جامعة بيرزيت

BIRZEIT UNIVERSITY

Faculty of Information Technology

Master Program of Computing

Dynamics of Nonlinear Difference Equation $x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + B x_n + C x_{n-k}}$

By:

Amer Ja'far

Supervisor:

Prof. Mohammad Saleh

M.Sc. Thesis

January 18, 2012

BIRZEIT UNIVERSITY

Faculty of Information Technology

Master Program of Computing

Dynamics of Nonlinear Difference Equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + B x_n + C x_{n-k}}$$

by:

Amer Ja'far

Supervisor:

Prof. Mohammad Saleh

"This Thesis was submitted in partial fulfillment of the requirements for the Master's Degree in Computing from the Faculty of Graduate Studies at Birzeit University, Palestine."

January 18, 2012

جامعة بيرزيت

BIRZEIT UNIVERSITY

Faculty of Information Technology

Master Program of Computing

Dynamics of Nonlinear Difference Equation $x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + B x_n + C x_{n-k}}$

by:

Amer Ja'far

This thesis was successfully defended and approved on January 18, 2012

Committee members:

Prof. Mohammad Saleh (Supervisor) ________________

Dr. Hasan Yousef (Member) ________________

Dr. Marwan Aloqeili (Member) ________________

إهداء...

إلى روح والديّ اللذين ربياني صغيرا..

إلى إخوتي وأخواتي..

إلى زوجتي..

إلى ابني وبناتي الغاليات.. داليا، رسيل، ميسان، حمزة ومآثر..

إلى كل من له فضل عليّ..

إلى كل من لا أنساهم من ذاكرتي..

عامر جعفر

Acknowledgement

Foremost, I would like to express my sincere gratitude to my advisor Prof. Mohammad Saleh for the continuous support of my MSc study and research, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis.

Besides my advisor, I would like to thank the rest of my thesis committee: Dr Marwan Aloqaili and Dr. Hasan Yousef for their encouragement, insightful comments, and hard questions.

My sincere thanks also goes to Dr. Khalid Takhman for offering me comments and hints in LaTeX. Also, I am grateful to my friend Hamdan Al-Sayed for enlightening me some points of research.

Last but not the least, I would like to thank my family: my sisters and brothers for their great encouragement and patience.

Contents

List of Figures

List of Tables

Abstract

The main goal of this thesis is to investigate the boundedness, invariant intervals, semi-cycles and global attractivity of all nonnegative solutions of the equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots$$

where the parameters β, γ, A, B and C and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0$ are non-negative real numbers, $k = \{1, 2, \ldots\}$.

We give a detailed description of the semi-cycles of solutions, and determine conditions that satisfy the global asymptotically stable of the equilibrium points.

In particular, this monograph is a generalization of the rational difference equation that was investigated in [13].

الملخص

الهدف الرئيسي لهذه الأطروحة هو دراسة السلوك الديناميكي من حيث الحدودية، الصفات الدورية، أنصاف الدورات، ونقاط الثبات للحلول الموجبة للمعادلة غير خطية المنفصلة:

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}, \quad n = 0,1,2,\ldots$$

حيث أن كل من المتغيرات β, γ, A, B, C والقيم الإبتدائية أعداد حقيقية غير سالبة، $k = \{1,2,\ldots\}$.

نركز في هذه الأطروحة على إعطاء وصف مفصل لأنصاف الدورات للحلول وتحديد الشروط التي تحقق الإستقرار الكامل لنقاط التوازن.

على وجه الخصوص، هذه الدراسة هي تعميم لدراسة المعادلة التي قام بها كل من Hu، Li و Stevic في [13].

CHAPTER 1

Introduction

1.1 Preliminary

The dynamic of any situation refers to how the situation changes over the course of time. A dynamical system is a physical setting together with rules for how the setting changes or evolves from one moment of time to the next. In simplest terms, a dynamical system is a system that changes over time.

For example, a moving body may be represented by state variable of velocity and position over time. Model of population dynamic, the system state variable may be the number of population that migrate, born and dead and the existing population.

Dynamical systems theory is an area of applied mathematics used to describe the behavior of complex dynamical systems, usually by employing differential equations or difference equations. When differential equations are employed, the theory is called continuous dynamical systems. When difference equations are employed, the theory is called discrete dynamical systems. When the time variable runs over a set which is discrete over some intervals and continuous over other intervals or is any arbitrary time-set such as a cantor set then one gets dynamic equations on time scales. Some situations may also be modeled by mixed operators such as differential-difference equations.

This theory deals with the long-term qualitative behavior of dynamical systems, and the studies of the solutions to the equations of motion of systems that are primarily mechanical in nature; although this includes both planetary orbits as well as the behavior of electronic circuits and the solutions to partial differential equations that arise in biology. Much of modern research is focused on the study of chaotic systems.

One basic goal of the mathematical theory of dynamical systems is to determine or characterize the long-term behavior of the system. Different kinds of dynamical systems have common characteristics: Symbolic, discrete, continuous, and fractal dynamic systems . In the study of dynamic systems we try to determine the future behavior given the past and present behaviors.

1.2 Equilibrium Points

Let us consider the difference equation

$$x(n+1) = f(x(n)) \tag{1.2.1}$$

When we study the dynamics of a difference equation, we attempt to do the following actions: determine equilibrium points and periodic points, analyze their stability and asymptotic stability, and determine periodic points. In this chapter we give rigorous definitions of all these notions for the difference equation (1.2.1).

Definition 1.1. A point $\bar{x}$ in the domain of $f(x)$ is said to be *an equilibrium point* of equation (1.2.1) if it is a *fixed point* of f, i.e., $f(\bar{x}) = \bar{x}$.

Graphically speaking, an equilibrium point of a map $f(x)$ is a point where the curve $y = f(x)$ intersects the diagonal line $y = x$.

Example 1.1. The equilibrium points of the cubic map $f(x) = x^3$ can be obtained by solving the equation $x^3 = x$ or $x^3 - x = 0$.
Hence, there are three equilibrium points -1, 0, and 1 for this map (see Figure 1.1).

1.3 Stability Theory

One of the main objectives in the theory of dynamical systems is the study of the behavior of orbits near equilibrium points, in other words, the

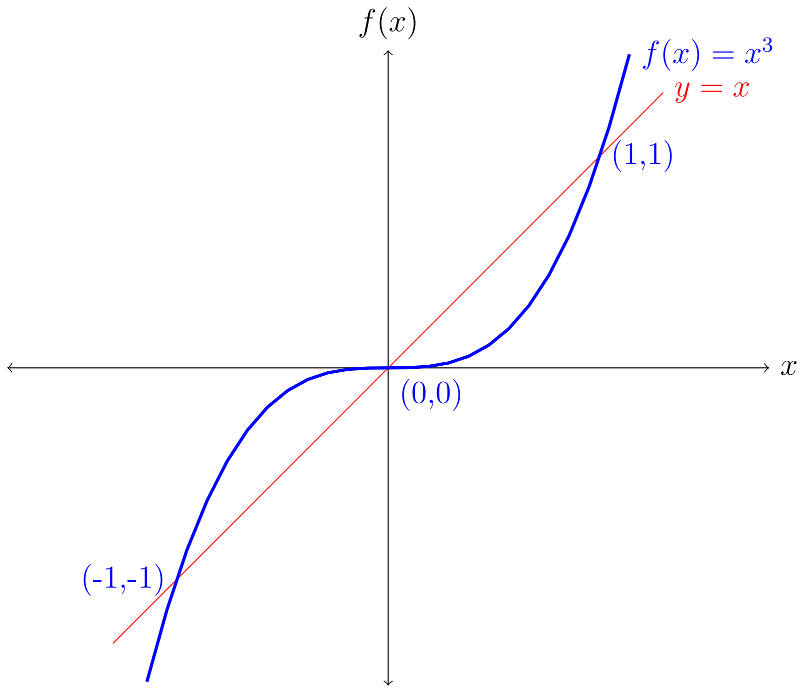

Figure 1.1: The equilibrium points of $f(x) = x^3$ are the intersection points with the diagonal line.

behavior of solutions of a difference equation near equilibrium points. Such a program of investigation is called *stability theory*, which henceforth will be our main focus in this chapter and the next. We begin our exposition by introducing the basic notions of stability.

Definition 1.2. [10] Let $\bar{x}$ be an equilibrium point of equation (1.2.1):

(a) The equilibrium point $\bar{x}$ of equation (1.2.1) is called *stable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|x_0 - \bar{x}| < \delta$ implies $|x_n - \bar{x}| < \epsilon$ for all $n > 0$.

(b) The equilibrium point $\bar{x}$ of equation (1.2.1) is called *attracting* if there exists $\eta > 0$ such that $|x_0 - \bar{x}| < \eta$ implies $\lim_{n \to \infty} x(n) = \bar{x}$.

If $\eta = \infty$, then $\bar{x}$ is called *a global attractor*.

(c) The equilibrium point $\bar{x}$ of equation (1.2.1) is called *globally asymptotically stable* if it is stable and a global attractor.

(d) The equilibrium point $\bar{x}$ is called *unstable* if it is *not stable.*

Example 1.2. Consider the difference equation

$$x_{n+1} = x_n^2 - 2\,x_n + 1$$

So $f(x) = x^2 - 2\,x + 1$. To find the equilibrium points, let

$$\bar{x} = \bar{x}^2 - 2\,\bar{x} + 1$$

which implies $\bar{x} = 1$ or $\bar{x} = 2$ are equilibrium points.

Suppose at some point the solution of a difference equation deviates from the equilibrium value. Will the solution return to the equilibrium value? This problem is called stability problem of the difference equation.

Suppose we are studying the growth of a population and suppose the population has reached a point where for all intents and purposes it is not changing. We say that the population is in equilibrium with its surrounding and the value of the population is the population equilibrium value. Now suppose there is a disaster and 10% of the population is suddenly killed. Will the population return to its original equilibrium value? Will it oscillate? Will it become extinct? Will the population find a new equilibrium value?

We called the equilibrium value is attracting or stable. Regardless the choice of x_0, the solution of a stable difference equation will stabilize itself even if it is temporarily perturbed from its course. The equilibrium is called unstable (repelling) if the solution is perturbed, it remains at its perturbed value and does not return to its original value[22].

The following theorems are used to characterize the stability of the equilibrium points.

Theorem 1.1. *Let $\bar{x}$ be an equilibrium point of the difference equation*

$$x_{n+1} = f(x_n) \tag{1.3.1}$$

where f is continuously differentiable at $\bar{x}$, then the following statements are true:

(i) If $|f'(\bar{x})| < 1$, then $\bar{x}$ is asymptotically stable.

(ii) If $|f'(\bar{x})| > 1$, then $\bar{x}$ is unstable.

In the difference equations, the equilibrium point $\bar{x}$ is said to be ***hyperbolic*** if $|f'(\bar{x})| \neq 1$, and ***unhyperbolic*** if $|f'(\bar{x})| = 1$.

The following two theorems treat stability of *unhyperbolic* points.

Theorem 1.2. *Let $\bar{x}$ be an equilibrium point of the difference equation (1.3.1), and suppose $f'(\bar{x}) = 1$. The following statements are true:*

(i) If $f''(\bar{x}) \neq 0$, then $\bar{x}$ is unstable.

(ii) If $f''(\bar{x}) = 0$ and $f'''(\bar{x}) > 0$ then $\bar{x}$ is unstable.

(ii) If $f''(\bar{x}) = 0$ and $f'''(\bar{x}) < 0$ then $\bar{x}$ is asymptotically stable.

Definition 1.3 (Schwarzian Derivative)**.** Let f be a continuous and differentiable function for three level in some interval I. The *Schwarzian Derivative* $Sf(x)$ of f at a point $x \in I$, where $f'(x) \neq 0$, is given by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2$$

Theorem 1.3. *Let $\bar{x}$ be an equilibrium point of the difference equation (1.3.1), and suppose $f'(\bar{x}) = -1$. The following statements are true:*

(i) If $Sf(\bar{x}) < 0$, then $\bar{x}$ is asymptotically stable.

(ii) If $Sf(\bar{x}) > 0$, then $\bar{x}$ is unstable.

where $Sf(x)$ is the Schwarzian Derivative *of f.*

Example 1.3. Consider the map $f(x) = x^2 + 3\,x$ on the interval $[-3, 3]$. Find the equilibrium points and then determine their stability.

Solution. The equilibrium points of f are obtained by solving the equation

$$x^2 + 3\,x = x$$

Thus, there are two equilibrium points: $\bar{x}_1 = 0$ and $\bar{x}_2 = -2$
So for $\bar{x}_1$, we have $f'(0) = 3 > 1$, which implies by theorem (1.1) that $\bar{x}_1 = 0$ is unstable.

For $\bar{x}_2 = -2$, we have $f'(-2) = -1$, which requires the employment of theorem (1.3). We observe that

$$Sf(-2) = -f'''(-2) - \frac{2}{2}\left[f''(-2)\right]^2 = -6 < 0$$

Hence, $\bar{x}_2 = -2$ is asymptotically stable.

1.4 The Cobweb Diagram

One of the most effective graphical iteration methods to determine the stability of equilibrium points is *the cobweb diagram*[1]. On the xy–plane, we draw the curve $y = f(x)$ and the diagonal line $y = x$ on the same plot. We start at an initial point x_0. Then we move vertically until we hit the graph of f at the point $(x_0, f(x_0))$. We then travel horizontally to meet the line $y = x$ at the point $(f(x_0), f(x_0))$. This determines $f(x_0)$ on the x-axis.

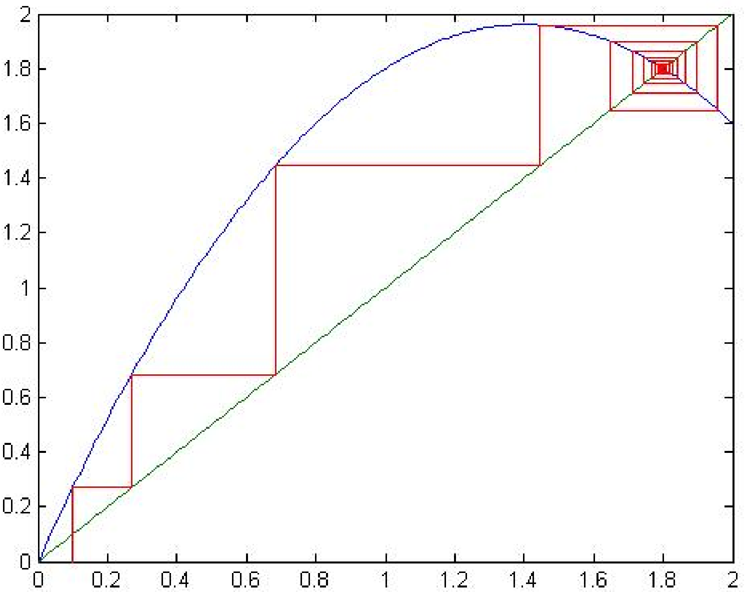

Figure 1.2: Cobweb (Stair step) diagram.

To find $f^2(x_0)$, we move again vertically until we strike the graph of f at the point $(f(x_0), f^2(x_0))$; and then we move horizontally to meet the line $y = x$ at

[1]It is also called the stair-step diagram.

the point $(f^2(x_0), f^2(x_0))$. Continuing this process, we can evaluate all of the points in the orbit of x_0, namely, the set $\{x_0, f(x_0), f^2(x_0), \ldots, f^n(x_0), \ldots\}$ (See Figure (1.2)).

1.5 Periodicity and General Background

The most important notion in the study of dynamical systems is the notion of periodicity. For example, the motion of a pendulum is periodic. Another example, the sensitivity of the suppliers to price is equal to the sensitivity of consumers to price, then prices oscillate between two values only.

