lot of other highly nontrivial results.

Remark 2.1.12: The preceding theorem fails to hold for non intervals or higher dimensional space or even for the unit circle S^1 as may be seen in the following example.

Example 2.1.13: Consider the rotation map $R_\lambda : S^1 \to S^1$ defined by

$$R_\lambda(\theta) = \theta + 2\pi\theta,$$

where λ is an irrational number let $\theta \in S^1$. Then

$$R_\lambda^m(\theta) \neq R_\lambda^n(\theta) \qquad if \quad m \neq n.$$

otherwise, $\theta + 2\pi m\lambda = \theta + 2\pi n\lambda$, which implies that $2\pi(m-n)\lambda = 1$. Thus λ is integer. But since λ is irrational we must have $m = n$. Hence the orbit $O^+(\theta)$ is an infinite set in S^1.

Since $O^+(\theta)$ is a bounded sequence, it must have a convergent subsequence. Therefor, for $\varepsilon > 0$ there exist positive integer r, s with $|R_\lambda^r(\theta) - R_\lambda^s(\theta)| < \varepsilon$. Without loss of generality, we may assume that $m = r - s > 0$. Since R_λ preserves arc length in S^1, it follows that:

$$|R_\lambda^m(\theta) - \theta| = |R_\lambda^s(R_\lambda^m(\theta)) - R_\lambda^s(\theta)|$$

$$= |R_\lambda^r(i\theta) - R_\lambda^s(\theta)| < \varepsilon$$

Now under R_λ^m, the arc of length less than ε connecting θ to $R_\lambda^m(\theta)$ is mapped to the arc, of less than ε, connecting $R_\lambda^m(\theta)$ to $R_\lambda^{2m}(\theta)$. This arc, in turn, is mapped to the arc of length less than ε joining $R_\lambda^{2m}(\theta)$ to $R_\lambda^{3m}(\theta)$, etc. So the points $\theta, R_\lambda^m(\theta), R_\lambda^{2m}(\theta), ...,$ partition S^1 into arcs of length less than ε. But since ε was arbitrary chosen, it follows that $O^+(\theta)$ intersects every open arc in S^1, and thus $O^+(\theta)$ is dense in S^1.

Observe that R_λ has no periodic points in S^1.

Now we will give some examples to note that there are no other trivialities in devaney's definition when restricted to interval. [10](example 2.1.14, 2.1.15, 2.1.18)

Example 2.1.14 : Let $f(x) = x$ on $(-\infty, \infty)$, clearly the periodic points of f are dense in R but doesn't need to have sensitive dependence on initial conditions.

for finite case take $I = [0, 1]$, $f : I \to I$, $f(x) = x$.

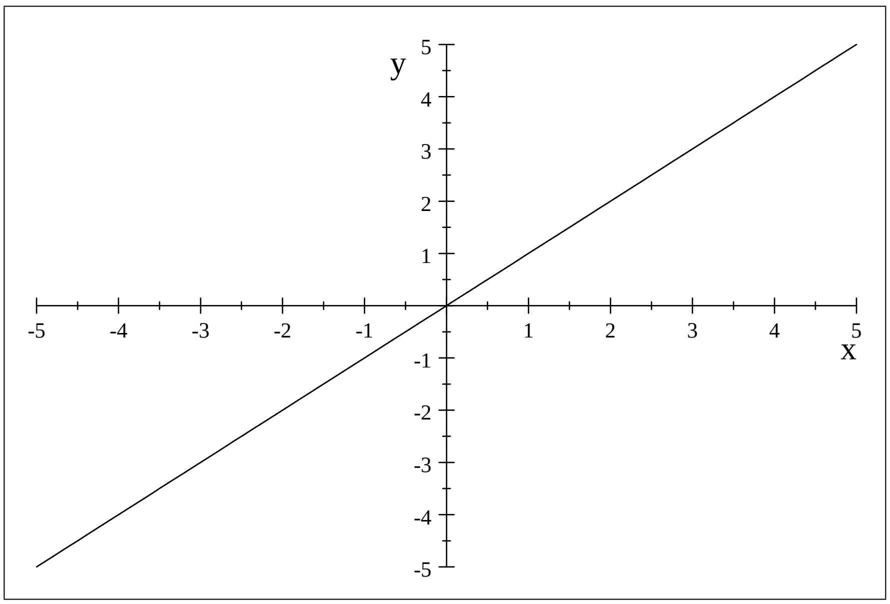


Figure 2.1.1: $f(x) = x$

Example 2.1.15 : Define on $I = R^+$ the function

$$f(x) = \begin{cases} 3x & , \quad 0 \le x < \frac{1}{3} \\ -3x + 2 & , \quad \frac{1}{3} \le x < \frac{2}{3} \\ 3x - 2 & , \quad \frac{2}{3} \le x < 1 \\ f(x-1) + 1 & , \quad x \ge 1 \end{cases}$$

This function has sensitive dependence on initial conditions and its periodic points are dense but it is not transitive to show that f has sensitive dependence on initial conditions, we need to prove the following two propositions.

Proposition 2.1.16: *Consider the continuous and differentiable map $f : I \to I$. If f has positive Lyapunove exponent then f has also sensitive dependence on initial conditions.*

Proof: First recall that we define the *Lyapunove exponent* of x say $\lambda(x)$ as:

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(x_i)|, \forall x_i \in I.$$

Now we make this motivation: consider the map $x_{n+1} = f(x_n)$ and let the point x_0, x_0' be originally displaced by $\delta = |x_0' - x_0|$. Then after n iterations of the map we get:

$$\delta x_n = |x_n' - x_n| = |f^n(x_0 - \delta) - f^n(x_0)| = \delta e^{n\lambda(x_0)} \text{.........}(1),$$

in the limits $\delta \to 0$ and $n \to \infty$. If we solve the last relation with respect to $\lambda(x_0)$ we get:

$$\lambda(x_0) = \lim_{n\to\infty} \lim_{\delta\to 0} \frac{1}{n} \log \left| \frac{f^n(x_0+\delta) - f^n(x_0)}{\delta} \right| = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{df^n(x_0)}{dx} \right|$$

$$= \lim_{n\to\infty} \frac{1}{n} \log \left| \prod_{i=o}^{n-1} f'(x_i) \right| = \lim_{n\to\infty} \sum_{i=0}^{n-1} \log |f'(x_i)|.$$

Consider a point $x_0 \in X$. Now we choose a point x_0' close to x_0.Then from relation (1) we have

$$\delta x_n = |f^n(x_0 + \delta) - f^n(x_0)| = |x_n' - x_n| = \delta x_0 e^{n\lambda(x_0)} = \delta,$$

where $\delta x_0 = |x_0' - x_0|$ and $x_0' = x_0 + \delta \Rightarrow e^{n\lambda(x_0)} = \frac{1}{\lambda(x_0)} \log \left| \frac{\delta}{\delta x_0} \right|$. Choosing some δ for given x_0 then $\exists x_0' \in N_\varepsilon(x)$ such that $\forall \varepsilon > 0$ after $m > n$ iterations we get

$$|f^m(x_0') - f^m(x_0)| = \delta x_0 e^{m\lambda(x_0)}$$

$$= \delta x_0 e^{(m-n)\lambda(x_0)} e^{n\lambda(x_0)} = e^{(m-n)\lambda(x_0)} \delta > \delta,$$

then f has sensitive dependence on initial conditions.■

Proposition 2.1.17: *Every expanding map $f : I \to I$ has sensitive dependence on initial conditions.*

Proof: Since the map f is expanding, we have $|f'(x_i)| > 1, \forall x_i \in I$. Now the Lyapunove exponent of f at the point x is given by

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{n=0}^{n-1} \log |f'(x_i)| .$$

Since

$$|f'(x_i)| > 1 \Rightarrow \log |f'(x_i)| > \log 1 = 0$$

$$\Rightarrow \sum_{n=0}^{n-1} |f'(x_i)| > 0 \Rightarrow \frac{1}{n} \sum_{n=0}^{n-1} |f'(x_i)| > 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{n=0}^{n-1} |f'(x_i)| > 0 \quad i.e. \lambda(x) > 0.$$

Hence we have that f is sensitive.■

Now we return to example 2.1.15 : f has sensitive dependence on initial conditions since $|f'(x_i)| = 3 \ \ \forall x \in I$. Note that f^n has $3^n - 2$ fixed points between any two integer values with distance between these points smaller than $(\frac{1}{3})^{n-1}$, so the periodic points are dense. But since $f([0,1]) = [0,1]$ the function is not topologically transitive. For finite case take $I = [0,2]$.

Example 2.1.18 : Let $f : I \to I$, where $I = [0, \frac{3}{4}]$,be defined as:

$$f(x) = \begin{cases} \frac{3}{2}x & , \quad 0 \le x < \frac{1}{2} \\ \frac{3}{2}(1-x) & , \quad \frac{1}{2} \le x \le \frac{3}{4} \end{cases}$$

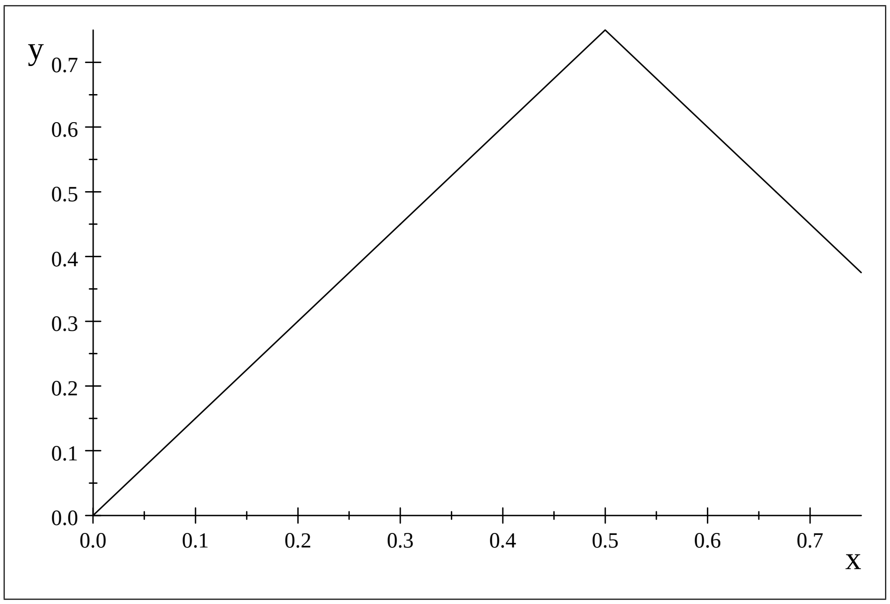


Figure 2.1.2: The graph of $f\,(x)$

Since $|f'(x_i)| > 1$, f is sensitive, but there can be no periodic points in $(0, \frac{3}{8})$ since any trajectory with initial value in this subinterval will not return there.

For infinite case take $f(x) = 2x$ on $\mathbb{R}^+$.

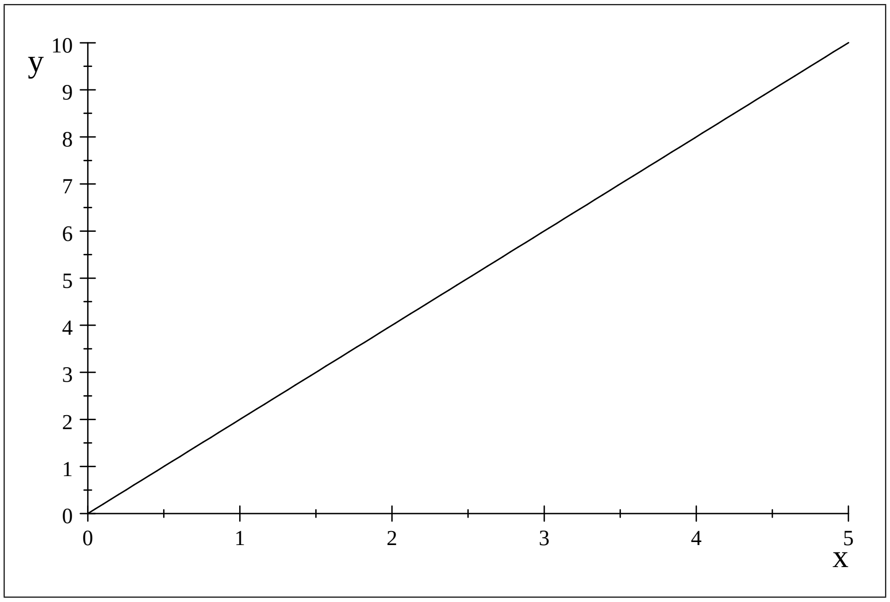


Figure 2.1.3 :$f\ (x) = 2x$

Before ending this section we will give one example which satisfies the three conditions of Devaney's definition.

Example 2.1.19: Consider the Bernolli shift map $B(x) : [0, 1) \to [0, 1)$ given by:

$$B(x) = 2x \bmod 1 = \begin{cases} 2x & , \quad 0 \leq x < 0.5 \\ 2x - 1 & , \quad 0.5 \leq x < 1 \end{cases}$$

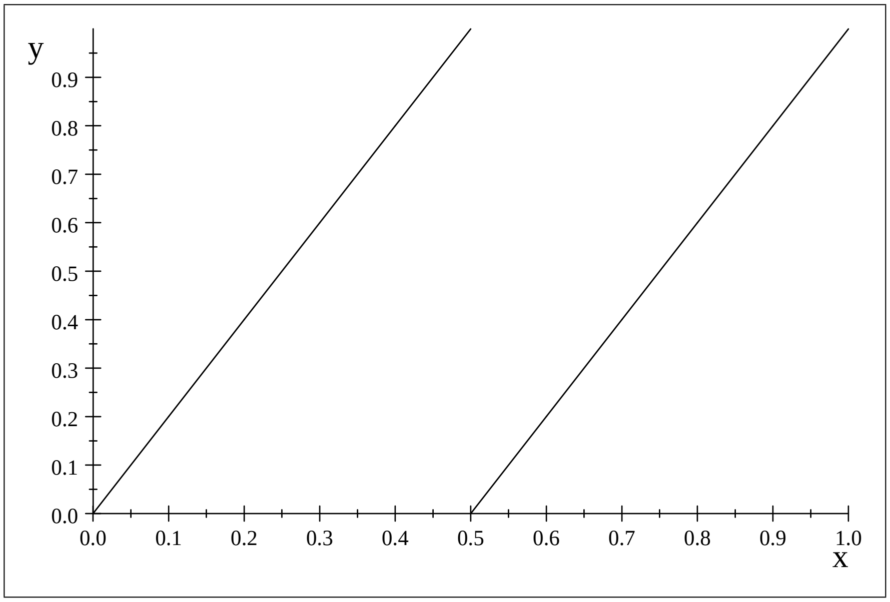


Figure 2.1.4: The graph of $B\left(x\right)$

We will show that $B(x)$ is transitive using symbolic dynamics. We let $\sum$ be the metric space of all infinite sequences containing $0's$ and $1's$ equipped with the metric

$$\rho(s,J)=\frac{1}{2^{i}}\left|S_{i}-J_{i}\right|,\ \forall S=(S_{0}S_{1}S_{2}...)$$

and $J=(J_{0}J_{1}J_{2}...)\in\sum$ and we define $\sigma:\sum\rightarrow\sum$ given by

$$\sigma(S_{0}S_{1}S_{2}...)=(S_{1}S_{2}S_{3}..)$$

Then there exist a point $x=(0100011011000001...)$ created by blocks of 0's and 1's, which has a dense orbit. So σ is transitive and then $B\left(x\right)$ is transitive $\left[2\right]$.

We have that

$$fix\left(B\right)=Per_{1}\left(B\right)=0\ ,\quad\left|Per_{1}\left(B\right)\right|=1=2^{1}-1.$$

The second iterated map B^2 is given by $B^2 = 4x \bmod 1$ and

$$Per_2(B) = \left\{0, \frac{1}{3}, \frac{2}{3}\right\} \Longrightarrow |Per_2(B)| = 3 = 2^2 - 1$$

Generalizing this result the $n'th$ iterated map is given by $B^n = 2^n x \bmod 1$. So

$$Per_n(B) = \left\{0, \frac{1}{2^n - 1}, \frac{2}{2^n - 1}, ..., \frac{2^n - 2}{2^n - 1}\right\}$$

and $|Per_n(B)| = 2^n - 1$.

Now $\lim\limits_{n\to\infty} |Per_n(B)| = \infty$, so $\forall \varepsilon > 0, N_\varepsilon(x)$ will contains a periodic point. Hence the periodic points of B are dense.

Also since $B(x) = 2x \bmod 1, B' = 2 \quad \forall x \in [0, 1)$ except for $x = 0.5$, so $B(x)$ has sensitive dependence on I.C.

2.2 Periodic Orbits and Turbulence

In this section we will derive a number of results of independent interest to give the notion of chaos in sense of turbulence. However we will prove the important theorem: sarkoviski theorem[15] which played an important rule in chaos theory. I follows chapters "I" and "II" in block and coppel's book: Dynamics in one dimension[4].

Lemma 2.2.1: *If J is a compact subinterval such that $J \subseteq f(J)$, then f has a fixed point.*

Proof: if $J = [a, b]$ if $f(a) = a$ or $f(b) = b$ then it is done. If not, then for some $c, d \in J$ we have $f(c) = a, f(d) = b$. Now consider $g(x) = f(x) - x$ on the interval $[c, d]$ .Thus $g(c) = f(c) - c = a - c < 0; g(d) = f(d) - d = b - d > 0$, and the result follows from the intermediate value theorem.■

Lemma 2.2.2: *If J, K are compact subintervals such that $K \subseteq f(J)$, then there is a compact subinterval $L \subseteq J$ such that $f(L) = K$.*

Proof: Let $K = [a, b]$ and let c be the greatest point in J for which $f(c) = a$. If $f(x) = b$ for some $x \in J$ with $x > c$, let d be the least. Then we can take $L = [c, d]$. Otherwise $f(x) = b$ for some $x \in J$ with $x < c$. Let c' be the greatest and let $d' \leq c'$ be the least $x \in J$ with $x > c'$ for which $f(x) = a$. Then we can take $L = [c', d']$.■

Lemma 2.2.3: *If $J_0, J_1, ..., J_m$ are compact subintervals such that $J_k \subseteq f(J_{k-1})$, $1 \leq k \leq m$, then there is a compact subinterval $L \subseteq J_0$ such that $f^m(L) = J_m$ and $f^k(L) \subseteq J_k$, $1 \leq k < m$.*

If also $J_0 \subseteq J_m$, then there exists a point y such that $f^m(y) = y$ and $f^k(y) \in J_k$,$0 \leq k < m$.

Proof: The first assertion holds for $m = 1$, by *Lemma 2:2:2.* We assume that $m > 1$ and that it holds for all smaller values of m. Then we can choose $L' \subseteq J_1$ So that $f^{m-1}(L') = J_m$ and $f^k(L') \subseteq J_{k-1}, 1 \leq k < m - 1$. We now choose $L \subseteq J_0$, so that $f(L) = L'$. The second assertion follows from the first by Lemma 2.2.1.■

An application of these ideas the following proposition.

Proposition 2.2.4: *Between any two points of a periodic orbit of period $n > 1$ there is a point of a periodic orbit of period less than n.*

Proof: let $a < b$ two adjacent points of the orbit of period n. Since there is one more point of the orbit to the left of b than to the left of a we must have $f^m(a) > a; f^m(b) < b$ for some m such that $1 \leq m < n$. It follows at once that $f^m(c) = c$ for some c such that $a < c < b$, assuming that f^m is defined throughout $[a, b]$. However, the same conclusion can be reached without this assumption. For if $J_k = \left[f^k(a), f^k(b)\right]$ is the closed interval with endpoints $f^k(a)$, $f^k(b)$ then $J_k = f(J_{k-1})$, $1 \leq k \leq m$. But $J_0 \subseteq J_m$, since $f^m(a) \geq b; f^m(b) \leq a$. The result now follows from *Lemma 2.2.3.* ■

Suppose that f has a periodic orbit of period $n > 1$. Let $x_1 < x_2 < ... < x_n$ be the distinct points of this orbit and set $I_j = [x_j, x_{j+1}]$, $1 \leq j < n$. We associate a *directed graph*, or *digraph* in the following way: The vertices of the direct graph are the subintervals $I_1, I_2, ..., I_{n-1}$ and there is an are $I_j \to I_k$ if I_k is contained in the closed interval $[f(x_j), f(x_{j+1})]$.

Properties of digraph:

[i] For any vertex I_j there is always at least one vertex I_k for which $I_j \to I_k$.

It is always possible to choose $k \neq j$ unless $n = 2$.

[ii] For any vertex I_k there is at least one vertex I_j for which $I_j \to I_k$, it is always possible to choose $j \neq k$ unless n is even and $k = \frac{n}{2}$.

[iii] The digraph always contains a loop.

Definition 2.2.5: *A cycle $J_0 \to J_1 \to ... \to J_{n-1} \to J_0$ of length n in the digraph will be said to be a fundamental cycle if J_0 contains an endpoint c such that $f^k(c)$ is an endpoint of J_k for $1 \leq k < n$.*

Proposition 2.2.6: *A fundamental cycle always exists and it is unique.*

Proof: without loss of generality, take $c = x_1$ so that $J_0 = I_1$. Suppose $J_0, ..., J_{i-1}$ have been defined. If $J_{i-1} = [a, b]$, so that $f^{i-1}(c)$ is either a or b, we must take J_i to be the uniquely determined interval $I_k \subseteq [f(a), f(b)]$ which has $f^i(c)$ as one endpoint. Then $J_n = J_0$ and we obtain a cycle of length n.■

Definition 2.2.7: *A cycle in a digraph is said to be primitive if it does not consist entirely of a cycle of smaller length described several times.*

Lemma 2.2.8: *Suppose f has aperiodic point of period $n > 1$. If the associated digraph contains a primitive cycle $J_0 \to J_1 \to ... \to J_{m-1} \to J_0$ of length m, then f has a periodic point y of periodic m such that $f^k(y) \in J_k$, $0 \leq k < m$.*

Proof : By *Lemma 2.2.3* there exists a point y such that $f^m(y) = y$ and $f^k(y) \in J_k$, $0 \leq k < m$. Since the cycle is primitive and distinct intervals J_k have at most one endpoint in common it follows that y has a period m, unless possibly $y = x_i$ for some i and n is a divisor of m. This possible only if the cycle is a multiple of the fundamental cycle since, given J_{k-1}, the requirements $f^k(y) \in J_k$ and $J_{k-1} \to J_k$ uniquely determine J_k.■

Example 2.2.9: Suppose c is a periodic point of period 3 with $f(c) < c < f^2(c)$. The corresponding directed graph has two vertices, namely the interval $I_1 = [f(c), c]$ and $I_2 = [c, f^2(c)]$ connected in the following way:

$$\curvearrowleft I_1 \rightleftarrows I_2$$

corresponding to the loop $I_1 \to I_1$ there is a fixed point of f and corresponding to the primitive cycle $I_1 \to I_2 \to I_1$ there is a point of period 2. For any positive integer $m > 2$ there is a point of period m, corresponding to the primitive cycle:

$$I_1 \to I_2 \to I_1 \to I_1 \to \ldots \to I_1 \text{ of length } \ m.$$

Thus there are orbits of period n for every $n \geq 1$.

Proposition 2.2.10: *If f has a period point of period > 1, then it has a fixed point and a periodic point of period 2.*

Proof: The first assertion follows at once from the fact that two digraph of a periodic orbit always contains a loop, i.e. if f has no fixed point then either $f(x) > x$ for all x or $f(x) < x$ and hence f has no periodic point. The second assertion: let n be the last positive integer greater than 1 such that f has a periodic point of period n. We will assume $n > 2$ and deduce a contradiction. In fact the fundamental cycle decomposes into two cycles of smaller length ,each of which is primitive. Since at least one of these has length greater than 1,it follows from *lemma 2.2.8* that there is a periodic point with period strictly between 1 and n.

Proposition 2.2.11: *Suppose* f *has a periodic orbit of odd period* $n >$ 1, *but no periodic orbit of odd period strictly between 1 and* n. *If* c *is the midpoint of the orbit of odd period,* n *then the points of this orbit have the order:*

$$f^{n-1}(c)<f^{n-3}(c)<\ldots<f^2(c)<c<f(c)<\ldots<f^{n-2}(c).$$

or the reverse order:

$$f^{n-2}(c) < \ldots < f(c) < c < f^2(c) < \ldots < f^{n-3}(c) < f^{n-1}(c).$$

In either case the associated digraph is given by the following figure where $J_1 = [c, f(c)],\ J_k = \left[f^{k-2}(c), f^k(c)\right]$ *for* $1 < k < n$:

$$\begin{tikzcd}
J_1 \arrow[loop left] \arrow[r] & J_2 \arrow[r] & J_3 \arrow[r] & \ldots \arrow[r] & J_{n-3} \arrow[r] & J_{n-2} \arrow[r] & J_{n-1} \arrow[d] \\
{} \arrow[u] & & {} \arrow[u] & & & {} \arrow[u] & {} \arrow[llllll]
\end{tikzcd}$$

Proof: The fundamental cycle decomposes into two smaller primitive cycles, one of which has odd length. This length must be one, since f has no orbit of odd period strictly between 1 and n.Thus the fundamental cycle has the form:

$$J_1 \to J_1 \to J_2 \to \ldots \to J_{n-1} \to J_1.$$

where $J_i \neq J_1$ for $1 < i < n$. If we had $J_i = J_k$, where $1 < i < k < n$, then by omitting the intermediate vertices we would obtain a smaller primitive cycle. Moreover, by excluding the loop at J_1 if necessary, we can arrange that its length is odd. Since this is contrary to hypothesis we conclude that $J_1, \ldots, J_{n-1}$ are all distinct and thus a permutation of $I_1, \ldots, I_{n-1}$. Similarly we cannot have $J_i \to J_k$ if $k > i+1$ or if $k = 1$ and $i \neq 1, n-1$. Suppose $J_1 = I_h = [a, b]$. Since J_1 is directed only to J_1 and J_2, the interval J_2

is adjacent to J_1 on the real line and f maps one endpoint of J_1 into an endpoint of J_1 and the other endpoint of J_1 into an endpoint J_2. Since the end points are not fixed points, there are just two possibilities:

either

$$x_h = a, x_{h+1} = f(a),\ x_{h-1} = f^2(a),$$

or:

$$x_{h+1} = b, x_h = f(b), x_{h+2} = f^2(b).$$

We consider only the first case, the argument in the second case being similar. For $n = 3$ the result follows immediately. Suppose $n > 3$. If $f^3(a) < f^2(a)$ then $J_2 \to J_1$, which is forbidden. Hence $f^3(a) > f^2(a)$. Since J_2 is not directed to J_k for $k > 3$ it follows that $J_3 = [f(a), f^3(a)]$ is adjacent to J_1 on the right. If $f^4(a) > f^3(a)$ then $J_3 \to J_1$, which is forbidden. Hence $f^4(a) < f^3(a)$ and since J_3 not directed to J_k for $k > 4$, $J_4 = [f^4(a), f^2(a)]$ is adjacent to J_2 on the left. Proceeding in this way we see that the order of the intervals J_i on the real line is given by:

$$\frac{J_{n-1}}{[f^{n-1}(a),f^{n-3}(a)]}\cdots\frac{J_4}{[f^4(a),f^2(a)]}\,\frac{J_2}{[f^2(a),a]}\,\frac{J_1}{[a,f(a)]}\,\frac{J_3}{[f(a),f^3(a)]}\cdots\frac{J_{n-2}}{[f^{n-4}(a),f^{n-2}(a)]}$$

Since the endpoint of J_{n-1} are mapped into a and $f^{n-2}(a)$ we have $J_{n-1} \to J_k$ if and only if k is odd. It is readily verified now that there are no others arcs in the digraph besides those already found.■

Proposition 2.2.12: *If f has a periodic point of odd period $n > 1$, then it has a periodic points of arbitrary even order and periodic points of arbitrary odd order $> n$.*

Proof: We may suppose that n is minimal so that the associated digraph is given by *proposition 2.2.11.* If $m < n$ is even then $J_{n-1} \to J_{n-m} \to$

$J_{n-m+1} \to ... \to J_{n-1}$ is primitive cycle of length m. If $m > n$ is even or odd then:

$$J_1 \to J_2 \to ... \to J_{n-1} \to J_1 \to J_1 \to ... \to J_1,$$

is a primitive cycle of length m. ■

Lemma 2.2.13 : *If c is a periodic point of f with periodic then for any positive integer h, c is a periodic point of f^h with periodic $\frac{n}{gcd(h,n)}$ where $gcd(h,n)$ denotes the greatest common divisor of h and n. Conversely, if c is a periodic point of f^h with period m then c is a periodic point of f with period $\frac{mh}{d}$, where d divides h and is relatively prime to m.*

Proof: Suppose c has period n for f and let $m = \frac{n}{gcd(h,n)}$. Then $f^{mh}(c) = c$. On the other hand, if $f^{kh}(c) = c$ then n divides kh and hence m divides k. Suppose c has period m for f^h. Then c has period n for f, where n divides mh.

Thus we can write $n = \frac{mh}{d}$. Then by what we have already proved, $\frac{n}{gcd(h,n)} = \frac{nd}{h}$ and hence $gcd(h,n) = \frac{h}{d}$. Thus $de = h$ for some e and $gcd(de, me) = e$.■

Now we are ready to state and prove the *"sarkoviskii theorem"*.

Theorem 2.2.14: *Let the positive integers be totally ordered in the following way*

$3 \prec 5 \prec 7 \prec 9 \prec ... \prec 2.3 \prec 2.5 \prec ... \prec 2^2.3 \prec 2^2.5 \prec ... \prec 2^3 \prec 2^2 \prec 2 \prec 1$

If f has a periodic orbit of periodic n and $n < m$ then f also has a periodic orbit of period m.

Proof: We give the proof initially for $f : I \to I$. Let $n = 2d^q, q$ is odd. Suppose first that $q = 1$ and $m = 2^e$ where $0 \leq e < d$. *By proposition 2.2.10* we may assume $e > 0$. The map $g = f^{\frac{m}{2}}$ has a periodic point of period 2^{d-e+1} by *Lemma 2.2.13* and hence also a periodic point of period 2 by *proposition 2.2.10.* This point has period m for by *Lemma* 2.2.13 again. Suppose next that $q > 1$. The remaining cases to be considered are $m = 2^d r$, where either (i) r: even (ii) r: odd , $r > q$.

The map $g = f^{2d}$ has a periodic point of period q and hence also has a periodic point of period r, by *proposition* 2.2.12. In case (i) this point has period $m = 2^d.r$ for f. In case (ii) its f-period is $2^e.r$ for some $e \leq d$. If $e = d$ we are finished. If $e < d$ we can replace n by $2^e.r$. Since $m = 2^e(2^{d-e}.r)$ it then follows from case (i) that f also has a periodic point of period m. We now give the proof for $f : I \to R$, let x_1 and x_n denote respectively the least and greatest points of a periodic orbit of f of period n. Then $k : [x_1, x_n] \cup [f[x_1, x_n]$ is a compact interval. Define a continuous map $g : K \to K$ by sitting $g(x) = f(x_1)$ if $x \leq x_1$, $g(x) = f(x)$ if $x \in [x_1, x_n]$, and $g(x) = f(x_n)$ if $x \geq x_n$. Since g has a periodic orbit of period n, g also has a periodic orbit of period m, by what we have already proved. Since this orbit of period m is contained in the interval $[x_1, x_n]$, it is also a periodic orbit of f.■

Definition 2.2.15 : *The map f is said to be Turbulent if there exist compact subintervals J,K with at most one common point such that*

$$J \cup K \subseteq f(J) \cap f(K).$$

If the subintervals J, K can be chosen disjoint, it is said to be *strictly turbulent.*

Remark 2.2.16: It follows from the definition that if "f" is *(strictly) turbulent* then "f^n" is (*strictly*) *turbulent* for every $n > 1$.

Remark 2.2.17: The map "f" may be turbulent but not strictly turbulent.

The following example shows this.

Example 2.2.18 : Let $f : [0, 1] \to [0, 1]$ be the piecewise linear map defined by

$$f(0) = 0, f(\frac{1}{2}) = 1, f(1) = 0$$

Then if we take $J = [0, \frac{1}{2}], [\frac{1}{2}, 1]$ then:

$$J \cup K \subseteq f(J) \cap f(K)$$

but it is seen that $J \cup K \subseteq f(J) \cap f(K)$ does not hold for any disjoint compact subintervals J, K.

Lemma 2.2.19: *If f is turbulent then there exist points a, b, $c \in I$ such that:*

$$f(b) = f(a) = a, f(c) = b$$

and either:

$$a < c < b$$

$$f(x) > a \quad \text{for} \quad a < x < b$$

$$x < f(x) < b \quad \text{for} \quad a < x < c$$

or:

$$b < c < a$$

$$f(x) < a \quad \text{for} \quad b < x < a$$

$$b < f(x) < x \quad \text{for } c < x < a.$$

Proof: Let $J = [\alpha, \beta]$ and $K = [\gamma, \delta]$; where $\beta \leq \gamma$ be compact subintervals such that:

$$J \cup k \subseteq f(J) \cap f(k).$$

If $\beta = \gamma$ we may assume that $f(\beta) \neq \beta$; since otherwise we can choose $J_0 \subset J$ so that $f(J_0) = f(J)$ and $J_0 \cap K = \phi$. Let a' be least fixed point of f in J and let b' be the greatest point of k for which $f(b') = a'$. Suppose first that $f(c') = b'$ for some $c' \in (a', b')$. Then we can take a to be the greatest fixed point of f in $[a', c'), b$ to be the least point of $(c', b']$ for which $f(b) = a$, and c to be the least point of $(a, c']$ for which $f(c) = b$. Suppose next that $f(x) < b$ for $a' \leq x \leq b'$. Then f takes the value δ in the intervals $[\alpha, a')$ and $(b, \delta]$, and $f(x) > x \geq \alpha$ for $\alpha \leq x < a'$. Thus $f(a'') = a''$ for some $a'' \in (b', \delta]$, $f(b'') = a''$ for some $b'' \in [\alpha, a']$; and $f(c'') = b''$ for some $c'' \in (a', \beta]$. Then, as before, we can take a to be the least fixed point of f in $(c'', a'']$, b to be the greatest point of $[b'', c'')$ for which $f(b) = a$, and c to be the greatest point of $[c'', a)$ for which $f(c) = b$.■

Lemma 2.2.20: *If f is turbulent, then f has periodic points of all periods.*

Proof : Let J, K be compact subintervals with at most one common point such that $J \cup K \subseteq f(J) \cap f(K)$ by *lemma 2.2.19* we may assume that if J and K are not disjoint then their common points is not periodic. Since $J \subseteq f(J), f$ has fixed point by $lemma$ 2.2.3, for any given $n > 1$ there exist a point $x \in J$ such that $f^n(x) = x$ and $f^k(x) \in K$ for $0 < k < n$. Evidently x has period n.■

Lemma 2.2.21: *If f has a periodic point of odd period $n > 1$, then f^2 is strictly turbulent and f^m is also strictly turbulent for every $m \geq n$.*

Proof: We may suppose n chosen so that f does not have an orbit of odd period strictly between 1 and n. Then by *proposition 2.2.11* the orbit of period n has the form:

$$f^{n-1}(x) < f^{n-3}(x) < \ldots < f^2(x) < x < f(x) < \ldots < f^{n-2}(x),$$

or its mirror image. Without loss of generality, assume it has the form displayed. Choose d between x and $f(x)$, so that $f(d) = x$ and hence $d < f^2(d)$. Choose a between $f^{n-1}(x)$ and $f^{n-3}(x)$, so close to $f^{n-1}(x)$ that $f^2(a) > d$. Choose b between a and $f^{n-3}(x)$, so close to $f^{n-3}(x)$ that $f^2(b) < a$, and choose c between $f^{n-3}(x)$ and d, so close to $f^{n-3}(x)$ that $f^2(c) < a$. Then $J = [a, b]$, $K = [c, d]$ are disjoint and:

$$f^2(J) \supseteq \left[f^2(b), f^2(a)\right] \supseteq [a, d],$$

$$f^2(K) \supseteq \left[f^2(c), f^2(d)\right] \supseteq [a, d].$$

Thus f^2 is strictly turbulent. Now choose $e \in (f^{n-1}(x), f^{n-3}(x))$ so that $f(e) = d$. If we set $\widehat{J} = [f^{n-1}(x), e]\,,\ \ \widehat{H} = [x, d]$, then

$$f^n(\widehat{J}) \supseteq \left[f^{n-1}(x), f^{n-2}(x)\right]$$

and

$$f^n(\widehat{H}) \supseteq f^{n-1}[x, d] \supseteq \left[f^{n-1}(x), f^{n-2}(x)\right].$$

Thus f^n is strictly turbulent.