Definition 1.4 (Periodicity)**.** Let b be in the domain of f, then b is called a *periodic point* of f if for some positive integer k, $f^k(b) = b$. Hence a point is k-periodic if it is a fixed point of f^k, that is, if it is an equilibrium point of the difference equation

$$x(n+1) = f^k(x(n))$$

The periodic orbit of b,

$$O(b) = \{b, f(b), f^2(b), \ldots, f^{k-1}(b)\},$$

is often called a *k-cycle.*

Definition 1.5 (eventually k-periodic)**.** Let b be in the domain of f, then b is called *eventually k-periodic* if for some positive integer m, $f^m(b)$ is a k-periodic point. In other words, b is *eventually k-periodic* if

$$f^{m+k}(b) = f^m(b)$$

Graphically, we can find the k-periodic point of such a function by finding the point for which the diagonal $y = x$ intersects the graph of $f^k(x)$ and then finding the x-coordinate of such a point.

Example 1.4. Take the equation

$$x(n+1) = x^2(n) - 1$$

Then $f(x) = x^2 - 1$. As we want to find the 2-periodic points, we must find $f^2(x)$.

We know that $f^2(x) = f(f(x)) = f(x^2 - 1) = x^4 - 2\,x^2$. We will plot f^2 and see where will it intersect $y = x$.

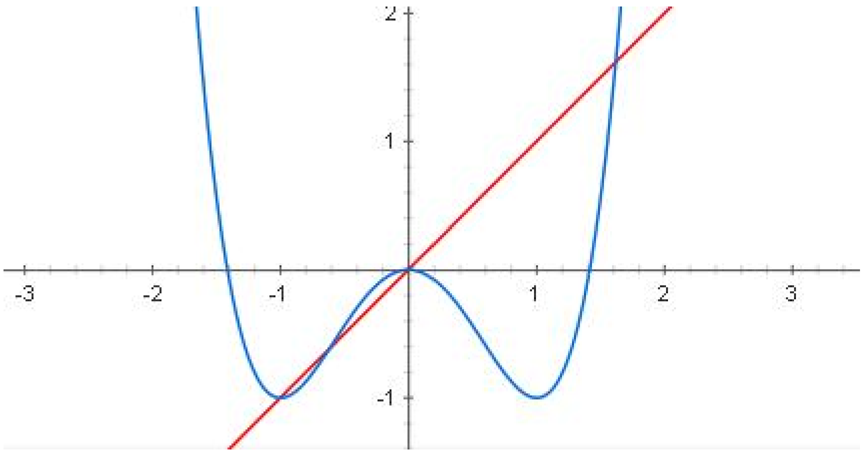

Figure 1.3: 2-periodic points of $x(n+1) = x^2(n) - 1$

As we see in the figure (1.3), the 2-periodic points of our equation are -0.1, -0.622, 0 and 1.618.

Definition 1.6. Let b be a k-period point of f. Then b is:

(i) *stable* if it is a stable fixed point of f^k.

(ii) *asymptotically stable* if it is an asymptotically stable fixed point of f^k.

(iii) *unstable* if it is an unstable fixed point of f^k.

Theorem 1.4. *Let $O(b) = \{b = x(0), x(1), \ldots, x(k-1)\}$ be a k-cycle of a continuously differentiable function f. Then the following statements hold:*

(i) The k-cycle $O(b)$ is asymptotically stable *if*

$$|f'(x(0))f'(x(1)), \ldots, f'(x(k-1))| < 1.$$

(ii) The k-cycle $O(b)$ is unstable *if*

$$|f'(x(0))f'(x(1)), \ldots, f'(x(k-1))| > 1.$$

CHAPTER 2

High Order Difference Equations

2.1 General theory of Linear Difference Equations

The standard form of a k^{th}-order linear difference equation is given by

$$y(n+k) + p_1(n)\, y(n+k-1) + \cdots + p_k(n)\, y(n) = g(n) \tag{2.1.1}$$

where $p_i(n)$ and $g(n)$ are real-valued functions defined for $n \geqslant n_0$ and $p_k(n) \neq 0$ for all $n \geqslant n_0$.

If $g(n)$ is identically zero, then (2.1.1) is said to be *a homogeneous equation*. Otherwise, it is called a *nonhomogeneous equation*.

By letting $n = 0$ in equation (2.1.1), we obtain $y(k)$ in terms of $y(k-1), y(k-2), \dots, y(0)$. Explicitly, we have

$$y(k) = -p_1(0)\,y(k-1) - p_2(0)\,y(k-2) - \cdots - p_k(0)\,y(0) + g(0)$$

Once $y(k)$ is computed, we can go to the next step and evaluate $y(k+1)$ by letting $n = 1$ in equation (2.1.1). By repeating this process, it is possible to evaluate all $y(n)$ for $n \geqslant k$.

Theorem 2.1. *[10] The initial value problems of equation (2.1.1) have a unique solution $\{y(n)\}_{n_0}^{\infty}$ or simply $y(n)$.*

2.1.1 Linear Homogeneous Equations

In the following, we are going to study the general theory of k^{th}-order linear homogeneous difference equations of the form

$$x(n+k) + p_1(n)\,x(n+k-1) + \cdots + p_k(n)\,x(n) = 0 \qquad (2.1.2)$$

Definition 2.1. [10] The functions $f_1(n), f_2(n), \dots, f_r(n)$ are said to be *linearly independent* for $n \geqslant n_0$ if whenever

$$a_1\,f_1(n) + a_2\,f_2(n) + \cdots + a_r\,f_r(n) = 0$$

for all $n \geqslant n_0$, then we must have $a_1 = a_2 = \cdots = a_r = 0$.

Otherwise, the functions are said to be *linearly dependent* if one or more of a_i's are not equal zero.

Definition 2.2. [10] A set of k linearly independent solutions of equation (2.1.2) is called a *fundamental set of solutions.*

Definition 2.3. [10] The *Casoratian* $W(n)$ of the solutions $x_1(n), x_2(n), \ldots, x_r(n)$ is given by

$$W(n) = \det \begin{bmatrix} x_1(n) & x_2(n) & \cdots & x_r(n) \\ x_1(n+1) & x_2(n+1) & \cdots & x_r(n+1) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(n+r-1) & x_2(n+r-1) & \cdots & x_r(n+r-1) \end{bmatrix}$$

Lemma 2.2. *Suppose that $p_k(n) \neq 0$ for all $n \geqslant n_0$, then the Casoratian $W(n) \neq 0$ for all $n \geqslant n_0$ if and only if $W(n_0) \neq 0$.*

Theorem 2.3. *The set of solutions $x_1(n), x_2(n), \ldots, x_k(n)$ of equation (2.1.2) is a* fundamental set *if and only if for some $n_0 \in \mathbb{Z}^+$, the Casoratian $W(n_0) \neq 0$.*

Example 2.1. Consider the third-order difference equation

$$x(n+3) + 3\,x(n+2) - 4\,x(n+1) - 12\,x(n) = 0$$

Show that the functions 2^n, $(-2)^n$, and $(-3)^n$ form a fundamental set of solutions of the equation.

Solution. First, let us verify that 2^n is a legitimate solution by substituting $x(n) = 2^n$ into the equation

$$2^{n+3} + 3\left(2^{n+2}\right) - 4\left(2^{n+1}\right) - 12\left(2^n\right) = 2^n\left[8 + 12 - 8 - 12\right] = 0.$$

In the same way, we verify that $(-2)^n$ and $(-3)^n$ are solutions of the equation.

Second, to affirm the linear independent of these solutions we construct the Casoratian

$$W(n) = \det \begin{bmatrix} 2^n & (-2)^n & (-3)^n \\ 2^{n+1} & (-2)^{n+1} & (-3)^{n+1} \\ 2^{n+2} & (-2)^{n+2} & (-3)^{n+2} \end{bmatrix}$$

Thus

$$W(0) = \det \begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & 3 \\ 4 & 4 & 9 \end{bmatrix} = -20 \neq 0.$$

By theorem (2.3) the solutions 2^n, $(-2)^n$ and $(-3)^n$ are linearly independent, and form a fundamental set of solutions.

Theorem 2.4 (Fundamental Theorem)**.** *[10] If $p_k(n) \neq 0$ for all $n \geqslant n_0$, then equation (2.1.2) has a fundamental set of solutions for $n \geqslant n_0$.*

Lemma 2.5. *[10] If $x_1(n), x_2(n), \ldots, x_r(n)$ are solutions of equation (2.1.2), then*

$$x(n) = a_1\, x_1(n) + a_2\, x_2(n) + \cdots + a_r\, x_r(n)$$

is also a solution of equation (2.1.2), where $a_1, a_2, \ldots, a_r$ are real numbers.

This lemma leads to define the general solution of the homogeneous equation as follows.

Definition 2.4. [10] Let $\{x_1(n), x_2(n), \ldots, x_k(n)\}$ be a fundamental set of solutions of equation (2.1.2). Then the general solution of equation (2.1.2) is given by $x(n) = \sum_{i=1}^{k} a_i x_i(n)$, for arbitrary constants a_i.

2.1.2 Linear Homogeneous Equations with Constant Coefficients

Consider the k^{th}-order difference equation

$$x(n+k) + p_1\, x(n+k-1) + p_2\, x(n+k-2) + \cdots + p_k\, x(n) = 0 \qquad (2.1.3)$$

where the p_i's are constants and $p_k \neq 0$.

Suppose that our solution is in the form of λ^n, where λ is either a real or a complex number. By substituting this value in (2.1.3), we get

$$\lambda^k + p_1 \lambda^{k-1} + \cdots + p_k = 0 \tag{2.1.4}$$

This equation is called the *characteristic equation* of equation (2.1.3), and its roots $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ are called the *characteristic roots*.

There are different cases of the characteristic roots, so the general solution of equation (2.1.3) has different situations depending on the cases of the characteristic roots:

Case(a): Suppose that the characteristic roots $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ are distinct. The fundamental set of solutions will be $\{\lambda_1^n, \lambda_2^n, \ldots, \lambda_k^n\}$, Consequently, the general solution of equation (2.1.3) is

$$x(n) = \sum_{i=1}^{k} a_i \lambda_i^n$$

where a_i's are constant numbers.

Example 2.2. Find the general solution of

$$x(n+2) - x(n+1) - 2\,x(n) = 0$$

Solution. To find the homogeneous solution, we solve the characteristic equation

$$\lambda^2 - \lambda - 2 = 0$$

$$\Rightarrow (\lambda + 1)(\lambda - 2) = 0$$

implies that $\lambda_1 = -1$ and $\lambda_2 = 2$

So the homogeneous solution is

$$x(n) = a_1\, \lambda_1^n + a_2\, \lambda_2^n = a_1\, (-1)^n + a_2\, 2^n.$$

Case(b): Suppose that the characteristic roots $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ all are equal, so the general solution is given by

$$x(n) = \sum_{i=1}^{k} \lambda_i^n (a_{i0} + a_{i1}\, n + \cdots + a_{i,m_i-1}\, n^{m_i-1})$$

where $a_{i0}, a_{i1}, \cdots, a_{i,m_i-1}$ are constant numbers.

Example 2.3. Find the general solution of

$$x(n+2) + x(n+1) + \frac{1}{4}\, x(n) = 0$$

Solution. To find the homogeneous solution, we solve the characteristic equation

$$\lambda^2 + \lambda + \frac{1}{4} = 0$$

$$\Rightarrow (\lambda + \frac{1}{2})(\lambda + \frac{1}{2}) = 0$$

implies that $\lambda_1 = \lambda_2 = \dfrac{-1}{2}$.

So the homogeneous solution is

$$\begin{aligned} x(n) &= a_1\, \lambda^n + a_2\, n\, \lambda^n \\ &= a_1 \left(\frac{-1}{2}\right)^n + a_2\, n \left(\frac{-1}{2}\right)^n \\ &= \left(\frac{-1}{2}\right)^n (a_1 + a_2\, n)\, . \end{aligned}$$

2.1.3 Linear Nonhomogeneous Equation

In the last two subsections we study the solution of the homogeneous difference equations. Here we will focus on solving the k^{th}-order linear nonhomogeneous equation

$$y(n+k) + p_1(n)\, y(n+k-1) + \cdots + p_k(n)\, y(n) = g(n) \tag{2.1.5}$$

where $p_k(n) \neq 0$ for all $n \geqslant n_0$. The sequence $g(n)$ is called the *forcing term*, the *external force*, the *control* or the *input* of the system.

The equation (2.1.5) represents a physical system in which $g(n)$ is the input and $y(n)$ is the output. The designing engineer uses $g(n)$ to force the system to behave in a specified way (control the output $y(n)$).

Theorem 2.6. *[10] If $y_1(n)$ and $y_2(n)$ are solutions of equation (2.1.5), then*

$$y(n) = y_1(n) - y_2(n)$$

is a solution of the corresponding homogeneous equation

$$y(n+k) + p_1(n)\, y(n+k-1) + \cdots + p_k(n)\, y(n) = 0 \tag{2.1.6}$$

Theorem 2.7. *[10] Any solution $y(n)$ of equation (2.1.5) may be written as*

$$y(n) = y_p(n) + \sum_{i=1}^{k} a_i\, y_i(n)$$

where $\{y_1(n), y_2(n), \ldots, y_k(n)\}$ is a fundamental set of solutions of the homogeneous equation (2.1.6).

Example 2.4. Find the general solution of

$$x(n+2) - 3\,x(n+1) + 2\,x(n) = 4^n - n^2$$

Solution. The general solution is

$$x_g(n) = x_h(n) + x_p(n)$$

where $x_h(n)$ is the homogeneous solution and $x_p(n)$ is the particular solution. To find the homogeneous solution, we solve the characteristic equation

$$\lambda^2 - 3\lambda + 2 = 0$$

implies

$$(\lambda - 1)(\lambda - 2) = 0$$

implies that

$$\lambda = 1 \quad \text{or} \quad \lambda = 2$$

Then the homogeneous solution is

$$x_h(n) = a\,\lambda^n + b\,\lambda^n$$

$$x_h(n) = a\; + b\,2^n$$

Now, to find the particular solution, let

$$x_p(n) = \alpha_1\,4^n + \alpha_2\,n^2 + \alpha_3\,n + \alpha_4$$

Substituting this potential solution into the the particular solution formula and equating coefficients as follows:

$$\begin{aligned}
x_p(n) \quad &= \quad \alpha_1\,4^n + \alpha_2\,n^2 + \alpha_3\,n + \alpha_4 \\
x_p(n+1) \quad &= \quad \alpha_1\,4^{n+1} + \alpha_2\,(n+1)^2 + \alpha_3\,(n+1) + \alpha_4 \\
x_p(n+2) \quad &= \quad \alpha_1\,4^{n+2} + \alpha_2\,(n+2)^2 + \alpha_3\,(n+2) + \alpha_4
\end{aligned}$$

Then, substitute the above values of x_p's in the general equation to get

$$\begin{aligned}\Big[\alpha_1\, 4^{n+2} + \alpha_2\,(n+2)^2 + \alpha_3\,(n+2) + \alpha_4\Big]\\ -3\,\Big[\alpha_1\, 4^{n+1} + \alpha_2\,(n+1)^2 + \alpha_3\,(n+1) + \alpha_4\Big]\\ +2\,\Big[\alpha_1\, 4^{n} + \alpha_2\, n^2 + \alpha_3\, n + \alpha_4\Big] = 4^n - n^2\end{aligned}$$

this formula can simplified to

$$\begin{aligned}\alpha_1\, 4^n\,\Big[4^2 - 3\,\times\,4 + 2\Big]\\ +\alpha_3\Big[n + 2 - 3\,n - 3 + 2\,n\Big]\\ +\alpha_2\,\Big[n^2 + 4\,n + 4 - 3\,n^2 - 6\,n - 3 + 2\,n^2\Big] = 4^n - n^2\end{aligned}$$

After doing simple algebraic calculations, we get

$$6\,\alpha_1\, 4^n - 2\,\alpha_2\, n + \alpha_2 - \alpha_3 = 4^n - n^2$$

implies that

$$\alpha_1 = \frac{1}{6}$$

$$\alpha_2 = 0$$

$$\alpha_3 = 0$$

So the particular solution is

$$x_p(n) = \frac{1}{6}\,4^n + \alpha_4$$

Thus the general solution of the difference equation is:

$$x_g(n) = (a + \alpha_4) + b\,2^n + \frac{1}{6}\,4^n$$

$$x_g(n) = A + b\,2^n + \frac{1}{6}\,4^n$$

where $A = a + \alpha_4$.