Moreover since

$$f\left[f^{n-1}(x), f^{n-2}(x)\right] \supseteq \left[f^{n-1}(x), f^{n-2}(x)\right]$$

f^m is strictly turbulent for every $m \geq n$.■

Lemma 2.2.22: *Suppose f is not turbulent let $c \in I$ and $n > 1$:*

$$\textit{If} \quad f^n(c) \leq c < f(c), \textit{then} \quad f(x) > \forall x \in [f^n(c), c]$$

$$\textit{If} \quad f(c) < c \leq f^n(c) \qquad \textit{then} \quad f(x) < x \forall x \in [c, f^n(c)].$$

Proof: Assume, on the contrary, that the interval $[f^n(c), c]$ contains a fixed point of f, and let z be the greatest. Then $f(x) > x$ for $z < x \leq c$. For a unique integer $m; (2 \leq m \leq n)$, we have

$$z < f^j(c), 0 \leq j \leq m, f^m(c) \leq z.$$

Then $c < f^{m-1}(c)$, since $f^m(c) \leq z < f^{m-1}(c)$. If we had $f^j(c) < f^{m-1}(c) < f^{j+1}(c)$ for some $j, 0 \leq j \leq n-2$, then $J = [z, f^j(c)]$ and $K = [f^j(c), f^{m-1}(c)]$ would satisfy

$$J \cup K \subseteq f(J) \cap f(K),$$

which is contrary to hypothesis. Consequently $f^j(c) < f^{m-1}(c)$ implies $f^{j+1}(c) < f^{m-1}(c)$. Since $c < f^{m-1}(c)$, this yields the contradiction $f^{m-1}(c) < f^{m-1}(c)$.■

We will say that a point $x \in I$ is a $U-point$ if $f(x) > x$ and a $D-point$ if $f(x) < x$.

Lemma 2.2.23: *Suppose if f is not turbulent and let $c \in I$. Then all $U-points$ in the orbit of c lie to the left of all $D-points$ in its orbit. If the orbit contains a fixed point, then it lies to the left of all $D-points$ and to the right of all $U-points$.*

Proof: If $f^j(c) \leq f^k(c) < f^{k+1}(c)$ for some j, k with $j > k$ then $f^j(c) < f^{j+1}(c)$, by *Lemma 2.2.22.* Similarly if $f^{j+1}(c) < f^j(c)$. It follows that any $U-points$ $f^j(c)$ lies to the left of any $D-points f^k(c)$, regardless of whether $k > j$ or $k < j$. If $f^h(c)$ is a fixed point then $f^i(c) = f^h(c)$, $\forall\ i > h$ and argument applies.■

Theorem 2.2.24: *Suppose that, for some $c \in I$ and some $n > 1$, $f^n(c) \leq c < f(c)$.*

If n is odd, then f has a periodic point of period q, for some odd q satisfying $1 < q \leq n$.

If n is even, then at least one of the following alternative holds:

(i) *f has a periodic point of period q for some odd q satisfying $1 < q \leq \frac{n}{2} + 1$,*

(ii) *$f^k(c) < f^j(c)$ $\forall$ even k and all odd j with $0 \leq j, k \leq n$.*

Proof: If $n = 2$ then (ii) holds. We assume $n > 2$ and the theorem holds $\forall$ smaller value of n (also when the inequalities for the points in the trajectory of c are all reversed). We may assume also that f is not turbulent, since otherwise f has periodic points of arbitrary period. Let $x_k = f^k(c), k \geq 0$. Then x_n is not a fixed point, by *Lemma 2.2.23.* Hence each point x_k, $0 \leq k \leq n$ is either a $U-point$ or a $D-point$, and both types occur. Moreover, if x_i is the greatest $U-point$ and x_j the least $D-point$ then $x_i < x_j$, again by *Lemma* $2, 2, 23$. Since $x_{i+1} \geq x_j$ and $x_{j+1} \leq x_i$, the interval (x_i, x_j) contains a fixed point of f. Let z be the least fixed point of f in this interval. Then $f(x) > x$ for $x_n \leq x < z$, by *Lemma 2.2.22.* If n is odd, or if n is even and (ii) does not hold, there is an $h < n$ such that x_h and x_{h+1} are both $\leq x_i$ or both $\geq x_j$. Evidently $x_h \neq x_j, x_i$. Let J be the interval $[x_h, x_i]$ if $x_h < x_i$ and the interval $[x_j, x_h]$ if $x_h > x_j$. Also for $k = 0, 1, 2, ..., n$, $k \neq h$, let J_k be the interval $[x_k, z]$ if $x_k \geq x_i$ and the interval $[z, x_k]$ if $x_k \geq x_j$. Then it is easily verified that $J_{k+1} \subseteq f(J_k), 0 \leq k < n$. Since $J_0 \subseteq J_n$, there exists a point $y \in J_0$ such that $f^n(y) = y$, $f^k(y) \in J_k, 1 \leq k < n$, by *Lemma 2.2.3.*

Then y has period m, where m divides n. Assume $m = 1$. Then y is a fixed point and it belongs to the intersection of all J_k. But J_h and J_i have at most the endpoint x_i in common, and J_h and J_j have at most the endpoint x_j in common. Hence we must have $1 < m \le n$. If n is odd then m is also odd, and thus the theorem is proved for this case. Suppose n is even. The periodic point y is $U-point$, since $y \in [x_0, z)$. If y_s is the greatest $U-point$ and y_t the least $D-point$ in the orbit of y then $y_s < y_t$, by *Lemma 2.2.23.* Moreover $f(x) > x$ for $y \le x \le y_s$, by *Lemma 2.2.22*, and hence $z > y_s$. On the other hand $z < y_t$, since y_t is a $D-point$. Since $f^h(y)$ and $f^{h+1}(y)$ lie on the same side of z, by construction the alternative (ii) does not hold for the orbit of y. Thus we are reduced to showing that if c itself is periodic with even period $n > 2$ then either (i) or (ii) holds. Suppose first that $n = 4$. if $x_3 < x_0 < x_1$, or $x_0 < x_1 < x_2$ or $x_0 < x_3 < x_2$, then f has a periodic point of period 3 by what we have already proved (with the inequality reversed in the last case). Suppose next that $\frac{n}{2}$ is odd. For any k let $k' = k + \frac{n}{2}$. Assume that, for some $k, x_{k'}$ and x_k lie on the same side of z. If x_k is closer than $x_{k'}$ to z, then either $x_{k'} < x_k < x_{k+1}$ or $x_{k'} > x_k > x_{k+1}$. Hence f has a periodic point of odd period $q(1 < q \le \frac{n}{2})$, by what we have already proved. If $x_{k'}$ is closer than x_k to z the same conclusion holds, since $x_k = x_{k+n}$. Thus we may assume that $x_{k'}$ and x_k are on opposite sides of z for every k. Without loss of generality we may also assume that c is the greatest $U-point$ in its orbit. Then $x_{1+\frac{n}{2}} < x_0 < x_1$. Hence by induction hypothesis, f has a periodic point of odd period $q(1 < q \le \frac{n}{2})$ unless $x_0, x_1, ..., x_{1+\frac{n}{2}}$ are all less that z and $x_1, x_3, ..., x_{\frac{n}{2}}$ all greater. But then $x_{\frac{n}{2}}, x_{2+\frac{n}{2}}, ..., x_{n-1}$ are all greater than z and $x_{1+\frac{n}{2}}, x_{3+\frac{n}{2}}, x_{n-2}$ are all less than z. Thus (ii) holds. Suppose finally that $\frac{n}{2}$ is even and $n > 4$. As in the previous case we may assume that x_k and $x_{k'+1}$ are on opposite sides of z for every k. But then x_k, x_{k+2} are on the same sides of z for every k, and hence (ii) holds.■

A trajectory $\{f^n(c)\}$ will be said to be alternative if either $f^k(c) < f^j(c)$ for all even k and all odd j or $f^k(c) > f^j(c)$ for all even k and all odd j.

Theorem 2.2.25: *If n is not a power of 2, the following statements are equivalent:*

(i) *f has a periodic point of period n.*

(ii) *f^n is strictly turbulent.*

(iii) *f^n is turbulent.*

Proof: Suppose first that n is odd. This implies $n > 1$, since n is not a power of z, then $(i) \implies (ii)$, by *Lemma 2.2.21*, and $(ii) \implies (iii)$ is trivial. It remain to show that $(iii) \implies (i)$. By *Lemma 2.2.19* we assume that there exist point $a < c < b$ such that:

$$f^n(c) = b, f^n(b) = f^n(a) = a.$$

If $f(a) \neq a$ then f has a periodic point of period n, by *Sarkovskii's theorem.* Thus we may suppose that $f(a) = a$. If $f(c) < c$ then f has aperiodic point of period, by *theorem 2.2.24* and *sarkoviskiv's theorem.* Thus we may suppose that $f(c) > c$. Since $a = f^{2n}(c) < c$ and $f(a) = a$, it follows from *Lemma 2.2.22* that f is turbulent. Hence f has periodic points of every period by *Lemma 2.2.20 .* Suppose next that $n = 2^d q$, $q > 1$ is odd and $d \geq 1$. Then the equivalence of (i),(ii) and (iii) for f follows from their equivalence for f^{2d}, by *Lemma* 2.2.13 and *Sarkoviskii's theorem.* ■

It is an immediate consequence of *Theorem 2.2.25* and *Lemma 2.2.20* that the following conditions are equivalent:

(i) *f has a periodic point whose period is not a power of 2.*

(ii) *f^m is strictly turbulent for some positive integer m.*

(iii) *f^n is turbulent for some positive integer n.*

Definition 2.2.26: *The map f will be said to be chaotic if one, and hence all three, of the above conditions is satisfied.*

We will denote this definition by *t-chaos.*

2.3 Lyapunove Chaos

The *Lyapunove* (or *Liapunove*) exponents measure the exponential rate at which nearby orbits are moving apart. In this section we give a precise definition and calculate the exponents in two examples and give a definition of chaos in sense of *Liapunove*.

Definition 2.3.1: *Let $f : R \to R$ be a continuous and differentiable map. Then $\forall x \in R$ we define the liapunove exponent of x say $\lambda(x)$ as:*

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log|f'(x_i)| \ , \ \forall x_i \in R$$

Definition 2.3.2:[14] *Consider the continuous and differentiable map $f : R \to R$ then f is said to be chaotic according to liapunove or $L-chaotic$ if:*

(i) *f is topologically transitive,*

(i) *f has a positive liapunove exponent.*

Remark 2.3.3: In a set of positive measure liapunove exponent can be found from the relation:

$$\lambda(x) = \int \log|f'(x)| \, \rho(x)dx$$

where $\rho(x)$ is the invariant measure (if f is ergodic then $\rho(x)$ is unique [14]).

Remark 2.3.4: In higher dimensions, for example in $\mathbb{R}^n$ the map will have n liapunove exponents, say: $\lambda_1, \lambda_2, ..., \lambda_n$ for a system of n variables. Then the map is $L-chaotic$ if the maximum liapunove exponent is positive i.e. $\max\{\lambda_1, \lambda_2, ..., \lambda_n\} > 0$.

In [16] (pages 21-23) it is explained a way how to calculate this exponent.

Proposition 2.3.5: *Consider the continuous and differentiable map f :R→ R. If f has a positive liapunove exponent then f has also sensitive dependence on initial conditions.*

Proof: See *Proposition 2.1.16.*

Proposition 2.3.6: *Every expanding map $f : R \to R$ has sensitive dependence on initial conditions.*

Proof: See $Proposition$2.1.17.

Example 2.3.7: Let $F_\mu(x) = \mu x(1 - x)$ for $\mu \geq 2 + \sqrt{5}$. Let be the invariant cantor set. Then for $x_0 \in \Lambda_\mu$, $\log(|F'(x_j)|) \geq \lambda_0 > 0$ for some λ_0. Thus we may not know an exact value, but it is easy to drive an inequality and know that the exponent is positive.[14]

2.4 Robinson's Chaos

Clark Robinson in his book [14] defined the chaos same as devany's definition but without condition two in devaney's definition i,e the map is chaotic if f has topologically transitive and has sensitive dependence.

Definition 2.4.1: *A map f on a metric space X is said to be chaotic on an invariant set Y if*

(i) *f is transitive on Y,*

(ii) *f has sensitive dependence on initial condition on Y.*

We will denoted this definition by $R - chaos$.

The question now is: why Robinson left out condition two in devany's definition in his definition. The answer can be found in Remark 5.2 in chapter"III" in his book.

Remark 2.4.2: In remark 5.2 in Robinson book he wrote: " Devany's gave an explicit definition of a chaotic invariant set in an attempt to clarify the notion of chaos. To our two assumptions, he adds the assumption that the periodic points are dense in Y. Although this property is satisfied by "uniformly hyperbolic" maps like the quadratic maps, it does not seem that this condition is at the heart of the idea that the system is chaotic. Therefore we leave out condition about periodic points in our definition of chaos".

Example 2.4.3: Consider the continuous map $f : X \to X$ defined by $f(e^{i\theta}) = e^{2i\theta}$ and $X = s^1 \backslash \left\{ e^{\frac{2\pi pi}{q}} : p, q \in \mathbb{Z}, q \neq 0 \right\}$ is a metric space equipped with the arc length metricized.

We showed that this example in "example 2.1.5" is not $D-chaotic$ but it $R-chaotic$ since the function f has sensitive dependence and topologically transitive, but has no (dense) periodic points.

2.5 Wiggins' Chaos

Stephen Wiggins defined chaos as Robinson in his books like: Introduction to Applied Nonlinear Dynamical systems and chaos [19] and chaotic transport in dynamical system[17]. I added this definition because many authors defined chaos in sense of *Wiggins.*

Definition 2.5.1: *"Wiggins' definition of chaos": Let $f : X \to X$ be a continuous map. Then the map f is said to be chaotic in sense of Wiggins or W - chaotic if :*

(1) *f is topologically transitive,*

(2) *f exhibits sensitive dependence on initial condition.*

Example 2.5.2: In [9] it is proved that the quadratic map $Q : [0,1] \to [0,1]$ given by

$$Q(x) = 4x(1-x)$$

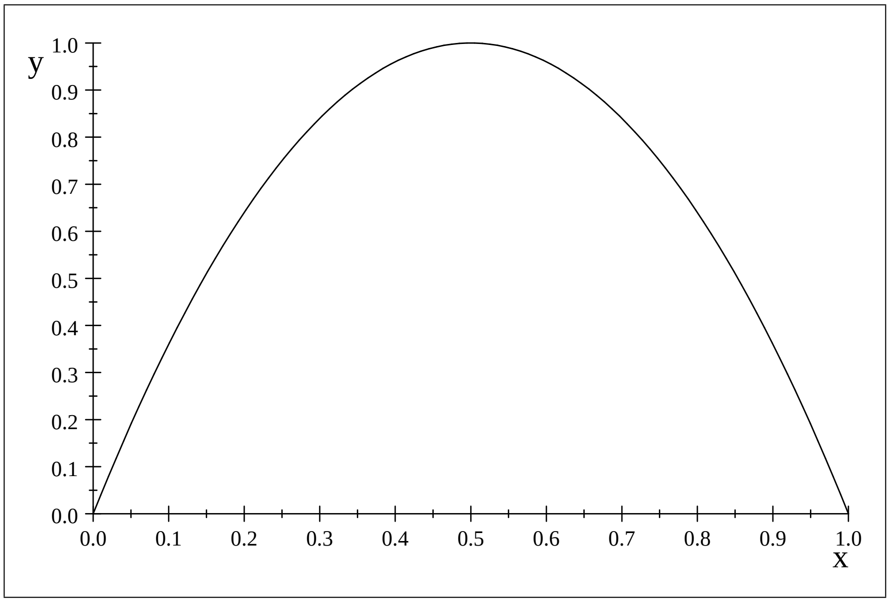


Figure 2.5.1: The graph of $Q(x)$

is D-chaotic, R-chaotic and W-chaotic: We will use this result to investigate the chaotic behavior of the map $G(\rho, \theta) = (4\rho(1-\rho), \theta + 1)$ using the polar coordinates (ρ, θ). The map G is defined on disk $D(0,1) = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$. After a finite number of iterations of the map G the image of a small disk in $D(0,1)$ will contain an open set $U \subset D(0,1)$ with a full radius. Also the rotation of 1 radian will spread U totally over $D(0,1)$ after a finite number of iterations. So G is transitive [20]. Now since the quadratic map F is sensitive on $[0,1]$ then G is also sensitive. Finally G has only a fixed point in the origin and does not have any periodic orbit of period $p > 1$. Basically G shrinks or stretches the distance of every point of $D(0,1)$ from the origin while rotating by angle of 1 radian. Since $\frac{1}{\pi}$ is irrational, no point x_n that belong to the orbit of x_0 can come back to the same ray which contains x_0. Hence G has no dense periodic points. So G is $W-chaotic$ but not $D-chaotic$.

2.6 Touhey's Chaos

The purpose of his article [12] is to introduce yet another definition of chaos. Simply: a map $f : X \to X$ is chaotic on X if every pair of non-empty open subsets of X shares a periodic orbit. In the same article he shows that the new definition of chaos is equivalent to the definition given by Devaney.

Definition 2.6.1: *Given a metric space X and continuous mapping f :X → X, we say that f is transitive if for any two non-empty open subsets U and V of X there exist $u \in U$ and a non-negative integer k such that $f^k(u) \in V$, that is every pair of non-empty open subsets of X shares a forward orbit.*

Although this is not the usual definition of topologically transitive but it equivalence to it.

Definition 2.6.2: *Given a metric space X and continuous mapping f:X → X, we say that f is chaotic or T-Chaotic on X if given U and V, non-empty open subsets of X, there exists a periodic point $p \in U$ and non-negative integer k such that $f^k(p) \in V$, that is every pair of non-empty open subsets of X shares a periodic orbit.*

Now we will show that this definition of chaos is equivalent to Devaney's definition of chaos .

Proposition 2.6.3[12] : *$f : X \to X$ is $T-chaotic$ if and only if f is $D-chaotic$.*

Proof : ($\Longrightarrow$)If f is *T-chaotic* on X then every pair of non-empty open sets shares a periodic orbit. In particular, every non-empty open set must contain a periodic point so the periodic points of f are dense in X. The transitivity of f follows from the definition of *T-chaos* since every pair of non-empty open sets shares a forward orbit.

($\Longleftarrow$)Now let f is $D-chaotic$. Given any pair of non-empty open sets $U, V \subset X$ transitivity ensures that there exists $u \in U$ and non-negative integer k such that $f^k(u) \in V$. Now define $W = f^{-k}(V) \cap U$. Note that W is open and non-empty since it is the intersection of two open sets and u is an element of both of them. It is also clear that W has the property that $f^k(W) \subset V$. But the periodic points of f are assumed to be dense in X, so the non-empty open set W must contain a periodic point p. Thus we have shown that there exists a periodic point $p \in W \subset U$ with the property that $f^k(p) \in f^k(W) \subset V$.■

2.7 Experimentalists' Chaos

According to many non-mathematician, particularly physical scientists, a dynamical systems $x_{n+1} = f(x_n)$ is chaotic in an invariant set X if f has in X sensitive dependence on initial conditions. Therefore, we may obtain very different orbits from two almost identical starting points. It follows that the evolution of the system is unpredictable, since it is practically impossible to know the initial conditions exactly. This is obviously an important feature of the experimentalists's chaos. An additional mevit is that sensitive dependence on initial conditions can be checked numerically.[8]

Definition 2.7.1: *A map $f : X \to X$ is experimentalist's chaos or SD-chaotic if f has in X sensitive dependence on initial conditions.*

However, despite the advantages, this definition of chaos is not satisfactory. The following example illustrates some of the problems which may arise.

Example2.7.2: Let $D = \{x \in \mathbb{R}^2 : \|x\| \leq 2\}$. Using polar coordinates define $F : D \to D$ by:

$$F(x) : F(\rho, \theta) = (\rho, \theta + \rho).$$

Notice that for every $\rho \in (0, 2]$ the set $C_\rho = \{x \in \mathbb{R}^2 : \|x\| = \rho\}$ is invariant and the dynamical system defined by F is a rotation in C_ρ. Consequently, it does not seem appropriate to label the system as chaotic in the invariant set C_ρ. However, the system has in C_ρ sensitive dependence on initial conditions with $r_0 = \rho$. In fact, let $x_0 = (\rho_0, \theta_0)$ and $d > 0$. Choose n so large that $\frac{\pi}{n} < d$ and $\rho_0 - \frac{2\pi}{n} > 0$. Let $y_0 = \left(\rho_0 - \frac{\pi}{8}, \theta_0\right)$ Then

$$\|x_0 - y_0\| < d$$

and:

$$x_n = (\rho_0, \theta_0 + n\rho_0);$$

$$y_0 = (\rho_0 - \frac{\pi}{2}, \theta_0 + n\rho_0 - \pi).$$

Consequently $\|x_n - y_n\| > r_0$ and the system is chaotic is C_ρ for every $\rho \in (0, 2]$. However, the system is non chaotic in the Disk D. We do not seem to have a satisfactory situation.[8]

Remark 2.8.2: In [21] Gulick in his book defined chaos as sensitivity sense.

2.8 Knudsen's Chaos

In his paper " *Chaos without Nonperiodicity* ", *Carsten Knudsen* proposed a new definition of chaos. The aim of this paper is to prove that sensitive dependence on the initial conditions and topological transitivity are stable properties under closure, as well as under restriction to dense invariant subsets. The consequence of the results is that chaos according to devaney may exist on bounded but non compact spaces without any non-periodic orbits.

Definition 2.8.1: *Let f be a continuous transformation of a bounded metric space X.* If *f has a dense orbit in X and f has sensitive dependence on initial conditions, then f is said to be $K-chaotic$.*

2.9 P-chaos

Let f be a continuous map from a compact metric space X to itself. The map f is called to be $P-chaotic$ if it has the pseudo-orbit-tracing property and the closure of the set of all periodic points for f is equal to X. [22]

Definition 2.9.1: *A continuum is a nondegenrate compact connected metric space.*

Definition 2.9.2: *A continuos map f from a nondegenerate compact metric space X to itself has the specification property if for any $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that for any $K \geq 2$, for any K points $x_1, x_2, ..., x_k \in X$, for any nonnegative integers $a_1 \leq b_1 < a_2 \leq b_2 < ... < a_k \leq b_k$ with $a_i - b_{i+1} \geq M$ for each $i = 2, 3, ..., K$ and for any $p \geq M + b_k - a_1$, there exists a points $y \in X$ such that $f^p(y) = y$ and $d(f^n(y), f^n(x_i)) \leq \varepsilon$ for all $a_i \leq n \leq b_i, 1 \leq i \leq K$.*

Definition 2.9.3:

$*$*Let* f *be a continuous map from a compact metric space* X *to itself. A sequence of points* $\{x_i : i \geq 0\}$ *is called a* $\delta - pseudo - orbit$ *for* f *if* $d(f(x_i), x_{i+1}) < \delta$ *for each* i.

$*$ *A sequence* $\{x_i : i \geq 0\}$ *is said to be* $\varepsilon{-}traced$ *by* $x \in X$ *if* $d(f^i(x), x_i) < \varepsilon$ *holds for each* $i \geq 0$.

$*$*A map* f *is said to have the* $pseudo - orbit - tracing - property$ *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that each* $\delta - pseudo - orbit$ *for* f *is* $\varepsilon - traced$ *by some point of* X.

Theorem 2.9.4: *Let* f *be continuous map from non-degenerate compact metric space* X *to itself and has the specification property then* f *is topologically mixing and the set of all periodic points for* f *is dense and f has positive topological entropy.*

Proof: See [29].(for the case $X = I$ see the next chapter)

Theorem 2.9.5: *Let* f *be continuous map from the unit interval to itself, if* f *is topologically mixing, then* f *has the specification property.*

Proof: See [27] or [64].(see next chapter section 6)

Theorem 2.9.6: *Let* $f : X \to X$ *be a homeomorphism of compact metric space. If* f *is topologically mixing, expansive, and has pseudo-orbit-tracing-property then* f *has specification property.*

Proof: Let $C > 0$ be an expansive constant. For ε with $0 < \varepsilon < \frac{c}{2}$ let $\delta > 0$ be a number in the definition of $pseudo - orbit - tracing - property$. Let $U = \{U_i\}$ be a finite open cover of X such that each U_i has the diameter $> \delta/2$. Since f is topologically mixing for $U_j, U_i \in U$ there is $M_{j,i} > 0$ such that for all $n > M_{i,j}$, $f^n(U_i) \cap U_j \neq \phi$. Let $M = \max\{M_{i,j}\}$ and take any finite points $x_1, x_2, ..., x_k \in X$ and any integers

$$a_1 \leq b_1 < a_1 \leq b_2 < ... < a_k b_k$$

satisfying

$$a_j - b_{j-1} \geq M \quad \text{for} \;\; 2 \leq j \leq K$$

and an integer

$$p \geq M + (b_k - a_1).$$

Define

$$a_{k+1} = b_{k+1} = p + a_1$$

$$x_{k+1} = f^{a_1 - a_{k+1}}(x_1).$$

For any $z \in X$ we denote as $U'(z)$ an open ball U' in U satisfying $z \in U$. Since $a_{j+1} - b_j \geq M$, by topologically mixing we have:

$$U(f^{a_{j+1}}(x_{j+1})) \cap f^{a_{j+1}-b_j}(U'(f^{b_j}(x_j))) \neq \phi$$

form which there is $y_j \in U'(f^{b_j}(x_j))$ such that

$$f^{a_{j+1}-b_j}(y_j) \in U'(f^{a_{j+1}}(x_{j+1})).$$

Define a sequence $\{z_i\}$ in X by:

$$z_i = f^i(x_j), a_j \leq i \leq b_j$$

$$z = f^{i-b_j}(y_j), b_j \leq i < a_{j+1}$$

$$z_i + p = z_i, i \in \mathbb{Z}.$$

Then $\{z_i\}$ is a $\delta - pseudo - orbit$. Since f has $pseudo - orbit - tracing - property$. There is $x \in X$ such that:

$$d(f^i(x), x_i) < \varepsilon \text{ for all } i \in \mathbb{Z}.$$

Since $z_i + p = z_i$ for $i \in \mathbb{Z}$, we have $d(f^{i+p}(x), z_i) < \varepsilon$ for $i \in \mathbb{Z}$ and so

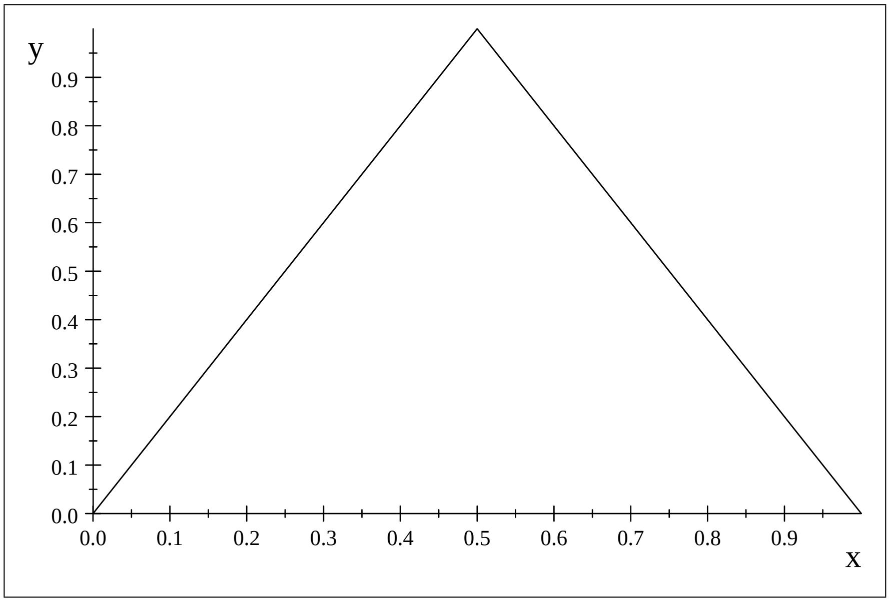
$$d(f^i \circ f^p(x), f^i(x)) < 2\varepsilon < C.$$

By expansivity we have $f^p(x) = x$, Therefore f has specification.■

Definition 2.9.7: *A continuous map f from a compact metric space X to itself is said to be $P-chaotic$ if f has $pseudo-orbit-tracing-property$ and $\overline{P(f)} = X$.*

Example 2.9.8: Tent map is $P-chaotic$. We show that T which defined by:

$$T(x) = \begin{cases} 2x & , \quad 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & , \quad \frac{1}{2} < x \leq 1 \end{cases}$$



Figute 2.9.1: Tent map

has the *pseudo − orbit − tracing − property*. Let $\varepsilon > 0, n \in \mathbb{N}$ and let $\{x_i : i \geq 0\}$ be a $\varepsilon/4 - pseudo - orbit$ for f. Since

$$\overline{(B(x_{i+1}\varepsilon/2))} \subset \overline{(B(f(x_i), \varepsilon))}$$

$$= f(\overline{(B(x, \varepsilon/2))}) \neq \phi,$$

we have

$$f^{-1}(\overline{(B(x_{i+1}, \varepsilon/2))}) \cap \overline{(B(x_i, \varepsilon/2))}.$$

Thus, there exists a point

$$x \in \bigcap_{i=0}^{n} f^{-i}(\overline{(B(x_i, \varepsilon/2))}) \neq \phi$$

and $\{x_i, i \geq 0\}$ is $\varepsilon - traced$ by x. Hence we see that f has the *pseudo − orbit − tracing − property*. Since for each open set U there exists K such that

$$f^k(U) = [0,1], \text{we have } \overline{(p(f))} = [0,1].$$

Corollary 2.9.9: *Let f be a $P-$chaotic map from a continuum (X,d) to itself and $\varepsilon > 0$. There exists $M \in \mathbb{N}$ such that for each $x, y \in X$ and each $K \geq M$, $d(x, z_k) < \varepsilon$ and $d(y, f^k(z_k)) < \varepsilon$, for some periodic point $z_k \in X$.*

Proof: See *Lemma 3.2* and *corollary 3.3* in [22].

Corollary 2.9.10: *Every P - chaotic map from a continuum to itself is mixing.*

Proof: Let f be a $P-$*chaotic* map from a continuum X to it self and let U, V be non-empty open subset of X. There exist $x \in U, y \in V$ and $\varepsilon > 0$ such that $B(x,\varepsilon) \subset U$ and $B(y,\varepsilon) \subset V$. By *corollary 2,2.9,* we have $M \in \mathbb{N}$ such that for each $K \geq M$, there exists $z_k \in X$ such that $d(x, z_k) < \varepsilon$ and $d(y, f^k(z_k)) < \varepsilon$. We see that $f^k(U) \cap V \neq \phi$ for all $K \geq M$, thus f is mixing.

Proposition 2.9.11: *Let f be a continuous map from a compact metric space X to it self. If f is $P-$chaotic then f^k is $P-$chaotic for each $K > 0$. Moreover if f^k is $P-$chaotic for some $K > 0$, then f is $P-$chaotic.*

Proof: See *proposition 3.1* in [22].

2.10 Martelli's Chaos

The next definition of chaos appears in Martellie's book " An Introduction To Discrete Dynamical System And Chaos"[18].

Definition 2.10.1: *The orbit of a point $x \in X$ is said to be unstable if there exists $r > 0$ such that for every $\varepsilon > 0$ there are $y \in X$ and $n \geq 1$ satisfying the two inequalities*

$$d(x, y) < \varepsilon, d(f^n(x), f^n(y)) > r.$$

Definition 2.10.2: *The map f is chaotic in the sense of Martelli if there exists $x_0 \in X$ such that:*

[i] The orbit of x_0 is dense in X.

[ii] The orbit of x_0 is unstable.

Example 2.10.3: Let $B(x) = 2x - [2x] = \begin{cases} 2x & , \quad 0 \leq x < \frac{1}{2} \\ 2x - 1 & , \quad \frac{1}{2} \leq x < 1 \\ 2x - 2 & , \quad x = 1 \end{cases}$

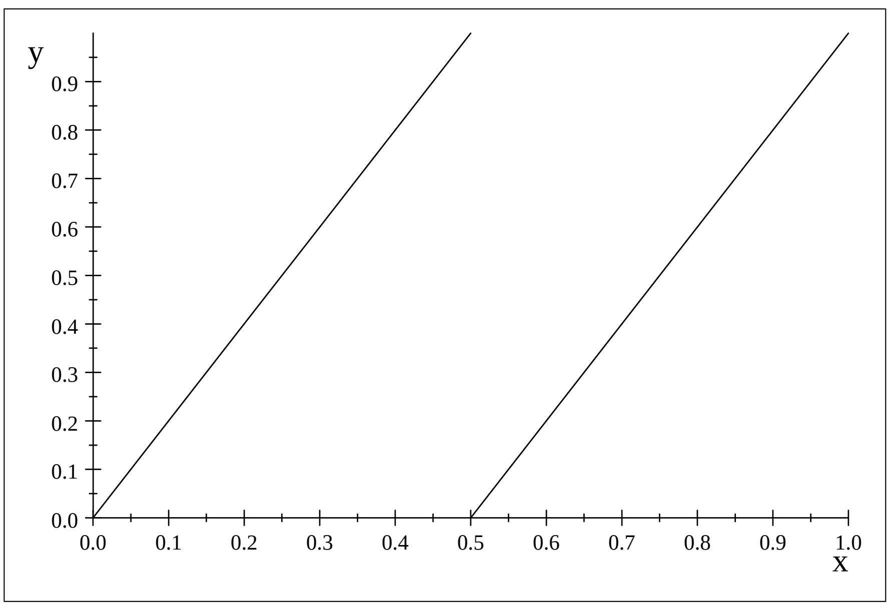


Figure 2.10.1 : $B(x) = 2x - [2x]$

where $[2x]$ denotes the greatest integer less that or equal to $2x$. Notice that B maps $[0, 1]$ into itself and it is discontinuous at $x = 0.5$ and $x = 1$.

The action of B and its iterates on the elements of $[0,1]$ is better understood if we write them with their binary expansion. Then, for $x \in [0, 0.5)$ we have

$$x = 0.0a_2a_3...,$$

while for $x \in [0.5, 1)$ we have

$$x = 0.1a_2a_3...,$$

where $a_i, i = 2, 3; ...,$ are either 0 or 1. In both cases we obtain

$$B(x) = 0.a_2a_3... \ .$$

Now can easily see that the orbit of

$$x_0 = 0.0100011011000001010100...$$

has the property $\overline{O(x)} = [0,1]$. Moreover $O(x_0)$ is unstable since $B'(x) = 2$ for $x \neq 0, 1$. Hence $B(x)$ is chaotic in the sense of Martilli.

We will denote the chaos in the sense of *Martilli* by $M-chaotic$ function.

Theorem 2.10.4: (Theorem 4.1 in [8]) : *Let $X \subset \mathbb{R}^q$ be closed and bounded and $F : X \to X$ be continuous. Then F is topologically transitive in X iff there exists $x_0 \in X$ such that $\overline{O(x_0)} = X$.*

Remark 2.10.5: We showed in *example 1.2.16* that the above theorem is not true over compact metric space, the space must be *thick*, and complete metric space with a *countable base*. But it is still true over closed bounded interval of $\mathbb{R}$.

Theorem 2.10.6: *Let $x_0 \in X$ be such that* $\overline{O(x_0)} = X$ and $f : X \to X$ be continuous*. Then f has sensitive dependence on initial conditions if and only if $O(x_0)$ is unstable with respect to X.*

Sketch of the proof: Sensitivity to initial conditions with respect to X clearly implies that $O(x_0)$ is unstable with respect to X. Given $y_0 \in X$ and $d > 0$, determine an iterate x_n of x_0 such that $\|x_n - y_0\| \leq d/2$. This can be done since $\overline{O(x_0)} = X$. Next one shows that for every $n > 1$ the orbit $O(x_n)$ has the same instability constant of $O(x_0)$, i.e. $r(x_n) = r(x_0)$. It follows that other some iterate y_p of y_0 is at least as far as $r(x_0)/3$ from x_{n+p}, or this separation happens for some iterate z_p of a point z_0 which is closer than d to both y_0 and x_n. In either case, we obtain that $r(y_0) \geq r(x_0/3)$.

2.11 Block-Coppel's Chaos

Now we return to "*Dynamics In One Dimension*" by Block and Coppel to give a new definition of chaos in sense of them and we will denoted it by $BC - Chaos$.