The values of A and b can be founded by using the initial conditions x_0 and x_1 of the general solution.

2.2 Limiting Behavior of the Solutions

To simplify our exposition we restrict our discussion to the second order difference equation

$$y(n+2) + p_1\,y(n+1) + p_2\,y(n) = 0 \tag{2.2.1}$$

Suppose that λ_1 and λ_2 are the characteristic roots of the equation. Then we have the following three cases[10]:

CASE 1: Suppose that λ_1 and λ_2 are distinct real roots. Then $y_1(n) = \lambda_1^n$ and $y_2(n) = \lambda_2^n$ are two linearly independent solutions of equation (2.2.1). If $|\lambda_1| > |\lambda_2|$, then we call $y_1(n)$ the dominant solution, and λ_1 the dominant characteristic root.

We will now show that the limiting behavior of the general solution

$$y(n) = a_1\,\lambda_1^n + a_2\,\lambda_2^n$$

is determined by the behavior of the dominant solution. So assume, without loss of generality, that $|\lambda_1| > |\lambda_2|$. Then

$$y(n) = \lambda_1^n \left(a_1 + a_2 \left(\frac{\lambda_2}{\lambda_1} \right)^n \right)$$

Since $\left|\frac{\lambda_2}{\lambda_1}\right| < 1$ it follows that $\left(\frac{\lambda_2}{\lambda_1}\right)^n \to 0$ as $n \to \infty$

Consequently, $\lim_{n\to\infty} y(n) = \lim_{n\to\infty} a_1\,\lambda_1^n$

There are six different situations that may arise here depending on the value of λ_1

1. $\lambda_1 > 1$: The sequence $\{a_1\,\lambda_1^n\} \to \infty$ *(unstable system).*
2. $\lambda_1 = 1$: The sequence $\{a_1\,\lambda_1^n\}$ is a *constant sequence.*
3. $0 < \lambda_1 < 1$: The sequence $\{a_1\,\lambda_1^n\}$ is *monotonically decreasing to zero* (stable system).
4. $-1 < \lambda_1 < 0$: The sequence $\{a_1\,\lambda_1^n\}$ is *oscillating around zero* (i.e., alternating in sign) and *converging to zero* (stable system).
5. $\lambda_1 = -1$: The sequence $\{a_1\,\lambda_1^n\}$ is *oscillating between two values* a_1 and $-a_1$.
6. $\lambda_1 < -1$: The sequence $\{a_1\,\lambda_1^n\}$ is *oscillating but increasing in magnitude* (unstable system).

Example 2.5. Find the general solution of

$$x(n+2) + \frac{1}{2}\,x(n+1) - \frac{1}{2}\,x(n) = 0$$

and determine the limiting behavior of the solution

Solution. To find the general solution, we solve the characteristic equation

$$\lambda^2 + \frac{1}{2}\,\lambda - \frac{1}{2} = 0$$

$$\Rightarrow (\lambda+1)(\lambda - \frac{1}{2}) = 0$$

implies that $\lambda_1 = -1$ and $\lambda_2 = \dfrac{1}{2}$

So the general solution is

$$x(n) = a_1\,\lambda_1^n + a_2\,n\,\lambda_2^n = a_1\,(-1)^n + a_2\,\left(\frac{1}{2}\right)^n.$$

We have $|\lambda_1| > |\lambda_2|$ and $\lambda_1 = -1$, so by applying the fifth item above, the sequence $\{a_1\,(-1)^n\}$ oscillates between the values a_1 and a_2.

Case 2: $\lambda_1 = \lambda_2 = \lambda$

The general solution of (2.2.1) is given by $y(n) = (a_1 + a_2\,n)\lambda^n$. Clearly, if $|\lambda| \geqslant 1$, the solution $y(n)$ diverges either monotonically if $\lambda \geqslant 1$ or by oscillating if $\lambda \leqslant -1$. However, if $|\lambda| < 1$, then the solution converges to zero, since $\lim\limits_{n\to\infty} n\,\lambda^n = 0$.

Example 2.6. Let us check the solution of the equation in example (2.3)

$$x(n+2) - x(n+1) + \frac{1}{4}\,x(n) = 0$$

In this case $\lambda_1 = \lambda_2 = \dfrac{-1}{2}$ and the solution was

$$x(n) = a_1\,\lambda^n + a_2\,n\,\lambda^n = \left(\frac{-1}{2}\right)^n (a_1 + a_2\,n)$$

We find that the solution converges to zero, since $|\lambda| = \left|\dfrac{-1}{2}\right| < 1$ and $\lim\limits_{n\to\infty} n\left(\dfrac{-1}{2}\right)^n = 0$.

Case 3: The last case when the roots λ_1 and λ_2 are complex roots. Set $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$. The general solution will be

$$y(n) = \alpha\,r^n\,\cos(n\theta - \omega)$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(\frac{b}{a})$.

The solution $y(n)$ clearly oscillates, since the cosine function oscillates. However, $y(n)$ oscillates in three different ways depending on the location of the conjugate characteristic roots:

(a) $r > 1$: Here λ_1 and $\lambda_2 = \overline{\lambda_1}$ are outside the unit circle. Hence $y(n)$ is oscillating but increasing in magnitude (unstable system).

(b) r = 1: Here λ_1 and $\lambda_2 = \overline{\lambda_1}$ lie on the unit circle. In this case $y(n)$ is oscillating but constant in magnitude.

(c) $r < 1$: Here λ_1 and $\lambda_2 = \overline{\lambda_1}$ lie inside the unit disk. The solution $y(n)$ oscillates but converges to zero as $n \to \infty$ (stable system).

Example 2.7. Consider the difference equation

$$x(n+2) + x(n+1) + x(n) = 0$$

The characteristic equation is

$$\lambda^2 + \lambda + 1 = 0$$

implies that $\lambda_1 = 1 - i$ and $\lambda_2 = 1 + i$, then the general solution will be

$$x(n) = \alpha \left(\sqrt{2}\right)^n \cos(n\theta - \omega)$$

Since $r = \sqrt{2} > 1$, the solution $x(n)$ is oscillating and increasing, so it is unstable.

The above discussion can be summarized in the following theorem.

Theorem 2.8. *[10] The following statements hold:*

(i) All solutions of equation (2.2.1) oscillate about zero if and only if the characteristic equation has no positive real roots.

(ii) All solutions of equation (2.2.1) converge to zero (i.e., the zero solution is asymptotically stable) if and only if $\max\{|\lambda_1|, |\lambda_2|\} < 1$.

Now, consider nonhomogeneous difference equations in which the input is constant, that is, equations of the form

$$y(n+2) + p_1\, y(n+1) + p_2\, y(n) = M \tag{2.2.2}$$

where M is a nonzero constant. Let the equilibrium point or solution be y^*, we have

$$y^* + p_1\, y^* + p_2\, y^* = M,$$

which implies

$$y^* = \frac{M}{1 + p_1 + p_2}$$

Since $y_p(n) = y^*$ is a particular solution of equation (2.2.2), then the general solution is given by

$$y(n) = y_p(n) + y_c(n)$$

And if we take $y_p(n) = y^*$, this concludes to the following theorem.

Theorem 2.9. *[10] The following statements hold:*

(i) All solutions of the nonhomogeneous equation (2.2.2) oscillate about the equilibrium solution y^* *if and only if none of the characteristic roots of the homogeneous equation (2.2.1) is a positive real number.*

(ii) *All solutions of equation (2.2.2) converge to* y^* *as* $n \to \infty$ *if and only if* $\max\{|\lambda_1|, |\lambda_2|\} < 1$, *where* λ_1 *and* λ_2 *are the characteristic roots of the homogeneous equation (2.2.1).*

The following theorem provides us with the explicit criteria for stability based on the values of the coefficients p_1 and p_2 of the difference equations (2.2.1) or (2.2.2).

Theorem 2.10. *[10] The conditions*

$$1 + p_1 + p_2 > 0, \quad 1 - p_1 + p_2 > 0, \quad 1 - p_2 > 0$$

are necessary and sufficient conditions for the equilibrium points (solution) of equations (2.2.1) and (2.2.2) to be asymptotically stable.

If we consider the above conditions,

$$1 + p_1 + p_2 > 0 \Rightarrow 1 + p_2 > -p_1$$

and

$$1 - p_1 + p_2 > 0 \Rightarrow 1 + p_2 > p_1$$

implies that

$$1 + p_2 > |p_1|$$

Also

$$1 - p_2 > 0 \Rightarrow 1 > p_2$$

by adding one for both sides, we get

$$2 > 1 + p_2$$

This discussion implies that the conditions in theorem (2.10) can be written under the following compact form

$$|p_1| < 1 + p_2 < 2.$$

CHAPTER 3

Dynamics of $x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}$

In this chapter we consider the main issue of this thesis, that is studying and investigating the difference equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{3.0.1}$$

where the parameters β, γ, A, B and C are non-negative real numbers with at least one parameter is non zero and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0$ are non-negative real numbers for which the solution is defined and $k \in \{1, 2, \ldots\}$.

My concentration is on boundedness, invariant intervals, periodic character, the character of semi-cycles and global asymptotic stability of zero and positive solutions of equation (3.0.1).

It is a good idea to overview the difference equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \qquad n = 0, 1, 2, \ldots \tag{3.0.2}$$

where the parameters β, γ, A, B and C are non-negative real numbers and the initial conditions x_{-1}, x_0 are non-negative real numbers.

This will help us to study and analyze equation (3.0.1) more theoretically. The characteristics of equation (3.0.2) has been investigated by Lin-Xia Hu et al. in [13]. They showed, the global stability of the zero and positive equilibrium points, invariant intervals and semi-cycle analysis. In addition, M. Kulenović and G. Ladas considered equation (3.0.2) in their monograph in [16].

The dynamical characteristics and the behavior of positive solutions of some higher order nonlinear difference equations have been investigated by many researchers.

DeVault et al. investigated in [6] the global stability and the periodic character of solutions of the difference equation

$$y_{n+1} = \frac{p + y_{n-k}}{qy_n + y_{n-k}}, \qquad n = 0, 1, 2, \ldots$$

where the parameters p and q are positive real numbers and the initial conditions $y_{-k}, \ldots, y_{-1}, y_0$ are arbitrary non-negative real numbers, $k \in \{1, 2, \ldots\}$.

Li and Sun in [18] and A. Farhat in [11] studied independently the dynamical characteristics, such as the global asymptotic stability, the invariant interval, the periodic and oscillatory characters of all positive solutions of the difference equation

$$x_{n+1} = \frac{px_n + x_{n-k}}{q + x_{n-k}}, \qquad n = 0, 1, 2, \ldots$$

where the parameters p and q are non-negative real numbers and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are non-negative real numbers, $k \in \{1, 2, \ldots\}$.

In [8], M.M. El-Afifi investigated the local and globally asymptotically stability and the semi-cycles of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{Bx_n + Cx_{n-1}}, \qquad n = 0, 1, 2, \ldots$$

where the parameters α, β, γ, B and C are non-negative real numbers with at least one parameter is non zero and the initial conditions x_{-1}, x_0 are non-negative real numbers.

A. Farhat studied the general case of the last equation in [11] and investigated the periodic character of the positive solution, the invariant intervals, the oscillation and the global stability of all solutions of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots$$

where the parameters α, β, γ, B and C are non-negative real numbers with at least one parameter is non zero and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are non-negative real numbers, $k \in \{1, 2, \ldots\}$.

In [19] Sebdani and Dehghan investigated characteristics such as periodicity, invariant interval, the character of semicycles, the global stability, and the boundedness of positive solutions of the difference equation

$$y_{n+1} = \frac{y_n + p y_{n-k}}{q + y_n}, \qquad n = 0, 1, 2, \ldots$$

where the parameters p and q are non-negative real numbers and the initial conditions $y_{-k}, \ldots, y_{-1}, y_0$ are non-negative real numbers, $k \in \{1, 2, \ldots\}$.

M. Saleh and S. Abu-Baha investigated in [20] the difference equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{B x_n + C x_{n-k}}, \qquad n = 0, 1, 2, \ldots$$

where the parameters β, γ, B and C are non-negative real numbers and the initial conditions $x_0, x_1, \ldots, x_k$ are non-negative real numbers with the solution is defined and $k \in \{1, 2, 3, \ldots\}$.
Their concentration was on invariant intervals, periodic character, the character of semicycles and global asymptotic stability of all positive solutions of the equation.

Also, Lin-Xia Hu et al. investigated in [14] the boundedness, invariant interval, semicycle and global attractivity of all positive solutions of the difference equation

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{A + B x_n + C x_{n-1}}, \qquad n = 0, 1, 2, \ldots$$

where the parameters $\alpha, \gamma, A, B, C \in (0, \infty)$ and the initial conditions x_{-1}, x_0 are non-negative real numbers.

3.1 Changing the Variables

Before investigating the characteristics of equation (3.0.1), it is more convenient to reduce the number of parameters by a change of variables.

To change the variables of equation (3.0.1), assume that $x_n = \frac{\gamma}{C} y_n$, and substitute it in the equation as the following:

$$\frac{\gamma}{C} y_{n+1} = \frac{\beta \frac{\gamma}{C} y_n + \frac{\gamma^2}{C} y_{n-k}}{A + B \frac{\gamma}{C} y_n + C \frac{\gamma}{C} y_{n-k}}$$

tacking a common factor $\frac{\gamma^2}{C}$ in the numerator and γ in the denominator,

$$\frac{\gamma}{C}y_{n+1} = \frac{\frac{\gamma^2}{C}\left[\frac{\beta}{\gamma}y_n + y_{n-k}\right]}{\gamma\left[\frac{A}{\gamma} + \frac{B}{C}y_n + y_{n-k}\right]}$$

then

$$\frac{\gamma}{C}y_{n+1} = \frac{\gamma\left[\frac{\beta}{\gamma}y_n + y_{n-k}\right]}{C\left[\frac{A}{\gamma} + \frac{B}{C}y_n + y_{n-k}\right]}$$

which implies

$$y_{n+1} = \frac{\frac{\beta}{\gamma}y_n + y_{n-k}}{\frac{A}{\gamma} + \frac{B}{C}y_n + y_{n-k}}$$

by assuming $p = \frac{\beta}{\gamma}$, $q = \frac{B}{C}$ and $r = \frac{A}{\gamma}$, we get the following equation:

$$y_{n+1} = \frac{py_n + y_{n-k}}{r + qy_n + y_{n-k}}, \qquad n = 0, 1, \ldots \tag{3.1.1}$$

3.2 Equilibrium Points

In this section we investigate the equilibrium points of the nonlinear rational difference equation

$$y_{n+1} = \frac{py_n + y_{n-k}}{r + qy_n + y_{n-k}}, \qquad n = 0, 1, \ldots \tag{3.2.1}$$

where the parameters p, q, r and the initial conditions $y_{-k}, \ldots, y_{-1}, y_0$ are positive real numbers, $k = 1, 2, \ldots$.