Definition 2.11.1: *A continuous map $f :\ X\rightarrow\ X$ on a compact metric space X is called chaotic in the sense of block and Coppel or BC-Chaotic if there exist an $m \in \mathbb{N}$ and a compact $f^m - invarint$ subset Y of X such that $f^m|_Y$ is semi-conjugate to the shift on $\sum$,i.e. if there exists a continuous surjection $h : Y \rightarrow \sum$ satisfying:*

$$h \circ f^m = \sigma \circ h \quad on \quad Y.$$

The next definition is equivalent to the above definition.[see 4] or [44].

Definition 2.11.2: *The map $f : X \rightarrow X$ is chaotic in sense of block and Coppel if there exist disjoint closed subsets X_0, X_1 of X and a positive integer m such that, if $\tilde{X} = X_0 \cup X_1$ and $g = f^m$ then:*

[*i*] $g(\tilde{X}) \subset \tilde{X}$,

[ii] *for every sequence* $\alpha = (a_0, a_1, a_2, ...)$ *of 0's and 1's there exist a point* $x = x_\alpha \in \tilde{X}$ *such that* $g^k(x) \in X_{a_k}$ *for all* $k \geq 0$.

Remark 2.11.3: If $m = 1$ in definition 2.11.1 then this notion of chaos is also known as chaos in the sense of *Coin Tossing* [45].

Remark 2.11.4: Note every $BC-chaotic$ map is chaotic in the sense of *Coin Tossing*. The next example shows that.

Example 2.11.5: (*example 2.2.5 in 46*) Consider the subset $T =$

$$\{(0, a_0, 0, a_1, 0, ...)|(a_0, a_1, ...) \in \sum\}$$

$$\cup$$

$$\{(a_0, 0, a_1, 0, a_2, ...)|(a_0, a_1, a_2, ...) \in \sum\}$$

of $\sum$. This set is $\sigma - invariant$ and the map $\sigma^2|_T$ is semi-conjugate to $\sigma : \sum \rightarrow \sum$. However, there is no $\sigma - invariant$ subset W of T such that $\sigma|_W$ is semi-conjugate to $\sigma : \sum \rightarrow \sum$. For more details see example 2.2.5 in [46].

Theorem 2.11.6: *Let* (X, d) be a *compact metric space suppose* $f : X \rightarrow X$ *is continuous then* f *is* $BC-chaotic$ *if and only if* f^n *is* $BC-chaotic$.

Proof: Let f be *BC-chaotic*, $m \in \mathbb{N}$, $Y \subset X$ compact and $f - invariant$ and let $f^m|_Y$ be semi conjugate to σ via $h : L \rightarrow \sum$. Then defining the continuous surjection $t : \sum \rightarrow \sum$ by:

$$(a_0, a_1, a_2, ...) \rightarrow (a_0, a_n, a_{2n}, ...)$$

and $\overline{h} = t \circ h$ we get

$$\overline{h} \circ (f) = \sigma \circ h \text{ on } Y$$

and f^n is $BC-chaotic$. The converse follows from thedefinition.$\blacksquare$

2.12 Li-Yorke's Chaos

We must remember that the use of the word chaos in Dynamical systems was introduced by *Li and Yorke*[9]. In their paper Li and Yorke proved that for a map on the real line which has a point with period three there exists an uncountable scrambled set.

Definition 2.12.1: *A continuous map $f : X \to X$ on a compact metric space (X, d) is called chaotic in the sense of Li and Yorke or "LY -chaotic" if there exists an uncountable subset S (called a scrambled set) of X with the following properties:*

$[i]$ $\limsup\limits_{n\to\infty} d(f^n(x), f^n(x)) > 0$, *for all* $x, y \in S, x \neq y$.

$[ii]$ $\liminf\limits_{n\to\infty} d(f^n(x), f^n(y)) = 0$, *for all* $x, y \in S, x \neq y$.

$[iii]$ $\limsup\limits_{n\to\infty} d(f^n(x), f^n(p)) > 0$, *for all* $x \in S, p \in X$ *where* p *is periodic.*

Condition $[iii]$ requires that for an orbit starting from a point in S does not approach asymptotically any periodic orbit (two orbits $\{x, f(x), f^2(x)\}$ and $\{y, f(y), f^2(y)\}$ approach asymptotically if $\lim\limits_{n\to\infty} |f^n(x) - f^n(y)| = 0$). The second condition requires that two arbitrary orbits starting from two different points in S can be close to each other but can not approach each other asymptotically. In [9] they proved the following theorem.

Theorem 2.12.2: *Let I be an interval and let $f : I \to I$ be a continuous map. If there exists $a \in I$ such that $b = f(a), c = f^2(a)$ and $d = f^3(a)$ satisfying $d \leq a < b < c$ (or $d \geq a > b > c$) then:*

[i] $\forall n = 1, 2, 3, ...$ *there exists a periodic point in* I *with period* n.

[ii] *There exists an uncountable scrambled set* $S \subset I$ *with no periodic points.*

Proof of [i]: Let K be a positive integer. For $K > 1$, let $\{I_n\}$ be the sequence of intervals $I_n = L$ for $n = 0, 1, ..., k-2$ and $I_{n+k} = H$, and define I_n to be periodic inductively, $I_{n+k} = I_n$ for $n = 0, 1, 2, ...$ If $k = 1$, let $I_n = L$ for all n. Let Q_n be the sets in the proof of *lemma 2.2.2*. Then notice that $Q_k \subset Q_0$ and $f^k(Q_k) = Q_0$ and so by *lemma* 2.2.1. $G = f^k$ has a fixed point p_k in Q_k. It is clear that p_k cannot have period less than k for f, other wise we would need to have $f^{k-1}(p_k) = b$, contrary to $f^{k+1}(p_k) \in L$. The point p is a periodic point of period k for f.

Proof of [ii]: Let μ be the set of sequences $M = \{M_n\}_{n=1}^{\infty}$ of intervals with

$$M_n = H \text{ or } \; M_n \subset L, \text{and } F(M_n) \supset M_{n+1} \qquad (12.1)$$

if $M_n = H$ then

$$n \text{ is the square of an integer and } \; M_{n+1}, M_{n+2} \subset L \; (12.2)$$

where $H = [a, b]$ and $L = [b, c]$. Of course if n is the square of an integer, then $n+1$ and $n+2$ are not, so the last requirement in (12.2) is redundant. For $M \in \mu$, let $p(M, n)$ denote the number of i's in $\{1, ..., n\}$ for which $M_i = H$. For each $r \in (3/4, 1)$ choose $M_r = \{M_n^r\}_{n=1}^{\infty}$ to be a sequences in μ such that

$$\lim_{n\to\infty} p(M^r, n^2)/n = r \qquad (12.3)$$

Let

$$\mu_0 = \{M^r : r \in (3/4, 1)\} \subset M.$$

Then μ_0 is uncountable since $M^{r_1} \neq M^{r_2}$ for $r_1 \neq r_2$. For each $M^r \in \mu_0$, by *Lemma 2.2.2* there exists a point x_r with $f^n(x_r) \in M_n^r$ for all n. Let $S = \{x_r : r \in (3/4, 1)\}$. Then S is also uncountable. For $x \in S$, Let $P(x, n)$ denote the number of i's in $\{1, ..., n\}$ for which $F^i(x) \in H$. We can never have $f^*(x_r) = b$, because then x_r would eventually have period 3, a contrary to (12.2).

Consequently

$$p(x_r, n) = P(M^r, n) \text{ for all } n$$

and so

$$d(x_r) = \lim_{n\to\infty} P(X_r, n^2) = r \text{ for all } r.$$

We claim that for $p, q \in S$, which $p = q$, there exist infinitely many n's such that $f^n(p) \in H$ and $f^n(q) \in L$ or vice verca. We may assume that $d(p) > d(q)$, then

$$P(p, n) - P(q, n) \to \infty$$

and so must be infinitely many n's such that $f^n(p) \in H$ and $f^n(q) \in L$. Since $f^2(b) = d \leq a$ and f^2 is continuous, there exist $\delta > 0$ such that

$$f^2(x) < (b + d)/2 \text{ for all } x \in [b - \delta, b] \subset H.$$

If $p \in S$ and $f^n(p) \in H$, then (12.2) implies $f^{n+2}(p) \in L$ and $f^{n+2}(p) \in L$. Therefore

$$f^n(p) < b - \delta.$$

If $f^n(q) \in L$ then $f^{n+2}(q) \geq b$ so $|f^n(p) - f^n(q)| > \delta$. For any $p, q \in S$, $p \neq q$, it follows

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| \geq \delta > 0.$$

This proves part of *Li -York theorem* (part [i] of the definition of chaos in the sense of Li and York). To prove $[ii]$ in LY - chaos. Since $f(b) = c, f(c) = d \leq a$, we may choose intervals $[b_n, c_n], n = 0, 1, 2, ...,$ such that

(a) $[b, c] = [b^0, c^0] \supset [b^1, c^1] \supset ... \supset [b^n, c^n] \supset ...,$

(b) $f(x) \in (b^n, c^n) \quad \text{for all } x \in (b^{n+1}, c^{n+1}) \quad ,$

(c) $f(b^{n+1}) = c_n \quad , \quad f(c^{n+1}) \ = \ b_n.$

Let

$$A = \bigcap_{n=0}^{\infty} [b^n, c^n], b^* = infA, c^* = supA,$$

then $f(b^*) = c^*$ and $f(c^*) = b^*$ because of (12.2). We must be more specific in our choice of the sequences M^r. In addition to our previous requirements on $M \in \mu$, we will assume that if $M_k = H$ for both $k = n^2$ and $(n+1)^2$ then

$$M_k = [b^{2n-(2j-1)} b^*] \text{ for } k = n^2 + (2j-1),$$

$$M_k = [c^*, c^{2n-2j}] \text{ for } K = n^2 + 2j \quad \text{where} \quad j = 1, .., n.$$

For the remaining k's which are not square of integers, we assume $M_k = L$. It easy to check that these requirements are consisted with (12.12.2) and (1), and that we can still choose M^r so as to satisfy (12.3). From the fact that $d(x)$ may be thought of as the limit of the fraction of n's for which $f^{n^2}(x) \in H$, it follows that for any $r^*, r \in (3/4, 1)$ there exist infinitely many n such that

$$M_k^r = M_k^{r^*} = H \quad \text{for both} \quad k = n^2 \text{ and } (n+1)^2$$

Let $x_r \in S$ and $x_{r^*} \in S$. Since $b^n \to b^*, c^n \to c^*$ as $n \to \infty$, for any $\varepsilon > 0$

there exists N with

$$|b^n - b^*| < \frac{\varepsilon}{2}, |c^n - c^*| < \frac{\varepsilon}{2} \text{ for all } n > N.$$

Then for any n with $n > N$ and $M_k^r = M_k^{r^*} = H$ for both $K = n^2$ and $(n+1)^2$ we have

$$f^{n^2+1}(x) \in M_k^r = [b^{2n-1}, b^*] \text{ with } k = n^2 + 1$$

and $f^{n^2+1}(x_r)$ and $f^{n^2+1}(x_{r^*})$ both be long to $[b^{2n-1}, b^*]$. Therefore

$$\left| f^{n^2+1}(x_r) - f^{n^2+1}(x_{r^*}) \right| < \varepsilon.$$

Since there are infinitely many n with this property,

$$\liminf_{n\to\infty} |f^n(x_{r^*}) - f^n(x)| = 0. \blacksquare$$

Remark 2.12.3: Condition $[iii]$ in the definition of LY -chaos is redundant can be seen as follows. Two approximately periodic points x, y can not satisfy both $[i]$ and $[ii]$ in the definition of a scrambled set. Consequently there exists at most one approximately periodic point in any set satisfying condition [i] and [ii] of this definition. Removing this point new set also satisfies $[iii]$ [44]. However see the next lemma that found in [4]: *VI lemma* 28.

Lemma 2.12.4: If x and y are approximately periodic then either

$$\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$$

or

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

Proof: Assume to the contrary that

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0,$$

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) = \rho > 0,$$

Choose ε so that $0 < \varepsilon < \rho/5$. There exist periodic points z and w and positive integer N such that, for all $n \geq N$,

$$d(f^n(x), f^n(z)) < \varepsilon,$$

$$d\left(f^n\left(y\right), f^n\left(w\right)\right) < \varepsilon.$$

Let m be the least common multiple of the periodic of z and w, and choose $\delta > 0$ so small that $d(x_1, x_2) < \delta$ for any x_1, x_2 implies that

$$d(f^k(x_1), f^k(x_2)) < \varepsilon.$$

For some $p \geq N$ we have $d\left(f^p\left(x\right), f^p\left(y\right)\right) < \delta$ and hence

$$d(f^{p+k}(x), f^{p+k}(y)) < \varepsilon \text{ for } \ k = 1, ..., m.$$

It follows that

$$d(f^{p+k}(z), f^{n+k}(w)) < 3\varepsilon, \ k = 1, ..., m.$$

Thus

$$d(f^n(z), f^n(w)) < 3\varepsilon \text{ for all } n \geq 0.$$

Hence for all $n \geq N$,

$$d(f^n(x), f^n(y)) < 5\varepsilon < \rho$$

which is a contradiction.■

The last lemma showes that a scrambled set contains at most one approximately periodic point.

Remark 2.12.5: *Li-Yorke's* chaos has two disadvantages. The first is that it can only be used on intervals on the real line and not in higher dimensional spaces. For example the rotation on R^2 of $120°$ has a periodic point with period three but it does not have an uncountable scrambled set. The second disadvantage is that it cannot be applied to maps even with one discontinuity since discontinuity is critical to $LY - chaos$.

Proposition 2.12.6: *Let $f : X \to X$ be continuous. Then for any $n \in N$ the map f is $LY - chaotic$ if and only if f^n is $LY - chaotic$.*

Proof: The set $S \subseteq X$ is a scrambled set with respect to f is and only if it is a scrambled set with respect to f^n. ■

Example 2.12.7: Consider the map $f : [-1, 1] \to [-1, 1]$ given by $f(x) = 2|x| - 1$

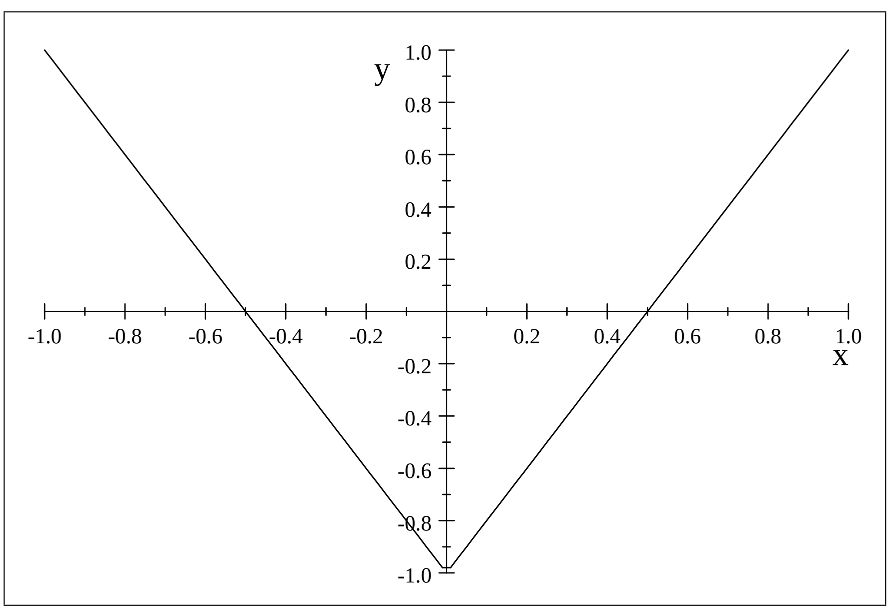


Figure 2.12.1: $f(x) = 2|x| - 1$

Now we can observe that $f(\frac{-7}{9}) = \frac{5}{9}$, $f(\frac{5}{9}) = \frac{1}{9}$ and $f(\frac{1}{9}) = \frac{-7}{9}$ so f has a periodic point of period 3 and f is $LY - chaotic$ [18].

Example 2.12.8: Consider the map $L : [0, 1] \to [0, 1]$ be given by

$$L(x) = \begin{cases} x + \frac{1}{3} & , \quad 0 \leq x \leq \frac{2}{3} \\ 1 - 7\left(x - \frac{2}{3}\right) & , \quad \frac{2}{3} \leq x \leq \frac{2}{3} + \frac{1}{8} \\ x - \frac{2}{3} & , \quad \frac{2}{3} + \frac{1}{8} \leq x \leq 1 \end{cases}$$

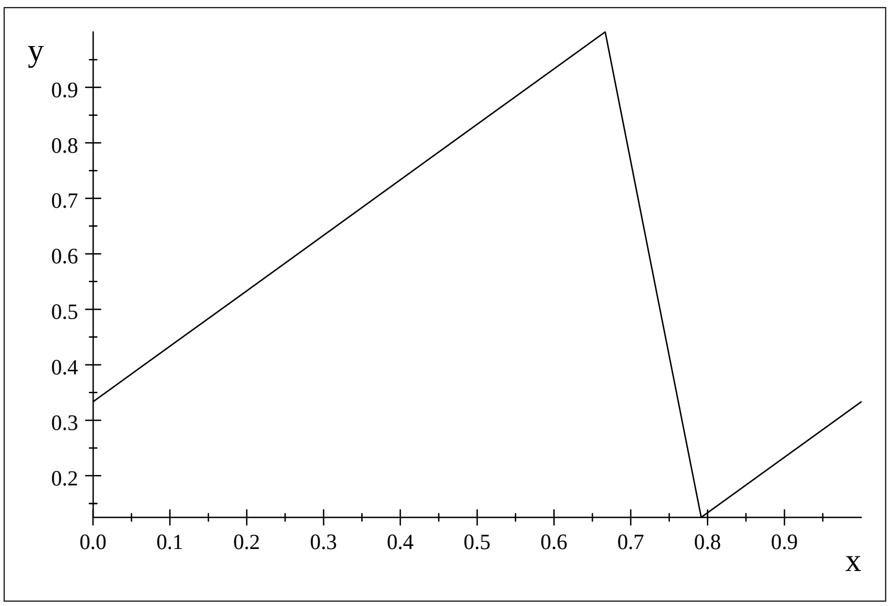


Figure 2.12.2: the graph of $L(x)$

Choosing the interval $I = [\frac{1}{8}, \frac{1}{3}]$ we can see that it is transported to the interval $J = [\frac{1}{3} + \frac{1}{8}, \frac{2}{3}]$. Continuing in the same way J is transported to $K = [\frac{2}{3} + \frac{1}{8}, 1]$ and finally K to I. So because of this all points in these intervals are period three points, hence the hypothesis of the definition is satisfied and there exists an uncountable scrambled set, hence it exhibits $Li-York-chaos$.

Example 2.12.9: The generalized logistic map $f(x) = ax(1-\frac{x}{b})$ with $a \in (3.84, 4)$ and $f(x) = max\left\{ax(1-\frac{x}{b}), 0\right\}$ for $a > 4$ both defined in the interval $I = [0, k]$ is LY *-chaotic.* The analysis of this example can be found in [9].

Example 2.12.10: Consider the map $f_a : [0,1) \to [01)$ given by $f_a(x) = frac(x-a)$, where $frac(x)$ denotes the fractional part of x and a is an irrational number. First we choose two arbitrary points x and y in $[0,1)$ such that $x < y$.

Then we have

$$\left|f_a^n(x) - f_a^n(y)\right| = \left|f_{na}(x) - f_{na}(y)\right| = \begin{cases} 1-|x-y| & , \quad x < frac\,(na) \\ |x-y| & , \quad otherwise \end{cases}$$

so from these we have:

$$\liminf_{n\to\infty} \left|f_a^n(x) - f_a^n(y)\right| = \min\left\{|x-y|, 1-|x-y|\right\} > 0$$

and from these their can not exist an uncountable scrambled set, so the map is not $LY-chaotic$.

2.13 Topological Entropy

The notion of topological entropy, introduced by Adleretal, [47] provides a numerical measure for the complexity of an endomorphism of a compact topological space. We intend to consider here some results which hold for the special case of a compact interval in particular a theorem of Misiurewicz [48], which implies that a continuous map is chaotic if and only if its topological entropy is positive. However, this is the introduction given by Block and Coppel in chapter V III in their book. We follow topically them with some changes.[4]

Let X be a compact topological space. An open cover α of X is a collection of open sets whose union is X. An open cover β is said to be a refinement of an open cover α, in symbols $\alpha < \beta$, if every open set of β is contained in some open set of α. We say that β is a subcover of α if every open set of β actually is an open set of α. If α and β are two open covers, their join $\alpha \vee \beta$ is the open cover consisting of all sets $A \cap B$ with $A \in \alpha, B \in \beta$. Thus $\alpha \cup \beta$ is a refinement of both α and β. Since X is compact, every open cover has a finite subcover. The entropy of an open cover α is defined to be $H(\alpha) = logN(\alpha)$, where $N(\alpha)$ is the minimum number of open sets in any finite subcover. Evidently $H(\alpha) \geq 0$, with equality if and only if $x \in \alpha$. Moreover it is easily seen that:

[i] if $\alpha < \beta$, then $H(\alpha) \leq H(\beta) \; and \; H(\alpha \vee \beta) = H(\beta)$,
[ii] $H(\alpha \vee \beta) \leq H(\alpha) + H(\beta)$,
[iii] if $\alpha < \beta$, then $f^{-1}\alpha < f^{-1}\beta$,
[iv] $f^{-1}(\alpha \vee \beta) = f^{-1}\alpha \vee f^{-1}\beta$,
[v] $H(f^{-1}\alpha) \leq H(\alpha)$, with equality if f is surjective.

From $[iv], [ii], [v]$ we obtain for any positive integers m, n

$$\begin{aligned} H(\alpha \vee ... \vee f^{-m-n+1}\alpha) &= H(\alpha \vee ... \vee f^{-m+1}\alpha \vee f^{-m}(\alpha \vee ... \vee f^{-n+1}\alpha)) \\ &\leq H(\alpha \vee ... \vee f^{-m+1}\alpha) + H(f^{-m}(\alpha \vee ... \vee f^{-n+1}\alpha)) \\ &\leq H(\alpha \vee ... \vee f^{-m+1}\alpha) + H(\alpha \vee ... \vee f^{-n+1}\alpha). \end{aligned}$$

$$h(f, a) = \lim_{n \to \infty} H(\alpha \vee ... \vee f^{-m+1}\alpha \vee f^{-m}(\alpha \vee ... \vee f^{-n+1}\alpha).$$

Lemma 2.13.1: Let a_n be a sequence of real numbers which is sub additive, i.e.

$$a_{m+n} \leq a_m + a_n \text{ for all } m, n$$

Then $\lim\limits_{n\to\infty} \frac{a_n}{n}$ exists and has value $C = inf \frac{a_n}{n}$.

Proof: For any fixed m set $n = qm + r$, where q, r nonnegative integers and $r < m$. It follows from the subadditivity that $a_n \leq qa_m + a_r$, If $n \to \infty$ for a fixed m then $\frac{q}{n} \to \frac{1}{m}$ and takes finitely many values. Hence

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_m}{m}.$$

Since this holds for arbitrary m we have $\limsup\limits_{n\to\infty} \frac{a_n}{n} \leq C$ But $\frac{a_n}{n} \geq C$ for every n, we also have $C \leq \liminf\limits_{n\to\infty} \frac{a_n}{n}$. The result follows.■

The limit $h(f, \alpha)$ is called the entropy of f relative to the cover α.

[vi] if $\alpha < \beta$, the $h(f, \alpha) \leq h(f, \beta)$.

[vii] if f is a homeomorphism, then $h(f^{-1}, \alpha) = h(f, \alpha)$, since

$$H(\alpha \vee ... \vee f^{-n+1}\alpha) = H(f^{n-1}(\alpha \vee ... \vee f^{-n+1}\alpha)),$$

$$= H(\alpha \vee f\alpha \vee ... \vee f^{n-1}\alpha),$$

$$= H(\alpha \vee (f^{-1})^{-1}\alpha \vee ... \vee (f^{-1})^{-n+1}\alpha).$$

The topological entropy of a continuous map $f : X \to X$ is defined to be $h(f) = sup_\alpha h(f, \alpha)$, where the supremum is taken over all open covers α.

Proposition 2.13.2: *If $f : X \to X$ is a continuous map then for any positive integer k, $h(f^k) = kh(f)$.*

Proof: For any open cover we have

$$h(f^k) \geq h(f^k, \alpha \vee f^{-1}\alpha \vee ... \vee f^{-k+1}\alpha)$$

$$= \lim_{n\to\infty} \frac{kH(\alpha \vee f^{-1}\alpha \vee ... \vee f^{-k+1}\alpha \vee ... \vee f^{-nk+1}\alpha)}{nk}$$

$$= kh(f, \alpha)$$

Hence $h(f^k) \geq kh(f)$. On the other hand, since

$$\alpha \vee (f^k)^{-1}\alpha \vee ... \vee (f^k)^{-n+1}\alpha < \alpha \vee f^{-1} \vee ... \vee f^{-nk+1}\alpha$$

We have

$$h(f, \alpha) = \lim_{n\to\infty} \frac{H(\alpha \vee f^{-1}\alpha \vee ... \vee f^{-nk+1}\alpha)}{nk}$$

$$\geq \lim_{n\to\infty} \frac{H(\alpha \vee (f^k)^{-1}\alpha \vee ... \vee (f^k)^{-n+1}\alpha)}{nk}$$

$$= h\frac{(f^k, a)}{k}$$

Hence $h(f^k) \leq kh(f)$. ■

Proposition 2.13.3: *If $f : X \to X$ is a homeomorphism, then $h(f^{-1}) = h(f)$.*

Proof: This follows at once from [vii]. ■

Proposition 2.13.4: *Let X, Y be compact topological spaces and let $f : X \to X$, $g : Y \to Y$ be continuous maps. If there exists a continuous maps $\phi : X \to Y$ such that $\phi(X) = Y$ and the diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

commutes, then h(g) $\leq$*h(f). Moreover, if* ϕ *is a homeomorphism, then* $h(g) = h(f)$.

Proof: If α is any open cover of Y then, since ϕ is surjective, $\phi^{-1}\alpha$ is an open cover of X. Moreover, since $\phi \circ f^k = g^k \circ \phi$,

$$h(g, \alpha) = \lim \frac{H(\alpha \vee g^{-1} \vee ... \vee g^{-n+1}\alpha)}{n}$$

$$= lim\frac{H(\phi^{-1}(\alpha \vee g^{-1}\alpha \vee ... \vee g^{-n+1}\alpha))}{n}$$

$$= lim\frac{H(\phi^{-1}(\alpha \vee \phi^{-1}g^{-1}\alpha \vee ... \vee \phi^{-1}g^{-n+1}\alpha)}{n}$$

$$= lim\frac{H(\phi^{-1}\alpha \vee f^{-1}\phi^{-1}\alpha \vee ... \vee f^{-n+1}\phi^{-1}\alpha)}{n}$$

$$= h(f, \phi^{-1}\alpha).$$

Hence $h(g) \leq h(f)$. If ϕ is a homeomorphism then $\phi^{-1} \circ g = f \circ \phi^{-1}$ and hence $h(f) \leq h(g)$.■

We consider first some ways of estimating the topological entropy $h(f)$ of a continuous map $f : I \to I$.

Proposition 2.13.5: Let $f : I \to I$ be a continuous map. If there exist disjoint closed intervals $J_1, ..., J_p$ such that

$$J_i \cup ... \cup J_p \subseteq f(J_i) \qquad (i = 1, ..., p)$$

then $h(f) \geq \log p$.

Proof: We can choose pairwise disjoint open intervals $G_1, ..., G_p$ with $J_i \subseteq G_i$ for $i = 1, ..., p$. By adjoining further open intervals $G_{p+1}, ..., G_q$, satisfying $G_i \cap J_k = \phi$ for $p+1 \leq i \leq q$ and $1 \leq k \leq p$, we obtain a finite open cover α. For any positive integer n and any i_k with $1 \leq i_k \leq p$ the set

$$J_{i_1 \dots i_n} = \left\{ x : x \in J_{i_1}; f(x) \in J_{i_2}, ..., f^{n-1}(x) \in J_{i_n} \right\}$$

is non empty. Each point in this set containes a unique element of

$$\alpha \vee f^{-1}\alpha \vee ... \vee f^{-n+1}\alpha$$

namely

$$G_{i_1} \cap f^{-1}(G_{i_2}) \cap ... \cap f^{-n+1}(G_{i_n}).$$

It follows that

$$H(\alpha \vee f^{-1} \vee ... \vee f^{-n+1}\alpha) \geq n \log p.$$

Hence $h(f, \alpha) \geq \log p$ and, a fortiori, $h(f) \geq \log p$. ■

The proof of the following lemmas (2.13.6 , 2.13.7 , 2.13.8) can be found in [49].

Lemma 2.13.6: *Let $A = (a_{ik})$ be a $p \times p$ matrix of non-negative real numbers. Then there exist $\lambda \geq 0$ and a non-zero vector $x = (x_k)$ with $x_k \geq 0$ $(k = 1, ..., p)$ such that $Ax = \lambda x$ and $|\mu| \leq \lambda$ for every other eigenvalue μ of A.*[49]

We will refer to λ as the maximal eigenvalue of A. A non-negative matrix A is said to be reducible if there exists a permutaion matrix P such that:

$$P^t AP = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$$

where B and D are square matrices of smaller size, and irreducible otherwise.

Lemma 2.13.7: *For any non-negative matrix A there exists a permutaion matrix P such that :*

$$P^t AP = \begin{bmatrix} A_{11} & 0 & ... & 0 \\ A_{21} & A_{22} & ... & 0 \\ & ... & ... & \\ A_{r1} & A_{r2} & ... & A_{rr} \end{bmatrix}$$

where each diagonal block A_{kk} $(k = 1, ..., r)$ is irreducible.[49]

Lemma 2.13.8: *Let A be a non-negative matrix with maximal eigenvalue λ. If A is irreducible, then there exists a positive integer h such that the eigenvalues of A with absolute value λ are :*

$$\lambda, \lambda\omega, ..., \lambda\omega^{h-1}$$

where $\omega = exp(2\pi i/h)$. [49]

We define the norm of a real or complex matrix $A = (a_{ik})$ to be

$$|A| = \sum_{i,k} |a_{ik}| \, .$$

The maximal eigenvalue is related to the norm in the following way.

Lemma 2.13.9: *Let A be a non-negative matrix with maximal eigenvalue λ. Then*

$$\lambda = \lim_{n\to\infty} |A^n|^{\frac{1}{n}}$$

Proof: Let $x = (x_k)$ be the non-negative eigenvector corresponding to the eigenvalue λ, so that:

$$\lambda x_i = \sum_k a_{ik} x_k$$

If we choose i so that $x_i = \max\, x_k$ we obtain

$$\lambda \le \sum_k a_{ik} \le |A|\,.$$

Applying this inequality to A^n, instead of A, we obtain $\lambda^n \le |A^n|$ and hence

$$\lambda \le \liminf_{n\to\infty} |A^n|^{\frac{1}{n}}$$

Let T be a non-singular matrix such that $J = T^{-1}AT$ is in Jordan normal form. If $\rho > \lambda$ then

$$J^n/\rho^n \to 0 \text{ as } n \to \infty$$

and hence also $A^n/\rho^n \to 0$. Thus $|A^n| < \rho^n$ for all large n and

$$\limsup_{n\to\infty} |A^n|^{\frac{1}{n}} \le \rho.$$

Since this holds for any $\rho > \lambda$, the result follows. ■

Lemma 2.13.10: *Let A be a non-negative matrix with maximal eigenvalue λ. Then*

$$\lambda = \limsup_{n\to\infty} (tr\ A^n)^{\frac{1}{n}}$$

Proof : If A is $p \times p$ matrix then $trA^n \le p\lambda^n$ and hence

$$\limsup_{n\to\infty} (tr\ A^n)^{\frac{1}{n}} \leq \lambda.$$

It remains to prove the reverse inequality.

Suppose first that A is irreducible. Then by *lemma 2.13.8*, the eigenvalue of A with absolute value λ are

$$\lambda, \lambda\omega, ..., \lambda\omega^{h-1}$$

where $\omega = exp(2\pi i/h)$. If we denote the remaining eigenvalues of A by $\lambda_{h+1}, ..., \lambda_p$ then

$$trA^n = \lambda^n + \lambda^n\omega^n + ... + \lambda^n\omega^{(h-1)n} + \lambda_{h+1}^n + ... + \lambda_p^n.$$

In particular if we take $n = kh$ to be a multiple of h then $\omega^n = 1$ and

$$(trA^n)^{\frac{1}{n}} = \lambda \left[h + \textstyle\sum_{i=h+1}^{p} (\lambda_i/\lambda)^n\right]^{\frac{1}{n}}$$

$$\to \lambda\ as\ k \to \infty.$$

In the general case we appeal to $Lemma 10$. For some k the irreducible diagonal block A_{kk} has maximal eigenvalue since

$$trA^n \geq tr(A_{kk})^n.$$

It follows from what we have just proved that

$$\limsup (trA^n)^{\frac{1}{n}} \geq \lambda. \blacksquare$$

Proposition 2.13.11: *Let $f : I \to I$ be a continuous map. Let $J_1, ..., J_p$ be closed intervals with pairwise disjoint interiors and let $A = (a_{ik})$ be the $p \times p$ matrix defined by $a_{ik} = 1$ if $J_k \subseteq f(J_i)$, $a_{ik} = 0$ otherwise. Then $h(f) \geq \log \lambda$, where λ is the maximal eigenvalue of A.*

Proof: Evidently we may suppose that $\lambda > 1$. By *lemma* 2.13.10 we have

$$log\lambda = lim\sup[\log tr(A^n)]/n.$$

since

$$max_i(A^n)_{ii} \le tr(A^n) \le p \max_i(A^n)_{ii}.$$

It follows that for some $i(1 \le i \le p)$,

$$log\lambda = lim[log(A^n)_{ii}]/n.$$

Thus for any μ with $1 < \mu < \lambda$ there exist arbitrarily large n such that there are more than μ^n paths of length n from J_i back to J_i. Evidently J_i must have non-empty interior. Hence there exist more than μ^n closed intervals K_j with pairwise disjoint interiors such that

$$K_j \subseteq J_i, f^n(K_j) = J_i.$$

By omitting the two intervals K_j which are furthest to the left and to the right and slightly shrinking the remaining intervals. We obtain at least $[\mu^n] - 1$ disjoint closed intervals L_j such that

$$\cup_i L_i \subseteq int f^n(L_m) \text{ for every } m.$$

Hence, by *Propositions 2.13.2 and 2.13.5*

$$h(f) = h(f^n)/n \ge [log(\mu^n - 2)]/n$$

$$= \log\mu + [\log(1 - 2/\mu^n)]/n$$

Since n can be arbitrarily large, it follows that $h(f) \ge \log\mu$. Since this holds for any $\mu < \lambda$, it follows that $h(f) \ge \log\lambda$.■

Corollary 2.13.12: *Let $f : I \to I$ be a continuous map. If f is turbulent, then $h(f) \geq \log 2$.* ■

Proposition 2.13.13: *Let $f : I \to I$ be a continuous map. For any open cover β, there exists a cover α consisting of finitely many disjoint intervals such that*

$$h(f, \beta) \leq h^*(f, \alpha).$$

Conversely, for any cover α consisting of finitely many disjoint intervals there exists an open cover β such that

$$h^*(f, \alpha) \leq h(f, \beta) + log3.$$

Proof: Let β be any open cover. We may suppose that β is the union of finitely many open intervals, since replacing β by a refinement does not decrease the entropy of f relative to the cover. Let α be a cover consisting of finitely many disjoint intervals such that each interval $A \in \alpha$ is contained in some open interval $B \in \beta$. For any chain $\{A_1, ..., A_n\}$ in α^n pick some $B_k \in \beta$ such that

$$A_k \subseteq B_k \qquad\qquad (1 \leq K \leq n).$$

Since the collection of all open sets

$$B_1 \cap f^{-1}(B_2) \cap ... \cap f^{-n+1}(B_n)$$

obtained in this way covers I, it follows that

$$H(\beta \vee f^{-1} \vee \beta \vee ... \vee f^{-n+1} B) \leq logC_n(\alpha).$$

and hence

$$h(f,\beta) \leq h^*(f,\alpha).$$

Now let α be any cover consisting of finitely many disjoint intervals. Let β be a finite open cover of I, consisting of the interior of the intervals in α together with small open intervals surrounding the end points of these intervals, chosen so that every open interval in β is contained in the union of at most there intervals in α (three intervals being needed if an interval in α contains only one point). Let σ_n denote a subcover of $\beta \vee f^{-1}\beta \vee ... \vee f^{-n+1}\beta$ of minimum cardinality. For any chain $\{A_1, ..., A_n\}$ in σ^n there exists a point x with $f^{j-1}(x) \in A_j$ for $j = 1, ..., n$. Then x belongs to some element of

$$B_1 \cap f^{-1}(B_2) \cap ... \cap f^{-n+1}(B_n).$$

Evidently

$$A_n \cap B_n \neq \phi \qquad\qquad (1 \leq K \leq n).$$

Since the number of different chains $\{A_1, ..., A_n\}$ which correspond in this way to the same sequence $\{B_1, ..., B_n\}$ is at most 3, it follows that

$$\log C_n(\alpha) \leq H(\beta \vee f^{-1}\beta \vee ... \vee f^{-n+1}\beta) + n \log 3$$

and hence

$$h^*(f,\alpha) \leq h(f,\beta) + log3. \blacksquare$$

Lemma 2.13.14: *If $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers, then*

$$\limsup_{n\to\infty}[log(a_n + b_n)]/n = max\left\{\limsup_{n\to\infty}(\log a_n)/n, \limsup_{n\to\infty}(\log b_n)/n\right\}$$

Proof: Let λ and μ denote the left and right sides of equality to be proved. Evidently $\lambda \geq \mu$. On the other hand, for any $\sigma > \mu$ there exists an integer p such that

$$a_n < e^{n\sigma}, b_n < e^{n\sigma} \quad \text{for} \quad \text{all} \quad n \geq p$$

It follows that

$$[log(a_n + b_n)]/n < \sigma + (log2)/n \quad for\ n \geq p,$$

and hence $\lambda \leq \sigma$. Since this holds for any $\sigma \geq \mu$, we must actually have $\lambda \leq \mu$.■

Lemma 2.13.15: *Let $f : I \to I$ be a continuous map. If J, K are intervals such that $f(J) \cap K \neq \phi$, then there exists an interval $L \subseteq J$ such that*

$$f(L) = f(J) \cap K.$$

Proof: Evidently $\mu = f(J) \cap K$ is an interval. Since the result is obvious if μ contains only a single point we may suppose that $\overline{\mu} = [a, b]$, where $a < b$. Choose $c, d \in \overline{J}$, and if possible $c, d \in J$, so that

$$f(c) = a, \quad f(d) = b.$$

If $c < d$, let c' be the greatest value greater than c and less than d such that $f(c') = a$ and let d' be the least value greater than c' such that $f(d') = b$. Then we can take L to be an appropriate (closed, open,or half open) interval with endpoints c' and d'. If $c > d$ the proof is analogous.■

Let α be a collection of finitely many disjoint intervals and let γ be the collection of those intervals $A \in \alpha$ for which $\limsup_{n\to\infty} [\log c_n(\alpha|A)]/n = h^*(f, \alpha)$. We now define inductively a sequence $\{\delta_n\}$ where each δ_n is a finite collection of disjoint intervals in the following way :

(i) $\delta_1 = \gamma$,

(ii) if δ_n is defined then for each interval $D \in \delta_n$ and each interval $A \in \gamma$ with $f^n(D) \cap \neq \phi$ we choose by lemma 2.13.15 an interval $E(D, A)$ such that $E(D, A) \subseteq D, f^n(E(D, A)) = f^n(D) \cap A$.

The intervals $E(D, A)$ are necessarily disjoint and we take δ_{n+1} to be the collection of all such intervals. By induction we see that if $\{A_1, ..., A_n\}$ is a chain in γ^n then there exists a unique $D_n \in \delta_n$ such that

$$D_n \subseteq A_1, f(D_n) \subseteq A_2, ..., f^{n-2}(D_n) \subseteq A_{n-1}, f^{n-1}(D_n) \subseteq A_n.$$

Then

$$f^{n-1}(D_n) = \{y_n \in A_n : \exists\ y_1, ..., y_{n-1},\ y_i \in A_i,\ f(y_i) = y_{i+1},\ i = 1, .., n-1\}$$

Conversely, each interval $D_n \in \delta_n$ is contained in a unique interval $D_k \in \delta_k\,(1 \leq k < n)$ and there is a unique interval $A_k \in \gamma$ such that $f^{k-1}(D_n) \subseteq A_k\,(1 < k \leq n)$. Since $D_1 = A_1 \in \delta_1$, it follows that $f^{k-1}(D_n) \subseteq A_k\,(1 \leq k \leq n)$. Then $\{A_1, ..., A_n\}$ is a chain in γ^n. Consequently the number of intervals in δ_n is exactly $c_n(\gamma)$, and the number of intervals in δ_n which are contained in an interval $A \in \gamma$ is $c_n(\gamma|A)$.

For any $A, B \in \gamma$ let $g(A, B, n)$ denote the number of interval $D \in \delta_n$ such that $D \subseteq A, f^n(D) \supseteq B$. This notion and definition of α, γ are understood in the statements of the next two results.

Proposition 2.13.16: *For any $A, B, C \in \gamma$ and any positive integers m, n*

$$g(A, B, m)g(B, C, n) \leq g(A, C, m+n).$$

Proof: Let D be an interval in m such that $D \subseteq A$ and $f^m(D) \supseteq B$, and let E be an interval in δ_n such that $E \subseteq B$ and $f^n(E) \supseteq C$. Then there exists a chain $\{A_1, ..., A_m\}$ in $\gamma^m|A$ such that $f^{k-1}(D) \subseteq A_k(1 \leq K \leq m)$, and a chain $\{B_1, ..., B_n\}$ in $\gamma^n|B$ such that $f^{k-1}(E) \subseteq B_k(1 \leq K \leq n)$. Hence $\{A_1, ..., A_m, B_1, ..., B_n\}$ is a chain in $\gamma^{m+n}|A$ and there exists $D' \in \delta_{m+n}$ such that

$$f^{k-1}(D') \subseteq A_k(1 \leq K \leq m), f^{m+k-1}(D') \subseteq B_k(1 \leq K \leq n)$$

Moreover $D' \subseteq D$, since D' is contained in a unique interval of δ_m and D is the interval of δ_m with the itinerary $\{A_1, ..., A_m\}$. Since

$$f^m(D) \supseteq E, f^{k-1}(D) \subseteq A_k(1 \leq K \leq m)$$

and

$$f^n(E) \supseteq C, f^{k-1}(E) \subseteq B_k(1 \leq K \leq n).$$

It follows that for every $y_{m+n+1} \in C$ there are points y_i with $y_i \in A_i$ for $i = 1, ..., m$ and $y_i \in B_{i-m}$ for $i = m+1, ..., m+n$ and $f(y_i) = y_{i+1}$ for $i = 1, ..., m+n$. Thus $f^{m+n}(D') \supseteq C$. Since D' is the only interval of δ_{m+n} with the itinerary $\{A_1, ..., A_m, B_1, ..., B_n\}$, the result follows. ■

Proposition 2.13.17: *If $h^*(f, \alpha) > log3$, then there exists $A \in \gamma$ such that*

$$\limsup_{n\to\infty}[\log g(A, A, n)]/n = h^*(f, \alpha).$$

Proof: Choose any $A \in \gamma$ and let μ be a real number such that $log3 < \mu < h^*(f, \alpha)$. We will show first that the following condition is satisfied:

[#] For every p there exists an integer $n \geq p$ such that:

$$c_{n+1}(\gamma|A)/3 \geq c_n(\gamma|A) > e^{n\mu}.$$

Assume on the contrary that there exists p such that, for $n \geq p$

$$[\log c_n (\gamma|A)] / n > \mu \ \ \textit{implies} \ \ c_{n+1} (\gamma|A) < 3c_n (\gamma|A) .$$

If for some $q \geq p$ we have log $\log c_n (\gamma|A) > \mu n$ for all $n \geq q$ then $c_{n+q} (\gamma|A) < 3^n c_q (\gamma|A)$ for all $n \geq 1$ and hence

$$\limsup_{n \to \infty} [\log c_n (\gamma|A)] / n \leq \log 3$$

which is a contradiction. Therefore (1/N) $\log c_N (\gamma|A) \leq \mu$, for infinitely many positive integers N. Suppose that for such an N we have

$$\mu < [\log c_n (\gamma|A)] / n \ \ \text{for} \ \ n = N + 1, ..., N + r$$

where $r \geq 1$. Since in general $c_{n+1} (\gamma|A) \leq s \ c_n (\gamma|A)$, where s is the number of intervals in γ it follows that

$$c_{N+r} (\gamma|A) < s.3^{r-1} c_N (\gamma|A) .$$

Taking logarithms, we obtain

$$(N + r)\mu < \log s + (r - 1) \log 3 + N_\mu,$$

i.e.

$$r(\mu - \log 3) < \log s - \log 3.$$

Thus $r \leq t$, for some positive integer t independent of N. It follows that

$$c_n (\gamma|A) < s.3^{t-1} e^{n\mu}$$

for all large n, and hence

$$\limsup_{n \to \infty} [\log c_n (\gamma|A)] / n \leq \mu,$$

which is again a contradiction. This establishes [#]. Now fix $D \in \delta_n$ with $D \subseteq A$. By the definition of δ_{n+1}, the number q of intervals in δ_{n+1} which are contained in D is equal to the number of intervals $C \in \gamma$ such that $f^n(D) \cap C \neq \phi$. Since $f^n(D)$ is an interval, at most two intervals of γ have non-empty intersection with $f^n(D)$ but are not contained in it. Hence the number of intervals $B \in \gamma$ such that $f^n(D) \supseteq B$ is at least $q-2$. Summing over all $D \in \delta_n$ with $D \subseteq A$, we obtain

$$\textstyle\sum_{B\in\gamma} g(A,B,n) \geq c_{n+1}(\gamma|A) - 2c_n\left(\gamma|A\right).$$

Combining this with [#], we see that for infinitely many n we have

$$\textstyle\sum_{B\in\gamma} g(A,B,n) \geq c_n\left(\gamma|A\right) > e^{n\mu}$$

Hence, since $\mu < h^*(f,\alpha)$ is arbitrary,

$$\limsup_{n\to\infty}\left[\log \textstyle\sum_{B\in\gamma} g\left(A,B,n\right)\right]/n \geq h^*\left(f,\alpha\right).$$

It follows from lemma 14 that for each $A \in \gamma$ there exists $B = \varphi(A) \in \gamma$ such that

$$\limsup_{n\to\infty}[\log g(A,\varphi(A),n)]/n \geq h^*(f,\alpha).$$

Since γ is finite, the map $\varphi : \gamma \to \gamma$ has a periodic point A_0. Let m be its period. By repeated application of Proposition 16 we obtain, for any positive integers n_i $(0 \leq i < m)$,

$$g\left(A_0, A_0, \textstyle\sum_{i=0}^{m-1} n_i\right) \geq \textstyle\prod_{i=0}^{m-1} g\left(\varphi^i\left(A_0\right), \varphi^{i+1}\left(A_0\right), n_i\right).$$

But for any $\lambda < h^*(f,\alpha)$ we can choose arbitrarily large n_i, so that

$$\log g\left(\varphi^i\left(A_0\right), \varphi^{i+1}\left(A_0\right), n_i\right) \geq \lambda n_i.$$

Then, putting $n = \sum n_i$, we have

$$\log g(A_0, A_0, n) \geq \lambda_n$$

Thus

$$\limsup_{n\to\infty} [\log g(A_0, A_0, n)]/n \geq h^*(f, \alpha).$$

since $g(A_0, A_0, n) \leq c_n(\gamma) \leq c_n(\alpha)$ the reverse inequality is obvious. ■

Theorem 2.13.18: *Let $f : I \to I$ be a continuous map. If f has topological entropy $h(f) > 0$ then for any λ with $0 < \lambda < h(f)$ and any $N > 0$, there exist pairwise disjoint closed intervals $J_1, ..., J_p$ and an integer $n > N$ such that $(1/n)\log p > \lambda$ and*

$$J_1 \cup ... \cup J_p \subseteq int\ f^n(J_i) \quad (i = 1, ..., p).$$

Proof: Suppose first that $h(f) < +\infty$. Choose $\varepsilon > 0$, so small that $h(f) > \lambda + 2\varepsilon$ and let r be a positive integer such $rh(f) > \varepsilon + log3$. There exists a finite open cover β of I such that

$$h(f^r, \beta) \geq h(f^r) - \varepsilon.$$

By *Proposition2.13.13*, there exists a cover consisting of finitely many disjoint intervals such that

$$h^*(f^r, \alpha) \geq h(f^r, \beta).$$

Hence

$$h^*(f^r, \alpha) \geq rh(f) - \varepsilon > log3.$$

By proposition 2.13.17, applied to f^r and α. There exists an interval $A \in \alpha$ and arbitrarily large integers m such that

$$[\log g(A, A, m)]/m \geq h^*(f^r, \alpha) - \varepsilon \geq rh(f) - 2\varepsilon.$$

Thus if we put $n = mr$ there exist P_n disjoint intervals D_i such that $D_i \subseteq A$ and $f^n(D_i) \supseteq A$, where

$$(\log P_n)/n \geq h(f) - 2\varepsilon/r > \lambda.$$

Since $P_n \to \infty$ as $n \to \infty$, the intervals A has non-empty interior and so have the intervals D_i. If we omit the two intervals which are furthest to the left and to the right and replace the remaining intervals D_i by slightly smaller closed intervals J_i then

$$J_1 \cup ... \cup J_{p_n - 2} \subseteq int\ f^n(J_i) \quad (i = 1, ..., P_n - 2)$$

and for all large n,

$$[\log(P_n - 2)]/n \geq [\log(1 - 2/p_n)]/n + h(f) - 2\varepsilon/r > \lambda.$$

Suppose next that $h(f) = +\infty$. If we choose $\varepsilon = r = 1$ and a finite open cover β such that $h(f, \beta) > \lambda + 2$ then the preceding argument carries through.■

2.14 Auslander and Yorke's Chaos (Rulle and Taken's Chaos)

Let (x, f) be a dynamical system. The map f is called lyapunov ε-unstable at a point $x \in X$ if for every neighborhood U of x, there is $y \in U$ and $n \geq 0$

with $d(f^n(x), f^n(y)) > \varepsilon$. The map f is called unstable at a point x if there is $\varepsilon > 0$ such that f is Lyapunove ε- unstable at x.[53],[50].

In [53] a system (X, f) with surjective f is called chaotic if every point is unstable and X contain a dense orbit. The instability of all points in a system implies that the system has no isolated points. So the chaos in a standard dynamical system is defined as "topological transitivity plus pointwise instability". Notice also that if a standard dynamical system has he property of pointwise instability then it can happen that there is no $\varepsilon > 0$ such that all points are $\varepsilon-$unstable with this ε. But if, additionally, the system has a dense orbit then pointwise instability implies uniform pointwise instability [2]. In general there is no connection between transitivity and pointwise instability. But on the interval transitivity implies pointwise instability, the converse is not true.[2]

Definition 2.14.1: *Let $\varepsilon > 0$. A map f on a set X is called lyapunov ε-unstable at a point $x \in X$ if for every neighbourhood U of x there is $y \in U$ and $n \geq 0$ with $d(f^n(x), f^n(y) > \varepsilon$. The map f is called unstable at a point x (or the point x itself is called unstable) if there is $\varepsilon > 0$ such that f is luapunov ε-unstable at x.*

Definition 2.14.2: *The map f is chaotic in the sense of Ruclle and Takens (or Auslander and Yorke) if :*

[*i*] *it is surjective,*

[*ii*] *every point is unstable (in the sense of lyapunove),*

[*iii*] *X contains a dense orbit.*

We will denote this chaos by *AY- chaos.*

2.15 Distributional Chaos Or Schweizer-Smital Chaos

Distributional chaos, briefly DC, is a more sophisticated version of Li -York - chaos. While the latter notion considers only external values of distances between pairs of trajectories, DC is based on their asymptotic distribution. DC was introduced in 1994 by *B.Schwiezer and J.Smital* for continuous maps of the interval [25], [54], but later it was generalized to compact metric spaces[55]. Given a continuous map $f : X \to X$, a positive integer n and a real parameter t, put

$$F_{xy}^{n}(t) = \frac{1}{n}\#\left\{n, 0 \leq m < n \text{ and } \delta_{xy}(m) < t\right\}.$$

Where $\delta_{xy}(m)$ stands for $d(f^m(x), f^m(y))$. We are interested in a asymptotic behavior of the function $F_{xy}^{n}(t)$ as n gets large. Accordingly we consider the

functions $F_{xy}(t)$ and $F_{xy}^{*}(t)$ defined by:

$$F_{xy}(t) = \liminf_{n\to\infty} F_{xy}^{n}(t) \ \ and \ F_{xy}^{*}(t) = \limsup_{n\to\infty} F_{xy}^{n}(t).$$

For any two points x and y, the function $F_{xy}(t)$ and $F_{xy}^{*}(t)$ are distribution functions such that $F_{xy}(t) \leq F_{xy}^{*}(t)$ for any real t. Changing their values at a countable set of points if necessary we may assume that they are left-continuous. We also adopt the convection that $F_{xy} < F_{xy}^{*}$ means $F_{xy}(t) \leq F_{xy}^{*}(t)$ for some, and hence for all, t in some interval. We refer to F_{xy} and F_{xy}^{*} as the *Lower* and *Upper distribution* of x and y respectively, It is easy to see that $F_{xy}(t) = F_{xy}^{*}(t) = 0$ for $t \leq 0$, and $F_{xy}(t) = F_{xy}^{*}(t) = 1$ for $t > \dim(X)$, therefore we assume that the distributions F_{xy}, F_{xy}^{*} are defined on the interval $T = [0, \dim(X)]$.

Definition 2.15.1: *Let S be a scrambled set. For any $x \neq y$ in S the map f is said to exhibit distributional chaos of type 1or DC1 if*

$$F^*_{xy}(t) = 1 \text{ for all } \ t > 0 \ \text{ and } \ F_{xy}(t) = 0 \text{ for some } t > 0$$

and of type 2 or DC2 if:

$$F^*_{xy}(t) = 1 \qquad \text{and} \qquad F_{xy}(t) \leq F^*_{xy}(t)$$

and of type 3 or DC3 if:

$$F^*_{xy} > F_{xy}$$

Obviously, DC1 implies DC2, and DC2 implies DC3. Moreover, $\lim\limits_{n\to\infty} F^*_{xy}(t) > 0$ implies that the trajectories of x and y are proximal, i.e. $\liminf\limits_{n\to\infty} \delta_{xy}(n) = 0$,while $F_{xy}(\varepsilon) < 1$ gives $\limsup\limits_{n\to\infty} \delta_{xy}(n) \geq \varepsilon$. Thus, either of DC1 and DC2 implies $Li - York - chaos$. The properties of L-Y-chaos and DC1, DC2, DC3 can be classified relative to the size of the corresponding *scrambled* set S. Originally, the *scrambled* set was supposed to be uncountable. On the compact interval I the size of S is not essential, since in any case there is a *2-point scrambled* set if and only if there is an *uncountable* perfect set S.

2.16 Kato's Chaos

Definition 2.16.1: *A map f is called accessible if for every pair of non-empty open sets U and V of X, there exist points $x \in U, y \in V$ and a positive integer n such that*

$$d(f^n(x), f^n(y)) < \varepsilon.$$

Definition 2.16.2: *The map f is chaotic in the sense of Kato if it is:*
[i] *sensitive on initial conditions,*

[ii] *accessible.*

Remark 2.16.3: The Robinson's chaoticity implies the Kato's chaoticity on a complete metric space, but the converse is not true in general (see [56]).

Remark 2.16.4: The definition of the Knudsen's chaos is equivalent to the definition of Kato's chaos on a compact metric space (see [57]).

2.17 ω-Chaos

The ω-set of the point x is the set of all limit points of the orbit of x, that is,

$$\omega(x, f) = \bigcap_{n \in \mathbb{N}} \overline{\{f^k(x) \mid k \geq n\}}$$

The following properties can be easily deduced from the definition.

Lemma 2.17.1: Let $f : I \to I$ be an interval map, $x \in I$ and $n \geq 1$. Then

[i] $\omega(x, f)$ is a closed set,
[ii] $\omega(f^n(x), f) = \omega(x, f)$,
[iii] $f(\omega(x, f)) = \omega(x, f)$,
[iv] $\omega(f^i(x), f^n) = f^i(\omega(x, f^n))$,

[v] $\omega(x, f) = \bigcup_{i=0}^{n-1} \omega(f^i(x), f^n)$ and if $\omega(x, f)$ is infinite $\omega(f^i(x), f^n)$ is infinite too.

Definition 2.17.2: [28] *Let $S \subset A$. We say that S is an ω-scrambled set if for any $x, y \in S$ with $x \neq y$,*

(1) $\omega(x,f)\backslash\omega(y,f)$ *is uncountable,*

(2) $\omega(x,f) \cap \omega(y,f)$ *is nonempty, and*

(3) $\omega(x,f)$ *is not contained in the set of periodic points.*

We say that f is $\omega-chaotic$, if there exists an uncountable ω-scrambled set.

Remark 2.17.3: J. Smitál has proved that in the case of a compact interval $\omega(x,f) \subset P(f)$ implies that $\omega(x,f)$ is finite. Thus, in this case, condition (3) is not needed in Definition 2.17.2.

3 Related Definitions of Chaos and Cross-links

In this chapter, we will examine the relationships between pairs of definitions. In some cases two or more of the conditions defined in Chapter 2 have been used to characterize a function as chaotic.

Thus, the comparisons presented in this chapter will help clarify previous potential definitions of chaos. The main results of Chapter 3, with the number on an arrow referring to the proposition in which a proof or a counterexample of the relationship is provided.

The Tent function (see Figure 3.0.1) $T : [0, 1] \to [0, 1]$ defined by

$$T(x) = \begin{cases} 2x & , \quad 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & , \quad \frac{1}{2} < x \leq 1 \end{cases}$$

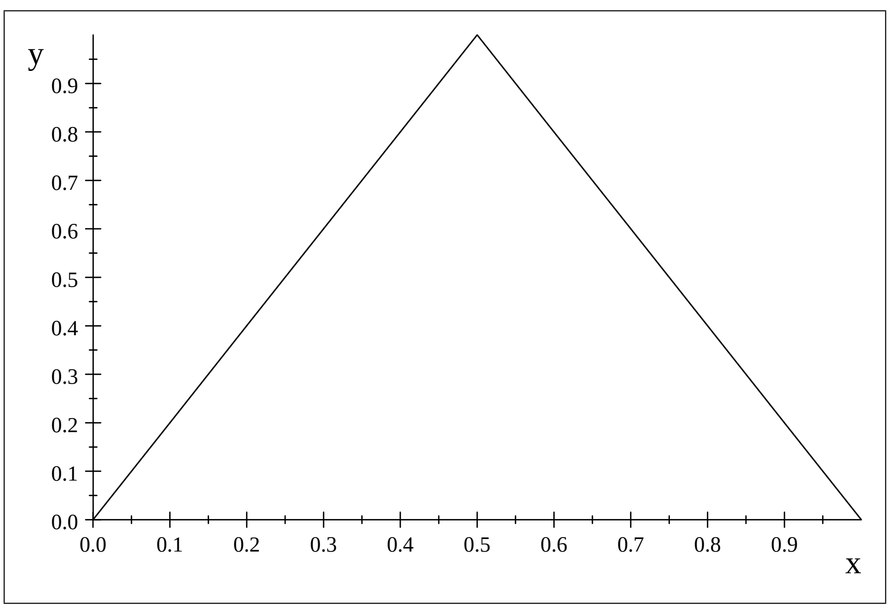


Figure 3.0.1: The tent map

In order to find the fixed points of T, we let $T(x) = x$, which implies either $2x = x$ or $2 - 2x = x$. Thus, the only two fixed points of T are $x = 0$ and $x = \frac{2}{3}$. To find an eventually fixed point of T, we need an x that is not 0 or $\frac{2}{3}$ such that some iterate of x equals either 0 or $\frac{2}{3}$.

Consider $x = \frac{1}{12}$

$$T(\frac{1}{12}) = 2 \times \frac{1}{12} = \frac{1}{6}$$

$$T(\frac{1}{6}) = 2 \times \frac{1}{6} = \frac{1}{3}$$

$$T(\frac{1}{3}) = 2 \times \frac{1}{3} = \frac{2}{3}$$

and $\frac{2}{3}$ is a fixed point. Thus, $T^3(\frac{1}{12}) = \frac{2}{3}$, and therefore $T^n(\frac{1}{12}) = \frac{2}{3}$ for all positive integers $n \geq 3$, so $x = \frac{1}{12}$ is an eventually fixed point.

Next we illustrate periodic points. Let $x = \frac{2}{3}$

$$T(\frac{2}{9}) = 2 \times \frac{2}{9} = \frac{4}{9}$$

$$T(\frac{4}{9}) = 2 \times \frac{4}{9} = \frac{8}{9}$$

$$T(\frac{8}{9}) = 2 - 2 \times \frac{8}{9}$$

Thus $T^3(\frac{2}{9}) = \frac{2}{9}$ and $T^n(\ \frac{2}{9}) \neq \frac{2}{9}$ for $n = 1$ and 2, so $x = \frac{2}{9}$ is a periodic point of T with period $n = 3$. Similarly, one can show that x $= \frac{4}{13}$ is a periodic point of T with period $n = 6$.

Now we will illustrate eventually periodic points. Let $x = \frac{1}{18}$

$$T(\frac{1}{18}) = 2 \times \frac{1}{18} = \frac{1}{9}$$

$$T(\frac{1}{9}) = 2 \times \frac{1}{9} = \frac{2}{9}$$

and $\frac{2}{9}$ is a periodic point, as we have just shown. Thus, x = $\frac{1}{18}$ is an eventually periodic point.

In some of the proofs to follow, we will associate each $x \in [0, 1]$ with a sequence $x_0x_1x_2x_3...$ by means of a function h that is related to T and which we now define. Let h map the interval $[0, 1]$ to the set of all sequences of 0's and 1's by

$$h(x) = \text{the sequence} \qquad x_0x_1x_2... \tag{0.1}$$

where

$$x_n = \begin{cases} 0 & : \quad 0 \leq T^n(x) \leq \frac{1}{2} \\ 1 & : \quad \frac{1}{2} \leq T^n(x) \leq 1 \end{cases}$$

and where $T^0(x) = x_0$ by definition. Notice that by the definition of h, if $h(x) = x_0x_1x_2x_3...$, then $h(T(x)) = x_1x_2x_3x_4...$, and by induction, $h(T^n(x)) = x_nx_{n+1}x_{n+2}....$

Thus, $h \circ T$ is a "left shift" on the set of all sequences of 0's and 1's, in the sense that $h(T(x)) = x_1x_2$ ..., so that under h, the sequence $x_0x_1x_2...$ is shifted to the left, with x_0 disappearing. We will associate a number $x \in [0, 1]$with its image under h by writing $x \sim x_0x_1x_2...$, where $h(x) = x_0x_1x_2....$ For example, $x = \frac{2}{9} \sim 001001...$, since

$$x = \frac{2}{9} \in \left[0, \frac{1}{2}\right] \quad \text{implies} \quad x_0 = 0$$

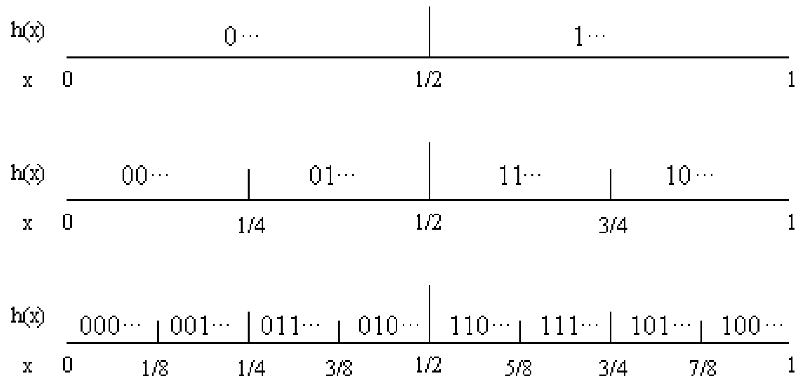


Figure 1: 3.0.2

$$T^1(x) = \frac{4}{9} \in \left[0, \frac{1}{2}\right] \text{ implies } \quad x_1 = 0$$

$$T^2(x) = \frac{8}{9} \in (\frac{1}{2}, 1] \text{ implies } x_2 = 1$$

and then the sequence repeats itself since $x = \frac{2}{9}$ is a periodic point with period $n = 3$.

Another important feature of the association of each $x \in [0,1]$ with its image under h is the division of the interval $[0,1]$ into blocks of length $\frac{1}{2^n}$ for every positive integer n (see Figure 3. 0.2).

For example, if $0 \leq x \leq \frac{1}{2}$, then $x_0 = 0$, whereas if $1/2 < x \leq 1$, then $x_0 = 1$. It follows that by considering x_0, we can determine which half of the $[0,1]$ interval $x \sim x_0 x_1 x_2 ...$ lies in.

Similarly,

$$\text{if} \quad 0 \leq x \leq 1/4, \text{then } 0 \leq T(x) \leq 1/2, \text{ so that } x_0 = 0, x_1 = 0.$$

$$\text{If } 1/4 < x \leq 1/2, \text{then } \ 1/2 < T(x) \leq 1, \text{ so that } \ x_0 = 0, x_1 = 1.$$

$$\text{If } \ 1/2 < x < 3/4, \text{then } \ 1/2 < T(x) \leq 1, \text{ so that } \ x_0 = 1, x_1 = 1.$$

$$\text{If } \ 3/4 \leq x \leq 1, \text{then } \ 0 \leq T(x) \leq 1/2, \text{ so that } \ x_0 = 1, x_1 = 0.$$

By considering x_0x_1, we can determine which quarter of the$[0, 1]$ interval $x \sim x_0x_1x_2$...lies in. Similarly, by considering x_0x_1... x_n, we can determine which subinterval of $[0, 1]$ of length$1/2^n$ that x lies in.

It can also be shown that h is one-to-one and onto the set of sequences of 0's and 1's,

excluding "finite" sequences of the form $x_0x_1x_2...\overline{0}...$ (that is, sequences in which all terms to the right of a given term are 0). We will denote by A this set of sequences which excludes finite sequences. Thus, in future examples we will construct sequences in A whose associated numbers in [0, 1] have specified properties for T.

The Tent function is an important function because it exhibits all of the characteristics that we will present in this Chapter that are associated with chaotic functions. Let us briefly explore one of the most important characteristics of the Tent function: long term iterates.

Let $x = \frac{2}{7}, y = \frac{9}{32}$, and $z = \frac{\pi}{11}$. Then $|x - y| < 0.005$ and $|x - z| < 0.002$, so x, y, and z are close together. However, if we look at the 29th iterates of these numbers, we see that they have separated over the interval $[0, 1]$:

$$T^{29}(\frac{2}{7}) = \frac{6}{7} \ , T^{29}(\frac{9}{32}) = 0, T^{29}(\frac{\pi}{11}) \approx 0.169955$$

We see that small differences in the initial starting points lead to large differences in higher iterates.

3.1 Transitivity, Total transitivity and Mixing

We are going to see that, for interval maps, the properties of total transitivity, topological weak mixing and topological mixing coincide, contrary to what happens in general. Moreover, the notions of transitivity and mixing are very close to each other. Indeed, if f is a transitive interval map which is not mixing then the interval can be divided into two subintervals on each of which f^2 is mixing. We also give some properties equivalent to mixing for interval maps. The results of this Section are classical (we here follow [58]), see also [4] .

Definition 3.1.1: *Let X be a compact metric space and $f{:}X \to X$ a continuous map. The transformation f is totally transitive if $f^n : X \to X$ is transitive for all integers $n \geq 1$.*

A *transitive set* is a closed set $E \subset X$ such that $f(E) \subset E$ and $T|_E$ is transitive.

Definition 3.1.2: *Let X be a compact metric space and $f : X \to X$ a continuous map. The map f is called (topologically) weakly mixing if $f \times f$ is transitive on $X \times X$.*

Definition 3.1.3: *Let X be a compact metric space and $f : X \to X$ a continuous map. The transformation f is (topologically) mixing if for* *all nonempty open sets U, V there exists $N \geq 0$ such that for all $n \geq N$, $f^n(U) \cap V \neq \phi$.*

It is well known that mixing implies weak mixing [60] . Moreover weak mixing implies total transitivity; this is a folklore result. It can easily be proved using the following well-known result, which is due to Furstenberg [61].

Proposition 3.1.4: *Let X be a compact metric space and $f : X \to X$ a continuous map. If $(X,\ f)$ is weakly mixing then for all integers $n \geq 1$, then the product system $(X^n, f \times ... \times f)$ (with n times f) is transitive.*

Proposition 3.1.5: *Let X be a compact metric space and $f : X \to X$ a continuous map. If $(X,\ f)$ is weakly mixing then for all integers $n \geq 1$ the system (X, f^n) is weakly mixing; in particular $(X; f)$ is totally transitive.*

Proof: Let $n \geq 1$ and U, U', V, V' non empty open sets in X. Define

$$W = U \times f^{-1}(U) \times ... \times f^{-(n-1)}(U) \times V \times f^{-1}(V) \times ... \times f^{-(n-1)}(V)$$

and

$$W = \underbrace{U' \times ... \times U'}_{n \text{ times}} \times \underbrace{V' \times ... \times V'}_{n \text{ times}}$$

The sets W, W' are open in X^{2n}, and $(X^{2n}, f \times ... \times f)$ (with $2n$ times f) is transitive by Proposition 3.1.4, thus there exists $k \geq 0$ such that $f^k(W) \cap W' \neq \phi$. It means that $f^{-(k+i)}(U) \cap U' \neq \phi$ and $f^{-(k+i)}(V) \cap V' \neq \phi$ for all $0 \leq i \leq n-1$. There exists $0 \leq i \leq n-1$ such that $k+i$ is a multiple of n; write $k+i = np$. One has then that

$$(f \times f)^{-np}(U \times V) \cap (U' \times V') \neq \phi$$

that is (X, f^n) is weakly mixing. This clearly implies that (X, f^n) is transitive.■

Theorem 3.1.6: Let X be a compact metric space and $f : X \to X$ a continuous map. If $(X,\ f)$ is mixing then it is weakly mixing. If (X, f) is weakly mixing then (X, f) is totally transitive.

Proposition 3.1.7: *An interval map $f : [a, b] \to [a, b]$ is mixing if and only if for all $\varepsilon > 0$ and all non degenerate subintervals J there exists an integer N such that $\forall n \in N$, $f^n(J) \supset [a + \varepsilon, b - \varepsilon]$.*

Proof: Suppose first that f is mixing and put $U_1 = (a, a + \varepsilon)$ and $U_2 = (b - \varepsilon, b)$. If J is a nonempty open interval, there exists N_1 such that $\forall n \in N_1, f^n(J) \cap U_1 \neq \phi$, because f is mixing. In the same way, there exists N_2 such that $\forall n \in N_2, f^n(J) \cap U_2 \neq \phi$. Therefore, for all $n \geq \max\{N_1, N_2\}$, $f^n(J)$ meets both U_1 and U_2, which implies that $f^n(J)$ $\supset [a + \varepsilon, b - \varepsilon]$ by connectedness. If J is a non degenerate subinterval, one considers $\text{Int}(J)$ which is not empty. Suppose now that for all $\varepsilon > 0$ and all non degenerate subinterval J there exists an integer N such that $\forall n \in N$, $f^n(J) \supset [a+\varepsilon, b-\varepsilon]$. Let U, V be two nonempty open sets in $[a, b]$. Let J, K be two non degenerate subintervals such that $J \subset U, K \subset V$ and neither a nor b is an endpoint of K, let $\varepsilon > 0$ such that

$$K \subset [a + \varepsilon; b - \varepsilon]$$

By assumption, there exists N such that $\forall n \in N$, $f^n(J) \supset [a + \varepsilon, b - \varepsilon]$, which implies that $f^n(U) \cap V \neq \phi$. Consequently, f is mixing.■

Definition 3.1.8: *Let $f : I \to I$ be an interval map, f is said λ-expanding if for every subinterval $[x, y]$ on which f is monotone one has $|f(x) - f(y)| \geq \lambda |x - y|$.*

Lemma 3.1.9: *Let $f : I \to I$ be a λ-expanding interval map. Suppose that $\lambda > N$, where N is a positive integer. If J is a non degenerate subinterval, then there exists an integer $n \geq 0$ such that $f^n(J)$ contains at least N distinct critical points.*

Proof: Let C_f be the set of critical points of f, it is a closed set. Put $\alpha = \lambda/N$ and consider a subinterval J. If J contains exactly k distinct critical points with $k \leq N-1$, then $J \backslash C_f$ has $k+1$ connected components, that we call $J_0, ..., J_k$. One has $|J_0| + ... + |J_k| = |J|$, thus there exists $0 \leq i \leq k$ such that $|J_i| \geq |J|/(k+1) \geq |J|/N$. The interval J_i does not contain any critical point thus $f|_{J_i}$ is monotone. By assumption one gets that $|f(J_i)| > \lambda |J_i|$, thus

$$|f(J)| \geq |f(J_i)| \geq \lambda |J_i| \geq \alpha |J| . \qquad (3.1)$$

Suppose that J is a non degenerate subinterval and that for all $n \geq 0, f^n(J)$ contains strictly less than N distinct critical points. According to Equation (1), $|f^n(J)| > \alpha^n |J|$ for all $n \geq 0$, which is a contradiction because $\alpha > 1$.■

Lemma 3.1.10: *Let $f : I \to I$ be an interval map and $a, b \in I$; $a < b$. Suppose that a is a fixed point of f and that f is increasing with a slope greater than $\lambda > 1$ on $[a, b]$. Then for all $\varepsilon > 0$ there exists $n \geq 0$ such that $f^n([a, a+\varepsilon]) \supset [a, b]$.*

Proof: If $a+\varepsilon \leq b$ then $f([a, a+\varepsilon]) \supset [a, a+\lambda\varepsilon]$ because a is a fixed point of f and f is increasing with a slope greater than $\lambda > 1$ on $[a, b]$. For the same reason, as long as a $a + \lambda^{k-1}\varepsilon \leq b$, one has $f^k([a, a+\varepsilon]) \supset [a, a+\lambda^k\varepsilon]$.