Definition 3.1. The equilibrium point $\bar{y}$ of the equation

$$y_{n+1} = f(y_n, y_{n-1}, \ldots, y_{n-k}), \qquad n = 0, 1, \ldots$$

is the point that satisfies the condition

$$\bar{y} = f(\bar{y}, \bar{y}, \ldots, \bar{y}).$$

To find the equilibrium points of equation (3.2.1) using the definition, let $f(\bar{y}, \bar{y}) = \bar{y}$, then we solve the following equation

$$\bar{y} = \frac{p\bar{y} + \bar{y}}{r + q\bar{y} + \bar{y}}$$

Cross multiplication, implies

$$\bar{y}r + (q+1)\bar{y}^2 = (p+1)\bar{y}$$

Rearranging the terms, we get

$$(r - p - 1)\bar{y} + (q+1)\bar{y}^2 = 0$$

Then,

$$\bar{y}\big[(r - p - 1) + (q+1)\bar{y}\big] = 0$$

hence, the equilibrium points of equation (3.2.1) are

$$\bar{y} = 0$$

and

$$\bar{y} = \frac{p + 1 - r}{q + 1}$$

$$\text{where} \quad p + 1 > r.$$

3.3 Linearization of the Difference Equation

Consider the k^{th}-order difference equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \qquad n = 0, 1, \ldots \tag{3.3.1}$$

Assume that f is continuously differentiable in some neighborhood around $\bar{x}$, then we can linearize equation (3.3.1) around $\bar{x}$. Thus, by chain rule, the linearized equation around $\bar{x}$ becomes

$$z_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial u_i}(\bar{x}, \ldots, \bar{x}) z_{n-i}$$

The characteristic equation is given by

$$\lambda^{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial u_i}(\bar{x}, \ldots, \bar{x}) \lambda^{n-i}$$

or

$$\lambda^{k+1} - \frac{\partial f}{\partial u_0}\lambda^k - \frac{\partial f}{\partial u_1}\lambda^{k-1} - \cdots - \frac{\partial f}{\partial u_k} = 0$$

To find the linearization of our equation (3.2.1) about the equilibrium point, consider

$$f(u, v) = \frac{pu + v}{r + qu + v}$$

Thus

$$\begin{aligned} f_u(u, v) &= \frac{p(r + qu + v) - q(pu + v)}{(r + qu + v)^2} \\ &= \frac{pr + pqu + pv - pqu - qv}{(r + qu + v)^2} \\ &= \frac{pr + (p - q)v}{(r + qu + v)^2} \end{aligned}$$

Which implies

$$\begin{aligned} f_u(\bar{y},\bar{y}) &= \frac{pr+(p-q)\bar{y}}{(r+q\bar{y}+\bar{y})^2} \\ &= \frac{pr+(p-q)\bar{y}}{(r+(q+1)\bar{y})^2} \end{aligned}$$

In the same way

$$\begin{aligned} f_v(u,v) &= \frac{(r+qu+v)-(pu+v)}{(r+qu+v)^2} \\ &= \frac{r+(q-p)u}{(r+qu+v)^2} \end{aligned}$$

Substituting $\bar{y}$ implies

$$\begin{aligned} f_v(\bar{y},\bar{y}) &= \frac{r+(q-p)\bar{y}}{(r+q\bar{y}+\bar{y})^2} \\ &= \frac{r+(q-p)\bar{y}}{(r+(q+1)\bar{y})^2} \end{aligned}$$

The linearized equation is

$$z_{n+1} = f_u(\bar{y},\bar{y})\, z_n + f_v(\bar{y},\bar{y})\, z_{n-k}$$

$$z_{n+1} = \frac{pr+(p-q)\bar{y}}{(r+(q+1)\bar{y})^2}\, z_n + \frac{r+(q-p)\bar{y}}{(r+(q+1)\bar{y})^2}\, z_{n-k}$$

i.e.

$$z_{n+1} - \frac{pr+(p-q)\bar{y}}{(r+(q+1)\bar{y})^2}\, z_n - \frac{r+(q-p)\bar{y}}{(r+(q+1)\bar{y})^2}\, z_{n-k} = 0 \tag{3.3.2}$$

and the characteristic equation is

$$\lambda^{n+1} - \frac{pr+(p-q)\bar{y}}{(r+(q+1)\bar{y})^2}\, \lambda^n - \frac{r+(q-p)\bar{y}}{(r+(q+1)\bar{y})^2}\, \lambda^{n-k} = 0$$

which implies

$$\lambda^{k+1} - \frac{pr+(p-q)\bar{y}}{(r+(q+1)\bar{y})^2}\, \lambda^k - \frac{r+(q-p)\bar{y}}{(r+(q+1)\bar{y})^2} = 0 \tag{3.3.3}$$

3.4 Local Stability of the Equilibrium Points

Our aim in this section is to investigate the local stability of the zero and positive equilibrium points of equation (3.2.1) for which we found the linearized equation in the last section.

First we present some powerful criterion and theorems to study the local stability of equilibrium points.

Definition 3.2. [19] Let $\bar{y}$ be an equilibrium point of equation (3.5.1).

(i) The equilibrium point $\bar{y}$ of equation (3.5.1) is called *locally stable* (or *stable*) if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $y_{-k}, \ldots, y_{-1}, y_0 \in I$ with

$$\sum_{i=-k}^{0} |y_i - \bar{y}| < \delta,$$

we have $|y_n - \bar{y}| < \epsilon$ for all $n \geqslant -k$.

(ii) The equilibrium point $\bar{y}$ of equation (3.5.1) is called locally *asymptotically stable* (*asymptotic stable*) if it is *locally stable*, and if there exists $\gamma > 0$ such that for all $y_{-k}, \ldots, y_{-1}, y_0 \in I$ with

$$\sum_{i=-k}^{0} |y_i - \bar{y}| < \gamma,$$

we have $\lim_{n\to\infty} y_n = \bar{y}$.

(iii) The equilibrium point $\bar{y}$ of equation (3.5.1) is called a *global attractor* if for every $y_{-k}, \ldots, y_{-1}, y_0 \in I$, we have $\lim_{n\to\infty} y_n = \bar{y}$.

(iv) The equilibrium point $\bar{y}$ of equation (3.5.1) is called *globally asymptotically stable* if it is *locally stable* and a *global attractor*.

(v) The equilibrium point $\bar{y}$ of equation (3.5.1) is called *unstable* if it is *not stable*.

(vi) The equilibrium point $\bar{y}$ of equation (3.5.1) is called a *source*, or a *repeller*, if there exists $r > 0$ such that for all $y_{-k}, \ldots, y_{-1}, y_0 \in I$ with $\sum_{n=-k}^{0} |y_i - \bar{y}| < \gamma$, there exists $N \geqslant 1$ such that $|y_N - \bar{y}| \geqslant r$.

Theorem 3.1. *[16] Let I be some interval of real numbers and let*

$$f : I^{k+1} \to I$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, \ldots, x_1, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-k}), \quad n = 0, 1, \ldots \tag{3.4.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Theorem 3.2. *[15] Assume that $a, b \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$. Then*

$$|a| + |b| < 1 \tag{3.4.2}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+1} - ay_n + by_{n-k} = 0, \quad n = 0, 1, \ldots \tag{3.4.3}$$

Suppose in addition that one of the following two cases holds:

(a) k odd and $b < 0$.

(b) k even and $ab < 0$.

Then (3.4.2) is also a necessary condition for the asymptotic stability of equation (3.4.3).

Theorem 3.3. *[15] Assume that $a, b \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$. Then*

$$|a| < 1 - b < 2$$

is a necessary and sufficient condition for the asymptotic stability of the difference equation

$$y_{n+1} - ay_n - by_{n-k} = 0, \quad n = 0, 1, \ldots \tag{3.4.4}$$

Theorem 3.4. *[16] Assume that all the roots of the characteristic equation of the above equation (3.4.4) lie inside the unit circle, then the equilibrium point is* locally asymptotically stable.

3.4.1 Local Stability of the Zero Equilibrium Point

To investigate the local stability of the zero equilibrium point, let us find the linearized equation about the zero equilibrium point associated with equation (3.2.1).

By substituting $\bar{y} = 0$ in the linearized equation (3.3.2), we find the linearized equation associated with equation (3.2.1) about $\bar{y} = 0$,

$$z_{n+1} - \frac{p}{r} z_n - \frac{1}{r} z_{n-k} = 0 \tag{3.4.5}$$

Now, let us apply theorem (3.3) to the above equation (3.4.5), so we have

$$\overbrace{\left|\frac{p}{r}\right| < 1 - \frac{1}{r}}^{\text{part (I)}} \hspace{-4em} \underbrace{\hspace{4em} < 2}_{\text{part (II)}} \tag{3.4.6}$$

It is clear that $\dfrac{p}{r}$ is positive since p and r are nonnegative, so

$$\frac{p}{r} < 1 - \frac{1}{r}$$

implies

$$p < r - 1$$

i.e.

$$p + 1 < r.$$

Now, the right hand inequality in (3.4.6),

$$1 - \frac{1}{r} < 2$$

then

$$\frac{r-1}{r} < 2$$

implies

$$-1 < r$$

which is true for all nonnegative values of r.

The above discussion yields the following theorem.

Theorem 3.5. *The zero equilibrium point of equation (3.2.1) is* locally asymptotically stable *under the condition* $p+1 < r$, *otherwise it is* unstable.

3.4.2 Local Stability of the Positive Equilibrium Point

In this subsection we prove the local stability of the positive equilibrium point under some conditions.

Theorem 3.6. *The positive equilibrium point* $\bar{y} = \frac{p+1-r}{q+1}$ *of equation (3.2.1) is* locally asymptotically stable *for all values of the parameters* p*,* q *and* r *provided that all roots of equation (3.3.3) lie inside the unit circle.*

Theorem 3.7. *Assume that* $p + 1 > r$*, then the positive equilibrium point* $\bar{y} = \frac{p+1-r}{q+1}$ *of equation (3.2.1) is* locally asymptotically stable *when*

$$q + r < 3p + 1 + qr + pq \tag{3.4.7}$$

Proof. First substitute $\bar{y} = \frac{p+1-r}{q+1}$ in the linearized equation (3.3.2) to get the linearized equation about $\bar{y} = \frac{p+1-r}{q+1}$,

$$z_{n+1} - \frac{pr + (p-q)\frac{p+1-r}{q+1}}{\left(r + (q+1)\frac{p+1-r}{q+1}\right)^2} z_n - \frac{r + (q-p)\frac{p+1-r}{q+1}}{\left(r + (q+1)\frac{p+1-r}{q+1}\right)^2} z_{n-k} = 0$$

simplifying the above equation by MATLAB program[1], we get

$$z_{n+1} - \frac{p - q + qr}{qp + q + p + 1} z_n - \frac{-p + q + r}{qp + q + p + 1} z_{n-k} = 0 \tag{3.4.8}$$

By applying theorem (3.3) on the linearized equation (3.4.8), we have

$$a = \frac{p - q + qr}{qp + q + p + 1}$$

and

$$b = \frac{-p + q + r}{qp + q + p + 1}$$

Now we need to verify the inequality

$$\overbrace{\left|\frac{p - q + qr}{qp + q + p + 1}\right| < 1 - \frac{-p + q + r}{qp + q + p + 1}}^{\text{first side}}\!\phantom{\left|\frac{p - q + qr}{qp + q + p + 1}\right| <\ } \underbrace{1 - \frac{-p + q + r}{qp + q + p + 1} < 2}_{\text{second side}} \tag{3.4.9}$$

[1]see the code of simplifying the linearized equation in the Appendix A.2

First, assume that the left of the first side is positive, then

$$\frac{p-q+qr}{qp+q+p+1} < 1 - \frac{-p+q+r}{qp+q+p+1}$$

$$\Longrightarrow \frac{p-q+qr}{qp+q+p+1} < \frac{qp+2p+1-r}{qp+q+p+1}$$

i.e.

$$p-q+qr < qp+2p+1-r$$

implies

$$r < p+1$$

which is the assumption.

Now, assume that the left of the first side of inequality (3.4.9) is negative, then

$$-\frac{p-q+qr}{qp+q+p+1} < \frac{qp+2p+1-r}{qp+q+p+1}$$

i.e.

$$-p+q-qr < qp+2p+1-r$$

$$\Longrightarrow q+r < 3p+1+qr+pq$$

as needed.

The second side of the inequality

$$1 - \frac{-p+q+r}{qp+q+p+1} < 2$$

then

$$\frac{qp+2p+1-r}{qp+q+p+1} < 2$$

i.e.

$$qp+2p+1-r < 2qp+2q+2p+2$$

implies

$$0 < qp + 2q + 1 + r$$

which is true for all positive values of p, q and r.

The proof is complete. □

3.5 Boundedness

In this section we give the boundedness of solutions of the difference equation (3.0.1). In the beginning we present the following definition.

Definition 3.3. We say that a solution x_n of a difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \qquad n = 0, 1, \ldots \tag{3.5.1}$$

is *bounded and persists* if there exist positive constants P and Q such that

$$P \leqslant x_n \leqslant Q \quad \text{for} \quad n = -k, -k+1, \ldots$$

Theorem 3.8. *Assume that B and C are greater than zero, then every non-negative solution of equation (3.0.1) is bounded from above by a positive constant.*

Proof. Firt case, assume $A = 0$, then by using equation (3.0.1), we can write the following:

$$\begin{aligned} x_{n+1} &= \frac{\beta x_n + \gamma x_{n-k}}{B x_n + C x_{n-k}} \\ &\leqslant \frac{\max(\beta, \gamma)(x_n + x_{n-k})}{\min(B, C)(x_n + x_{n-k})} \\ &= \frac{\max(\beta, \gamma)}{\min(B, C)} \end{aligned}$$

The second case, assume $A > 0$, then

$$\begin{aligned}
x_{n+1} &= \frac{\beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}} \\
&\leqslant \frac{\max(\beta,\gamma) + \beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}} \\
&\leqslant \frac{\max(\beta,\gamma)(1 + x_n + x_{n-k})}{\min(A,B,C)(1 + x_n + x_{n-k})} \\
&= \frac{\max(\beta,\gamma)}{\min(A,B,C)}
\end{aligned}$$

The proof is complete. □

3.6 Invariant Intervals

Here we investigate the invariant intervals of equation (3.2.1), and the following definition will be a key concept in the monograph.

Definition 3.4 (Invariant Interval)**.** An *Invariant Interval* of the difference equation (3.5.1) is an interval with the property that if $k+1$ consecutive terms of the solution fall in I then all subsequent terms of the solution also belong to I. In other words, I is an *invariant interval* for (3.5.1) if $x_{N-k+1}, \ldots, x_{N-1}, x_N \in I$ for some $N \geqslant 0$, then $x_n \in I$ for every $n > N$.

Assume that $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of equation (3.2.1), then the following identities are easily established:

$$y_{n+1} - 1 = (p-q)\frac{y_n - \frac{r}{p-q}}{r + qy_n + y_{n-k}}, \qquad n \in \mathbb{N}_0, \tag{3.6.1}$$

$$y_{n+1} - \frac{p}{q} = \frac{q-p}{q}\frac{y_{n-k} - \frac{pr}{q-p}}{r + qy_n + y_{n-k}}, \qquad n \in \mathbb{N}_0 \tag{3.6.2}$$

If $p = q$ then the numerator in (3.6.1) can be written in the following form

$$(p-q)\,y_n - r = -r.$$

Thus if $p+1 > r$ and $p = q$, the unique equilibrium is $\bar{y} = \frac{p+1-r}{p+1}$ and the following identities hold:

$$y_{n+1} - 1 = \frac{-r}{r + p\,y_n + y_{n-k}}, \qquad n \in \mathbb{N}_0, \tag{3.6.3}$$

$$y_{n+1} - \bar{y} = \frac{r}{p+1}\frac{p\,(y_n - \bar{y}) + (y_{n-k} - \bar{y})}{r + p\,y_n + y_{n-k}}, \qquad n \in \mathbb{N}_0 \tag{3.6.4}$$

When $p = q + r$, the unique equilibrium is $\bar{y} = 1$ and identity (3.6.1) becomes

$$y_{n+1} - 1 = \frac{r\,(y_n - 1)}{r + q\,y_n + y_{n-k}}, \qquad n \in \mathbb{N}_0, \tag{3.6.5}$$

And when $q = p + qr$, the unique equilibrium is $\bar{y} = \frac{p}{q}$ and identity (3.6.2) becomes

$$y_{n+1} - \frac{p}{q} = \frac{r\left(y_{n-k} - \frac{p}{q}\right)}{r + q\,y_n + y_{n-k}}, \qquad n \in \mathbb{N}_0 \tag{3.6.6}$$

Theorem 3.9. *[13] Assume that $f(x,y)$ is defined as*

$$f(x,y) = \frac{p\,x + y}{r + q\,x + y} \tag{3.6.7}$$

Then the following statements hold true:

(i) $f(x,x)$ *is* strictly increasing *in x in* $[0,\infty)$.