Since $\lambda > 1$, there exists an integer n such that $a + \lambda^{n-1}\varepsilon \leq b < a + \lambda^n \varepsilon$, hence $f^n([a, a+\varepsilon]) \supset [a, b]$.■

Example 3.1.11: Here is a basic example of a mixing transformation, which is a piecewise linear map with a constant slope (in absolute value).

Let p be an integer, $p \geq 2$. We define the map $f_p : [0,1] \to [0,1]$ by:

$f_p(x) = px - 2k$ if $x \in \left[\frac{2k}{p}, \frac{2k+1}{p}\right]$, $0 \leq k \leq \frac{p-1}{2}$,

$f_p(x) = -px + 2k + 2$ if $x \in \left[\frac{2k+1}{p}, \frac{2k+2}{p}\right]$, $0 \leq k \leq \frac{p-2}{2}$.

f_p is piecewise monotone and its slope is $\mp p$ on each interval of monotonicity. Moreover, the image of a non degenerate interval is non degenerate.

Let J be a non degenerate interval. By Lemma 3.1.9, there exists n such that $f_p^n(J)$ contains $p-1$ distinct critical points. If $p \geq 3$, $f_p^n(J)$ contains at least one critical point whose image is 0; if $p = 2$, $f_p^n(J)$ contains $\frac{1}{2}$ and $f_p^2(\frac{1}{2}) = f(1) = 0$. In both cases, f_p^{n+2} is a non degenerate interval containing 0. By Lemma 3.1.10, there exists an integer m such that $f_p^{n+m+2}(J) \supset \left[0, \frac{1}{p}\right]$, thus $f_p^{n+m+3}(J) \supset [0,1]$.■

If the interval map f is transitive, the following Proposition shows that either f is totally transitive, or the interval can be divided into two subintervals on which f^2 is totally transitive. The next Proposition shows that total transitivity implies mixing. These two results were proven by Barge and Martin [62, 63]. The proof, which is different, can be found in [64]. In the demonstrations below we follow the ideas of Barge and Martin.

Proposition 3.1.12: *Let $f : [a,b] \to [a,b]$ be a transitive interval map. Then one of the following cases holds:*

i) f is totally transitive (that is, f^n is transitive for all $n \geq 1$).

ii) There exists $c \in (a,b)$ such that, if $J = [a,c]$ and $K = [c,b]$, then $f(J) = K$ and $f(K) = J$. Moreover, $f^2|_J$ and $f^2|_K$ are totally transitive and c is the unique fixed point of f.

Proof: There exists a point x_0 of dense orbit and its ω-limit set $\omega(x_0, f)$ is equal to the whole interval $[a, b]$. Fix an integer $n \geq 1$ and put $W_i = \omega(f^i(x_0); f^n)$ for $0 \leq i \leq n-1$.

As $[a, b] = W_0 \cup ... \cup W_{n-1}$, at least one of the W_i's has a nonempty interior according to Baire's Theorem. Moreover, $f(W_i) = W_{i+1 \bmod n}$ for $0 \leq i \leq n-1$, hence each W_i has a nonempty interior. If $Int(W_i) \cap Int(W_j)$ $\neq \phi$; then $W_i = W_j$. Indeed it implies that there exists $k \geq 0$ such that $f^{kn+i}(x_0) \in Int(W_i) \cap Int(W_j)$; W_j is invariant by f^n, thus $f^{\acute{k}n+i}(x_0) \in W_j$ for all $k' \geq k$, so $W_i \subset W_j$. Similarly one has $W_j \subset W_i$ thus $W_i = W_j$.

Define ε_n as the set of connected components of the sets Int (W_i); $0 \leq i \leq n-1$. The elements of ε_n are disjoint open intervals the union of which is dense in $[a, b]$. For every $C \in \varepsilon_2, f(\overline{C})$ is a closed non degenerate interval , and it is contained in some W_i, thus there exists $C' \in \varepsilon_n$ such that $f(\overline{C}) \subset C'$. Moreover, if one fixes $C \in \varepsilon_n$, then for all $C' \in \varepsilon_n$ there exists $k \geq 1$ such that $f^k(\overline{C}) \cap C' \neq \phi$; because the orbit of x_0 is dense, thus $f^k(\overline{C})\overline{\in C'}$. This implies that ε_n is finite and its elements are cyclically permuted under the action of f. Write $\varepsilon_n = \{C_1, ..., C_p\}$ with $f(\overline{C_i}) \subset \overline{C_{i+1 mod p}}$ for $1 \leq i \leq p$; the inclusions are indeed equalities because the set $\cup C_i$ is dense hence $\cup f(C_i)$ is dense too.

If for every integer $n \geq 1$ the number of elements of ε_n is 1 then $\omega(x_0, f^n) = [a, b]$ and f is totally transitive; this is the case (i) of the Proposition.

Suppose that for a given n the number p of elements of ε_n is strictly greater than 1. We are going to show that $p = 2$. Let c be a fixed point of f . If there exists $C \in \varepsilon_n$ with $c \in C$, then $f(\overline{C}) = \overline{C}$, which is impossible because the elements of ε_n are cyclically permuted. Similarly, c cannot be an endpoint of $[a, b]$. Consequently, the point c is necessarily a common endpoint of two distinct elements C and C' of ε_n. One has then $f(\overline{C}) = \overline{C'}$ and $f(\overline{C'}) = \overline{C}$, which is possible only if $p = 2$; this also implies that n is even. If we write $J = [a, c]$ and $K = [c, b]$, we have $\varepsilon_n = \{$Int (J) , Int $(K)\}$

and

$$f(J) = K \quad , \quad f(K) = J \tag{3.2}$$

Moreover, c is the unique fixed point of f and c is not an endpoint. As n is even, one has
$\omega(x_0; f^n) \subset \omega(x_0; f^2)$; combining this with Equation (3.2), we get that

$$\varepsilon_2 = \{Int(J), Int(K)\}.$$

Therefore both $f^2|_J$ and $f^2|_K$ are transitive. If $f^2|_J$ is not totally transitive, then what is above shows that $f^2|_J$ has a unique fixed point $\acute{c}$ and $a < \acute{c} < c$. But c is already a fixed point of $f^2|_J$, thus $f^2|_J$ is totally transitive, as well as $f^2|_K$. This is the case (ii) of the Proposition.■

Proposition 3.1.13 : *Let $f : I \to I$ be an interval map. If f is totally transitive then it is mixing.*

Proof: Write $I = [a, b]$. Let J be a non degenerate subinterval and $\varepsilon > 0$. There exists a periodic point $x \in J$. One can find a periodic point $x_1 \in (a, a + \varepsilon)$ such that $f^n(x_1) \notin \{a, b\}$ for all $n \geq 0$. Define

$$y_1 = min\{f^n(x_1) \mid \ n \geq 0\}$$

and

$$z_1 = max\{f^n(x_n)|n \geq 0\}$$

In the same way, there exists a periodic orbit contained in (a, b) such that the maximal point z_2 belongs to $(b - \varepsilon, b)$; let y_2 be the minimal point in the orbit of z_2. Let k be a common multiple of the periods of x_1, y_1 and y_2. Put $g = f^k$ and $K = \bigcap_{n=0}^{+\infty} g^n(J)$. The set K is an interval because for all n, $g^n(J)$ contains x, which is a fixed point for g. Moreover, g is transitive by assumption; this implies that K is dense in $[a, b]$ so it contains the points y_1, y_2, z_1, z_2. Let p_i and q_i $(i = 1, 2)$ such that $y_i \in g^{p_i}(J)$ and $z_i \in g^{q_i}(J)$, and put $N = max\{p_1, p_2, q_1, q_2\}$. The points y_i and z_i are fixed by g, thus y_i, z_i belong to $g^N(J)$ and by connectedness $[y_i, z_i] \subset g^N(J)$. By choice of y_i, z_i, the interval $g^N(J) = f^{kN}(J)$ contains the whole orbit of y_i, thus $[y_i, z_i] \subset \ f^n(J)$ for all $n \geq kN$. Since $y_1 < a+\varepsilon$ and $z_2 > b-\varepsilon$, this implies that $[a-\varepsilon, b+\varepsilon]$ $\subset f^n(J)$ for all$n \geq kN$. This means that f is mixing by Proposition 3.1.7.

The next Proposition is a consequence of the above results.

Corollary 3.1.14: *Let $f : I \to I$ be a transitive interval map. Then f is mixing if and only if it has a periodic point of odd period different from 1.*

Proof: Suppose first that f is mixing. The set of fixed points of f is a closed set of empty interior. Let J be a non degenerate closed subinterval included in $Int(I)$ and containing no fixed point. Then, there exists an integer N such that $f^n(J) \supset J$ for all $n \geq N$. Let $n \geq N$ be an odd integer, one gets that f^n has a fixed point $x \in J$. Let p be the period of x for ; p divides n, thus p is odd; in addition $p > 1$ because $x \in J$.

Suppose now that f is transitive but not mixing. The map f is in the case (ii) of Proposition 3.1.12: there exist two subintervals J, K such that $f(J) = K$, $f(K) = J$ and $J \cup K = I$. Consequently, any periodic point has an even period, except the common endpoint of J and K which is a fixed point. By refutation, a transitive map with a periodic point of odd period different from 1 is mixing.■

Summary Theorems:

Theorem 3.1.15: *Let $f : [a, b] \to [a, b]$ be a transitive interval map. Then one of the following cases holds:*

1) f is mixing.

2) there exists $c \in (a,b)$ such that, if $J = [a,c]$ and $K = [c,b]$, then $f(J) = K$ and $f(K) = J$, in addition, c is the unique fixed point of f and both $f^2|_J$ and $f^2|_K$ are mixing.

Theorem 3.1.16: *Let $f : [a,b] \to [a,b]$ be an interval map. Then the following properties are equivalent:*

1) f is transitive and has a periodic point of odd period different from 1,

2) f^2 is transitive,

3) f is totally transitive,

4) f is weakly mixing,

5) f is mixing,

6) For all $\varepsilon > 0$ and all non degenerate subinterval J, there exists an integer N such that for all $n \geq N$, $f^n(J) \supset [a+\varepsilon, b-\varepsilon]$.

Example 3.1.17: Let $g : [0,1] \to [0,1]$ be define by:

$$g(x) = \begin{cases} \frac{1}{2} + 2x & , \quad 0 \leq x \leq \frac{1}{4} \\ \frac{3}{2} - 2x & , \quad \frac{1}{4} \leq x \leq \frac{1}{2} \\ 1 - x & , \quad \frac{1}{2} \leq x \leq 1 \end{cases}$$

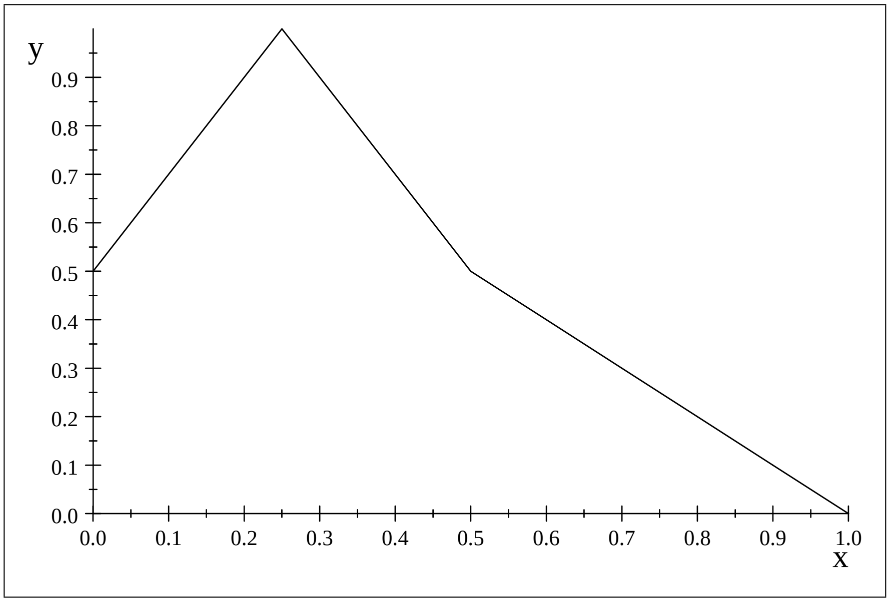


Figure 3.1.1: The graph of G(x)

Let $J = \left[0, \frac{1}{2}\right]$ and $K = \left[\frac{1}{2}, 1\right]$. Then $g(J) = K$ and $g(K) = J$. The map $g^2|_K$ is equal to the tent map f^2 of Example 3.1.11 up to a scaling thus $g^2|_K$ is mixing. The map $g^2|_J$ is similar to $g^2|_K$ except it is upside down, thus a similar proof shows that $g^2|_J$ is mixing. If U is a non empty open set then either $U \cap J \neq \phi$; and for all n large enough one has $g^{2n}(U) \supset J$ by Theorem 3.1.14, or $U \cap K \neq \phi$; and for all n large enough one has $g^{2n}(U) \supset K$. In both cases there exists n such that $g^n(U) \cup g^{n+1}(U) = [0, 1]$ thus g is transitive.

Now we end this section by showing that the tent map is transitive.

Example 3.1.18: In order to show that T is transitive, we will first show that there is an $x \in [0, 1]$ such that the orbit of x under T is dense in $[0, 1]$.

Consider the sequence s:

$$s = \underbrace{01}_{\text{1 block}}\underbrace{00011011}_{\text{2 block}}\underbrace{000001010011100101110111}_{\text{3 block}}...$$

where s is composed of all blocks of singles, doubles, triples, etc. of 0's and 1's. Since s is a non-finite sequence of 0's and 1's, $s \in A$. Then, since h is onto A, there exists some $x^* \in [0,1]$ such that $h(x^*) = s$, which implies that

$$x^* \sim x_0x_1x_2x_3x_4... = 01000110110000010100111001011101111... \quad .$$

Therefore,

$$x^* \in [0, 1/2] \quad \text{since} \quad x* \sim 0x_1x_2... \quad ,$$

$$T(x^*) \in [1/2, 1] \quad \text{since} \quad T(x^*) \sim 1x_2x_3... \quad ,$$

$$T^2(x^*) \in [0, 1/4] \text{ since } T^2(x^*) \sim 00x_4x_5... \quad ,$$

$$T^4(x^*) \in [1/4, 1/2] \quad \text{since} \quad T^4(x^*) \sim 01x_6x_7... \quad ,$$

$$T^6(x^*) \in [3/4, 1] \text{ since } T^6(x^*) \sim 10x_8x_9... \quad ,$$

etc. In general for any interval $L = [\frac{k}{2^n}, \frac{k+1}{2^n}]$, there exist a positive integer m such that $T^m(x) \in L$. Thus, the orbit of x under T is dense in $[0, 1]$.

Now we will show T is transitive. Let U and V be non-empty open intervals in $[0, 1]$. Since the orbit of x is dense, there exists a positive integer n such that $y \equiv T^n(x) \in U$. But then, again since the orbit of x is dense, there exists a positive integer m such that $T^{n+m}(x) = T^m(y) \in V$. Thus, T is transitive.

3.2 Transitivity and Density of Periodic Points

By [**10**], Vellekoop and Berglund showed that for continuous maps on an interval in $\mathbb{R}$, transitivity implies that the set of periodic points is dense.

The converse is note true: Define on $I = \mathbb{R}^+$ the function

$$f(x) = \begin{cases} 3x & , \quad 0 \leq x < \frac{1}{3} \\ -3x+2 & , \quad \frac{1}{3} \leq x < \frac{2}{3} \\ 3x-2 & , \quad \frac{2}{3} \leq x < 1 \\ f(x-1)+1 & , \quad x \geq 1 \end{cases}$$

The set of periodic points are dense but f is not transitive [example 2.1.15].

Example 3.2.1: Tent map has a dense set of periodic points.

Note that any number in [0, 1] of the form $\frac{\text{even integer}}{\text{odd integer}}$ is a periodic point for T (shown in [5]; in Section 3.0 we looked at the example x $= \frac{2}{9}$). Thus, to show that T has a dense set of periodic points, it suffices to show that the numbers of the form $\frac{\text{even integer}}{\text{odd integer}}$ are dense in [0, 1].

Let U be a non-empty open set in $[0,1]$. Then U must contain some interval $[a,b]$. Let $d = b-a$ and choose a positive odd integer n such that $n > \frac{2}{d}$. Then for any positive integer l,

$$\frac{l}{n} - \frac{l-1}{n} = \frac{1}{n} < \frac{d}{2}.$$

Since $[a,b] \subseteq U$ has length d, there must exist a positive integer k, $2 \leq k \leq n-1$ such that $\frac{k}{n}$ and $\frac{k-1}{n}$ are both in $[a,b] \subseteq U$. Then either k or $k-1$ is even, so that either $\frac{k}{n}$ or $\frac{k-1}{n}$ is of the form $\frac{\text{even integer}}{\text{odd integer}}$, which implies there exists a number of the form $\frac{\text{even integer}}{\text{odd integer}} \in [a,b] \subseteq U$. Thus, U contains a periodic point, and therefore T has a dense set of periodic points.

3.3 Uniform Sensitivity and Transitivity

Recall **Theorem 2.1.3** [6]**:** Let $f : X \to X$ be a continuous map where X is a metric space .Then if f is topologically transitive and has dense periodic point then f exhibits sensitive dependence on initial conditions.

And when $X = I$ by [10] transitivity $\Longrightarrow$ sensitivity.

The converse is not true.

Example 3.3.1: Consider the map $f : R \to R$ given by $f(x) = 2x$

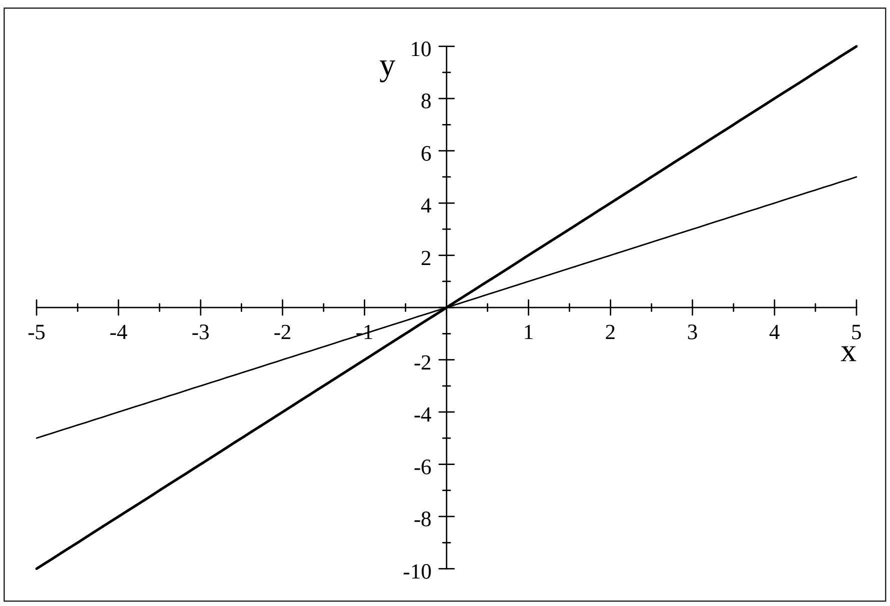


Figure 3.3.1 : f(x)=2x with line x

Clearly f is sensitive, but we can observe from the graph that for $x > 0, f^n(x) \to \infty$ when $n \to \infty$. On the other hand for for $x < 0,\ f^n(x) \to -\infty$ so there does not exist an orbit that goes from $x < 0$ to $x > 0$ or vice versa and the map can be decomposed into two open disjoint sets. So the f is not transitive.

Example 3.3.2: The tent map has sensitive dependence on initial conditions.

The following proof is due to Gulick in [21]. First recall that f has SDIC if there is an $\varepsilon > 0$ such that for any x in the domain and for any $\delta > 0$, there exist a y in the domain and a positive integer n that satisfy $|x - y| < \delta$ and $|f^n(x) - f^n(y)| > \varepsilon$.

Choose ε in the definition of SDIC to be1/4, let $x \in [0, 1]$, and let $\delta > 0$. We will show that there exist a dyadic rational v, an irrational w, and a positive integer m such that either

$$|T^m(x) - T^m(v)| > 1/4 \qquad \text{or} \qquad |T^m(x) - T^m(w)| > 1/4$$

It is shown in [21] that for any dyadic rational v, there exists a positive integer n such that $T^n(v) = 0$, and, since 0 is a fixed point, it follows that $T^k(v) = 0$ for all positive integers k such that $k \geq n$. It is also shown in [21] that any irrational w is not fixed or eventually fixed, or even eventually periodic for T. Then, if some iterate of w is in $(0, 1/2)$, some future iterate of w must be greater than $1/2$, since T doubles each number in $(0, 1/2)$. So there exists some integer $m > n$ such that $T^m(w) > 1/2$. Also, since dyadic rationals and irrationals are dense in $[0, 1]$, for any δ neighborhood U of x, there exist a dyadic rational v and an irrational w in both U and $[0, 1]$.

The idea behind this proof is that for any x, there is a dyadic rational and an irrational close to x. Eventually iterates of v will be 0, whereas there are infinitely many iterates of w that will be greater than $1/2$. Since an iterate of x cannot be close to both 0 and $1/2$, the associated iterates of x must be a certain distance (in this case, the distance is $1/4$) from either the associated iterates of v or the associated iterates of w.

Formally, let n be a positive integer such that $T^n(v) = 0$, and let $m > n$ be a positive integer such that $T^m(w) > 1/2$. Since $m > n$, $T^m(v) = 0$, and thus

$$|T^m(v) - T^m(w)| > 1/2.$$

Using the triangle inequality, we see that

$$1/2 < |T^m(v) - T^m(w)| \leq |T^m(v) - T^m(x)| + |T^m(x) - T^m(w)|.$$

Thus either

$$1/4 < |T^m(v) - T^m(x)|$$

or

$$1/4 < |T^m(x) - T^m(w)|.$$

Therefore, T has SDIC.

This definition of sensitivity is called uniform sensitivity.

3.4 Uniform Sensitivity and Pointwise Sensitivity

Definition 3.4.1: *A function f has pointwise sensitive dependence on initial conditions (PSDIC) if for every x in the domain of f, there is an $\varepsilon_x > 0$ such that for any $\delta > 0$, there exist a y in the domain and a positive integer n that satisfy $|x - y| < \delta$ and $|f^n(x) - f^n(y)| > \varepsilon_x$.*

Note that y and n depend on x, δ, and ε_x, and that ε_x depends on x, but in uniform definition, ε does not depend on x. This means that if a function has *USDIC*, then it automatically has *PSDIC*.

Proposition 3.4.2: *USDIC implies PSDIC.*

Proof: The only difference between the definitions of USDIC and PSDIC relates to the conditions on ε. In the definition of USDIC, the ε is independent of x, whereas in the definition of PSDIC, the ε is dependent on x. Thus, this implication is trivial.

Example 3.4.3: PSDIC does not imply USDIC.
Define a function $h_n : [0, 1/2^n] \to [0, 1/2^n]$ for $n = 1, 2, 3, ...$ such that:

$$h_n = \begin{cases} 2x & , \quad x \in \left[0, \frac{1}{2^{n+2}}\right] \\ -2x + \frac{1}{2^n} & , \quad x \in \left(\frac{1}{2^{n+2}}, \frac{1}{2^{n+1}}\right] \\ 2x - \frac{1}{2^n} & , \quad x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right] \end{cases}$$

Now define a function $H : [0, 1) \to [0, 1)$ by letting

$$H(x) = h_1(x) \text{ for } x \in [0, 1/2),$$

$$H(x) = h_2(x - 1/2) + 1/2 \text{ for } x \in [1/2, 3/4),$$

$$H(x) = h_3(x - 3/4) + 3/4 \text{ for } x \in [3/4, 7/8),$$

etc, so that

$$H(x) = h_n(x - \frac{2^{n-1} - 1}{2^{n-1}}) + \frac{2n - 1 - 1}{2^{n-1}} \text{ for } x \in [\frac{2^{n-1} - 1}{2^{n-1}}, \frac{2^n - 1}{2^n})$$

We will prove that H has *PSDIC* but not *USDIC* by considering each interval of H where H is equal to h_n for some positive integer n, possibly shifted up and to the right. We will see that on $[0, 1/2^n), h_n$ has *PSDIC* for each $n = 0, 1, 2, ...$ by first showing that if $x \in (\frac{1}{2^{n+1}}, \frac{1}{2^n})$, then eventually for some positive integer m, $h_n^m(x)$ will be in the interval $[0, 1/\left(2^{n+2}\right))$. On the interval $[0, \frac{1}{2^{n+2}})$, h_n is a modified Tent function and therefore has *PSDIC.*

First we will show that if $x \in (\frac{1}{2^{n+1}}, \frac{1}{2^n})$, then $h_n(x) < x$. Consider $x \in (\frac{1}{2^{n+1}}, \frac{1}{2^n})$. Then

$$h_n(x) - x = 2x - \frac{1}{2^n} - x$$

$$= x - \frac{1}{2^n}$$

$$< 0, \text{since} \quad x < \frac{1}{2^n}.$$

Thus, since $h_n(x) < x$ and $h_n(x) \in [0, \frac{1}{2^n})$ for $x \in (\frac{1}{2^{n+1}}, \frac{1}{2^n})$, $\{h_n^m(x)\}_{m=0}^{\infty}$ is a decreasing sequence for $x \in (\frac{1}{2^{n+1}}, \frac{1}{2^n})$, with no fixed points in the interval $(\frac{1}{2^{n+1}}, \frac{1}{2^n})$. So eventually for some positive integer m, $h_n^m(x) \in [0, \frac{1}{2^{n+1}}]$. Therefore, to prove that h_n has *PSDIC*, it is sufficient to consider h_n on the interval $[0, \frac{1}{2^{n+1}}]$. But once we are considering h_n on the interval $[0, \frac{1}{2^{n+1}}]$, the argument showing that h_n has *PSDIC* is analogous to the argument showing that the Tent function has USDIC in Example 3.3.2, letting ε_x in the definition of PSDIC be $\frac{1}{2^{n+3}}$. In any δ-neighborhood of any $x \in [0, \frac{1}{2^{n+1}}]$, we can find an irrational number v and a dyadic rational number w such that for some positive integer m,

$$|h_n^m(x) - h_n^m(v)| > \varepsilon_x$$

or

$$|h_n^m(x) - h_n^m(w)| > \varepsilon_x.$$

So on $[0, \frac{1}{2^n})$, h_n has *PSDIC* for each $n = 0, 1, 2, ...$ Shifting h_n up and to the left will not alter the conditions required for *PSDIC*, and thus H has *PSDIC* on every interval of the form $[\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^n-1}{2^n})$ in $[0, 1]$. Therefore, H has *PSDIC*. Note that although we can show that for each positive integer n, h_n has *USDIC*, the ε_x in the proof above depends on n, and therefore H does not necessarily have *USDIC*.

In fact, we will now show that H does not have *USDIC*. Let $\varepsilon > 0$ be arbitrary. Choose a positive integer n such that $\frac{1}{2^n} < \varepsilon$. Now consider h_n. For any $z \in [0, \frac{1}{2^{n+1}}], h_n(z)$ remains in the interval $[0, \frac{1}{2^n}]$. Therefore, let $x = 0$, and let $\delta < \frac{1}{2^n}$. Then for any $y \in [0, \frac{1}{2^n}]$ such that $|x - y| < \delta$, we have $h_n^m(y) \in [0, \frac{1}{2^n})$ for all positive integers m, which implies

$$|h_n^m(x) - h_n^m(y)| = |0 - h_n^m(y)| < \frac{1}{2^n} < \varepsilon$$

for every positive integer m. Now since H is a transposition of h_n on some subinterval of the form $[\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^{n-1}}{2^n})$, H does not have *USDIC*.

3.5 Extreeme Sensitivity and Uniform Sensitivity

Definition 3.5.1: *A function f has extreme sensitive dependence on initial conditions (ESDIC) if there is an $\varepsilon > 0$ such that for any x in the domain and for any $\delta > 0$, there exists a y in the domain such that $|x - y| < \delta$, $\limsup_{n\to\infty} |f^n(x) - f^n(y)| \geq \varepsilon$, and $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$.*

This definition is due to Du in [65].

Example 3.5.2: Tent map has *ESDIC*.

Proof: We will show that T has extreme sensitive dependence on initial conditions with $\varepsilon = 1/4$ in the definition of *ESDIC*. Let $x \in [0, 1], x \sim x_0x_1x_2...$, a non-finite sequence. Let

$$x'_n = \begin{cases} 1 \quad , \quad \text{if} \quad x_n = 0 \\ 0 \quad , \quad \text{if} \quad x_n = 1 \end{cases}$$

and choose $y \in [0,1]$ such that

$$y \sim s = x'_0 x'_1 x_2 x_3 x'_4 x'_5 x_6 x_7 ... x_{15} x'_{16} x_{17} x_{18} x_{19} ... x_{63} x'_{64} x'_{65} x_{66} ...,$$

where the y_m, y_{m+1} terms have primes on them, for $m = 0, 1, 2, ...$ Assume that this manipulation of the sequence associated with x does not produce a finite sequence. We will consider the case that s is finite after we examine the non-finite sequence case.

Since A is the set of all non-finite sequences and s is not a finite sequence by assumption, s is in A, and thus a $y \in [0.1]$ such that $h(y) = s$, or such that $y \sim s$, can be chosen since h is onto A. Consider the blocks of length $\frac{1}{4}$ of the interval $[0,1]$ and the corresponding first two terms of the associated sequence (see Figure 3.0.2).

Let

$$z \in [0,1], z \sim z_0 z_1 z_2 ...,$$

and let

$$w \sim z'_0 z'_1 z_2 ....$$

If $z_0 z_1 = 00$, then

$$w \sim z'_0 z'_1 z_2 ... = 11... \quad ,$$

so that $|z - w| > \frac{1}{4}$. If $z_0 z_1 = 01$, then $w \sim z'_0 z'_1 z_2 ... = 10...$, and $|z - w| > \frac{1}{4}$. Similarly, if $z_0 z_1 = 11$ or 10, then $|z - w| > \frac{1}{4}$.

Now consider $|x - y|$. By the statements above concerning z and w, we can see that

$$|x - y| > \frac{1}{4}, \quad \text{since} \quad y \sim x'_0 x'_1 ... \ .$$

We also have that

$$|T^4(x)-T^4(y)|>\frac{1}{4},\ \text{since}\qquad T^4(y)\sim x'_4x'_5...$$

$$|T^{16}(x)-T^{16}(y)|>\frac{1}{4},\ \text{since}\ \ T^{16}(y)\sim x'_{16}x'_{17}....$$

More generally, for any positive integer n,

$$|T^{4^n}(x)-T^{4^n}(y)|>\frac{1}{4},\text{since}\qquad T^{4^n}(y)\sim x'_{4^n}x'_{4^n+1}....$$

Thus,

$$\limsup_{n\to\infty}|T^n(x)-T^n(y)|>\frac{1}{4}.$$

Now it remains to show that $\liminf_{n\to\infty}|T^n(x)-T^n(y)|=0$. But

$$|T^2(x)-T^2(y)|<\frac{1}{4},\text{since}\ \ T^2(y)\sim x_2x_3...,$$

$$|T^6(x)-T^6(y)|<\frac{1}{2^{10}},\text{since}\qquad T^6(y)\sim x_6x_7...x_{15}....$$

For any positive integer n, we observe that the number of identical initial terms of $T^{4^n+2}(y)$ and $T^{4^n+2}(x)$ is $4^{n+1}-4^n-2$. Thus

$$|T^{4^n+2}(x)-T^{4^n+2}(y)|<\frac{1}{2^{(4^{n+1}-4^n-2)}}\to 0\quad\text{as}\quad n\to\infty.$$

Thus,

$$\liminf_{n\to\infty}|T^n(x)-T^n(y)|=0.$$

If the sequence s is finite, then we can still use the same ideas as just discussed, but we

will need to slightly modify the sequence, because a $y\in[0,1]$ such that $y\sim s$ does not necessarily exist, since the sequence is not in A. Instead, we consider the sequence

$t=x'_0x'_1x_2x_3x'_4x'_5\ \ 1\ \ x_7x_8...x_{15}x'_{16}x'_{17}\ \ 1\ \ x_{19}x_{20}...x_{63}x'_{64}x'_{65}\ \ 1\ \ x_{67}x_{68}...$

(We have replaced x_{4^n+2} for $n = 1, 2, 3, ...$ with the number one to guarantee that t is non-finite). Now $t \in A$, and therefore there exists a $y \in [0, 1]$ such that $h(y) = t$, or $y \sim t$, since h is onto A.

With such a sequence, we clearly still have that for any positive integer n,

$$|T^{4^n}(x) - T^{4^n}(y)| > \frac{1}{4}, \text{since} \;\; T^4(y) \sim x'_{4^n} x'_{4^n+1} \ldots .$$

The argument considering the $\liminf_{n\to\infty} |T^n(x) - T^n(y)|$ is slightly altered.

$$|T^7(x) - T^7(y)| < \frac{1}{2^9}, \;\text{since}\;\; T^7(y) \sim x_7 x_8 \ldots x_{15} \ldots,$$

$$|T^{19}(x) - T^{19}(y)| < \frac{1}{2^{45}}, \text{since} \;\;\; T^{19}(y) \sim x_{19} x_{20} \ldots x_{63} \ldots$$

For any positive integer n, we observe that the number of identical initial terms of $T^{4^n+3}(y)$ and $T^{4^n+3}(x)$ is $4^{n+1} - 4^n - 3$. Thus

$$|T^{4^n+3}(x) - T^{4^n+3}(y)| < \frac{1}{2^{4^{n+1}-4^n-3}} \to 0 \;\text{ as }\;\; n \to \infty,$$

and $\liminf_{n\to\infty} |T^n(x) - T^n(y)| = 0$.

So in both cases, T has *ESDIC*.

Proposition 3.5.3: *ESDIC implies USDIC.*

Proof: Let $f : X \to X$ have *ESDIC*, let $x \in X$, and let U be any open neighborhood of x. Then by the definition of *ESDIC*, there exists $\delta > 0$ and there is a $y \in U$ such that

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| \geq \varepsilon.$$

So clearly there exists a positive integer n such that

$$|f^n(x) - f^n(y)| \geq \frac{\varepsilon}{2},$$

and thus f has *USDIC*.

Example 3.5.4: *USDIC does not imply ESDIC.*

Proof: In [65] Du shows that a function with *USDIC* does not necessarily have *ESDIC* by letting $S = \{x|x = x_0x_1x_2..., \text{ where } \ x_n = 0 \text{ or } \ 1\}$ and defining the metric d on S by $d(x,y) = \sum_{n=0}^{\infty} \frac{x_n - y_n}{2^{n+1}}$. Finally, define the shift map σ by $\sigma(x) = \sigma(x_0x_1x_2...) = x_1x_2....$ Du shows that σ has *USDIC* but not $ESDIC$.

3.6 Specification and Mixing

We saw that a transitive interval map has dense periodic points. If in addition the map is mixing then it satisfies the specification property, which roughly means that there exist periodic points whose orbits approach pieces of orbits arbitrarily chosen. This result is due to Blokh [64].

We first shows three Lemmas then we prove the specification property for mixing interval maps.

Lemma 3.6.1: *Let $f : I \to I$ be an interval map. Consider $x \in I$, $0 < \varepsilon < |I|/2$ and$n \geq 0$. Then there exist closed subintervals $J_0, ..., J_n$ such that*

* $f(J_i) = J_{i+1}$ *for* $0 \leq i \leq n-1$,

$*f^i(x) \in J_i$,

$*J_i \subset [f^i(x) - \varepsilon, f^i(x) + \varepsilon]$,

$*$ there exists $0 \leq i \leq n$ such that the interval J_i contains either $f^i(x) - \varepsilon$ or $f^i(y) + \varepsilon$.