(ii) Assume $p = q$, then $f(x,y)$ is strictly increasing *in each of its arguments.*

(iii) Assume $p > q$, then $f(x,y)$ is strictly increasing *in each of its arguments for $x < \frac{r}{p-q}$ and it is* strictly increasing *in x and* decreasing *in y for $x \geqslant \frac{r}{p-q}$.*

(iv) *Assume $p < q$, then $f(x,y)$ is* strictly increasing *in each of its arguments for $y < \frac{pr}{q-p}$ and it is* strictly increasing *in y and* decreasing *in x for $y \geqslant \frac{pr}{q-p}$.*

Proof. (i) Note that $f(x,x) = \dfrac{(p+1)x}{r+(q+1)x}$ is strictly increasing function in the interval $[0,\infty)$, since the derivative of f

$$f'(x,x) = \frac{r(p+1)}{(r+(q+1)x)^2}$$

is always positive for $x \geqslant 0$.

(ii)-(iv) By calculating the partial derivatives of the function $f(x,y)$, we have that

$$f_x(x,y) = \frac{pr-(q-p)y}{(r+qx+y)^2} \quad \text{and} \quad f_y(x,y) = \frac{r-(p-q)x}{(r+qx+y)^2},$$

from which these statements easily follow.

□

Theorem 3.10. *Assume that $p > q$, $p+1 > r$, and that $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of equation (3.2.1). Then the following statements are true:*

(i) *$y_n \leqslant \frac{p}{q}$ for all $n \in \mathbb{N}$.*

(ii) *If $p \geqslant q+r$ and for some $N \geqslant 0$, $y_N > \frac{r}{p-q}$, then $y_n > 1$ for all $n > N$.*

(iii) *If $p = q+r$ and for some $N \geqslant 0$, $y_N = 1$, then $y_n = 1$ for all $n > N$.*

(iv) *If $p \leqslant q+r$ and for some $N \geqslant 0$, $y_N < \frac{r}{p-q}$, then $y_n < 1$ for all $n > N$.*

(v) If $p \leqslant q + \frac{qr}{p}$*, then equation (3.2.1) possesses an* invariant interval $[0, \frac{p}{q}]$ *and* $\bar{y} \in [0, \frac{p}{q}]$*, moreover, the interval* $[0, 1]$ *is also* an invariant interval *for equation (3.2.1) and* $\bar{y} \in (0, 1)$.

(vi) If $q + \frac{qr}{p} < p < q + r$*, then equation (3.2.1) possesses an* invariant interval $[0, \frac{r}{p-q}]$ *and* $\bar{y} \in [0, \frac{r}{p-q}]$*, moreover, the interval* $[0, 1]$ *is also an* invariant interval *for equation (3.2.1) and* $\bar{y} \in (0, 1)$.

(vii) If $p > q + r$*, then equation (3.2.1) possesses an* invariant interval $[\frac{r}{p-q}, \frac{p}{q}]$ *and* $\bar{y} \in [\frac{r}{p-q}, \frac{p}{q}]$*, moreover, the interval* $[0, \frac{p}{q}]$ *is also* an invariant interval *for equation (3.2.1) and* $\bar{y} \in (0, \frac{p}{q})$.

Proof. (i) By writing the identity (3.6.2) in the form

$$y_{n+1} - \frac{p}{q} = \frac{q-p}{q} \frac{y_{n-k} + \frac{pr}{p-q}}{r + qy_n + y_{n-k}}, \qquad n \in \mathbb{N}_0$$

and since $p > q$, implies that

$$y_{n+1} - \frac{p}{q} \leqslant 0$$

which implies $y_{n+1} \leqslant \dfrac{p}{q}$ for all $n \in \mathbb{N}$.

(ii) Since $p \geqslant q + r$, implies that $\frac{r}{p-q} \leqslant 1$. By using the identity (3.6.1)

$$y_{N+1} - 1 = (p-q) \frac{y_N - \frac{r}{p-q}}{r + qy_N + y_{N-k}},$$

and the assumption $p > q$, then the right side of this identity is greater than zero, implies $y_{N+1} > 1$.

For the next term y_{N+2}

$$y_{N+2} - 1 = (p-q) \frac{y_{N+1} - \frac{r}{p-q}}{r + qy_{N+1} + y_{N-k+1}},$$

then the right side of the identity greater than zero since $y_{N+1} > 1 \geqslant \frac{r}{p-q}$, implies $y_{N+2} > 1$. By induction $y_n > 1$ for all $n > N$.

(iii) Since $p = q + r$, this implies that $\frac{r}{p-q} = 1$. By using the identity (3.6.1)

$$y_{N+1} - 1 = (p-q)\frac{y_N - \frac{r}{p-q}}{r + qy_N + y_{N-k}},$$

and the assumption $y_N = \frac{r}{p-q}$, then the right side of the identity equal zero, implies $y_{N+1} = 1$.

For the next term y_{N+2}

$$y_{N+2} - 1 = (p-q)\frac{y_{N+1} - \frac{r}{p-q}}{r + qy_{N+1} + y_{N-k+1}},$$

then the right side of the identity equal zero since $y_{N+1} = \frac{r}{p-q} = 1$, implies $y_{N+2} = 1$. By induction $y_n = 1$ for all $n > N$.

(iv) Since $p \leqslant q + r$, implies that $\frac{r}{p-q} \geqslant 1$. By using the identity (3.6.1)

$$y_{N+1} - 1 = (p-q)\frac{y_N - \frac{r}{p-q}}{r + qy_N + y_{N-k}},$$

and the assumption $p > q$, then the right side of the identity less than zero, implies $y_{N+1} < 1$.

For the next term y_{N+2}

$$y_{N+2} - 1 = (p-q)\frac{y_{N+1} - \frac{r}{p-q}}{r + qy_{N+1} + y_{N-k+1}},$$

then the right side of the identity less than zero since $y_{N+1} < 1 \leqslant \frac{r}{p-q}$, implies $y_{N+2} < 1$. By induction $y_n < 1$ for all $n > N$.

(v) Since $q < p \leqslant q + \frac{qr}{p}$, from the left side $1 < \frac{p}{q}$, and from the right side $p - q \leqslant \frac{qr}{p}$ implies $\frac{p-q}{r} \leqslant \frac{q}{p}$. Then we have that $1 < \frac{p}{q} \leqslant \frac{r}{p-q}$.

Since $f(x,y)=\dfrac{px+y}{r+qx+y}$ is nondecreasing in x and y for each $x,y\in(0,\frac{p}{q}]$, then

$$y_1=f(y_0,y_{-k})\leqslant f\left(\frac{r}{p-q},\frac{r}{p-q}\right)=1,$$

which implies that $y_1\in[0,1]\subset[0,\frac{p}{q}]$, and

$$y_2=f(y_1,y_{-k+1})\leqslant f\left(1,\frac{r}{p-q}\right)\leqslant f\left(\frac{r}{p-q},\frac{r}{p-q}\right)=1,$$

By the induction we have that $y_n\in[0,1]\subset[0,\frac{p}{q}]$ for every $n\in\mathbb{N}$.

On the other hand, the condition $1<\frac{r}{p-q}$ is equivalent to $\frac{p+1-r}{q+1}<1$, that is $\bar{y}<1$ from which it follows that $\bar{y}\in(0,1)$.

(vi) Similar to the above, since $q+\frac{qr}{p}<p<q+r$, implies $\frac{qr}{p}<p-q<r$, then $\frac{q}{p}<\frac{p-q}{r}<1$, then we have $1<\frac{r}{p-q}<\frac{p}{q}$.

And the function $f(x,y)$ in nondecreasing in x and y for each $x,y\in(0,\frac{r}{p-q}]$, which implies that

$$y_1=f(y_0,y_{-k})\leqslant f\left(\frac{r}{p-q},\frac{r}{p-q}\right)=1,$$

which implies that $y_1\in[0,1]\subset[0,\frac{r}{p-q}]$, and

$$y_2=f(y_1,y_{-k+1})\leqslant f\left(1,\frac{r}{p-q}\right)\leqslant f\left(\frac{r}{p-q},\frac{r}{p-q}\right)=1,$$

which implies that $y_2\in[0,1]\subset[0,\frac{r}{p-q}]$. By the induction we have that $y_n\in[0,1]\subset[0,\frac{r}{p-q}]$ for every $n\in\mathbb{N}$.

Now note that the condition $p<q+r$ implies that

$$\frac{p+1-r}{q+1}<\frac{q+r+1-r}{q+1}=1,$$

which means that $\bar{y}\in(0,1)$, as desired.

(vii) It is easy to see that the function $f(x, y)$ is strictly increasing in x for each fixed $y \in (0, \infty)$, and nonincreasing in y for each fixed $x \geqslant \frac{r}{p-q}$. From this and (i), we have

$$1 = f\left(\frac{r}{p-q}, \frac{p}{q}\right) \leqslant y_1 = f(y_0, y_{-k}) \leqslant \frac{p}{q}.$$

Since $p > q + r$, implies $\frac{r}{p-q} < 1$ and since $p > q \Rightarrow 1 < \frac{p}{q}$. So we have that $\frac{r}{p-q} < 1 < \frac{p}{q}$. From this, we have that $y_1 \in [1, \frac{p}{q}] \subset [\frac{r}{p-q}, \frac{p}{q}]$. By the induction it follows that $y_n \in [1, \frac{p}{q}] \subset [\frac{r}{p-q}, \frac{p}{q}]$, for every $n \in \mathbb{N}$.

Further, we have that $p > q + r$ implies that

$$\bar{y} = \frac{p + 1 - r}{q + 1} > \frac{q + r + 1 - r}{q + 1} = 1,$$

as well as $\dfrac{p + 1 - r}{q + 1} < \dfrac{p}{q}$, which means that $\bar{y} \in (1, \frac{p}{q})$, as desired.

$\square$

Theorem 3.11. *Assume that $p = q + r$, $p + 1 > r$, and that $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of equation (3.2.1). Then the following statements are true:*

(i) $y_n \leqslant \frac{p}{q}$ for all $n \in \mathbb{N}$.

(ii) If for some $N \geqslant 0$, $y_N > 1$, then $y_n > 1$ for all $n > N$.

(iii) If for some $N \geqslant 0$, $y_N < 1$, then $y_n < 1$ for all $n > N$.

Proof. By using theorem (3.10), the proof is direct consequence of the assumptions and identity (3.6.5). $\square$

Theorem 3.12. *Assume that $q > p$, $p+1 > r$, and that $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of equation (3.2.1). Then the following statements are true:*

(i) $y_n \leqslant 1$ *for all* $n \in \mathbb{N}$.

(ii) *If for some* $N \geqslant 0$, $y_N > \frac{pr}{q-p}$, *then* $y_{N+k+1} > \frac{p}{q}$.

(iii) *If for some* $N \geqslant 0$, $y_N = \frac{pr}{q-p}$, *then* $y_{N+k+1} = \frac{p}{q}$.

(iv) *If for some* $N \geqslant 0$, $y_N < \frac{pr}{q-p}$, *then* $y_{N+k+1} < \frac{p}{q}$.

(v) *If* $q \leqslant p+pr$, *then equation (3.2.1) possesses an invariant interval* $[0,1]$ *and* $\bar{y} \in [0,1]$, *moreover, the interval* $[0, \frac{p}{q}]$ *is also an invariant interval for equation (3.2.1) and* $\bar{y} \in (0, \frac{p}{q})$.

(vi) *If* $p + pr < q < p + qr$, *then equation (3.2.1) possesses an invariant interval* $[0, \frac{pr}{q-p}]$ *and* $\bar{y} \in [0, \frac{pr}{q-p}]$, *moreover, the interval* $[0, \frac{p}{q}]$ *is also an invariant interval for equation (3.2.1) and* $\bar{y} \in (0, \frac{p}{q})$.

(vii) *If* $q > p + qr$, *then equation (3.2.1) possesses an invariant interval* $[\frac{pr}{q-p}, 1]$ *and* $\bar{y} \in [\frac{pr}{q-p}, 1]$, *moreover, the interval* $[\frac{p}{q}, 1]$ *is also an invariant interval for equation (3.2.1) and* $\bar{y} \in (\frac{p}{q}, 1)$.

Proof. (i) If we write identity (3.6.1) in the form

$$y_{n+1} - 1 = (p-q)\frac{y_n + \frac{r}{q-p}}{r + qy_n + y_{n-k}}, \qquad n \in \mathbb{N}_0,$$

and use the assumption $q > p$, then the right side of the above identity is less than zero, which means $y_{n+1} - 1 \leqslant 0 \Rightarrow y_{n+1} \leqslant 1$.

(ii) By using the identity (3.6.2)

$$y_{n+k+1} - \frac{p}{q} = \frac{q-p}{q} \frac{y_n - \frac{pr}{q-p}}{r + qy_{n+k} + y_n}, \qquad n \in \mathbb{N}_0$$

and the assumptions $y_N > \frac{pr}{q-p}$ and $q > p$, implies that the right side of the identity is greater than zero, which implies that $y_{N+k+1} > \frac{p}{q}$.

(iii) As in (ii), by using the identity (3.6.2) and the assumptions $y_N = \frac{pr}{q-p}$ and $q > p$, implies that the right side of the identity is zero, which implies that $y_{N+k+1} = \frac{p}{q}$.

(iv) Similarly, by identity (3.6.2) and the assumptions $y_N < \frac{pr}{q-p}$ and $q > p$, implies that the right side of the identity is less that zero, which implies that $y_{N+k+1} < \frac{p}{q}$.

(v) Since $p < q \leqslant p + pr$, from left side $\frac{p}{q} < 1$ and from right side $q - p \leqslant pr \Rightarrow 1 \leqslant \frac{pr}{q-p}$, so we have that $\frac{p}{q} < 1 \leqslant \frac{pr}{q-p}$. By theorem (3.9) (iv) the function $f(x, y)$ is strictly increasing in y for each fixed $x \in (0, \infty)$, and nondecreasing in x for each fixed $y \in (0, 1]$, we have that

$$y_1 = f(y_0, y_{-k}) \leqslant f\left(\frac{pr}{q-p}, \frac{pr}{q-p}\right) = \frac{p}{q} < 1,$$

which implies that $y_1 \in [0, \frac{p}{q}] \subset [0, 1]$.

By the induction $y_n \in [0, \frac{p}{q}] \subset [0, 1]$ for every $n \in \mathbb{N}$.

On the other hand, we have that

$$\bar{y} = \frac{p+1-r}{q+1} < \frac{p}{q} < 1,$$

as desired.

(vi) Since $p + pr < q < p + qr$, implies $pr < q - p < qr$ then $1 < \frac{q-p}{pr} < \frac{qr}{pr}$, so we have that $\frac{p}{q} < \frac{pr}{q-p} < 1$. By theorem (3.9) (iv) as above, we have that

$$y_1 = f(y_0, y_{-k}) \leqslant f\left(\frac{pr}{q-p}, \frac{pr}{q-p}\right) = \frac{p}{q} < 1,$$

which implies that $y_1 \in [0, \frac{p}{q}] \subset [0, \frac{pr}{q-p}]$.

By the induction $y_n \in [0, \frac{p}{q}] \subset [0, \frac{pr}{q-p}]$ for every $n \in \mathbb{N}$.