In addition, if $x + \varepsilon \in I$, then one can choose $J_0 \subset [x, x + \varepsilon]$, and if $x - \varepsilon \in I$ one can choose $J_0 \subset [x - \varepsilon, x]$.

Proof: Write $x_k = f^k(x)$ for $k \geq 0$. We show the Lemma by induction on n.

Case n = 0: since $\varepsilon < |J|/2$, the interval I contains either $x - \varepsilon$ either $x + \varepsilon$. If $x + \varepsilon \in I$, we can put $J_0 = [x, x + \varepsilon]$, and if $x - \varepsilon \in I$ we can put $J_0 = [x - \varepsilon, x]$.

Suppose that the Lemma is true at rank $n - 1$, and write $J_0, ..., J_{n-1}$ the subintervals given by the Lemma. If $f(J_{n-1}) \subset [x_n - \varepsilon, x_n + \varepsilon]$, put $J_n = f(I_{n-1})$; the intervals $(J_0, ..., J_n)$ are suitable. Otherwise, $f(J_{n-1}) = f^n(J_0)$ is not included in $[x_n - \varepsilon, x_n + \varepsilon]$; by connectedness $f^n(J_0)$ contains either $x_n - \varepsilon$ or $x_n + \varepsilon$. We suppose that $J_0 \subset [x, x + \varepsilon]$, the case when $J_0 \subset [x - \varepsilon, x]$ being similar.

Put

$$y = min\{y > x | f^n(y) \in \{x_n - \varepsilon, x_n + \varepsilon\}\} < x + \varepsilon.$$

In this way, $f^n([x, y])$ equals either $[x_n - \varepsilon, x_n]$ or $[x_n, x_n + \varepsilon]$. Put $J'_0 = [x, y]$ and $J'_i = f^i(J'_0)$ for $1 \leq i \leq n$. Then the intervals $(J'_0, ..., J'_n)$ are suitable because $J'_i \subset J_i \subset [x_i - \varepsilon, x_i + \varepsilon]$ for $0 \leq i \leq n - 1$ and J'_n contains either $x_n - \varepsilon$ or $x_n + \varepsilon$. This ends the induction.■

Lemma 3.6.2: *Let $f : [a, b] \to [a, b]$ be a mixing interval map such that a is a non accessible fixed point. Let $0 < \varepsilon < (b-a)/2$. Then there exists $\delta > 0$ such that, for all $x \in [a, a + \delta]$ and all $n \geq 0$, there exist closed subintervals $J_0, ..., J_n$ satisfying*

$*$ *$f(J_i) = J_{i+1}$ for $0 \leq i \leq n-1$,*

$*$ *$J_0 \subset [a+\delta, b-\delta]$,*

$*$ *$J_i \subset [f^i(x) - \varepsilon, f^i(x) + \varepsilon]$ for $0 \leq i \leq n$,*

$*$ *there exists $0 \leq i \leq n$ such that $|J_i| \geq \varepsilon/4$.*

Proof: By continuity of f, there exists $\eta > 0$ such that $f(y) < a + \varepsilon$ for all $a \leq y \leq a+\eta$. By transitivity, $f([a, a+\varepsilon]) \not\subset [a, a+\varepsilon]$ thus there exists $z \in (a, a+\varepsilon]$ such that f(z)$\geq$ a+ε. Moreover there exists a fixed point c such that $a < c < min\{a, a+\eta, a+\varepsilon/2\}$, let $\delta = c - a$. Note that $a + \varepsilon < b - \delta$ because $\delta < \varepsilon/2 < (b-a)/4$. Let $K = [c, a+\varepsilon]$, one has $c, z \in K$ thus by connectedness $f(K) \supset K$.

Let $x \in [a, a+\delta] = [a, c]$ and $n \geq 0$, write $x_k = f^k(x)$ for all $k \geq 0$. Let $0 \leq m \leq n$ be the greatest integer such that $x_0, ..., x_m \in$ [a, c]. Note that $K \subset [x_i - \varepsilon, x_i + \varepsilon]$ for all $0 \leq i \leq m$. Chain of m intervals $(K, ..., K)$, we get subintervals $J_0, ..., J_m$ such that $J_m = K$, $J_i \subset K$ and $f(J_i) = J_{i+1}$ for $0 \leq i \leq m-1$. If $m = n$ then the proof is finished because the length of K is $a + \varepsilon - c > \varepsilon/2$. If $m < n$ then $x_{m+1} > c$ by choice of m and

$x_{m+1} = f(x_m) < a + \varepsilon$ by choice of $c < a + \eta$, hence $x_{m+1} \in K$. Since $|K| \geq \varepsilon/2$, the set K contains either $x_{m+1} - \varepsilon/4$ or $x_{m+1} + \varepsilon/4$. Then Lemma 3.6.1 applies and gives intervals $J'_{m+1}, ..., J'_n$ satisfying

$*J'_{m+1} \subset K$,

$*f(J'_i) = J_{i+1}\ for\ \ m+1 \leq i \leq n-1$,

$*x_i \in J'i\ \ for\ \ m+1 \leq i \leq n$,

$*J'_i \subset [x_i - \varepsilon/4, x_i + \varepsilon/4]$,

there exists $m+1 \leq i \leq n$ such that the interval J'_i contains either $x_i - \varepsilon/4$ or $x_i + \varepsilon/4$, thus the length of J'_i is at least $\varepsilon/4$.

$(J_0, ..., J_m = K, J'_{m+1})$ is a chain of intervals, thus there is subintervals $J'_0, ..., J'_m$ such that $J'_i \subset J_i$, $f(J'_i) = f(J'_{i+1})$ for $0 \le i \le m-1$ and $f(J'_m) = J'_{m+1}$. The sequence $(J'_0, ..., J'_n)$ satisfies the required properties.

Lemma 3.6.3: *Let $f : I \to I$ be an interval map and I_0 a subinterval of I. Suppose that for every non degenerate subinterval J there exists an integer N such that for all $n \ge N$, $f^n(J) \supset I_0$. Then for every $\varepsilon > 0$ there exists an integer N such that, for all subintervals J of length at least ε and all $n \ge N$, one has $f^n(J) \supset I_0$.*

Proof: Write $I = [a, b]$. Let p be an integer such that $(b-a)/p < \varepsilon/2$. For $0 \le k \le p-1$, put $J_k = (a + \frac{k(b-a)}{p}, a + \frac{(k+1)(b-a)}{p})$. By assumption, for all $0 \le k \le p-1$ there exists an integer N_k such that for all $n \ge N_k$, $f^n(J_k) \supset I_0$. Put $N = max\{N_0, ..., N_{p-1}\}$. Let J be a subinterval of length at least ε; it contains an interval J$_k$ for some k, thus for all $n \ge N$, $f^n(J) \supset I_0$.$\blacksquare$

Theorem 3.6.4 (Blokh): *If $f : I \to I$ is a mixing interval map, then it satisfies the specification property.*

Proof: First notice that if f^2 has the specification property then so has f by continuity; in addition, if f is mixing then f^2 is mixing too (Theorem 3.1.14). Therefore, we can suppose that the non accessible endpoints are fixed points, even if we may consider f^2 instead of f.

Let $0 < \varepsilon < |I|/4$ and write $I = [a, b]$. If both a and b are accessible, put $I_0 = [a, b]$. Otherwise, consider $\delta > 0$ the real given by Lemma 3.6.2 and put either $I_0 = [a + \delta, b]$ or $I_0 = [a, b - \delta]$ or $I_0 = [a + \delta, b - \delta]$ depending on a or b or both being non accessible fixed points (if both a and b are non accessible, taker the smaller of the two). By assumption f is mixing thus, according to the choice of I_0, for every non degenerate subinterval J there exists an integer N such that for all $n \ge N$, $f^n(J) \supset I_0$. Let N denote the integer given by Lemma 3.6.3 for $\varepsilon/4$, if J is a subinterval with $|J| \geq \varepsilon/4$ then for all $n \geq N$, $f^n(J) \supset I_0$.

If $J_0, ..., J_k$ are intervals satisfying $f(J_i) = J_{i+1}$ for $0 \leq i \leq n-1$ and if $|J_j| \geq \varepsilon/4$ for some j then

$$\forall n \geq N, f^n(J_k) \supset I_0 \quad , \quad (4.1)$$

This is due to the fact that $f^n(J_k) = f^{n+k-j}(J_j)$ and $n+k-j \geq N$.

Let $x \in I$ and $n \geq 0$. Then there exist subintervals $J_0, ..., J_n$ such that:

$*J_0 \subset I$,

$*J_i \subset [f^i(x) - \varepsilon, f^i(x) + \varepsilon]$ for $0 \leq i \leq n$,

(6.2)

$*f(J_i) = J_{i+1}$ for all $0 \leq i \leq n-1$,

$*$there exists $0 \leq i \leq n$ such that $|J_i| \geq \varepsilon/4$.

We split the proof of (6.2) depending on $x \in I_0$ or not. If $x \in I_0$, then either $x - \varepsilon \in I_0$ or $x + \varepsilon \in I_0$, and the subintervals $J_0, ..., J_n$ are obtained by applying Lemma 3.4.1. If a is not accessible and if $x \in [a, a+\delta]$ then Lemma 3.6.2 gives the suitable subintervals because $J_0 \subset [a+\delta, b-\delta] \subset I_0$ and one of the J_i's has a length at least $\varepsilon/4$. The same is true if $x \in [b-\varepsilon, b]$ with b a non accessible endpoint.

Now we show the following property by induction on p: Let $x_1, ..., x_p$ be points of I and $m_1 \leq n_1 < m_2 \leq n_2 < ... < m_p \leq n_p$ integers satisfying $m_{i+1} - n_i \geq N$ for all $1 \leq i \leq p-1$. Then there exist closed intervals $J_{m_1}, J_{m_1+1}, ..., J_{n_p}$ such that

$*J_{m_1} \subset I_0$,

$*f(J_i) = J_{i+1}$ for all $m_1 \leq i \leq n_p - 1$, (6.3)

$*J_i \subset [f^i(x_k) - \varepsilon; f^i(x_k) + \varepsilon]$ for all $1 \leq k \leq p$ and $m_k \leq i \leq n_k$,

$*f^n(J_{n_p}) \supset I_0$ for all $n \geq N$.

Case $p = 1$: we apply (6.2) with $x = f^{m_1}(x_1)$ and $n = n_1 - m_1$; the last condition is reached thanks to Equation (3.1).

Suppose that the property is true at rank $p-1$ and write $J_{m_1}, ..., J_{n_{p-1}}$ the corresponding intervals. We apply (3.2) with $x = f^{m_p}(x_p)$ and $n = n_p - m_p$ and we call the resulting intervals $J'_{m_p}, ..., J'_{n_p}$. According to Equation (3.1), $f^n(J'_{n_p}) \supset I_0$ for all $n \geq N$. Put

$$J_i = f^{i-n_{p-1}}(J_{n_{p-1}}) \quad \text{for all} \quad n_{p-1} < i \leq m_p.$$

By assumption, $m_p - n_{p-1} \geq N$, thus $J_{m_p} = f^{m_p - n_{p-1}}(J_{n_{b-1}}) \supset I_0 \supset J'_{m_p}$ by Equation (3.1). So $(J_{m_1}, ..., J_{m_{p-1}}, J'_{m_p})$ is a chain of intervals and we can restrict the intervals $J_{m_1}, .., J_{n_{p-1}}$ to intervals $J'_{m_1}, ..., J'_{n_{p-1}}$ such that $f(J'_i) = f(J'_{i+1})$ for $m_1 \leq i \leq m_{p-1}$. Then the sequence $J'_{m_1}, ..., J'_{n_p}$ satisfies (6.3).

It is now easy to prove that f has the specification property. Let $x_1, ..., x_p$ be points of I and $m_1 \leq n_1 < m_2 \leq n_2 < ... < m_p \leq n_p$ integers satisfying $m_{i+1} - n_i \geq N$ for $1 \leq i \leq p-1$. Let q be an integer greater than or equal to $n_p - m_1 + N$. We build the intervals $J_{m_1}, ..., J_{n_p}$ satisfying (6.3). For all $n \geq N, f^n(J_{n_p}) \supset I_0$, thus $f^q(J_{m_1}) = f^{q-n_p+m_1}(J_{n_p}) \supset I_0 \supset J_{m_1}$. There exists a point $x \in J_{m_1}$ such that $f^q(x) = x$. Put $y = f^{q-m_1}(x)$ so that $f^{m_1}(y) = x \in J_{m_1}$. We have $f^q(y) = y$ and

$$\forall 1 \leq k \leq p, \forall m_k \leq i \leq n_k, f^i(y) \in J_i[f^i(x_k) - \varepsilon, f^i(x_k) + \varepsilon]$$

it is exactly the specification property.■

Remark 3.6.5: *The specification property implies mixing [60], thus these two notions are equivalent for interval maps.*

3.7 Blending and Transitivity

In [35] Cranell suggests blending as an alternative of transitivity hypothesis in Devaney's definition of chaos. Blending is also a purely topological condition as transitivity.

Definition 3.7.1: *Consider the continuous* $f : X \to X$ *on a metric space* X. *Then* f *is said to be a strongly blending map if for every non-empty open sets* $U, V \subset X$ $\exists n > 0$ *such that* $f^n(U) \cap f^n(V)$ *contains an open set.*

Also f is said to be a *weakly blending* map if for every non-empty open sets $U, V \subset X$ $\exists n > 0$ such that $f^n(U) \cap f^n(V) \neq \phi$.

Now we will give two main theorems that give a relation between blending (strong and weak) with topological transitivity. The first theorem is valid for any subset of $\mathbb{R}^n$ and its proof will be given. On the other hand the second theorem is valid only for compact spaces in the real line. Its proof is omitted and can be found in [35].

Theorem 3.7.2: *Consider the continuous map* $f : X \subset R^n \to X$ *and let* f *has dense periodic points. If* f *is strongly blending then* f *is transitive.*

Proof: Consider two non empty open subsets $U, V \subset X$. Then $\exists n > 0$ such that $M \subset f^n(U) \cap f^n(V)$, where M is an open set in X. Now let $Y := f^{-n}(M) \cap V$. Then since f is a continuous map and Y is also open (as intersection of two open sets) we can choose a periodic point q in Y with period $m > n$. But then $f^n(q) \in M$ and there exist $y \in U$ such that $f^n(q) = f^n(y)$. Finally we have $f^m(y) = f^{m-n}(f^n(q)) = f^m(q) = q$ and so $q \in f^m(U) \cap V \neq \phi$ and the theorem is proved.

Before giving the second theorem we will define the *repelling fixed point.*

Definition 3.7.3: *Consider the continuous and differentiable map $f : X \to X$ and let $p \in X$ to be a fixed point for the map f. Then p is said to be a repelling fixed point for the map f if $|f'(p)| > 1$.*

Theorem 3.7.4: *Let $f : I \to I$ be a continuous map on the compact interval I. If f has a repelling fixed point and f is also transitive then f is weakly blending.*

Proof: see [35]

Remark 3.7.5: The converse of theorem 3.7.2 does not hold since there exist transitive maps without being strongly blending.

Example 3.7.6: Consider the map $f : S^1 \to S^1$ defined by $f(\theta) = \theta + k$, where $\frac{k}{\pi}$ is an irrational number. Then f is transitive but not strongly blending (also f is not weakly blending).

Remark 3.7.7: The converse of theorem 3.7.4 is also not true since there exist weakly blending maps without being transitive.

Example 3.7.8: Let the map F be the odd extension of the Tent map in $[-1, 1]$ i.e.

$$F(x) = \begin{cases} -(2x-2) & , \quad -1 \leq x \leq -\frac{1}{2} \\ 2x & , \quad -\frac{1}{2} < x < \frac{1}{2} \\ 2-2x & , \quad \frac{1}{2} \leq x \leq 1 \end{cases}$$

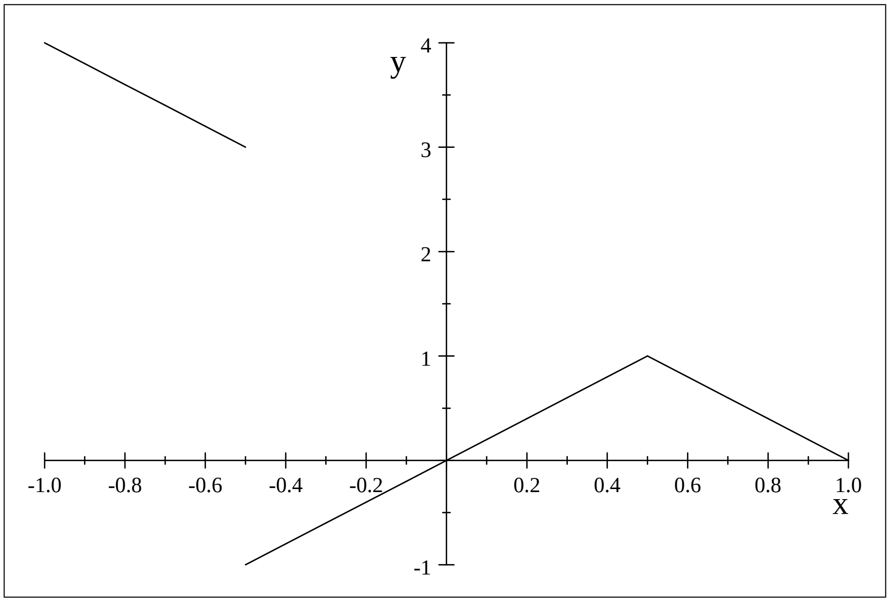


Figure 3.7.1:$F\left(x\right)$

then F is not transitive because the interval $(0,1)$ is not mapped onto any subinterval of $(-1,0)$.

But it is weakly blending since every open subinterval of $[-1,1]$ eventually maps onto another subinterval which contains the fixed point of F which is located at the origin.

3.8 Lyapunov Exponent and Sensitivity

Recall that the *Lyapunov exponent* $\lambda(x)$ of f at x is defined by $\lambda(x) = lim_{n\to\infty}\frac{1}{n}\ln|(f^n)'(x)|$, if the limit exists. To find a formula that is more tractable, notice that

$$\lambda(x) = \lim_{n\to\infty}\frac{1}{n}\ln|(f^n)'(x)|$$

$$= \lim_{n\to\infty} \frac{1}{n} \ln|f'(f^{n-1}(x)) \cdot (f^{n-1})'(x)|, \quad \text{by the chain rule}$$

$$= \lim_{n\to\infty} \frac{1}{n} (\ln|f'(x_{n-1})| + ln|(f^{n-1})'(x)|)$$

$$= \lim_{n\to\infty} \frac{1}{n} (\ln|f'(x_{n-1})| + ln|f'(f^{n-2}(x)) \cdot (f^{n-2})'(x)|), \text{by the chain rule again}$$

$$= \lim_{n\to\infty} \frac{1}{n} (\ln|f'(x_{n-1})| + ln|f'(x_{n-2})| + ln|(f^{n-2})'(x)|)$$

$$\vdots$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} ln|f'(x_k)|, \tag{8.1}$$

if the limit exists. There are examples of continuous functions for which the limit does not necessarily exist for all x (or for any x, for that matter) in the domain of f, so that the *Lyapunov exponent* does not necessarily exist for such x. In fact, Tent is such a function, since $T'(\frac{1}{2})$ does not exist. In some definitions, a function f is considered chaotic if the limit exists for a dense set of x in the domain of f, and if for these values of x, the Lyapunov exponent of f is positive. We will say f has a *positive Lyapunov exponent* (*PLE*) if $\lambda(x)$ exists for a dense set of x and if $\lambda(x) > 0$ for all x in the domain of f such that $\lambda(x)$ exists. Moreover, f is *PLE* chaotic if f has a positive *Lyapunov exponent.*

Example 3.8.1: Tent map has *positive Lyapunov exponent.*

Let $x \in [0, 1]$.

$$T'(x) = \begin{cases} 2 & , \quad 0 < x < \frac{1}{2} \\ -2 & , \quad \frac{1}{2} < x < 1 \end{cases}$$

So $|T'(x_k)| = 2$ if $T'(x_k)$ exists. Thus, if $\lambda(x)$ exists,

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |T'(x_k)|$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln(2)$$

$$= \lim_{n\to\infty} \frac{1}{n}(n)(\ln(2))$$

$$= \lim_{n\to\infty} \ln(2)$$

$$= \ln(2)$$

and $\ln(2) > 0$. Now it only remains to show that the Lyapunov exponent actually exists for a dense set of values in the domain of T. But if x is irrational, then $T^n(x) \neq \frac{1}{2}$, or 1 (the only possible values where the derivative of T fails to exist) for any positive integer n, so $\lambda(x)$ exists. The set of irrational numbers is dense in [0, 1], and thus, T has *PLE*.

Example 3.8.2: USDIC does not imply PLE (and therefore PSDIC does not imply PLE).

Let $F : [0, \infty) \to [0, \infty)$ be defined by $F(x) = (\sqrt{x} + 1)^2$

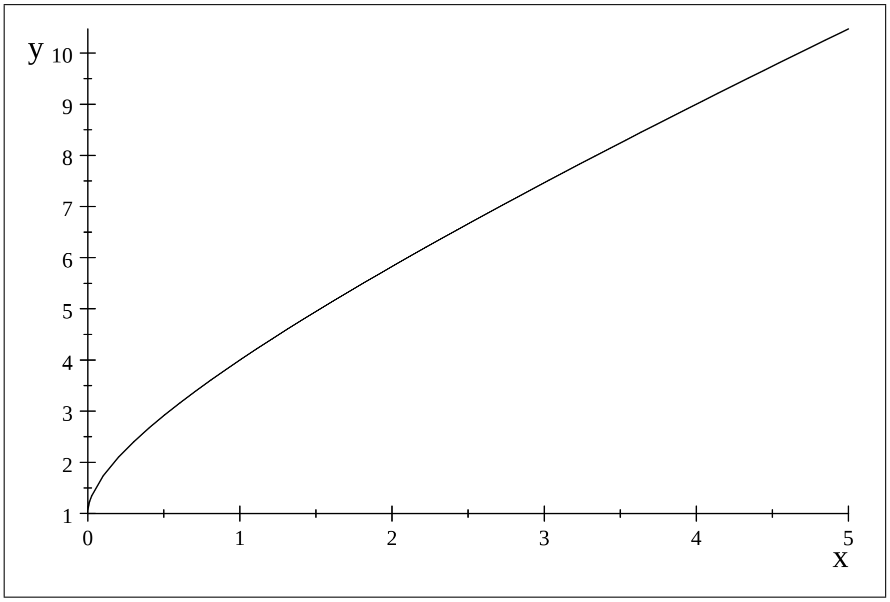


Figure 3.8.1:$F(x) = (\sqrt{x} + 1)^2$

This function and the following proof are from Pennings in [66],[59] .

First we will show that F has *USDIC*.

By an induction argument we will show that $F^n(x) = (\sqrt{x} + n)^2$ for all positive integers n.

Base case: $n = 1$

Then $F^n(x) = F^1(x) = (\sqrt{x} + 1)^2$.

Now assume that $F^n(x) = (\sqrt{x}+n)^2$ for all n such that $1 \leq n \leq k$, where k is an arbitrary positive integer. Consider the case $n = k + 1$.

$$F^n(x) = F^{k+1}(x) = F(F^k(x))$$

$$= F((\sqrt{x} + k)^2), \text{by the inductive hypothesis}$$

$$= (\sqrt{(\sqrt{x} + k)^2} + 1)^2$$

$$= (\sqrt{x} + k + 1)^2$$

Thus, by induction, for all positive integers n,

$$F^n(x) = (\sqrt{x} + n)^2. \tag{8.1}$$

Let $x \in [0, \infty)$, and let $\delta > 0$ be arbitrary. Then by (8.2),

$$|F^n(x) - F^n(x + \delta)| = |(\sqrt{x} + n)^2 - (\sqrt{x} + \delta + n)^2|$$

$$= |2n\sqrt{x} - \delta - 2n\sqrt{x + \delta}|$$

$$= \delta + 2n(\sqrt{x + \delta} - \sqrt{x})$$

As n increases to infinity, so will $\delta + 2n(\sqrt{x} + \delta - \sqrt{x}) = |F^n(x) - F^n(x + \delta)|$, and thus $|F^n(x) - F^n(x + \delta)| > \varepsilon$ for a large enough n, regardless of the choice of x and ε. Thus, F has *USDIC*.

Now we will show that F does not have *PLE*.

For all $x \in [0, \infty)$,

$$F'(x) = 2(\sqrt{x} + 1) \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x}}$$

$$= \frac{\sqrt{x} + 1}{\sqrt{x}}.$$

By (8.1), $x_k = F^k(x_0) = (\sqrt{x_0} + k)^2$, so we have $\sqrt{x_k} = \sqrt{x_0} + k$. Therefore,

$$F'(x_k) = \frac{\sqrt{x_k} + 1}{\sqrt{x_k}} = \frac{\sqrt{x_0} + k + 1}{\sqrt{x_0} + k}.$$

Thus,

$$\sum_{k=0}^{n-1} \ln|F'(x_k)| = \sum_{k=0}^{n-1} \ln\left|\frac{\sqrt{x_0}+k+1}{\sqrt{x_0}+k}\right|$$

$$= \ln(\frac{\sqrt{x_0}+1}{\sqrt{x_0}}) + \ln(\frac{\sqrt{x_0}+2}{\sqrt{x_0}+1}) + ... + \ln(\frac{\sqrt{x_0}+n}{\sqrt{x_0}+n-1})$$

$$= (\ln(\sqrt{x_0}+1) - \ln(\sqrt{x_0})) + (\ln(\sqrt{x_0}+2) - \ln(\sqrt{x_0}+1)) + ...$$

$$+(\ln(\sqrt{x_0}+n) - \ln(\sqrt{x_0}+n-1))$$

$$= \ln(\sqrt{x_0}+n) - \ln(\sqrt{x_0}) \tag{8.2}$$

Therefore,

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln|F'(x_k)|$$

$$= \lim_{n\to\infty} \frac{(\ln(\sqrt{x_0}+n) - \ln(\sqrt{x_0}))}{n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{x_0}+n}}{n}$$

$$= 0$$

So the Lyapunov exponent at an arbitrary x_0 is 0, and thus F does not have *PLE*.

Example 3.8.3: *PLE* does not imply *PSDIC* (and thus *PLE* does not imply *USDIC* or *ESDIC*).

Proof : Let $G : [0,1] \to [0,1]$ as follows:

$$G(x) = \begin{cases} 0 & , \quad x = 0 \\ 4x - 3(\frac{2^n-1}{2^n}) & , \quad x \in \left(\frac{2^n-1}{2^n}, \frac{2^{n+2}-3}{2^{n+2}}\right], n = 0, 1, 2, ... \\ -2x + 3(\frac{2^{n+1}-1}{2^{n+1}}) & , \quad x \in \left(\frac{2^{n+2}-3}{2^{n+2}}, \frac{2^{n+1}-1}{2^{n+1}}\right] \\ 1 & , \quad x = 1 \end{cases}$$

Note that G is defined so that $G'(x) = 4$ for $x \in (0, \frac{1}{4}), (\frac{1}{2}, \frac{5}{8}), (\frac{3}{4}, \frac{13}{16}), (\frac{7}{8}, \frac{29}{32}), ...$ and $G'(x) = -2$ for $x \in (\frac{1}{4}, \frac{1}{2}), (\frac{5}{8}, \frac{3}{4}), (\frac{13}{16}, \frac{7}{8}), ...$ (G is an infinite number of lopsided, shrinking tents along the line y = x.)

Now consider $\lambda(x)$ for $x \in [0,1]$ such that x is not a dyadic rational. Then

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \ln |(G^n)'(x)|$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |G'(x_k)|$$

But if $G'(x_k)$ exists, then $G'(x_k) = 4$ or -2, so $\ln |G'(x_k)| = \ln(2)$ or $\ln(4)$. Thus for all x where the Lyapunov exponent exists,

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |G'(x_k)|$$

$$\geq \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln(2)$$

$$= \lim_{n\to\infty} \frac{1}{n} . n \ln(2)$$

$$= \ln(2)$$

$$> 0$$

So clearly $\lambda(x) > 0$ wherever $\lambda(x)$ exists. Now we must show that $\lambda(x)$ does exist for a dense set of $x \in [0, 1]$. The only points where G' fails to exist are the points $x = \frac{2^n-1}{2^n}$ or $x = \frac{2^{n+2}-3}{2^{n+2}}$ for any positive integer n. Thus, clearly when x is not a dyadic rational, $G^n(x)$ will never be of the form $\frac{2^n-1}{2^n}$ or $\frac{2^{n+2}-3}{2^{n+2}}$ for any n, so $G'(x_k)$ will always exist, which implies that $\lambda(x)$ exists and that G has *PLE*.

Next we will show that $x \leq G(x)$, and that then G does not have *PSDIC*. The function G is a piecewise linear function, so the only possible local extrema are at the endpoints of the intervals on which G is piecewise defined. Since $G(x)$ is increasing for $x \in (\frac{2^n-1}{2^n}, \frac{2^{n+2}-3}{2^{n+2}}), n = 0, 1, 2, ...$(the slope on these intervals is 4) and decreasing for $x \in (\frac{2^{n+2}-3}{2^{n+2}}, \frac{2^{n+1}-1}{2^{n+1}}), n = 0, 1, 2, ...$(the slope on these intervals is -2), clearly the local minima occur at numbers of the form $x = \frac{2^{n+1}-1}{2^{n+1}},\ n = -1, 0, 1, 2, ...$

Let $x = \frac{2^{n+1}-1}{2^{n+1}}$for an arbitrary $n = -1, 0, 1, 2, ...$ If $n = -1$, then $x = 0$ and $G(x) = 0 = x$.

For all other values of n,

$$G(x) = G(\frac{2^{n+1}-1}{2^{n+1}}) = -2(\frac{2^{n+1}-1}{2^{n+1}}) + 3(\frac{2^{n+1}-1}{2^{n+1}}) = \frac{2^{n+1}-1}{2^{n+1}} = x$$

Thus, at the local minima, $G(x) = x$, and therefore $G(x) \geq x$ for all other values of $x \in [0, 1]$. So we have shown that for any $x \in [0, 1]$, $x \leq G(x)$. Note that this implies that $x \leq G^n(x)$ for all positive integers n.

Now we will show that G does not have *PSDIC* by showing that G does not have the requirements for *PSDIC* at $x = 1$. Choose any ε, with $0 < \varepsilon < 1$. Let $0 < \delta < \varepsilon$. Then if $|1 - y| < \delta$ and $y \in [0, 1]$, we must have $y \in (1 - \delta, 1]$. Also, for any positive integer n, we have $y \leq G^n(y)$ by the above argument and $G^n(1) = 1$ by the definition of G. Thus,

$$|G^n(1) - G^n(y)| = |1 - G^n(y)|$$

$$\leq |1 - y|$$

$$< \delta$$

$$< \varepsilon$$

So there does not exist an y and a positive integer n such that $|1-y| < \delta$ and

$$|G^n(1) - G^n(y)| > \varepsilon.$$

Thus, G does not have *PSDIC* at 1, and so G does not have *PSDIC*.

In this example, G has a *positive Lyapunov exponent* at a dense set of x in the domain but fails to have *PSDIC* at only one element in the domain, namely $x = 1$.

3.9 Expansivity and Sensitivity

Expansivity is a condition related directly with sensitive dependence on initial conditions but these two conditions are clearly not equivalent as we will explain later on.

Definition 3.9.1: *Consider a metric space X equipped with the metric d and the map $f : X \to X$. Then f is said to be expansive if there exists a positive number $c > 0$ such that if $x, y \in X$ and $x \neq y$, then $\exists n > 0$ such that $d(f^n(x), f^n(y)) \geq c$. The positive number c is is called an expansive constant for f.*

Remark 3.9.2: Clearly from the definition 3.9.1 expansivity implies sensitivity because in expansivity all nearby points of x separate by at least c (in sensitivity condition we need only one point to have this property). If a map is expansive then any two orbits become at least a fixed distance apart. On the other hand trivially sensitivity does not imply expansivity.

3.10 Shared Periodic Orbit and Transitivity

A function f has a shared periodic orbit if for every pair of non-empty open intervals U and V in X, there is a periodic point $p \in U$ such that $f^n(p) \in V$ for some positive integer n.

Example 3.10.1: Tent map has a shared periodic orbit.

Let U and V be non-empty open intervals in $[0,1]$. There exist positive integers k and n such that $[\frac{k}{2^n}, \frac{k+1}{2^n}] \subseteq U$. Similarly, there exist positive integers l and m, with $m > n$, such that $[\frac{l}{2^m}, \frac{l+1}{2^m}] \subseteq V$. We can construct a sequence $s = x_0x_1x_2...$ in a manner similar to the one in example 3.1.18 by specifying $x_0, x_1, x_2, ..., x_{a+b}$ such that $h(x^*) = s$ for some $x^* \in [0,1]$ and

$$T^a(x^*) \in [\frac{k}{2^n}, \frac{k+1}{2^n}] \subseteq U, T^{a+b}(x^*) \in [\frac{l}{2^m}, \frac{l+1}{2^m}] \subseteq V$$

where h is as defined in (0.1). If we then let $s = \overline{x_0x_1x_2...\ x_{a+b}}$, x^* will be periodic. Thus, T has a shared periodic orbit.

Proposition 3.10.2: *A shared periodic orbit implies transitivity.*

Proof: Let $f : X \to X$ have a shared periodic orbit, and let U and V be non-empty open intervals in X. Then by the definition of shared periodic orbit, there is a periodic point $p \in U$ and a positive integer n such that $f^n(p) \in V$. Thus since $f^n(p) \in f^n(U)$, we have $f^n(p) \in (f^n(U) \cap V)$, which implies $f^n(U) \cap V \neq \phi$, so f is transitive.

Proposition 3.10.3: *Transitivity implies a shared periodic orbit.*

Proof: Let $f : X \to X$ be transitive, and let U and V be non-empty open intervals in X. By the definition of transitivity, there exists $x \in U$ such that $f^n(x) \in V$ for some positive integer n. Since V is an open set, there exists an $\varepsilon > 0$ such that

$$(f^n(x) - \varepsilon, f^n(x) + \varepsilon) \subseteq V.$$

By hypothesis f is continuous, which implies that f^n is continuous. By the definition of continuity, there exists some $\delta > 0$ such that if $|x - y| < \delta$, then $|f^n(x) - f^n(y)| < \varepsilon$, which implies that

$$f^n(y) \in (f^n(x) - \varepsilon, f^n(x) + \varepsilon) \subseteq V.$$

Let $W \equiv ((x - \delta, x + \delta) \cap U)$. Then W is open. Since f is transitive, f has a dense set of periodic points. So there exists a periodic point $p \in W$. But

$$f^n(W) \subseteq (f^n(x) - \varepsilon, f^n(x) + \varepsilon) \subseteq V,$$

so $f^n(p) \in V$. Since $W \subseteq U$, we know that $p \in U$. Thus, we have found a periodic point p such that $p \in U$ and $f^n(p) \in V$. Therefore, f has a shared periodic orbit.

3.11 Mixing and Sensitivity

Proposition 3.11.1: *Consider a metric space X equipped with the metric d and the continuous map $f : X \to X$. If f is a mixing map then it is* *sensitive.*

Proof: Consider a positive number d and two points say $p, q \in X$ such that $d(p,q) > 4d$. Consider also the balls $B_d(p)$ and $B_d(q)$ both with radius d and centers p and q respectively. Now we take a point $x \in X$ and we choose an open neighbourhood of x say $N_\varepsilon(x)$ for some $\varepsilon > 0$. Now since f is a mixing map then $\exists n_1, n_2 \in N$ such that

$$f^n(N_\varepsilon(x)) \cap B_d(p) \neq \phi \forall n > n_1$$

and

$$f^n(N_\varepsilon(x)) \cap B_d(q) \neq \phi \ \forall n > n_2.$$

If we choose $n > \max\{n_1, n_2\}$ then $\exists y_1, y_2 \in N_\varepsilon(x)$ such that

$$f^n(y_1) \in B_d(p)$$

and

$$f^n(y_2) \in B_d(q).$$

Then we have

$$d(f^n(y_1), f^n(y_2)) \geq 2\delta.$$

Also from the triangular inequality we get

$$(f^n(y_2), f^n(x)) \geq \delta \qquad \text{or} \quad d(f^n(y_1), f^n(x)) \geq \delta.$$

So f is sensitive with sensitivity constant 4δ.