On the other hand, we have that

$$\bar{y} = \frac{p+1-r}{q+1} < \frac{p}{q} < \frac{pr}{q-p},$$

as desired.

(vii) Since $q > p + qr > p$, we have that $\frac{pr}{q-p} < \frac{p}{q} < 1$. On the other hand, by theorem (3.9) (iv) we have that for this case, the function $f(x, y)$ is strictly increasing in y for each fixed $x \in (0, \infty)$, and nonincreasing in x for each fixed $y > \frac{pr}{q-p}$. From this and (i) it follows that

$$\frac{p}{q} = f\left(1, \frac{pr}{q-p}\right) \leqslant y_1 = f(y_0, y_{-k}) \leqslant 1,$$

which implies that $y_1 \in [\frac{p}{q}, 1] \subset [\frac{pr}{q-p}, 1]$.

By the induction $y_n \in [\frac{p}{q}, 1] \subset [\frac{pr}{q-p}, 1]$ for every $n \in \mathbb{N}$.

On the other hand, $q > p + qr$ implies that

$$\frac{p}{q} < \bar{y} = \frac{p+1-r}{q+1} < \frac{p+1-r}{p+1} < 1,$$

as claimed.

The proof is complete. $\square$

3.7 Existence of Two Cycles

In this section we study the necessary and sufficient conditions of equation (3.7.1) to have a prime period two solution and we exhibit all prime period-two solutions of the equation.

First we present the definition of the prime period two solution.

Definition 3.5. We say that a solution $\{x_n\}_{n=-k}^{\infty}$ of the difference equation (3.5.1) is *periodic* if there exists a positive integer p such that $x_{n+p} = x_n$. The smallest such positive integer p is called the *prime period* of the solution of the difference equation.

We study here the periodic solution of our equation,

$$x_{n+1} = \frac{px_n + x_{n-k}}{r + qx_n + x_{n-k}}, \qquad n = 0, 1, 2, \dots \tag{3.7.1}$$

Lets assume that the two periodic nonnegative solutions of our equation will be in the form

$$\dots, \phi, \psi, \phi, \psi, \dots$$

- If k is odd then

  $$x_{n+1} = x_{n-k}$$

  So we get

  $$\psi = \frac{p\,\phi + \psi}{r + q\,\phi + \psi}$$
  $$\phi = \frac{p\,\psi + \phi}{r + q\,\psi + \phi}$$

  This yields

  $$\psi(r + q\,\phi + \psi) = p\,\phi + \psi \tag{3.7.2}$$

$$\phi(r + q\,\psi + \phi) = p\,\psi + \phi \tag{3.7.3}$$

By subtracting equation (3.7.3) from equation (3.7.2) we get the following

$$r(\psi - \phi) + (\psi^2 - \phi^2) = p(\phi - \psi) + (\psi - \phi)$$

$$\Rightarrow\ (\psi - \phi)(r + p - 1) + (\psi^2 - \phi^2) = 0$$

implies

$$(\psi - \phi)(r + p - 1 + (\psi + \phi)) = 0$$

Then either $\psi = \phi$ or $\psi + \phi = 1 - (r + p)$. Then in this case there is no two periodic nonnegative solution for equation (3.7.1) unless $\psi + \phi = 1 - (r + p)$.

Lets now take k to be even and see what we will get.

- If k is even then

$$x_n = x_{n-k}$$

So we get

$$\psi = \frac{p\,\phi + \phi}{r + q\,\phi + \phi}$$
$$\phi = \frac{p\,\psi + \psi}{r + q\,\psi + \psi}$$

This yields

$$\psi(r + q\,\phi + \phi) = \phi(p + 1) \tag{3.7.4}$$

$$\phi(r + q\,\psi + \psi) = \psi(p + 1) \tag{3.7.5}$$

By subtracting equation (3.7.5) from the equation (3.7.4) we get the following

$$\begin{aligned} r(\psi - \phi) + (\psi - \phi)(p + 1) &= 0 \\ (\psi - \phi)(r + p + 1) &= 0 \end{aligned}$$

Then either $\psi = \phi$ or $r+p+1=0$ which is impossible since r and p are nonnegative variables. Then in this case, there exists no two periodic nonnegative solution for our equation (3.7.1).

From the above discussion, we have the following theorem.

Theorem 3.13. *There exists no two periodic nonnegative solution for the difference equation*

$$x_{n+1} = \frac{px_n + x_{n-k}}{r + qx_n + x_{n-k}}, \qquad n = 0, 1, 2, \ldots$$

unless if k is odd and $r + p < 1$.

3.8 Semi-cycle Analysis

In this section we will study the semi-cycles behavior of solutions of equation (3.2.1) relative to the equilibrium point $\bar{y}$.

Here we give the definitions for the positive and negative semi-cycle of the solution of a difference equation, relative to an equilibrium point $\bar{y}$.

Definition 3.6. [15] Let $\{y_n\}_{n=-k}^{\infty}$ be a nonnegative solution of equation (3.5.1). A *positive semi-cycle* of a solution $\{y_n\}_{n=-k}^{\infty}$ of equation (3.5.1) consists of a "string" of terms $\{y_l, y_{l+1}, \ldots, y_m\}$, all greater than or equal to the equilibrium $\bar{y}$, with $l \geqslant -k$ and $m \leqslant \infty$ and such that

$$\text{either } l = -k, \text{ or } l > -k \text{ and } y_{l-1} < \bar{y}$$

and

$$\text{either } m = \infty, \text{ or } m < \infty \text{ and } y_{m+1} < \bar{y}.$$

Definition 3.7. [15] Let $\{y_n\}_{n=-k}^{\infty}$ be a nonnegative solution of equation (3.5.1). A *negative semi-cycle* of a solution $\{y_n\}_{n=-k}^{\infty}$ of equation (3.5.1) consists of a "string" of terms $\{y_l, y_{l+1}, \ldots, y_m\}$, all less than the equilibrium $\bar{y}$, with $l \geqslant -k$ and $m \leqslant \infty$ and such that

$$\text{either } l = -k, \text{ or } l > -k \text{ and } y_{l-1} \geqslant \bar{y}$$

and

$$\text{either } m = \infty, \text{ or } m < \infty \text{ and } y_{m+1} \geqslant \bar{y}.$$

Definition 3.8. A solution $\{y_n\}$ of equation (3.5.1) is called *non-oscillatory* if there exists $N \geqslant -k$ such that $y_n > \bar{y}$ for all $n \geqslant N$ or $y_n < \bar{y}$ for all $n \geqslant N$.

And a solution $\{y_n\}$ is called *oscillatory* if it is *not non-oscillatory.*

Theorem 3.14. *[16] Assume that $f \in C[(0,\infty) \times (0,\infty), (0,\infty)]$ is such that $f(x,y)$ is* increasing *in both arguments. Let $\bar{x}$ be a positive equilibrium of equation (3.5.1). Then, every oscillatory solution of equation (3.5.1) has semicycles of length k.*

Proof. When $k = 1$, the proof is presented as theorem (1.7.3) in [16]. We just give the proof of the theorem for $k = 2$, the other cases for $k \geqslant 3$ are similar and we omit them.

Assume that $\{x_n\}$ is an oscillatory solution with three consecutive terms

$$x_{N-1} \geqslant \bar{x} \ , x_N \geqslant \bar{x} \ \text{ and } \ x_{N+1} \geqslant \bar{x}$$

with at least one of the inequalities being strict. The proof in the case of negative semicycle is similar and is omitted. Then by using the increasing character of $f(x, y)$ we obtain:

$$x_{N+2} = f(x_{N+1}, x_{N-1}) > f(\bar{x}, \bar{x}) = \bar{x}$$

which shows that the next term x_{N+2} also belongs to the positive semicycle. It follows by induction that all future terms of this solution belong to this positive semicycle, which is a contradiction. The proof is complete. $\square$

Theorem 3.15. *[16] Assume that $f \in C[(0, \infty) \times (0, \infty), (0, \infty)]$ is such that $f(x, y)$ is* increasing *in x for each fixed y, and is* decreasing *in y for each fixed x. Let $\bar{x}$ be a positive equilibrium of equation (3.5.1). Then, except possibly for the first semicycle, every solution of equation (3.5.1) has semicycles of length at least $k + 1$.*

Proof. When $k = 1$, the proof is presented as theorem (1.7.4) in Ref.[16]. We just give the proof of the theorem for $k = 2$, the other cases for $k \geqslant 3$ are similar and can be omitted.

Assume that $\{x_n\}$ is an oscillatory solution with three consecutive terms

$$x_{N-1}, x_N, x_{N+1}$$

such that

$$x_{N-1} < \bar{x} < x_{N+1}$$

or

$$x_{N-1} > \bar{x} > x_{N+1}$$

We will assume that

$$x_{N-1} < \bar{x} < x_{N+1}$$

the other case is similar and will be omitted.

Then by using decreasing character of f we obtain

$$x_{N+2} = f(x_{N+1}, x_{N-1}) > f(\bar{x}, \bar{x}) = \bar{x}$$

Now, if $x_N \geqslant \bar{x}$ then the result follows. Otherwise $x_N < \bar{x}$. Hence

$$x_{N+3} = f(x_{N+2}, x_N) > f(\bar{x}, \bar{x}) = \bar{x}$$

which shows that it has at least *three* terms in the positive semicycle. □

Theorem 3.16. *[16] Assume that $f \in C[(0,\infty)\times(0,\infty),(0,\infty)]$ is such that $f(x,y)$ is* decreasing *in x for each fixed y, and is* increasing *in y for each fixed x. Let $\bar{x}$ be a positive equilibrium of equation (3.5.1). Then, except possibly for the first semicycle, every solution of equation (3.5.1) has semicycles of length k.*

Proof. When $k = 1$, the proof is presented as theorem (1.7.1) in Ref.[16]. We just give the proof of the theorem for $k = 2$, the other cases for $k \geqslant 3$ are similar and we omitted them. Let $\{x_n\}$ be a solution of equation (3.5.1) with at least three semicycles, then there exists $N \geqslant 0$ such that either

$$x_{N-1} < \bar{x} \leqslant x_{N+1}$$

or

$$x_{N-1} \geqslant \bar{x} > x_{N+1}$$

We will assume that

$$x_{N-1} < \bar{x} \leqslant x_{N+1}$$

the other case is similar and will be omitted.

Then by using the monotonic character of $f(x,y)$ we have

$$x_{N+2} = f(x_{N+1}, x_{N-1}) < f(\bar{x}, \bar{x}) = \bar{x}$$

and

$$x_{N+3} = f(x_{N+2}, x_N) > f(\bar{x}, \bar{x}) = \bar{x}$$

thus

$$x_{N+2} < \bar{x} < x_{N+3}$$

The proof is complete. $\square$

By using the theorems (3.9)-(3.12) together with theorems (3.14)-(3.16), it is easy to obtain the following results concerning semicycle analysis.

Theorem 3.17. *Assume that $p = q$, $p+1 > r$, and $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of equation (3.2.1). Then the following statements are true:*

(i) Except possibly for the first semicycle, every oscillatory solution of equation (3.2.1) has semicycles of length k.

(ii) If $(y_{-k} - \bar{y})(y_{-k+1} - \bar{y}) \cdots (y_0 - \bar{y}) > 0$, then $\{y_n\}_{n=-k}^{\infty}$ is not an oscillatory solution.

Theorem 3.18. *Assume that $p > q$, $p+1 > r$, and $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of equation (3.2.1). Then the following statements are true:*

(i) If $p \leqslant q + \frac{qr}{p}$, then except possibly for the first semicycle, every oscillatory solution of equation (3.2.1) has semicycles of length k.

(ii) *If* $q + \frac{qr}{p} < p < q + r$, *then except possibly for the first semicycle, every oscillatory solution of equation (3.2.1) which lies in the invariant interval* $[0, \frac{r}{p-q}]$ *has semicycles of length* k.

(iii) *If* $p > q + r$, *then every oscillatory solution of equation (3.2.1) which lies in the invariant interval* $[\frac{r}{p-q}, \frac{p}{q}]$ *has semicycles of length at least* k.

(iv) *If* $p = q + r$, *then equation (3.2.1) does not have oscillatory solutions.*

Theorem 3.19. *Assume that* $q > p$, $p+1 > r$, *and* $\{y_n\}_{n=-k}^{\infty}$ *is a nonnegative solution of equation (3.2.1). Then the following statements are true:*

(i) *If* $q \leqslant p + pr$, *then except possibly for the first semicycle, every oscillatory solution of equation (3.2.1) has semicycles of length* k.

(ii) *If* $p+pr < q < p+qr$, *then every oscillatory solution of equation (3.2.1) which lies in the invariant interval* $[0, \frac{pr}{q-p}]$ *has semicycles of length* k.

(iii) *If* $q > p + qr$, *then every oscillatory solution of equation (3.2.1) which lies in the invariant interval* $[\frac{pr}{q-p}, 1]$ *has semicycles of length* k.

(iv) *If* $q = p + qr$, *then every oscillatory solution of equation (3.2.1) has semicycles of length* k. *In particular, equation (3.2.1) does not have oscillatory solutions with* $(y_{-k} - \frac{p}{q})(y_{-k+1} - \frac{p}{q}) \cdots (y_0 - \frac{p}{q}) > 0$.

3.9 Analysis of Global Stability

In this section we will consider the global asymptotic stability for the zero and positive equilibrium points of equation (3.2.1). In section (3.4), we investigated local stability of the zero and positive equilibrium points, so it is sufficient to investigate the globally attractivity of the zero and positive equilibrium points.

Now, we present some theorems which will be used in this section.

Theorem 3.20. *[6, 16] Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \qquad n = 0, 1, \ldots \tag{3.9.1}$$

where $k \in \{1, 2, \ldots\}$. Let $I = [a, b]$ be some interval of real numbers and assume that

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(u, v)$ is nondecreasing in each of its arguments u and v.

(b) The equation

$$f(y, y) = y$$

has a unique positive solution.

Then equation (3.9.1) has a unique equilibrium point $\bar{y} \in [a, b]$ and every solution of equation (3.9.1) converges to $\bar{y}$.

Proof. Set

$$m_0 = a \quad \text{and} \quad M_0 = b$$

and for $i = 1, 2, \ldots$ set

$$M_i = f(M_{i-1}, M_{i-1}) \quad \text{and} \quad m_i = f(m_{i-1}, m_{i-1})$$

Now observe that for each $i \geqslant 0$,

$$m_0 \leqslant m_1 \leqslant \cdots \leqslant m_i \leqslant \cdots \leqslant M_i \leqslant \cdots \leqslant M_1 \leqslant M_0,$$

and

$$m_i \leqslant y_l \leqslant M_i \quad \text{for} \quad l \geqslant 2i+1.$$

Set

$$m = \lim_{i\to\infty} m_i \quad \text{and} \quad M = \lim_{i\to\infty} M_i.$$

Then clearly

$$m \leqslant \liminf_{i\to\infty} y_i \leqslant \limsup_{i\to\infty} y_i \leqslant M$$

and by the continuity of f,

$$m = f(m,m) \quad \text{and} \quad M = f(M,M).$$

In view of (b),

$$m = M = \bar{y},$$

from which the result follows. □

Theorem 3.21. *[6, 16] Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \qquad n = 0, 1, \ldots \tag{3.9.2}$$

where $k \in \{1, 2, \ldots\}$. Let $I = [a,b]$ be some interval of real numbers and assume that

$$f : [a,b] \times [a,b] \to [a,b]$$

is a continuous function satisfying the following properties:

(a) $f(u,v)$ is nondecreasing *in $u \in [a,b]$ for each $v \in [a,b]$ and* nonincreasing *in $v \in [a,b]$ for each $u \in [a,b]$.*

(b) If $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$m = f(m, M) \quad \text{and} \quad M = f(M, m),$$

then $m = M$.

Then equation (3.9.2) has a unique equilibrium $\bar{y}$ *and every solution of equation (3.9.2) converges to* $\bar{y}$.

Theorem 3.22. *[6, 16] Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \qquad n = 0, 1, \ldots \tag{3.9.3}$$

where $k \in \{1, 2, \ldots\}$. *Let* $I = [a, b]$ *be some interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(u, v)$ *is* nonincreasing *in* $u \in [a, b]$ *for each* $v \in [a, b]$ *and* nondecreasing *in* $v \in [a, b]$ *for each* $u \in [a, b]$.

(b) If $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$m = f(M, m) \quad \text{and} \quad M = f(m, M),$$

then $m = M$.

Then equation (3.9.3) has a unique equilibrium $\bar{y} \in [a, b]$ *and every solution of equation (3.9.3) converges to* $\bar{y}$.

Proof. Set

$$m_0 = a \quad \text{and} \quad M_0 = b$$

and for $i = 1, 2, \ldots$ set

$$m_i = f(M_{i-1}, m_{i-1}) \quad \text{and} \quad M_i = f(m_{i-1}, M_{i-1})$$

Now observe that

$$m_0 \leqslant m_1 \leqslant \cdots \leqslant m_i \leqslant \cdots \leqslant M_i \leqslant \cdots \leqslant M_1 \leqslant M_0,$$

and

$$m_i \leqslant y_l \leqslant M_i \quad \text{for } l \geqslant 2i+1.$$

Set

$$m = \lim_{i\to\infty} m_i \quad \text{and} \quad M = \lim_{i\to\infty} M_i.$$

Then clearly

$$m \leqslant \liminf_{i\to\infty} y_i \leqslant \limsup_{i\to\infty} y_i \leqslant M$$

and by the continuity of f,

$$m = f(M, m) \quad \text{and} \quad M = f(m, M).$$

In view of (b) $m = M = \bar{y}$. The proof is complete. □

Theorem 3.23. *[6, 16] Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \qquad n = 0, 1, \ldots \tag{3.9.4}$$

where $k \in \{1, 2, \ldots\}$. Let $I = [a, b]$ be some interval of real numbers and assume that

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(u,v)$ *is nonincreasing in each of its arguments* u *and* v.

(b) If $(m,M) \in [a,b] \times [a,b]$ *is a solution of the system*

$$m = f(M,M) \quad \textit{and} \quad M = f(m,m),$$

then $m = M$.

Then equation (3.9.4) has a unique equilibrium $\bar{y} \in [a,b]$ *and every solution of equation (3.9.4) converges to* $\bar{y}$.

Proof. Set

$$m_0 = a \quad \text{and} \quad M_0 = b$$

and for $i = 1, 2, \ldots$ set

$$m_i = f(M_{i-1}, M_{i-1}) \quad \text{and} \quad M_i = f(m_{i-1}, m_{i-1})$$

Now observe that

$$m_0 \leqslant m_1 \leqslant \cdots \leqslant m_i \leqslant \cdots \leqslant M_i \leqslant \cdots \leqslant M_1 \leqslant M_0,$$

and

$$m_i \leqslant y_l \leqslant M_i \quad \text{for } l \geqslant 2i+1.$$

Set

$$m = \lim_{i\to\infty} m_i \quad \text{and} \quad M = \lim_{i\to\infty} M_i.$$

Then clearly

$$m \leqslant \liminf_{i\to\infty} y_i \leqslant \limsup_{i\to\infty} y_i \leqslant M$$

and by the continuity of f,

$$m = f(M,M) \quad \text{and} \quad M = f(m,m),$$

and so $m = M = \bar{y}$.

The proof is complete. $\square$

3.9.1 The Global Stability of the Zero Equilibrium

Here we study the global stability of the *zero equilibrium point* under the condition $p+1<r$.

Theorem 3.24. *Assume that $p+1<r$, then the zero equilibrium of equation (3.2.1) is* globally asymptotically stable*.*

Proof. Consider the function

$$f(x,y) = \frac{px+y}{r+qx+y}$$

By using theorem (3.9) (ii,iii,iv), note that $f(x,y)$ is nondecreasing in each of its arguments in the intervals

$$I = \begin{cases} [0, \frac{r}{p-q}] & \text{where } p>q \\ [0, \frac{pr}{q-p}] & \text{where } p<q \\ [0, \infty] & \text{where } p=q \end{cases}$$

Now, let $(m,M) \in I \times I$ is a solution of the system

$$m = f(M,M) \quad \text{and} \quad M = f(m,m),$$

then

$$m = \frac{pM+M}{r+qM+M} \quad \text{and} \quad M = \frac{pm+m}{r+qm+m},$$

$$\begin{aligned} rm + qmM + mM &= pM + M \\ rM + qmM + mM &= pm + m \end{aligned}$$

this yields $(M - m)(r + p + 1) = 0$, then the only solution is $m = M$.

Then both conditions of theorem (3.20) hold, therefore, every solution of equation (3.2.1) converges to $\bar{y}$ in the interval I.

As $\bar{y}$ is locally asymptotically stable under the condition $p + 1 < r$ as shown in subsection (3.4.1), then by definition (3.2), it is *globally asymptotically stable* on the interval I. □

3.9.2 The Global Stability of the Positive Equilibrium

Here we study the global stability of the *positive equilibrium point* remembering that we studied the local stability of the *positive equilibrium point* under the following conditions:

$$p + 1 > r, \tag{3.9.5}$$

and

$$q + r < 3p + 1 + qr + pq. \tag{3.9.6}$$

Theorem 3.25. *Assume that $p > q$, (3.9.5) and (3.9.6) hold, then:*

(i) If $p \leqslant q + \frac{qr}{p}$, then the unique positive equilibrium of equation (3.2.1) on the interval $[0, \frac{p}{q}]$ is globally asymptotically stable.

(ii) If $q + \frac{qr}{p} < p < q + r$, then the unique positive equilibrium of equation (3.2.1) on the interval $[0, \frac{r}{p-q}]$ is globally asymptotically stable.

(iii) If $q + r < p \leqslant 1 + r$, then the unique positive equilibrium of equation (3.2.1) on the interval $[\frac{r}{p-q}, \frac{p}{q}]$ is globally asymptotically stable.

Proof. (i) Consider the function

$$f(x, y) = \frac{px + y}{r + qx + y}$$

By using theorem (3.9) (iii), note that $f(x, y)$ is nondecreasing in each of its arguments in the interval $[0, \frac{r}{p-q}]$.
By the assumption $q < p \leqslant q + \frac{qr}{p}$, implies that $\frac{p}{q} \leqslant \frac{r}{p-q}$, so $f(x, y)$ is nondecreasing in each of its arguments in the interval $[0, \frac{p}{q}] \subset [0, \frac{r}{p-q}]$.

Now, let $(m, M) \in [0, \frac{p}{q}] \times [0, \frac{p}{q}]$ is a solution of the system

$$m = f(M, M) \quad \text{and} \quad M = f(m, m),$$

then

$$m = \frac{pM + M}{r + qM + M} \quad \text{and} \quad M = \frac{pm + m}{r + qm + m},$$

$$\begin{aligned} rm + qmM + mM &= pM + M \\ rM + qmM + mM &= pm + m \end{aligned}$$

this yields $(M - m)(r + p + 1) = 0$, then the only solution is $m = M$.

Therefore, both conditions of theorem (3.20) hold, then every solution of equation (3.2.1) converges to $\bar{y}$ in the interval $[0, \frac{p}{q}]$.

As $\bar{y}$ is locally asymptotically stable under the conditions (3.9.5) and (3.9.6), then it is *globally asymptotically stable* on the interval $[0, \frac{p}{q}]$ by the definition (3.2).

(ii) As we have seen above, by theorem (3.9) (iii) the function $f(x, y)$ is nondecreasing in each of its arguments in the interval $[0, \frac{r}{p-q}]$.

Let $(m, M) \in [0, \frac{r}{p-q}] \times [0, \frac{r}{p-q}]$ is a solution of the system

$$m = f(M, M) \quad \text{and} \quad M = f(m, m),$$

then

$$m = \frac{pM + M}{r + qM + M} \quad \text{and} \quad M = \frac{pm + m}{r + qm + m},$$

$$\begin{aligned} rm + qmM + mM &= pM + M \\ rM + qmM + mM &= pm + m \end{aligned}$$

this yields $(M - m)(r + p + 1) = 0$, then the only solution is $m = M$.

Therefore, both conditions of theorem (3.20) hold, then every solution of equation (3.2.1) converges to $\bar{y}$ in the interval $[0, \frac{r}{p-q}]$.

Since $\bar{y}$ is locally asymptotically stable under the conditions (3.9.5) and (3.9.6), then it is *globally asymptotically stable* on the interval $[0, \frac{r}{p-q}]$ by the definition (3.2).

(iii) By theorem (3.9) (iii) the function $f(x, y)$ is nondecreasing in x and nonincreasing in y in the interval $[\frac{r}{p-q}, \infty)$. By the assumption $q + \frac{qr}{p} < p < q + r$, implies $\frac{qr}{p} < p - q < r$ then $1 < \frac{r}{p-q} < \frac{p}{q}$, i.e. the interval $[\frac{r}{p-q}, \frac{p}{q}] \subset [\frac{r}{p-q}, \infty)$.

Let $(m, M) \in [\frac{r}{p-q}, \frac{p}{q}] \times [\frac{r}{p-q}, \frac{p}{q}]$ is a solution of the system

$$m = f(m, M) \quad \text{and} \quad M = f(M, m),$$

then

$$m = \frac{pm + M}{r + qm + M} \quad \text{and} \quad M = \frac{pM + m}{r + qM + m},$$

$$\begin{aligned} rm + qm^2 + mM &= pm + M \\ rM + qM^2 + mM &= pM + m \end{aligned}$$

this yields

$$(M - m)\left[r + q(M + m) - p + 1\right] = 0$$

Then either $m = M$ or $M + m = \dfrac{p - (1 + r)}{q}$ which contradicts the assumption $p < 1 + r$. So the only solution is $m = M$.

Then both conditions of theorem (3.21) hold, therefore every solution of equation (3.2.1) converges to $\bar{y}$ in the interval $[\frac{r}{p-q}, \frac{p}{q}]$.

By definition (3.2), $\bar{y}$ is locally asymptotically stable under the conditions (3.9.5) and (3.9.6), then it is *globally asymptotically stable* on the interval $[\frac{r}{p-q}, \frac{p}{q}]$.

$\square$

Theorem 3.26. *Assume that $p < q$, (3.9.5) and (3.9.6) hold, then:*

(i) If $q \leqslant p + pr$, then the unique positive equilibrium of equation (3.2.1) on the interval $[0, \frac{p}{q}]$ *is* globally asymptotically stable*.*

(ii) If $p + pr < q < p + qr$, then the unique positive equilibrium of equation (3.2.1) on the interval $[0, \frac{pr}{q-p}]$ *is* globally asymptotically stable*.*

(iii) If $q > p + qr$, then the unique positive equilibrium of equation (3.2.1) on the interval $[\frac{pr}{q-p}, 1]$ *is* globally asymptotically stable*.*

Proof. (i) Consider the function

$$f(x, y) = \frac{px + y}{r + qx + y}$$

By using theorem (3.9) (iv), note that $f(x,y)$ is nondecreasing in each of its arguments in the interval $[0, \frac{pr}{q-p}]$.

By the assumption $p < q \leqslant p + pr$, implies that $\frac{pr}{q-p} \geqslant 1 > \frac{p}{q}$, so $f(x,y)$ is nondecreasing in each of its arguments in the interval $[0, \frac{p}{q}] \subset [0, \frac{pr}{q-p}]$.

Now, let $(m, M) \in [0, \frac{p}{q}] \times [0, \frac{p}{q}]$ is a solution of the system

$$m = f(M, M) \quad \text{and} \quad M = f(m, m),$$

then

$$m = \frac{pM + M}{r + qM + M} \quad \text{and} \quad M = \frac{pm + m}{r + qm + m},$$

$$\begin{aligned} rm + qmM + mM &= pM + M \\ rM + qmM + mM &= pm + m \end{aligned}$$

this yields $(M - m)(r + p + 1) = 0$, then the only solution is $m = M$.

Then both conditions of theorem (3.20) hold, therefore, every solution of equation (3.2.1) converges to $\bar{y}$ in the interval $[0, \frac{p}{q}]$.

As $\bar{y}$ is locally asymptotically stable under the conditions (3.9.5) and (3.9.6), then it is *globally asymptotically stable* on the interval $[0, \frac{p}{q}]$ by the definition (3.2).

(ii) As we see above, by theorem (3.9) (iv) the function $f(x,y)$ is nondecreasing in each of its arguments in the interval $[0, \frac{pr}{q-p}]$.

Let $(m, M) \in [0, \frac{pr}{q-p}] \times [0, \frac{pr}{q-p}]$ is a solution of the system

$$m = f(M, M) \quad \text{and} \quad M = f(m, m),$$

then

$$m = \frac{pM + M}{r + qM + M} \quad \text{and} \quad M = \frac{pm + m}{r + qm + m},$$

$$\begin{aligned} rm + qmM + mM &= pM + M \\ rM + qmM + mM &= pm + m \end{aligned}$$

this yields $(M - m)(r + p + 1) = 0$, then the only solution is $m = M$.

Therefore, both conditions of theorem (3.20) hold, then every solution of equation (3.2.1) converges to $\bar{y}$ in the interval $[0, \frac{pr}{q-p}]$.

Since $\bar{y}$ is locally asymptotically stable under the conditions (3.9.5) and (3.9.6), then it is *globally asymptotically stable* on the interval $[0, \frac{pr}{q-p}]$ by the definition (3.2).

(iii) By theorem (3.9) (iv) the function $f(x, y)$ is nonincreasing in x and nondecreasing in y in the interval $[\frac{pr}{q-p}, 1] \subset [\frac{pr}{q-p}, \infty)$.

Let $(m, M) \in [\frac{pr}{q-p}, 1] \times [\frac{pr}{q-p}, 1]$ is a solution of the system

$$m = f(M, M) \quad \text{and} \quad M = f(m, m),$$

then

$$m = \frac{pM + M}{r + qM + M} \quad \text{and} \quad M = \frac{pm + m}{r + qm + m},$$

$$\begin{aligned} rm + qmM + mM &= pM + M \\ rM + qmM + mM &= pm + m \end{aligned}$$

this yields $(M - m)(r + p + 1) = 0$, then the only solution is $m = M$.

Therefore, both conditions of theorem (3.22) hold, then every solution of equation (3.2.1) converges to $\bar{y}$ in the interval $[\frac{pr}{q-p}, 1]$.

By definition (3.2), $\bar{y}$ is locally asymptotically stable under the conditions (3.9.5) and (3.9.6), then it is *globally asymptotically stable* on the interval $[\frac{pr}{q-p}, 1]$.

$\square$

Theorem 3.27. *Assume that* $p = q$, *(3.9.5) and (3.9.6) hold, then the unique positive equilibrium of equation (3.2.1) is* globally asymptotically stable.

Proof. Consider the function

$$f(x,y) = \frac{px+y}{r+px+y}$$

By using the derivative of $f(x,y)$ with respect to x and y, then $f(x,y)$ is nonincreasing in each of its arguments.

Now, let $(m,M) \in [0,\infty] \times [0,\infty]$ is a solution of the system

$$m = f(M,M) \quad \text{and} \quad M = f(m,m),$$

then

$$m = \frac{pM+M}{r+pM+M} \quad \text{and} \quad M = \frac{pm+m}{r+pm+m},$$

$$\begin{aligned} rm + qmM + mM &= pM + M \\ rM + qmM + mM &= pm + m \end{aligned}$$

this yields $(M-m)(r+p+1) = 0$, then the only solution is $m = M$.

Therefore, both conditions of theorem (3.23) hold, then every solution of equation (3.2.1) converges to $\bar{y}$.