3.12 Summary

The following graphs summarize the results in chapter 3

$$\begin{tikzcd}
Transitivity \arrow[r] \arrow[d, Rightarrow] & Dense\ periodic\ points \\
USDIC & Shared\ periodic\ orpit \arrow[l] \arrow[u, Rightarrow]
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{array}{c}\textit{Transitivity} \ + \\ \textit{periodic point of} \\ \textit{odd period different from 1}\end{array} \arrow[r, shift left] \arrow[d, Rightarrow, shift right] & Mixing \arrow[l, shift left] \arrow[d, Rightarrow, shift right] \\
Totally\ transitive \arrow[r, shift left] \arrow[u, Rightarrow, shift right] & weakly\ mixing \arrow[l, shift left] \arrow[u, Rightarrow, shift right]
\end{tikzcd}$$

$$\begin{tikzcd}
PSDIC \arrow[d, Rightarrow] & & \\
USDIC \arrow[d, Rightarrow] & Specification \arrow[r, shift left] & Mixing \arrow[l, shift left] \\
ESDIC & &
\end{tikzcd}$$

$$\begin{array}{ccc}
\mathit{Expansivity} & \longrightarrow & \mathit{Sencitivity} \\
 & & \\
\begin{array}{c}\mathit{Trasitivity}\ + \\ \mathit{repelling} \\ \mathit{fixed\ point}\end{array} & \longrightarrow & \begin{array}{c}\mathit{Weakly} \\ \mathit{blending}\end{array} \\
 & & \\
\begin{array}{c}\mathit{Blending}\ + \\ \mathit{dense} \\ \mathit{periodic\ point}\end{array} & \longrightarrow & \mathit{Trasitivity}
\end{array}$$

4 Relationships Between Definitions of Chaos when X=I

In this chapter we discuss the mutual relations between the notions of chaos described in the chapter two for the special case of interval maps. In fact, throughout the present section we consider continuous maps $f : I \to I$ from a nontrivial compact interval $I = [a, b]$, $a < b$, into itself. The main results of this chapter say that in this case B/C-chaos and D-chaos and most other forms are equivalent while, on the other hand, B/C-chaos and D-chaos are sufficient for L/Y-chaos.

4.1 Devaney versus Block-Coppel

Recall that a continuous map $f : I \to I$ on a nontrivial compact interval I is

t-chaotic if and only if one of the following equivalent conditions is satisfied:

(i) f^m is turbulent for some $m \in N$,

(ii) f^m is strictly turbulent for some $m \in N$,

(iii) f has a periodic point whose period is not a power of 2.

Three more results are needed in order to reach the goals of this section:

Proposition 4.1.1: *f is L/Y-chaotic if and only if not every point in I is approximately periodic.*

Proof: See [4, p.145]).

Lemma 4.1.2: *f is B/C-chaotic if and only if there exists a $c \in I$ such that $\omega(c, f)$ contains a periodic orbit as a proper subset.*

Proof : See [4, VI Proposition 6]).

Lemma 4.1.3: *Let J and K be two compact subintervals of I having the property $K \subseteq f(J)$. Then there exists a compact subinterval L of J such that $f(L) = K$ and that f maps the endpoints of L onto the endpoints of K.*

Proof: Let $K = [a, b]$ for two points $a, b \in I$ and let c be the largest point in J with $f(c) = a$. If there exists an $x \in J, x > c$, with $f(x) = b$, let d be the smallest x with this property. Then with $L := [c, d]$ the claim follows. On the other hand, if there exists an $x \in J, x < c$, with $f(x) = b$ we define c' as the largest x with this property. Let d' be the smallest $x \in (c', c](\subset J)$ satisfying $f(x) = a$. Then the interval $L := [c', d']$ has the claimed property and the proof of the lemma is complete.

Theorem 4.1.4[44] : *A continuous map $f : I \to I$ on an interval I is D-chaotic if and only if it is B/C-chaotic.*

Proof: $\Rightarrow$ Let f be *D-chaotic* with compact *D-chaotic* set $Y \subseteq I$. Then Y is infinite since $f|_Y$ has sensitive dependence on initial conditions. Furthermore, since $f|_Y$ is transitive there is a $c \in Y$ with $\omega(c, f) = Y$, and because of the relation $\overline{P(f|_Y)} = Y$ the map $f|_Y$ has a periodic orbit. As a finite set this periodic orbit is a proper subset of $Y = \omega(c, f)$, and this implies (by Lemma 4.1.3) that f is *B/C-chaotic*.

$\Leftarrow$ Now suppose f is *B/C-chaotic*. Then the map f^m is strongly turbulent for some $m \in N$, i.e. there exist two disjoint compact subintervals X_0 and X_1 of I with the property that for $g := f^m$ we have

$$X_0 \cup X_1 \subseteq g(X_0) \cap g(X_1). \tag{4.1}$$

The idea of proceeding from here is to first derive from (4.1) the existence of a compact $g - invariant$ subset X of $X0 \cup X1$ with the property that the map $g|_X : X \to X$ is semi-conjugate to the shift via a continuous surjection $s : X \to \sum$ and then to show that there exists a compact $g - invariant$ subset Z of X on which g is *D-chaotic*. We carry out this program in 5 steps.

Step 1: Construction of X and s: Starting with the above X_0 and X_1 and using mathematical induction, for each $\alpha = (a_1, a_2, ...) \in \sum$ Lemma 4.1.3 yields a sequence of compact, pairwise disjoint intervals $X_{a_1 a_2 \ldots a_k}, (a_1, a_2, ..., a_k) \in \{0, 1\}^k, k \geq 1$ in $X_0 \cup X_1$ having the following properties:

$$X_{a_1 a_2 \ldots a_k} \subseteq X_{a_1 a_2 \ldots a_{k-1}}, g(X_{a_1 a_2 \ldots a_k}) = X_{a_2 a_3 \ldots a_k} \qquad 4.2$$

and g maps endpoints of $X_{a_1 a_2 \ldots a_k}$ onto endpoints of $X_{a_2 a_3 \ldots a_k}$. Then for each $\alpha = (a_1, a_2, ...) \in \sum$ the set

$$X_\alpha := \bigcap_{k=1}^{\infty} X_{a_1 \ldots a_k} \tag{4.3}$$

is either a singleton or a nontrivial compact interval. Furthermore we have

$$X_\alpha \cap X_\beta = \phi \quad \text{for all } \alpha, \beta \in \textstyle\sum, \alpha \neq \beta \tag{4.4}$$

since the sets $X_{a_1 a_2 \ldots a_k}, (a_1, a_2, ..., a_k) \in \{0,1\}^k$, are pairwise disjoint and

$$g(X_\alpha) = X_{\sigma(\alpha)} \quad \text{for all} \quad \alpha \in \sum . \qquad (4.5)$$

Next we define the set

$$X' = \bigcup_{\alpha \in \Sigma} X_\alpha \qquad (4.6)$$

which turns out to be strongly $g - invariant$ and compact. Also the set

$$X := \{x \in I \mid x \text{ is an endpoint of } \ X \ \text{ for some } \ \alpha \in \Sigma\}$$

is compact (even if $X_\alpha = \{x\}$ for some x we call x an endpoint of X_α). From (4.2) and (4.5) we conclude that for any $\alpha \in \Sigma$ the map g maps the endpoints of X_α onto the endpoints of $X_{\sigma(\alpha)}$ and that X is strongly $g - invariant$. On X we define the map

$$s : X \to \Sigma, \quad x \mapsto \alpha \text{ if } x \in X_\alpha.$$

Obviously, this map is well defined, continuous and onto and each point of Σ is the $s - image$ of at most two points of X. Finally, because of (4.5) and the definition of s we have

$$s \circ g|_X = \sigma \circ s \ \text{ on } X. \qquad (4.7)$$

Step 2: Construction of Z: For any $\alpha \in \Sigma$ the set X_α defined in (4.3) is a nonempty compact interval. Since the X_α's are pairwise disjoint (see [4]) there exist at most countably many α's in Σ such that X_α is not a singleton. Therefore the set

$$R := \{x \in X \mid X_\alpha = \{x\} \text{ for some } \alpha \in \Sigma\}$$

is nonempty and consists of all but countably many points of X. Because of (4.5) the set R is $g - invariant$ and the set

$$Z := \overline{R}$$

and the map $g|_Z : Z \to Z$ are well defined.

Step 3: Transitivity of $g|_Z$: Let U be an arbitrary open nonempty subset of Z. Then there exists a point $x \in U \cap R$ and some $\alpha = (a_1, a_2, ...) \in \Sigma$ with $X_\alpha = \{x\}$.

Because of definition (4.3) of X and the openness of U in Z there exists a $k \in N$ with

$$Z \cap X_{a_1 a_2 \ldots a_k} \subseteq U.$$

Therefore, in order to prove the transitivity of $g|_Z$ it suffices to prove the relation

$$g^k(Z \cap X_{a_1 a_2 \ldots a_k}) = Z. \tag{4.8}$$

Since $Z = \overline{R}$ and since the set $Z \cap X_{a_1 a_2 \ldots a_k}$ is compact, it even suffices to find a g^k-pre-image of an arbitrary point $y \in R$ in the set $Z \cap X_{a_1 a_2 \ldots a_k}$. Due to the definition of R, for any $y \in R$ there exists a $\beta = (b_1, b_2, ...) \in \Sigma$ with $\{y\} = X_\beta$. With the aid of this we define

$$\gamma := (a_1, a_2, ..., a_k, b_1, b_2, ...) \in \Sigma$$

and use (4.5) to get the relation

$$g^k(X_\gamma) = X_\beta = \{y\}. \tag{4.9}$$

If X_γ consists of a single point we get the inclusion $X_\gamma \subseteq Z$ and the claim (4.8) is proved, since X_γ is a subset of $X_{a_1 a_2 ... a_k}$. If, on the other hand, X_γ is a nontrivial interval then at least one of its endpoints is contained in Z. This can be shown as follows: For any $n \in N$ there exists (because of (4.3)) a number $m_n \in N$ with

$$X_{a_1 a_2 ... a_k} b_1 b_2 ... b_{m_n} \subseteq \{x \in I \mid dist(x, X) < \frac{1}{n}\}, \tag{4.10}$$

and since the set

$$\{(a_1, a_2, ..., a_k, b_1, b_2, ..., b_{m_n}, *, *, ...) \in \Sigma | * \in \{0, 1\}\}$$

is uncountable we can find a point γ_n in this set such that $X_{\gamma_n} = \{y_n\}$. By (4.4) the sets X_γ and X_{γ_n} are disjoint and by (4.10) the distance of the point y_n from at least one of the endpoints of X$_\gamma$ is less than $\frac{1}{n}$ (since $y_n \in X_{a_1 a_2 ... a_k} b_1 b_2 ... b_{m_n}$). Because the relation $X_{\gamma_n} \subset R$ holds for all $n \in N$, the sequence $(y_n)_n \in N$ in R converges, to one of the endpoints of X_γ. On the other hand, because of $Z = \overline{R}$ this endpoint is contained in Z and it is mapped via g^k to y according to (4.9). In both cases we thus can find a g^k-preimage of the point y in $Z \cap X_{a_1 a_2 ... a_k}$, and this proves claim (4.8).

Step 4: $\overline{P(g|_Z)} = Z$: Let U again be an arbitrary open nonempty set in Z and x a point in $U \cap R$ with $\{x\} = X_\alpha$ for some $\alpha = (a_1, a_2, ...) \in \Sigma$. As in the proof of Step 3, given any $n \in N$ there is an $m_n \in N$ with

$$X_{a_1a_2...a_{m_n}} \subseteq \{x \in I \mid dist(x, X_\alpha) < \frac{1}{n}\}. \tag{4.11}$$

We now consider the periodic point

$$\gamma_n := (a_1, a_2, ..., a_{m_n}, ...) \in \Sigma$$

and notice that because of $\sigma^{m_n}(\gamma_n) = \gamma_n$ and (4.5) we get $g^{m_n}(X_{\gamma_n}) = X\gamma_n$. Furthermore, the two endpoints of X_{γ_n} are periodic with respect to g, since g^{m_n} maps the endpoints of X_{γ_n} onto the endpoints of $g^{m_n}(X_{\gamma_n})(= X_{\gamma_n})$. In case X_{γ_n} is a nontrivial interval then at least one of its (periodic) endpoints is contained in$Z$. This can be seen as in the previous Step 3. So in any case, for any $n \in N$ we get a g-periodic point $x_n \in X_{\gamma_n} \cap Z$ and the sequence $(x_n)_{n\in N}$ converges to x because of $X_{\gamma_n} \subseteq X_{a_1...a_{m_n}}$ and (4.11). This implies the relation $x \in \overline{P(g|_Z)}$ and completes the proof of Step 4.

Step 5: Conclusion: The set Z is infinite (because R is infinite), and therefore the map g is *D-chaotic* on Z, then f is $D-chaotic$ on $Y := \bigcup_{i=0}^{n-1} f^i(Z)$. This completes the proof of the Theorem .■

4.2 Entropy versus Li-Yorke

Theorem 4.2.1[70]**:** *On intervals, the entropy type of chaos implies the Li and Yorke chaos, and the converse is not true.*

Proof: Let us assume that f^m has a *horseshoe (* We say that a continuous map $g : I \to I$ has a horseshoe if there exist $a < c < b$ in I such that

$[a,b] \subseteq g([a,c]) \cap g([c,b]))$. For some $m \geq 1$, so f^m has a point of period 3, thus there are some points $y < r < s < z$ such that: $[y,z] \subseteq f^{2m}([y,r])$ and $[y,z] \subseteq f^{2m}([s,z])$ (see section two in chapter two). We write

$$I_0 := [y,r], \quad I_1 := [s,z] \qquad \text{and} \qquad g := f^{2m}.$$

We claim: $\forall n \geq 1, \forall u_0, u_1, ..., u_n \in \{0,1\}$, there exists a non-empty compact interval $I_{u_0 u_1 ... u_n} \subseteq I_{u_0 ... u_{n-1}}$ such that:

$$g(I_{u_0 ... u_n}) = I u_1 ... u_n.$$

We prove this claim by induction on n. For $n = 1$, we have: $I_{u_1} \subseteq g(I_{u_0})$, so, by lemma 2.2.2, there exists a compact interval $I_{u_0 u_1} \subseteq I_{u_0}$ such that $g(I_{u_0 u_1}) = I_{u_1}$, and then necessarily $I_{u_0 u_1}$ is non-empty. Let us then assume that our claim is true until an integer $n \geq 1$.

By the induction hypothesis: $I_{u_1 ... u_{n+1}} \subseteq I_{u_1 ... u_n} = g(I_{u_0 ... u_n})$, so there exists a compact subinterval $I_{u_0 ... u_{n+1}} \subseteq I_{u_0 ... u_n}$ such that

$$g(I_{u_0 ... u_{n+1}}) = I_{u_1 ... u_{n+1}},$$

and necessarily $I_{u_0 ... u_{n+1}}$ is non-empty, which shows that the claim is true for $n+1$, and achieves the proof by induction.

We then set, for $u \in \{0,1\}^{\mathbb{N}}$,

$$I_u := \bigcap_{n \geq 0} I_{u_0 ... u_n}.$$

I_u is non-empty, otherwise, because of the compacticity of $[y,z]$, there would exist

$N \geq 0$ such that

$$\bigcap_{n=0}^{N} I_{u_0 ... u_n} = I_{u_0 ... u_N} = \phi,$$

which is not.

Let us remark that, for all $u \in \{0,1\}^{\mathbb{N}}$, and for all $n \geq 0$,

$$(x \in I_{u_0 \ldots u_n}) \implies (\forall k \in \{0, \ldots, n\}, g^k(x) \in I_{u_k}).$$

Indeed, for $k \in \{0, \ldots, n\}$,

$$g^k(x) \in g^k(I_{u_0 \ldots u_n}) = I_{u_k \ldots u_n} \subseteq I_{u_k}.$$

We then easily see that if $x \in I_u$, for all $n \geq 0, g^n(x) \in I_{u_n}$. This implies in particular that for $u, v \in \{0,1\}^{\mathbb{N}}$, $(u \neq v) \implies (I_u \cap I_v = \phi)$. Moreover, I_u, as intersection of compact intervals, is a single point or an interval $[c, d], c < d$. Let $C := \{u \in \{0,1\}^{\mathbb{N}} : l(I_u) := length(Iu) > 0\}$, $C = \bigcup_{n \geq 0} C_n$, where

$$C_n := \{u \in \{0,1\}^{\mathbb{N}} : l(I_u) > 1/(n+1)\}.$$

Taking $u^1, u^2, \ldots, u^k$ distinct elements of C_n, since $I_{u^1} \cup \ldots \cup I_{u^k} \subseteq [y, z]$, we have:

$$z - y \geq l(I_{u^1} \cup \ldots \cup I_{u^k}) = l(I_{u^1}) + \ldots + l(I_{u^k}) > k/(n+1),$$

i.e.

$$k < (n+1)(z-y),$$

meaning that C_n is finite, and then C is *countable*. For $u \in \{0,1\}^{\mathbb{N}} \backslash C$, we then write $I_u =: \{x_u\}$, and fixing $a \in \{0,1\}^{\mathbb{N}} \backslash C$, we set:

$$S := \{x_u, u := u(b) := a_0 b_0 a_0 a_1 b_0 b_1 a_0 a_1 a_2 b_0 b_1 b_2 \ldots \notin C, b \in \{0,1\}^{\mathbb{N}}\}.$$

We will show that S is a scrambled set of f. Since $\{0,1\}^{\mathbb{N}}$ is uncountable, since $b \in \{0,1\}^{\mathbb{N}} \mapsto u(b) \in \{0,1\}^{\mathbb{N}}$ is injective,since C is countable, and since

$u \in \{0,1\}^{\mathbb{N}} \backslash C \rightarrowtail x_u \in [y,z]$ is injective, S is uncountable.

Now let $x_u, x_v \in S, x_u \neq x_v$, i.e. there exists $i \geq 0$ such that $b_i \neq b'_i$, where

$$u =: a_0 b_0 a_0 a_1 b_0 b_1 a_0 a_1 a_2 b_0 b_1 b_2 ...$$

and

$$v =: a_0 b'_0 a_0 a_1 b'_0 b'_1 a_0 a_1 a_2 b'_0 b'_1 b'_2 ....$$

Since for all $n \geq 0$ and $k \in \{0, ..., n\}$,

$$u_{2(1+2+...+n)+k} = u_{n(n+1)+k} = a_k = v_{n(n+1)+k},$$

$$u_{2(1+2+...+n)+n+1+k} = u_{(n+1)^2+k} = b_k$$

and

$$v_{(n+1)^2+k} = b'_k$$

we have, for all $n \geq i$,

$$g^{(n+1)^2+i}(x_u) \in I_{b_i} \quad \text{and} \quad g^{(n+1)^2+i}(x_v) \in I_{b'_i},$$

so

$$|g^{(n+1)^2+i}(x_u) - g^{(n+1)^2+i}(x_v)| \geq s - r > 0.$$

From $(|g^{(n+1)^2+i}(x_u) - g^{(n+1)^2+i}(x_v)|)_{n \geq 0}$, we now can extract a convergent subsequence, showing that:

$$\limsup_{n \to \infty} |f^n(x_u) - f^n(x_v)| \geq s - r > 0.$$

Let us now assume that $\alpha := \liminf_{n \to \infty} |f^n(x_u) - f^n(x_v)| > 0$ (α exists because the sequence is bounded). Since $a \notin C$, I_a is a point, then there exists $k \geq 0$ such that $l(I_{a_0 \ldots a_k}) =: \beta < \alpha$. We have, for all $n \geq k$, since

$$x_u \in I_{u_0 \ldots u_{n(n+1)+n}}, g^{n(n+1)}(x_u) \in I_{u_{n(n+1)} \ldots u_{n(n+1)+n}} = I_{a_0 \ldots a_n} \subseteq I_{a_0 \ldots a_k},$$

and similarly:

$$g^{n(n+1)}(x_v) \in I_{a_0 \ldots a_k},$$

hence:

$$|g^{n(n+1)}(x_u) - g^{n(n+1)}(x_v)| \leq \beta.$$

Now, from the sequence $(|g^{n(n+1)}(x_u) - g^{n(n+1)}(x_v)|)_{n \geq 0}$, we can extract a subsequence converging to γ, say. We have: $\gamma \leq \beta < \alpha$, which is absurd.

Finally,

$$\liminf_{n \to \infty} |f^n(x_u) - f^n(x_v)| = 0$$

and this achieves the first part of the proof.

For the second part we will give a counterexample later.■

4.3 Block- Coppel versus t-chaos

Proposition 4.3.1: *If f non-chaotic (t - chaotic) then for any $x \in I$, the limit set $\omega(x, f)$ contains a unique minimal set M,i.e. $M = \omega(y, f)$ for every $y \in \omega(x, f)$.*

Proof: See Proposition 7 in [4], chapter V I.■

Theorem 4.3.2:[4] *Suppose that $f : X \to X$ then f is $BC-chaotic$ if and only if f is $t-chaotic$.*

Proof: Suppose f is $BC-chaotic$, choose $\alpha \in \Sigma$ so that $\omega(\alpha, \sigma) = \Sigma$ and $x \in \tilde{X}$ so that $h(x) = \alpha$. If we set

$$L = \omega(x, f^m)$$

then

$$h(L) = \Sigma$$

Thus there exists $x' \in L$ so that $h(x') = \alpha$. But if f were chaotic then $L' = \omega(x', f^m)$ would be a minimal set by proposition 2.11.6 and hence

$$L \subseteq SR(f)$$

($SR(f)$ denote the set of all strongly recurrent points). Since $h(L') = \Sigma$ and $h(SR(f)) \subseteq SR(\sigma)$ this yields a contradiction. Now suppose that f is $t-chaotic$ map. Let I_0, I_1 be disjoint compact subintervals such that :

$$I_0 \cup I_1 \subseteq f(I_0) \cap f(I_1).$$

Let $I_{a_1 a_2}$ be a subinterval of I_{a_1} of minimal length such that $f(I_{a_1 a_2}) = I_{a_2}$,where $a_1, a_2 = 0$ or 1. Proceeding inductively, let $I_{a_1 \ldots a_k}$ be a subinterval of $I_{a_1 \ldots a_{k-1}}$ of minimal length such that $f(Ia_1 \ldots a_k) = Ia_2 \ldots a_k$, where $a_1, \ldots, a_k = 0$ or 1. It is readily seen that

$$I_{a_1 \ldots a_k} \cap I_{b_1 \ldots b_k} = \phi \quad \text{if} \quad (a_1, \ldots, a_k) \neq (b1, \ldots, bk).$$

For any infinite sequence $\alpha = (a_1, a_2, \ldots)$ of 0's and 1's. Let $I = \bigcap_{k=1}^{\infty} I_{a_1 \ldots a_k}$. Then I_α either a compact interval or a single point. Moreover $I_\alpha \cap I_\beta = \phi$ if $\alpha \neq \beta$, and hence I_α is an interval for most countably many values of α. Since f is continuous:

$$f(I_\alpha) = \bigcap_{k=2}^{\infty} f(I_{a_1 \ldots a_k}) = \bigcap_{k=2}^{\infty} I_{a_2 \ldots a_k} = I_\sigma(\alpha)$$

where $\sigma(\alpha) = (a_2, a_3, ...)$. Thus if I_α is a point, so also is $I_{\sigma(\alpha)}$. Let

$$\tilde{X} = \bigcup_{\alpha \in \Sigma} I_\alpha$$

We will show that $\tilde{X}$ is closed, and hence compact, subset of $I_0 \cup I_1$. Suppose $x_n \to x$, where $x_n \in I_{\alpha_n}$. By restricting attention to a subsequence we may assume that $\alpha_n \to \alpha = (a_1, a_2, ...)$.Then for any given k, $I_{\alpha_n} \subseteq I_{a_1,..,a_k}$ for all large n. Thus $x \in I_{a_1...a_k}$ for every k, and hence $x \in I_\alpha$. Since $I_\alpha \cap I_\beta = \phi$ if $\alpha \neq \beta$, it follows that if $\alpha_n \to \alpha$ and $\alpha_n \neq \alpha$ for all large n, then any limit point of a sequence $x_n \in I_{\alpha_n}$ must be an endpoint of I_α. Note if I_α is a point, we consider this point to be an endpoint. Hence the set X of all endpoints of all I_α is also closed. Since $I_{a_1...a_k}$ was chosen of minimal length. f maps the two endpoints of $I_{a_1...a_k}$ onto the two endpoint of $I_{a_2...a_k}$. Therefore, if I_α and $I_{\sigma(\alpha)}$ are intervals, f maps the two endpoints of I_α onto the two endpoints of $I_{\sigma(\alpha)}$. it follows that $f(X) = X$. Define a map h of X onto Σ by setting $h(x) = \alpha$ if $x \in I_\alpha$. Then each point of Σ is the image of at most two points of X (and at most countably many points of Σ are the image of two points of X). Moreover the map h is continuous. For let $\delta_k > 0$ be the least distance between any two of the 2^k intervals $I_{a_1...a_k}$.If $x \in I_\alpha, y \in I_\beta$ and $|x - y| < \delta_k$ then $d(\alpha, \beta) < 2^{-k}$. Finally, since $f(I_\alpha) = I_{\sigma(\alpha)}$, we have

$$h \circ f(x) = \sigma \circ h(x) \text{ for every } x \in X. \blacksquare$$

4.4 Knudsen,Robinson,Wiggnis versus Devaney's

By definition and *Theorem 2.1.10 Robinson* (or *Wiggnis*) and *Devaney's chaos* are equivalent if $X = I$. By *Proposition* 1.2.18 and *Theorem* 2.1.10 again *Knudsen* and *Devaney's chaos* are equivalent.

4.5 Touhey versus Devaney

Proposition 4.5.1: *Let X be a metric space and $f : X \to X$ be a mapping. Then the following are equivalent :*

[1] *f is T-chaotic on X.*

[2] *f is D-chaotic on X.*

[3] *any finite collection of non-empty open sets of X shares a periodic orbit.*

[4] *any finite collection of non-empty open sets of X shares infinitely many periodic orbits* .[12]

Proof: We have already shown that [1] $\Leftrightarrow$ [2] in chapter two section 6 (proposition 2.6.3). Now [1 *or* 2] $\Rightarrow$ [3] : let N be the number of non-empty open subsets in our collection. If $N = 1$, the result follows from the density of periodic points, if $N = 2$ it follows from definition of chaotic mapping. We proceed by induction on N. Thus assume that assertion holds for $N = n$. We will show that it holds for $n + 1$non-empty open subsets. There is no loss of generality to assume that the collection consists of $n + 1$ disjoint subsets. If the sets are not disjoint then some pair of non-empty open subsets intersects in an open subset. Replacing the pair by their intersection yields a collection of n non-empty open subsets that by our induction hypothesis shares a periodic orbit. Clearly this orbit is shared by the original collection of $n+1$ subsets. Now from our disjoint collection choose a subsets and call it V. The remaining n subsets must share a periodic orbit and like all periodic orbits this orbit has a primitive period which we designate by M. From these remaining n non-empty open subsets choose any subset and call it U_0. Thus we must have $p \in U_0$ where p is a periodic point of primitive period $M > n - 1$, with the property that $O_f^+(p)$ intersect each of our n subsets. We now label each of the remaining $n - 1$ non-empty open subsets in the following manner. As we iterate the point p it first intersect one of the $n-1$ open subsets for some value k_1, of the iterate, $0 < k_1 < M$. Let this subset be designated by U_1 i.e. , $f^{k_1}(p) \in U_1$. Continuing in this fashion we arrive at the next iterate, $f^{k_2}(p), 0 < k_1 < k_2 < M$, intersecting one of the remaining $n-2$ open subsets. This subsets is designated U_2 . Eventually we will have labeled each of the n open subsets so that $f^{k_i}(p) \in U_i$ for all $i = 0, 1, ..., n-1$ where $0 = k_0 < k_1 < ... < k_{n-1} < M$. Now we define another collection of non-empty open subsets with a particularly nice property. Let $W_0 \equiv U_{n-1}$. Clearly $f^{k_{n-1}}(p) \in W_0$. Now consider

$$W_1 \equiv f^{-[k_{n-1}-k_{n-2}]}(W_0) \cap U_{n-2}.$$

We claim that W_1 is a non-empty open subset contained in U_{n-2}. It is open because it is the intersection of two open subsets and it is obviously in U_{n-2} That is non-empty follows from the facts that $f^{k_{n-1}}(p) \in W_0$ and $f^{k_{n-2}}(p) \in U_{n-2}$, and hence

$$f^{k_{n-2}}(p) = f^{-[k_{n-1}-k_{n-2}]}\left(f^{k_{n-1}}(p)\right) \in f^{-[k_{n-1}-k_{n-2}]}(W_0),$$

which implies that $f^{k_{n-2}}(p) \in W_1$. Also note that W_1 has the particularly nice property that $f^{[k_{n-1}-k_{n-2}]}(W_1) \subset W_0$. Continuing in this fashion, we define

$$W_i \equiv f^{-\left[k_{n-i}-k_{n-(i+1)}\right]}(W_{i-1}) \cap U_{n-(i+1)} \quad \text{for} \quad i = 1, 2, ..., n-1.$$

Each W_i is again non-empty open and contained in $U_{n-(i+1)}$. In addition we have the particularly nice property that

$$f^{\left[k_{n-i}-k_{n-(i+1)}\right]}(W_i) \subset W_{i-1} \quad \text{for } i = 1, 2, ..., n-1.$$

It is easy to find a periodic orbit that wends itself through our original collection of $n+1$ non-empty open subsets, $\{V, U_0, U_1, ..., U_{n-1}\}$. Since V and W_{n-1} are both open, they share a periodic orbit. Thus, there exists a periodic point $p' \in V$ and a positive integer q such that $f^q(p') \in W_{n-1} \subset U_0$. But then, by our particularly nice property, the subsequent iterates of p' must pass through all of the U_i's.

$$f^q(p') = f^{[q+k_0]}(p') \in W_{n-1} \subset U_0$$

$$f^{q+k_1}(p') = f^{[k_1-k_0]}\left(f^{[q+k_0]}(p')\right) \in f^{[k_1-k_0]}(W_{n-1}) \subset W_{n-2} \subset U_1$$

$$\vdots$$

$$f^{q+k_i}(p') = f^{[k_i-k_{i-1}]}\left(f^{[q+k_{i-1}]}(p')\right) \in f^{[k_i-k_{i-1}]}(W_{n-i}) \subset W_{n-(i+1)} \subset U_i$$

$$\vdots$$

$$f^{q+k_{n-1}}(p') = f^{[k_{n-1}-k_{n-2}]}\left(f^{[q+k_{n-2}]}(p')\right) \in f^{[k_{n-1}-k_{n-2}]}(W_1) \subset W_0 = U_{n-1}$$

Thus, the forward orbit of p', $O_f^+(p')$, intersects each of $V, U_0, U_1, ..., U_{n-1}$.

Now [1] $\Rightarrow$ [3] $\Rightarrow$ [4] : Assume the existence of a finite collection $\{U_i\}_{i=1,...,n}$ of non-empty open subsets that share only a finite number of periodic orbits. Define P to be the set consisting of the union of the points in these shared periodic orbits. Since each periodic orbit contains a finite number of points, the union of finitely many such orbits must be finite. Hence P is a finite set. We now define another collection of non-empty open subsets $\{V_i\}_{i=1,...,n}$

by $V_i \equiv U_i \backslash P$. It's clear that each $V_i \subset U_i$. And each V_i is non-empty and open and open since removing the finite set of points ,P, from the open set U_i leaves us with a non-empty open sets. Thus by ([1] $\Rightarrow$ [3]) there must be a periodic orbit shared by the collection $\{V_i\}_{i=1,\ldots,n}$. This new orbit is clearly not contained in P. On the other hand,this orbit obviously passes through the original collection $\{U_i\}_{i=1,\ldots,n}$ of non-empty open subsets since each $V_i \subset U_i$. This contradiction proves our result.

Now [4] $\Rightarrow$ [1] : If any finite collection of non-empty open sets contained in X shares infinitely many periodic orbits it is clear that any pair of open sets shares a periodic orbit.■

4.6 t-chaos versus Entropy

Proposition 4.6.1: *A continuous map $f : I \to I$ is $t-chaotic$ if only if it has topological entropy $h(f) > 0$.*

Proof: If f is t-chaotic then f^n is strictly turbulent for some $n > 0$. Hence, by $Corollary$ 2.13.12,

$$h(f^n) \geq \log 2 \qquad \text{and} \qquad h(f) \geq (\log 2)/n.$$

Conversely, suppose $h(f) > 0$. Then in the statement of *Theorem 2.13.18* we must have $p > 1$. Hence f^n is turbulent and f is $t-chaotic$.■

4.7 P-chaos versus DC-chaos, Devaney and Entropy

Theorem 4.7.1: *Every $P-$chaotic map from a continuum to itself is chaotic in the sense of Devaney.*[22]

Theorem 4.7.2: *Every $P-$ chaotic map from a continuum to itself is $DC1$.*[22]

Theorem 4.7.3: *Every $P-$ chaotic map from a continuum to itself has positive topological entropy.*[22]

Proof: Let f be a $P-chaotic$ map from a continuum X to itself. And let $Orb(p,f)$ and $Orb\,(q,f)$ be periodic orbits with periods m and n, respectively. Since f has the $pseudo-orbit-tracing-property$, for any $\varepsilon>0$ with $\varepsilon<1/3d(Orb(p,f),Orb(q,f))$, there exists $\delta>0$ such that each $\delta-pseudo-orbit$ is $\varepsilon-traced$. By the transitivity of f, there exists $x\in X$ such that $\{x,f^{\ell}(x)\}\in B(p,\delta)$ and $f^k(x)\in B(q,\delta)$ for some positive integers $k<\ell$. Denote

$$P=(p,f(p),...,f^{m-1}(p))$$

and

$$Q=(x,f(x),...,f^{k-1}(x),q,f(q),...,f^{n-1}(q),f^k(x),f^{k+1}(x),...,f^{\ell-1}(x)).$$

And put $P'=(\ell+n)\times P$ and $Q'=m\times Q$. Since the length of P is m and the length of Q is $\ell+n$, the length $m(\ell+n)$ of P' is equal to that of Q'. Let A be the set of all sequences $A_1A_2...$, where each A_i is an element of $\{P',Q'\}$ for $i\geq 1$. Note that each element of A is a δ-pseudo-orbit for f. Since $d(P',Q')>3\varepsilon$, any distinct elements $A_1A_2...$ and $A_1'A_2'...$ of A are $\varepsilon-traced$ by distinct points x and x' satisfying

$$d(f^i(x),f^i(x'))>\varepsilon \quad \text{for some} \quad i\geq 0, \quad \text{respectively.}$$

Hence for each k $\geq$1, there exists an *$(km(\ell + n), \varepsilon)$-separated set* for f with cardinality at least 2^k. Therefore

$$h(f) = \lim_{k\to\infty} 1/km(\ell+n)\log s_{km(\ell+n)}(\varepsilon, X)$$

$$\begin{aligned} &\geq \lim_{k\to\infty} 1/km(\ell+n)\log 2^k \\ &= 1/m(\ell+n)\log 2 > 0.\blacksquare \end{aligned}$$

Remark 4.7.4: Not every D-chaotic map is P-chaotic.

Example 4.7.5: Let $f:[0,1]\to[0,1]$ be the piecewise linear function defined by

$$f(0) = 1/2, f(1/8) = 0, f(1/4) = 1, f(3/8) = 0,$$

$$f(5/8) = 1, f(3/4) = 0, f(7/8) = 1, f(1) = 1/2.$$

i.e.

$$f(x) = \begin{cases} -4x+\frac{1}{2} & , \quad 0 \leq x \leq \frac{1}{8} \\ 8x-1 & , \quad \frac{1}{8} < x \leq \frac{1}{4} \\ -8x+3 & , \quad \frac{1}{4} < x \leq \frac{3}{8} \\ 4x-\frac{3}{2} & , \quad \frac{3}{8} < x \leq \frac{5}{8} \\ -8x+6 & , \quad \frac{5}{8} < x \leq \frac{3}{4} \\ 8x-6 & , \quad \frac{3}{4} < x \leq \frac{7}{8} \\ -4x+\frac{9}{2} & , \quad \frac{7}{8} < x \leq 1 \end{cases}$$

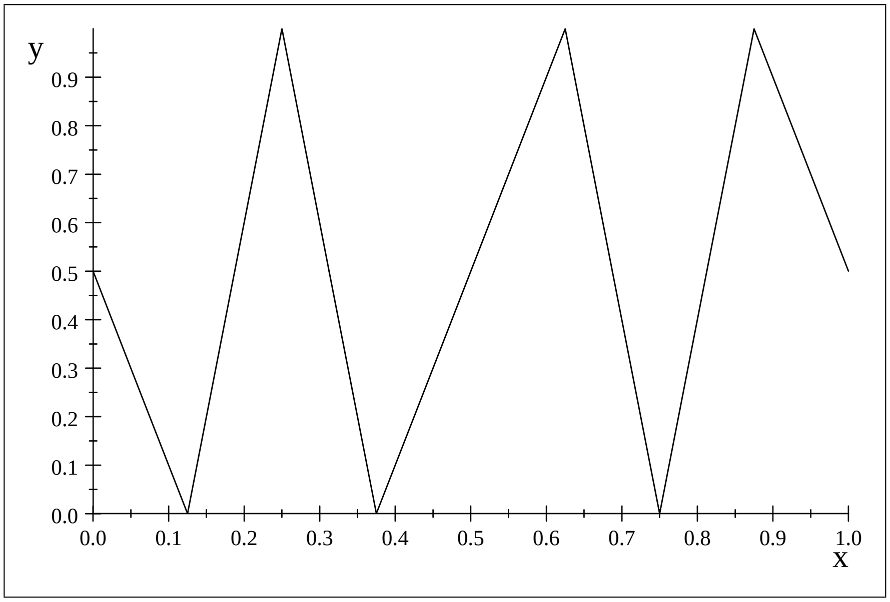


Figure 4.7.1: The graph of $f\,(x)$

We can show that f is topologically mixing and so transitive, but it is not P-chaotic.

Fix $\varepsilon > 0$ with $\varepsilon < 4^{-3}$. Let $\delta > 0$ with $\delta < \varepsilon, x_0 = 0, x_1 = 1/2 + \delta/2$ and $x_i = f^{i-1}(x_1)$ for each $i \geq 2$. We see that $\{x_i : i \geq 0\}$ is a $\delta - pseudo - orbit$ for f. Suppose that there exists a point x which ε-traces $\{x_i : i \geq 0\}$. We see that $x \in [0, \varepsilon)$, thus, $f(x) \in (1/2 - 4\varepsilon, 1/2]$ and $f^2(x) \in (1/2 - 4^2\varepsilon, 1/2]$. There exists $i_0 = \min\{i \geq 2 : f^i(x) \leq 1/2 - 3/4^3\}$. Also, there exists $i_1 = min\{i \geq 2 : x_i \geq 1/2 + 3/4^3\}$. Let $i_2 = min\{i_0, i_1\}$. We have $d(f^{i_2}(x), x_{i_2}) \geq 3/4^3$, thus, this is a contradiction. We see that f has not the $pseudo - orbit - tracing - property$.

4.8 Martelli versus Devaney

Theorem 4.8.1: *For a continuous map $f : I \to I$, Martelli's and Devaney's notions of chaos are equivalent.*

Proof : By *Theorem 2.10.4* and *Theorem 2.10.6* the result follows.■

4.9 Experimentalists' Chaos versus Devaney

It follows from definition that Devaney implies Experimentalists' chaos and the converse is not true by example 2.1.15.

4.10 Lyapunove versus Devaney

It follows from definition that lyapunove implies Devaney and the converse is not true, see example 2.1.15.

4.11 $\omega-$chaos and DC-chaos versus Entropy

Theorem 4.11.1[28]**:** *σ satisfies the following statement: There is an uncountable ω-scrambled set S such that:*

$$\bigcap_{x\in s} \omega\left(x,f\right) \neq \phi \tag{11.1}$$

Lemma 4.11.2[28]**:** *If f and p are semiconjugate, i.e. there is a continuous onto map $h : X \to Y$ such that $h \circ f = p \circ h$, then $h(\omega(x, f)) = \omega(h(x), p)$ for each $x \in X$.*

Theorem 4.11.3: *If f is countable to one semiconjugate to p with semiconjugacy $h : X \to Y$, then p satisfies the statement in Theorem 4.11.1 which implies that f satisfies the statement. Also we can take an ω-scrambled set in X from the preimage under h of some ω-scrambled set in Y.*

Proof: Since p satisfies the statement, there is an uncountable ω-scrambled set $S(p)$ in Y with $\bigcap_{y \in S(p)} \omega(y, p) \neq \phi$. Let $y_0 \in \bigcap_{y \in S(p)} \omega(y, p)$. For each $y \in S(p)$, choose one point $x = x(y) \in h^{-1}x(y)$ and let $T = \{x(y) : y \in S(p)\}$. By Lemma 4.11.2, $\omega(x, f) \cap h^{-1}(y_0) \neq \phi$ for every $x \in T$. Since h is countable to one, there exists $x_0 \in h^{-1}(y_0)$ such that $x_0 \in \omega(x, f)$ for uncountably many $x \in T$. Then $S(f) = \{x \in T : x_0 \in \omega(x, f)\}$ is an uncountable ω-scrambled set with $\bigcap_{y \in S(p)} \omega\,(y, f) \neq \phi$.■

Theorem 4.11.4: *Suppose that f^m satisfies the statement (11.1) , and let $S(f^m)$ be an $\omega -$ scrambled set as in statement (11.1). Suppose also that for any $x \in S(f^m)$ the following conditions are satisfied:*

(1) *$\omega(x, f^m)$ contains a finite, nonzero number of infinite minimal sets.*

(2) *$\omega(x, f^m)$ contains only countably many points which are not in these minimal sets.*

Then f satisfies statement (11.1).

Proof: For any $x \in S(f^m)$ and any $f^m - minimal$ set M either $M \subset \omega(x, f)$ or $M \cap \omega(x, f) = \phi$. By Lemma 2.17.1 and hypothesis 1, $\omega(x, f)$ contains only finitely many $f^m - minimal$ sets. Since $S(f^m)$ is uncountable, there exists an uncountable subset $S_1(f^m)$ such that $\omega(x, f)$ contains the same number of $f^m - minimal$ sets for each $x \in S_1(f^m)$. For $x, y \in S_1(f^m)$,say

$x\tilde{}y$ if $\omega(x, f)$ and $\omega(y, f)$ contain the same $f^m - minimal$ sets. It is easy to see that this is an equivalence relation. Note that for distinct x and $: \omega(x, f^m)\backslash\omega(y, f^m)$ contains an infinite $f^m - minimal$ set by Definition and hypothesis (2). Thus each equivalence class is finite. Let $S(f)$ be a subset of $S_1(f^m)$ which contains exactly one representative of each equivalence class. Then $S(f)$ is uncountable. Also for any pair of distinct points $x, y \in S(f), \omega(x, f)\backslash\omega(y, f)$ contains an infinite minimal set and hence is uncountable.■

Now let I denote a compact interval, and we suppose that $f : I \to I$ is a continuous map. Let $C(2^\infty)$ denote the set of maps f with no periodic points of periods not a power of two. Let $P, AP, R,$ and Λ denote the sets of *periodic points*, *almost periodic points, recurrent points*, and *ω-limit points*, respectively. Let $\Lambda^2 = \bigcup_{x\in\Lambda} \omega(x, f)$.

Proposition 4.11.5 [67]: *$\omega(x, f)$ contains only one minimal set for $f \in C(2^\infty)$.*

Proposition 4.11.6 [68]: *$\Lambda \setminus \Lambda^2$ is countable for any continuous map $f : I \to I$.*

Proposition 4.11.7[68]: *$\Lambda^2 = R = AP$ for $f \in C(2^\infty)$.*

Lemma 4.11.8: *For $f \in C(2^\infty)$ and $x \in I$, if $\omega(x) \subset \Lambda^2$, then $\omega(x)$ is a minimal set.*

Proof: By Proposition 4.11.5, ω(x) contains a unique minimal set M. For any $y \in \omega(x)$, we have $\omega(y) \subset \omega(x)$, since $\omega(x)$ is a closed invariant set. Then $y \in \omega(x) \subset \Lambda^2 = AP$ by Proposition 4.11.7. Thus $\omega(y)$ is a minimal set, and $y \in \omega(y)$. Hence $M = \omega(y)$ and $y \in M$. Since y was arbitrary, $\omega(x) = M$.■

Proposition 4.11.9 [69]: *If f has zero entropy, then $f \in C(2^\infty)$.*

Proposition 4.11.10: *Let y and z be distinct points of I. If $\{y, z\}$ is an $\omega -$ scrambled set, then f has positive entropy.*

Proof: Suppose that f does not have positive entropy. Then, by Proposition 4.11.9, $f \in C\left(2^{\infty}\right)$. Suppose one of $\omega(y)$ and $\omega(z)$ is contained in Λ^2. By Lemma 4.11.8, if ω(y) and ω(z) have nonempty intersection, then one is contained in the other. This contradicts the definition of an ω-scrambled set. So, both $\omega(y) \cap (\Lambda \backslash \Lambda^2)$ and $\omega(z) \cap (\Lambda \backslash \Lambda^2)$ are nonempty. By Proposition 4.11.5, $\omega(y)$ contains a unique minimal set $M(y)$, and $\omega(z)$ contains a unique minimal set $M(z)$. It follows from Proposition 4.11.7 that $\omega(y) \cap \Lambda^2 = M(y)$ and $\omega(z) \cap \Lambda^2 = M(z)$.From the second condition of Definition 2.17.1, we know that $\omega(y) \cap \omega(z) \neq \phi$. Let $u \in \omega(y) \cap \omega(z)$. Then $M(y) = \omega(u) = M(z)$ by Proposition 4.11.5 and Lemma 4.11.8. Thus,

$$\omega(y) \cap \Lambda^2 = M(y) = M(z) = \omega(z) \cap \Lambda^2$$

and

$$\omega(y) \backslash \omega(z) \subset \Lambda \backslash \Lambda^2.$$

By Proposition 4.11.6, $\Lambda \backslash \Lambda^2$ is countable, and hence $\omega(y) \backslash \omega(z)$ is countable. This contradicts the definition of an $\omega - scrambledset$. Therefore f has positive entropy.■

Proposition 4.11.11:[71] *If f has positive entropy then there exists a closed set $X \subset I$ and $m > 0$ such that $f^m(X) = X$ and $f^m|_X$ is at most two-to-one semiconjugate to the one-sided shift map σ. Furthermore, there are only countably many points in Σ which have 2 preimages, and if one of the preimages is periodic, then so is the other.*

Proposition 4.11.12 [5]: *The one-sided shift map is chaotic in the sense of Devaney with a chaotic set Σ_2.*

Proposition 4.11.13: *If $f : I \to I$ has positive entropy, then there is a positive integer m such that f^m is chaotic on I in the sense of Devaney.*

Proof: Let $m > 0$ and $X \subset I$ be as in Proposition 4.11.11. By Proposition 4.11.12, $D(\sigma) = \Sigma_2$ is a chaotic set. Let $s \in D(\sigma)$ satisfy $\overline{Orb(s, \sigma)} = D(\sigma)$. Let $x \in X$ be a preimage of s under the semiconjugacy in Proposition 4.11.11, and let $D(f^m) = \overline{Orb\,(x, f^m)}$. Then $D\,(f^m) \subset X$, and $D(f^m)$ contains at least one preimage of each point in $D(\sigma)$. It is not hard to show that the periodic points in $D(f^m)$ are dense in $D(f^m)$ and $f^m|_{D(f^m)}$ has sensitive dependence on initial conditions. Thus $D(f^m)$ is a chaotic set for f^m.■

Proposition 4.11.14: *If for some $m > 0$, f^m is chaotic in the sense of Devaney, then f is also chaotic in the sense of Devaney. Furthermore, if $D(f^m)$ is a chaotic set for f^m, then $\bigcup_{i=0}^{m-1} f^i\,(D\,(f^m))$ is a chaotic set for f.*

Theorem 4.11.15: *Let f be a continuous map of a compact interval I to itself. The following statements are equivalent:*

(I) *f has positive topological entropy.*

(II) *There is an uncountable ω-scrambled set S such that*

$$\bigcap_{x \in s} \omega\,(x, f) \neq \phi$$

(III) *f is ω-chaotic.*

(IV) *There is an ω-scrambled set containing exactly two points.*

(V) *f is chaotic in the sense of Devaney.*

(VI) *There is a chaotic set D and an uncountable ω-scrambled set $S \subset D$.*

Proof: (II) $\Rightarrow$(III) and (III) $\Rightarrow$(IV) are obvious. (IV) $\Rightarrow$ (I) is proved in Proposition 4.11.10.

Now let us prove that (I)$\Rightarrow$ (II). By Proposition 4.11.11, there is a closed subset X of I and a positive integer m such that $f^m(X) = X$, and $f^m|_X$ is at most two-to-one semiconjugate (via a semiconjugacy h) to the one-sided shift map σ. Also, there are at most countably many points in Σ_2 which have two preimages under h. By Theorem 4.11.1, σ satisfies statement (11.1), so, by Theorem 4.11.3, $f^m|_X$ satisfies the statement (11.1). Let $S(\sigma)$ be the $\omega-$scrambled set constructed in Theorem 4.11.1, and $S(f^m)$ be the ω-scrambled set constructed in Theorem 4.11.3. Let $x \in S(f^m)$, and let $h(x) = s \in S(\sigma)$.Then $\omega(s, \sigma)$ contains a unique infinite minimal set M, and there are only countably many points of $\omega(s, \sigma)$ not in this minimal set. Since $h^{-1}(M)$ is a closed invariant set as $h(f^m(h^{-1}(M))) = \sigma(h(h^{-1}(M))) = \sigma(M) = M$ implies that $f^m(h^{-1}(M)) \subset h^{-1}(M))$, $h^{-1}(M)$ must contain a minimal set $\tilde{M}$. Then h maps $\tilde{M}$ onto M since $h(\tilde{M})$ is a closed, invariant subset of M, and hence, $\tilde{M}$ is infinite. Because there are only countably many points in Σ_2 which have two preimages and $\omega(s, \sigma)\backslash M$ is countable, we have that $\omega(x, f^m)\backslash \tilde{M}$ is also countable. Since, x was arbitrary, the hypothesis of Theorem 4.11.4 is satisfied,and hence, statement (II) holds.

(VI) $\Rightarrow$(V) is obvious. (V) $\Rightarrow$(I) follows from Propositions 4.11.5 and 4.11.9 It remains to show that (I)$\Rightarrow$ (VI). Suppose that f has positive topological entropy. By Proposition 4.11.13 there is an integer $m > 0$ such that f^m is chaotic in the sense of Devaney. Let $D(f^m)$ be a chaotic set for f^m as in the proof of Proposition 4.11.13. Set

$$D\left(f\right) = \bigcup_{i=0}^{m-1} f^i\left(D\left(f^m\right)\right).$$

By Proposition 4.11.14, $D(f)$ is a chaotic set for f. Clearly $D(f^m) \subset D(f)$. Let S(σ) be the ω-scrambled set for σ constructed in Theorem 4.11.1. Let $S(f^m)$ be the collection of the preimages, under the semiconjugacy as in Proposition 4.11.11, of the points in S(σ) which have unique preimages. Since there are only countably many points in Σ_2 which have two preimages, $S(f^m)$ must be uncountable. Using Theorem 4.11.3 and its proof, it is easy to see that $S(f^m)$ is an ω-scrambled set for f^m. By the proof of Proposition 4.11.13, $D(f^m)$ contains at least one preimage of each point in Σ_2. Since each point in $S(f^m)$ is the unique preimage of some point in $S(\sigma), S(f^m) \subset D(f^m)$. Let $S(f)$ be the ω-scrambled set for f constructed as in Theorem 4.11.4. Then

$$S(f) \subset S(f^m) \subset D(f^m) \subset D(f).$$

This completes the proof. ■

Theorem 4.11.16[54]**:** *For a continuous map $f : I \to I$, all implications between various notions of chaos can be displayed as follows:*

$$PTE \Leftrightarrow \omega - Chaos \Leftrightarrow DC1 \Leftrightarrow DC2 \Leftrightarrow DC3 \implies LYC$$

4.12 Li-York and Devaney

Here we will give an example of a function which is chaotic in the sense of Li-Yorke and D-chaotic.[44]

Example 4.12.1: The piecewise linear map $g : [0,1] \to [0,1]$ with

$$g(0) = 0,\ g(\frac{1}{2}) = 1,\ g(1) = 0$$

is known as the (standard) tent map. Its graph is a "tent" with peak of height 1 at the point $\frac{1}{2}$. In order to modify this map to get a family of maps suitable for our purposes we cut the peak at any height $\lambda \in [0,1]$ and consider the family of truncated tent maps defined by

$$g_\lambda : [0,1] \to [0,1], \quad x \to min\{\lambda, g(x)\}, \quad \lambda \in [0,1].$$

It is apparent that for any $0 \leq \lambda < \ell \leq 1$ the maps g_λ and g_ℓ coincide on the set

$$J_\lambda = [0, \frac{\lambda}{2}] \cup [1 - \frac{\lambda}{2}], \lambda \in [0, 1]$$

and that (periodic) orbits of g_ℓ in J_λ are also (periodic) orbits of g_λ and vice versa. Furthermore, since g_λ is constant on the open interval

$$K_\lambda := (\frac{\lambda}{2}, 1 - \frac{\lambda}{2}), \lambda \in [0, 1],$$

the map g_λ has at most one periodic point in $\bar{K}_\lambda$.

For the original tent map $g(= g_1)$ the set $\left\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right\}$ is obviously a 3-periodic orbit and therefore, by Sarkovskii's Theorem, it has 2^n-periodic points for all $n \in N$. Furthermore, it is easy to see that

$|\{x \in [0, 1] \mid x$ is m-periodic with respect to $g\}| \leq 2^m$ for all $m \in N$. (12.1) Therefore the number

$\lambda_n := \min\{\lambda \in [0,\ 1] \mid g$ has a 2^n-periodic orbit in $[0,\lambda\ ]\}$

is well defined and λ_n is a 2^n-periodic point of g for any $n \in N$. Because of the relation $g(K_{\lambda_n}) = (\lambda_n, 1]$ we have $\mathrm{O}(\lambda_n,\ \mathrm{g}) \subset \mathrm{J}_{\lambda_n}$, and therefore λ_n is also periodic with respect to g_{λ_n} having the same periodic orbit as for g. By Sarkovskii's Theorem we have the identity

$\{2^i \mid i = 0, 1, ..., n\} = \{k \in N \mid x$ is k-periodic w.r. to g_{λ_n} for some $x \in [0, 1]\}$ (12.2) because otherwise there were an m-periodic orbit M of g_{λ_n} for some $m \in N$ with $m \prec 2^n$. Since g_{λ_n} has at most one periodic point in $\bar{K}_{\lambda_n}$ (the point λ_n) the inclusion $M \subseteq J_{\lambda_n}$ holds and with $\rho := \max M < \lambda_n$ the map g_ρ and hence also g_{λ_n} has a 2^n-periodic orbit in $[0, \rho] \cap J_\rho$. This contradicts the minimality of λ_n. The sequence $(\lambda_n)_{n\in\mathbb{N}}$ is strongly increasing because otherwise there would exist numbers $n, m \in N, m > n$ with $\lambda_m \leq \lambda_n$ such that the map $\mathrm{g}\lambda_m$ has a 2^n-periodic orbit in $[0, \lambda_m)$ and this would again contradict the minimality of λ_n. On the other hand, the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is bounded above by 1 and therefore it has a limit

$$\lambda^* = \lim_{n\to\infty} \lambda_n$$

which is smaller then $\frac{6}{7}$ since the map $g_{\frac{6}{7}}$ has periodic points of any period $n \in N$ (by Sarkovskii's Theorem). In addition, λ^* is greater than $\frac{4}{5}$, since $\lambda_2 = \frac{4}{5}$. Indeed, it has been mentioned that $\lambda^* = 0.8249080....$

The map g_{λ^*} is L/Y-chaotic but not B/C-chaotic:In [4, VI Example 29] it has been shown that not all points in [0, 1] are approximately periodic with respect to g_{λ^*} , and therefore g_{λ^*} is L/Y-chaotic by Proposition .4.1.1 On the other hand, assuming to the contrary that g_{λ^*} is B/C-chaotic, by Theorem 2.2.25 there exists an odd number $q > 1$ such that g_{λ^*} has a $q2^k$-periodic orbit P for some $k \geq 0$.In case $p := \max P < \lambda^*$ there is an $n \in N$ with $\lambda_n > p$ such that P is a periodic orbit of g_{λ_n}. This contradicts (12.2). If, on the other hand, $p = \lambda^*$, by Sarkovskii's Theorem the map g_{λ^*} has a $(q + 2)2^k$-periodic orbit Q. Because of $\max Q < \lambda^*$ this again leads to a contradiction.

4.13 Summary

We can summarize the results presented in this chapter with the following theorem.

Theorem 4.13.1: *For a continuous map $f : I \to I$ the following conditions are equivalent:*

(1) f is topologically chaotic, i.e., has positive topological entropy,

(2) f is DC1- chaotic,

(3) f is DC2- chaotic,

(4) f is DC3- chaotic,

(5) f is ω-chaotic,

(6) f is chaotic in the sense of Martelli,

(7) f is chaotic in the sense of Devaney,

(8) f is chaotic in the sense of Block and Coppel,

(9) f is chaotic in the sense of Robinson (Wiggins),

(10) f is chaotic in the sense of Touhey,

(11) f is chaotic in the sense of Kato,

(12) f is chaotic in the sense of Knudsen,

(13) f is $t-$chaotic .

All previous properties imply that f is chaotic in the sense of Li and Yorke, but the converse is not true.

Kolyada and Snoha in [2] gives the following theorem where the proof of it follows from definition and part of it can be found in [23] or [60] .

Theorem 4.13.2: *Let (X, f) be a dynamical system. Then the following are equivalent:*

(1) f is topologically transitive ,

(2) for every pair of nonempty open sets U and V in X, there is a nonnegative integer n such that $f^n(U) \cap V \neq \phi$,

(3) for every nonempty open set U in X, $\bigcup_{n=1}^{\infty} f^n(U)$ is dense in X,

(4) for every nonempty open set U in X, $\bigcup_{n=0}^{\infty} f^n(U)$ is dense in X,

(5) for every pair of nonempty open sets U and V in X, there is a positive integer n such that $f^{-n}(U) \cap V \neq \phi$,

(6) for every pair of nonempty open sets U and V in X, there is a nonnegative integer n such that $f^{-n}(U) \cap V \neq \phi$,

(7) for every nonempty open set U in X, $\bigcup_{n=1}^{\infty} f^{-n}(U)$ is dense in X,

(8) for every nonempty open set U in X, $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is dense in X,

(9) if $E \subset X$ is closed and $f(E) \subset E$ then $E = X$ or E is nowhere dense in X,

(10) if $U \subset X$ is open and $f^{-1}(U) \subset U$ then $U = \phi$ or U is dense in X,

(11) there exists a point $x \in X$ such that $\omega(x, f) = X$,

(12) there exists a G_δ-dense set $A \subset X$ such that $\omega(x, f) = X$ whenever $x \in A$,

(13) the set $tr(f)$ is G_δ-dense,

(14) the map f is onto and the set $tr(f)$ is nonempty,

(15) $\Omega(f) = X$ *and* $tr(f)$ *is nonempty,where*

$$\Omega(f) = \left\{x \in X \;\mid \text{for every neighbourhood } U \text{ of } x, \exists n \geq 1,\ f^{-n}(U) \cap U \neq \phi \right\}$$

(16) there is a point $x \in X$ *such that the set* $\{f^n(x) : n = 1, 2, ...\}$ *is dense in* X*.*

From the above two theorems and theorems in chapter two and four we get the following big theorem.

Theorem 4.13.3: *For a continuous map* $f : I \to I$ *the following conditions are equivalent:*

(1) f *has positive topological entropy.*

(2) *There is an uncountable* ω*-scrambled set* S *such that*

$$\bigcap_{x \in s} \omega(x, f) \neq \phi$$

(3) f *is* ω*-chaotic.*

(4) There is an ω*-scrambled set containing exactly two points.*

(5) f *is chaotic in the sense of Devaney.*

(6) *There is a chaotic set* D *and an uncountable* ω*-scrambled set* $S \subset D$*.*

(7) f *is DC1- chaotic,*

(8) f *is DC2- chaotic,*

(9) f is DC3- chaotic,

(10) f is chaotic in the sense of Martelli,

(11) f is chaotic in the sense of Block and Coppel,

(12) f is chaotic in the sense of Robinson (Wiggins),

(13) f is chaotic in the sense of Touhey,

(14) f is chaotic in the sense of Kato,

(15) f is chaotic in the sense of Knudsen,

(16) f has a periodic point whose period is not a power of 2.

(17) f^m is strictly turbulent for some positive integer m.

(18) f^n is turbulent for some positive integer n.

(19) any finite collection of non-empty open sets of I shares a periodic orbit.

(20) any finite collection of non-empty open sets of I shares infinitely many periodic orbits .

(21) for every pair of nonempty open sets U and V in I, there is a nonnegative integer n such that $f^n(U) \cap V \neq \phi$,

(22) for every nonempty open set U in I, $\bigcup_{n=1}^{\infty} f^n(U)$ is dense in I,

(23) for every nonempty open set U in I, $\bigcup_{n=0}^{\infty} f^n(U)$ is dense in I,

(24) for every pair of nonempty open sets U and V in I, there is a positive integer n such that $f^{-n}(U) \cap V \neq \phi$,

(25) for every pair of nonempty open sets U and V in I, there is a nonnegative integer n such that $f^{-n}(U) \cap V \neq \phi$,

(26) for every nonempty open set U in I, $\bigcup_{n=1}^{\infty} f^{-n}(U)$ is dense in I,

(27) for every nonempty open set U in I, $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is dense in I,

(28) if $E \subset I$ is closed and $f(E) \subset E$ then $E = X$ or E is nowhere dense in I,

(29) if $U \subset I$ is open and $f^{-1}(U) \subset U$ then $U = \phi$ or U is dense in I,

(30) there exists a point $x \in I$ such that $\omega(x, f) = I$,

(31) there exists a G_δ-dense set $A \subset I$ such that $\omega(x, f) = I$ whenever $x \in A$,

(32) the set $tr(f)$ is G_δ-dense,

(33) the map f is onto and the set $tr(f)$ is nonempty,

(34) $\Omega(f) = I$ and $tr(f)$ is nonempty.

Again, all previous properties imply that f is chaotic in the sense of Li and Yorke, but the converse is not true.

The following figure summarize the results presented in this chapter.

$$
\begin{array}{ccccc}
L-chaos & & P-chaos & \begin{array}{c}\Longrightarrow \\ \not\Longleftarrow\end{array} & DC1 \\
 & \searrow \quad \not\nwarrow & \Downarrow \quad \not\Uparrow & & \Updownarrow \\
AY-chaos & \begin{array}{c}\Longleftarrow \\ \not\Longrightarrow\end{array} & D-chaos & \Longleftrightarrow & DC2 \\
 & \swarrow \quad \not\nearrow & \Updownarrow & & \Updownarrow \\
SD-chaos & \begin{array}{c}\Longleftarrow \\ \not\Longrightarrow\end{array} & K-chaos & \Longleftrightarrow & DC3 \\
\Uparrow \quad \not\Downarrow & & \Updownarrow & & \Updownarrow \\
M-chaos & \Longleftrightarrow & R-chaos & \Longleftrightarrow & PTE \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
kato-chaos & \Longleftrightarrow & t-chaos & \Longleftrightarrow & BC-chaos \\
\Updownarrow & & \Updownarrow & & \Downarrow \quad \not\Uparrow \\
\omega-chaos & \Longleftrightarrow & T-chaos & \begin{array}{c}\Longrightarrow \\ \not\Longleftarrow\end{array} & LY-chaos
\end{array}
$$

5 The Relations of Definitions of Chaos in General Case

In the case of the real interval we have presented theorems relating various notions of chaos. In particular, we stated some equivalences.(here we follow [39] and [54])

What happens in general?

Things change completely and many problems are still open.[39]

We have seen in Theorem 4.13.1 that all notions presented, with the exclusion of Li–Yorke chaos, are equivalent to topological chaos, i.e. the function involved has positive topological entropy. In [72] it has been proven that, in general distributional chaos need not imply positivity of entropy, while the converse is an open problem. Continuing, there is a recent paper [73] where the authors solve a long-standing open question by proving that positive topological entropy implies Li–Yorke chaos. Since Devaney's (and Martelli's) chaos is based on the notion of transitivity, it is natural to ask for the relation between topological entropy and transitivity. Quoting [74] the question whether the positivity of the entropy implies transitivity does not make sense since transitivity is a global characteristic, while the positivity of the entropy may be caused by the behavior of the function on an invariant subset of the space." In the other direction, the problem in its generality appears still open [74].

Open problems are also the relations between the positivity of topological entropy and Block–Coppel and ω-chaos.

Now, we compare distributional chaos with the other notions (except for positive entropy). According to Theorem 4.13.1 Li–Yorke chaos does not imply in general the distributional one, while in [75] an example is produced of a map on R^2 which is distributionally chaotic but not chaotic in Li–Yorke sense. Thus, the two notions are independent.

Open problems are the relations between distributional chaos and Martelli's, Devaney's, Block–Coppel and ω-chaos.

It is now the turn of Li–Yorke chaos. By Theorem 4.13.1 Li–Yorke chaos does not imply in general ω-chaos, while in [76] it has been proven that an ω-chaotic map is always Li–Yorke chaotic.

We already know that in the case of the real interval Li–Yorke chaos does not imply Devaney's. Jie–Hua Mai in [77] proved that the reverse implication holds.

A question arises: does Martelli's or Block–Coppel chaos imply Li–Yorke?. The problem is still open.[39]

Relations between ω-chaos and Martelli's, Devaney's and Block–Coppel are not known to the authors[39].

We compare now the two very similar notions of Martelli's and Devaney's chaos. As already seen in chapter 2, Devaney's chaos implies Martelli's. On the other hand it is proven in [18] that the function

$$F(\rho,\theta)=\begin{cases}(2\rho,\theta+1) & , \quad 0\leq\rho<0.5\\ (2-2\rho,\theta+1) & , \quad 0.5\leq\rho\leq 1\end{cases}$$

which maps the unit disk of the plane into itself, is chaotic in the sense of Martelli, but not in the sense of Devaney.

Open problems appear to be the relations of Martelli's and Devaney's chaos with Block–Coppel chaos.

The Robinson's chaoticity implies the Kato's chaoticity on the complete metric space,but the converse is not true in general (see [56]). The definition of the Knudsen's chaos is equivalent to the definition of the Kato's chaos on a compact metric space [57] .

Now we consider a general compact metric space X [54]. The next theorem gives a survey. Since the implications strongly depend on the size of scrambled sets, we restrict ourself to two-point scrambled sets in Theorem 5.1, and uncountable scrambled sets in Theorem 5.2.

Theorem 5.1[54]**:** *Let f be a continuous map of a compact metric space, and let "scrambeld set"means a two-point scrambled set. Then*

$$DC1 \implies DC2 \implies DC3 \ \&\ LYC,$$

$$\omega Chaos \implies LYC,$$

and

$$PTE \implies LYC.$$

There are no other implications, with possible exception for PTE $\implies$ DC3.

Proof: The implications $DC1 \implies DC2 \implies DC3$ & LY-chaos follows by definition, PTE $\implies$ LY-chaos was proved in [Blanchard et al. 2002]. Its validity was an open problem for more than 15 years, cf. [Smital 1986]. The implication ω-chaos $\implies$ LY-chaos was proved by [Lampart 2002].

Next we give references to examples disproving particular implications:

$LY - chaos \not\Longrightarrow PTE$. [Smital 1986], even for $X = I$.

$LY - chaos \not\Longrightarrow \omega - chaos$. [Smital 1986] and [S. Li 1993], even for $X = I$.

$LY - chaos \not\Longrightarrow DC3$. [Schweizer, Smital 1994], even for $X = I$.

$PTE \not\Longrightarrow \omega - chaos$ – take as X any minimal set supporting positive topological entropy.

$DC1 \not\Longrightarrow PTE$ [Liao, Fan 1998], or [Wang, Liao 1999], or [Forti et al. 1999].

$DC2 \not\Longrightarrow DC1$.[Forti et al. 1999].

$DC3 \not\Longrightarrow LY - chaos$. [Babilonov´a 1999].

$PTE \not\Longrightarrow DC1$. [Pikula 2001].

$PTE \not\Longrightarrow DC2$. [Sklar et al. 2002].

$\omega - chaos \not\Longrightarrow PTE$. [Balibrea et al. 2003].

ω-*chaos* $\not\Longrightarrow$ *DC3* [Balibrea et al. 2003].

Theorem 5.2:[54] *Let f be a continuous map of a compact metric space, and let "scrambeld set"mean an uncountable scrambled set. Then:*

$$DC1 \Longrightarrow DC2 \Longrightarrow DC3 \ \&\ LY - chaos,$$

and

$$PTE \Longrightarrow LY - chaos.$$

There are no other implications, with possible exception for PTE $\Longrightarrow$ DC3, DC3 $\Longrightarrow$ LYC,DC3 $\Longrightarrow$ DC2, and DC2 $\Longrightarrow$ DC1.

Proof: The implications $DC1 \Longrightarrow DC2 \Longrightarrow DC3 \ \&\ LYC$ follows by definition, $PTE \Longrightarrow LYC$ was proved in [Blanchard et al. 2002].

Next we give references to examples disproving particular implications:

$LY - chaos \not\Longrightarrow PTE$. [Smital 1986], even for $X = I$.

$LY - chaos \not\Longrightarrow \omega - chaos$. [Smital 1986] and [S. Li 1993], even for $X = I$.

$LY - chaos \not\Longrightarrow DC3$. [Schweizer, Smítal 1994], even for $X = I$.

$DC1 \not\Longrightarrow PTE$.[Liao, Fan 1998], or [Wang, Liao 1999], or [Forti et al. 1999].

$\omega - chaos \not\Longrightarrow LY - chaos$.[Pikula 2001].

6 Conclusion

In Chapter 2 and 3 we have given a number of definitions that can be put into three groups: geometric (relating to sensitive dependence on initial conditions), analytic (utilizing derivatives, such as PLE), and topological (relating to mixing qualities of the iterates, such as transitive). Purely mathematically, all of these definitions are very interesting, but we are also concerned with which definitions are most suitable in different situations.

On the one hand, for physical applications one can usually assemble only a small number of observational data. It therefore follows that the most general form of sensitive dependence on initial conditions (PSDIC) might be most natural to use. Indeed, if one is actually getting data for weather prediction, and similarly, if one obtains data from a double pendulum, then USDIC and ESDIC and also analytic and topological conditions may well not be applicable. When people outside of mathematics speak of chaos, they are likely thinking of PSDIC; USDIC and ESDIC are important alternate definitions of chaos, but are more valuable as mathematical constructs.

However, mathematically speaking, PSDIC has problems. For example, $f(x) = 2x$ has PSDIC, though one would not think of $f(x) = 2x$ as a chaotic function. Thus when considering functions isolated from applications, PSDIC is not a satisfactory condition for considering a function to be chaotic.

On the other hand, if one has a function with a given formula, or a system of differential equations, then derivatives (or their higher dimension analogues) become accessible and can yield much information, not only about separation of iterates but also about the rate at which separation occurs. Thus, the Lyapunov exponent is an important definition. Again, however, using a PLE as the sole characteristic to define a function as chaotic is not satisfactory.

Afterwards various definitions of chaos were proposed. They do not coincide in general and none of them can be considered as the unique good definition of chaos. You may ask What is chaos then?. It relies generally on the idea of unpredictability or instability, i.e., knowing the trajectory of one point does not say what happens elsewhere. The map $f : X \to X$ is said sensitive to initial conditions if near every point x there exist arbitrarily close points y such that the distance between $f^n(x)$ and $f^n(y)$ is greater than a given $\delta > 0$ for some n. The chaos in the sense of Li-Yorke (see above) asks for more instability but only on a subset. For Devaney, chaos is seen as a mixing of unpredictability and regular behaviors: a system is chaotic in the sense of Devaney if it is transitive, sensitive to initial conditions and has a dense set of periodic points. Others put as a part of their definition that the entropy should be positive, which means that the number of different trajectories of length n, up to some approximation, grows exponentially fast. To get something uniform, the system is often assumed to be transitive, roughly speaking it means that it cannot be decomposed into two parts (with non empty interiors) which do not interact under the action of the transformation. This basic assumption has actually strong consequences for systems on one-dimensional spaces. For a continuous interval map, it implies most of the other notions linked to chaos: sensitivity to initial conditions, dense periodic points, positive entropy, chaos in the sense of Li-Yorke. This leads to look for (partial) converses: for instance, if the interval map f is sensitive to initial conditions then for some integer n the map f^n is transitive on a subinterval.

Finally, relationships between the definitions of chaos are very interesting and important, and therefore have been used in combination in some definitions of chaos. One must be careful, however, when combining these characteristics. However in chapter four we give the relationships between the definitions in compact interval and in chapter five we follow two papers to give the relations in general case.

7 References

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