As $\bar{y}$ is locally asymptotically stable under the conditions (3.9.5) and (3.9.6), then it is *globally asymptotically stable* by the definition (3.2). $\square$

CHAPTER 4

Special Cases of $\beta\gamma ABC = 0$

In this chapter we will study the character of solutions of equation (3.0.1), where one or two of the parameters in equation (3.0.1) are zeros. There are many equations that arise by considering zero parameters.

Observe that some of these equations are quite interesting and have been studied by some researchers, while few of them are meaningless such as the case when all the parameters in the denominator or the numerator are zero.

4.1 One of the Parameters $\beta\gamma ABC = 0$

In this section we will study the characteristics of the solutions of equation (3.0.1), where one of the parameters β, γ, A, B or C equals zero.

There are five such equations, namely:

$$x_{n+1} = \frac{\gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}, \qquad n = 0, 1, \ldots \tag{4.1.1}$$

$$x_{n+1} = \frac{\beta x_n}{A + Bx_n + Cx_{n-k}}, \qquad n = 0, 1, \ldots \tag{4.1.2}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \qquad n = 0, 1, \ldots \tag{4.1.3}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + Cx_{n-k}}, \qquad n = 0, 1, \ldots \tag{4.1.4}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + Bx_n}, \qquad n = 0, 1, \ldots \tag{4.1.5}$$

where the parameters β, γ, A, B and C are positive real numbers and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0$ are arbitrary nonnegative real numbers.

4.1.1 Characteristics of $x_{n+1} = \frac{\gamma x_{n-k}}{A+Bx_n+Cx_{n-k}}$

The change of variables $x_n = \frac{\gamma}{C} y_n$ reduces equation (4.1.1) to the difference equation

$$y_{n+1} = \frac{y_{n-k}}{p + qy_n + y_{n-k}}, \qquad n = 0, 1, \ldots \tag{4.1.6}$$

where $p = \frac{A}{\gamma}$ and $q = \frac{B}{C}$.

To find the equilibrium points of equation (4.1.6), assume that

$$\bar{y} = \frac{\bar{y}}{p + q\bar{y} + \bar{y}}$$

then

$$\bar{y} = 0 \quad \text{or} \quad \bar{y} = \frac{1-p}{q+1}$$

By assuming $f(x,y) = \dfrac{y}{p+qx+y}$, then the linearization of equation (4.1.6) is

$$z_{n+1} - \frac{-q\bar{y}}{(p+q\bar{y}+\bar{y})^2} z_n - \frac{p+q\bar{y}}{(p+q\bar{y}+\bar{y})^2} z_{n-k} = 0$$

Theorem 4.1. *The following statements are true:*

(i) Assume $p > 1$, then the zero equilibrium point of equation (4.1.6) is globally asymptotically stable.

(ii) Assume $p < 1$ and $q < 1$, then the positive equilibrium point $\bar{y} = \frac{1-p}{q+1}$ of equation (4.1.6) is globally asymptotically stable.

Proof. We find the linearized equation about zero equilibrium

$$z_{n+1} - \frac{1}{p} z_{n-k} = 0 \tag{4.1.7}$$

The proof of local stability follows immediately by applying theorem (3.3) on equation (4.1.7).

The linearized equation about positive equilibrium is

$$z_{n+1} - \frac{pq-q}{q+1} z_n - \frac{p+q}{q+1} z_{n-k} = 0 \tag{4.1.8}$$

and then apply theorem (3.3) to satisfy the local stability.

By applying theorem (3.22) on the properties of the function $f(x,y) = \dfrac{y}{p+qx+y}$, this yields the global asymptotic stability of the two equilibrium points separately. □

Theorem 4.2. *Assume that $p < 1$ and k odd, then equation (4.1.6) has prime period two solution. If k even then equation (4.1.6) has no prime period two solution.*

Proof. Let

$$\ldots, \psi, \phi, \psi, \phi, \ldots$$

be a period two solution of equation (4.1.6), where ψ and ϕ are two arbitrary positive and distinct real numbers.

- If k is odd, then $y_{n+1} = y_{n-k}$ and ψ, ϕ satisfy the following system:

  $$\psi = \frac{\psi}{p + q\phi + \psi} \quad \text{and} \quad \phi = \frac{\phi}{p + q\psi + \phi}$$

  simplifying the relation in the system to get

  $$(\psi - \phi)\left[p + (\psi + \phi) - 1\right] = 0$$

  implies that

  $$(\psi + \phi) = 1 - p,$$

  but ψ and ϕ are nonnegative, then

  $$1 - p > 0 \ \Rightarrow \ p < 1 \text{ as expected.}$$

- If k is even, then $y_n = y_{n-k}$ and ψ, ϕ satisfy the following system:

  $$\psi = \frac{\phi}{p + q\phi + \phi} \quad \text{and} \quad \phi = \frac{\psi}{p + q\psi + \psi}$$

  simplifying the relation in the system to get

  $$(\psi - \phi)(p + 1) = 0$$

  implies that $\psi = \phi$ or $p + 1 = 0$ which is impossible.

The proof is complete. □

4.1.2 Characteristics of $x_{n+1} = \frac{\beta x_n}{A+Bx_n+Cx_{n-k}}$

The change of variables $x_n = \frac{\beta}{C} y_n$ reduces equation (4.1.2) to the difference equation

$$y_{n+1} = \frac{y_n}{p + qy_n + y_{n-k}}, \qquad n = 0, 1, \ldots \tag{4.1.9}$$

where $p = \frac{A}{\beta}$ and $q = \frac{B}{C}$.

Let

$$\bar{y} = \frac{\bar{y}}{p + q\bar{y} + \bar{y}}$$

this yields that the equilibrium points are

$$\bar{y} = 0 \quad \text{or} \quad \bar{y} = \frac{1-p}{q+1}.$$

Assuming the function $f(x, y) = \dfrac{x}{p + qx + y}$, then the linearized equation associated with equation (4.1.9) about zero equilibrium point is

$$z_{n+1} - \frac{1}{p} z_n = 0$$

and the linearized equation about the positive equilibrium point is

$$z_{n+1} - \frac{1+pq}{q+1} z_n - \frac{p-1}{q+1} z_{n-k} = 0$$

By applying the theorems (3.3) and (3.21), we can construct the following theorem.

Theorem 4.3. *The following statements are true:*

(i) Suppose that $p \geqslant q$, then the zero equilibrium point of equation (4.1.9) is globally asymptotically stable.

(ii) *Suppose that* $p < 1$ *and the initial conditions of equation (4.1.9) are nonnegative, then the positive equilibrium point* $\bar{y} = \frac{1-p}{q+1}$ *is* globally asymptotically stable.

4.1.3 Characteristics of $x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}$

The change of variables $x_n = \frac{\gamma}{C} y_n$ reduces equation (4.1.3) to the difference equation

$$y_{n+1} = \frac{py_n + y_{n-k}}{qy_n + y_{n-k}}, \qquad n = 0, 1, \ldots \tag{4.1.10}$$

where $p = \frac{\beta}{\gamma}$ and $q = \frac{A}{\gamma}$.

To avoid a degenerate situation, assume that $p \neq q$.

To find the equilibrium points of equation (4.1.10), assume that

$$\bar{y} = \frac{p\bar{y} + \bar{y}}{q\bar{y} + \bar{y}}$$

implies that the only equilibrium point is

$$\bar{y} = \frac{p+1}{q+1}$$

By assuming $f(x, y) = \dfrac{px + y}{qx + y}$, then the linearized equation associated with equation (4.1.10) about $\bar{y} = \frac{p+1}{q+1}$ is

$$z_{n+1} - \frac{p-q}{qp+q+p+1} z_n - \frac{q-p}{qp+q+p+1} z_{n-k} = 0.$$

Equation (4.1.10) investigated in [20] by M. Saleh and S. Abu-Baha. Here are the main results.

Theorem 4.4. *The following statements are true:*

(i) Assume that $p < q$, and k is odd, then the positive equilibrium of equation (4.1.10) is globally asymptotically stable *when $q < pq + 3p + 1$.*

(ii) Assume that $p > q$, and $q < pq + 3p + 1$, then the positive equilibrium of equation (4.1.10) is globally asymptotically stable.

Theorem 4.5. *The following statements are true:*

(i) If $p > q$, then equation (4.1.10) has no nonnegative prime period-two solutions.

(ii) If $p < q$, then equation (4.1.10) has nonnegative prime period-two solutions if and only if k is odd and $q > pq + 3p + 1$.

4.1.4 Characteristics of $x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + C x_{n-k}}$

The change of variables $x_n = \frac{\gamma}{C} y_n$ reduces equation (4.1.4) to the difference equation

$$y_{n+1} = \frac{p y_n + y_{n-k}}{q + y_{n-k}}, \qquad n = 0, 1, \ldots \tag{4.1.11}$$

where $p = \frac{\beta}{\gamma}$ and $q = \frac{A}{\gamma}$.

To find the equilibrium points of equation (4.1.11), assume that

$$\bar{y} = \frac{p\bar{y} + \bar{y}}{q + \bar{y}}$$

implies that the equilibrium points

$$\bar{y} = 0 \quad \text{or} \quad \bar{y} = p - q + 1$$

Assuming that the function $f(x, y) = \dfrac{px + y}{q + y}$, then the linearized equation associated with equation (4.1.11) about zero equilibrium point is

$$z_{n+1} - \frac{p}{q} z_n - \frac{1}{q} z_{n-k} = 0$$

and the linearized equation about the positive equilibrium point is

$$z_{n+1} - \frac{p}{p+1} z_n - \frac{q-p}{p+1} z_{n-k} = 0.$$

Equation (4.1.11) investigated by Li et al in [18] and A. Farhat in [11] independently. Here are the main results of their studies.

Theorem 4.6. *The zero equilibrium point of equation (4.1.11) is* globally asymptotically stable *under the condition $p + 1 < q$. Moreover, the positive equilibrium point $\bar{y} = p - q + 1$ is* globally asymptotically stable *under the condition $-1 < p - q < 1$.*

Theorem 4.7. *(i) Assume that $p < q$. If $y_n \leqslant \frac{q}{p}$, then $y_{n+1} < \frac{q}{p}$. Furthermore, every nonnegative solution of equation (4.1.11) eventually enters the interval $[0, \frac{q}{p}]$.*

(ii) Assume that $p > q$. If $y_n \geqslant \frac{q}{p}$, then $y_{n+1} < \frac{q}{p}$. Furthermore, every nonnegative solution of equation (4.1.11) eventually enters the interval $[\frac{q}{p}, \infty]$.

Theorem 4.8. *The equation (4.1.11) has no nonnegative solution with prime period two under any condition.*

4.1.5 Characteristics of $x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + B x_n}$

The change of variables $x_n = \frac{\beta}{B} y_n$ reduces equation (4.1.5) to the difference equation

$$y_{n+1} = \frac{y_n + p y_{n-k}}{q + y_n}, \qquad n = 0, 1, \ldots \tag{4.1.12}$$

where $p = \frac{\gamma}{\beta}$ and $q = \frac{A}{\beta}$.

To find the equilibrium points, assume that

$$\bar{y} = \frac{p\bar{y} + \bar{y}}{q + \bar{y}}$$

then

$$\bar{y} = 0 \quad \text{or} \quad \bar{y} = p - q + 1$$

Equation (4.1.12) investigated by Mazrooei-Sebdani et al. in [19] and Yan et al. in [23] also by Jaberi et al in [7], and the following are main results of their studies.

Theorem 4.9. *(i) If $p + 1 \leqslant q$, then the zero equilibrium point of equation (4.1.12) is* globally asymptotically stable.

(ii) If $p < q < p + 1$, then the positive equilibrium point $\bar{y} = p - q + 1$ is locally asymptotically stable.

Theorem 4.10. *The following statements are true:*

(i) Assume k is odd, then equation (4.1.12) has prime period two solution

$$\ldots, \psi, \phi, \psi, \phi, \ldots$$

if and only if $p = q + 1$.

(ii) Assume k is even, then equation (4.1.12) does not have prime period two solution.

4.2 Two of the Parameters $\beta\gamma ABC = 0$

In this section we will study the character of solution of equation (3.0.1), where two parameters of β, γ, A, B and C are zero. There are eight cases for this equation, namely:

$$x_{n+1} = \frac{\gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \qquad n = 0, 1, \dots \tag{4.2.1}$$

$$x_{n+1} = \frac{\gamma x_{n-k}}{A + Cx_{n-k}}, \qquad n = 0, 1, \dots \tag{4.2.2}$$

$$x_{n+1} = \frac{\gamma x_{n-k}}{A + Bx_n}, \qquad n = 0, 1, \dots \tag{4.2.3}$$

$$x_{n+1} = \frac{\beta x_n}{Bx_n + Cx_{n-k}}, \qquad n = 0, 1, \dots \tag{4.2.4}$$

$$x_{n+1} = \frac{\beta x_n}{A + Cx_{n-k}}, \qquad n = 0, 1, \dots \tag{4.2.5}$$

$$x_{n+1} = \frac{\beta x_n}{A + Bx_n}, \qquad n = 0, 1, \dots \tag{4.2.6}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Cx_{n-k}}, \qquad n = 0, 1, \dots \tag{4.2.7}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Bx_n}, \qquad n = 0, 1, \dots \tag{4.2.8}$$

4.2.1 Characteristics of $x_{n+1} = \frac{\gamma x_{n-k}}{Bx_n + Cx_{n-k}}$

The change of variables $x_n = \frac{\gamma}{By_n}$ reduces equation (4.2.1) to the difference equation

$$y_{n+1} = p + \frac{y_{n-k}}{y_n}, \qquad n = 0, 1, \ldots \tag{4.2.9}$$

where $p = \frac{C}{B}$.

Equation (4.2.9) investigated by El-Owaidy et al in [9] and M. Saleh & M. Aloqeili in [21] independently.

4.2.2 Characteristics of $x_{n+1} = \frac{\gamma x_{n-k}}{A + Cx_{n-k}}$

The change of variables $x_n = \frac{1}{y_n}$ reduces equation (4.2.2) to the difference equation

$$y_{n+1} = \frac{A}{\gamma} y_{n-k} + \frac{C}{\gamma}, \qquad n = 0, 1, \ldots \tag{4.2.10}$$

and this equation is linear first order difference equation.

4.2.3 Characteristics of $x_{n+1} = \frac{\gamma x_{n-k}}{A + Bx_n}$

The change of variables $x_n = \frac{\gamma}{B} y_n$ reduces equation (4.2.3) to the difference equation

$$y_{n+1} = \frac{y_{n-k}}{p + y_n}, \qquad n = 0, 1, \ldots \tag{4.2.11}$$

where $p = \frac{A}{\gamma}$.

To find the equilibrium points of equation (4.2.11), let

$$\bar{y} = \frac{\bar{y}}{p + \bar{y}}$$

then the equilibrium points are

$$\bar{y} = 0 \quad \text{or} \quad \bar{y} = 1 - p$$

Assuming the function $f(x, y) = \dfrac{y}{p + x}$, then the linearized equation associated with equation (4.2.11) about zero equilibrium point is

$$z_{n+1} - \frac{1}{p} z_{n-k} = 0$$

and the linearized equation about the positive equilibrium point is

$$z_{n+1} - (p - 1) z_n - z_{n-k} = 0$$

Now, by applying the theorems (3.3) and (3.22), we can construct the following two theorems.

Theorem 4.11. *Assume that $p > 1$, then the zero equilibrium point of equation (4.2.11) is* globally asymptotically stable.

Theorem 4.12. *Assume that $p < 1$, then the zero and positive equilibrium points of equation (4.2.11) are both* unstable.

Theorem 4.13. *Assume that $p = 1$, then equation (4.2.11) has solution with prime period two when k is odd.*

Proof. Let

$$\ldots, \psi, \phi, \psi, \phi, \ldots$$

be a period two solution of equation (4.2.11), where ψ and ϕ are two arbitrary positive and distinct real numbers.

- If k is odd, then $y_{n+1} = y_{n-k}$ and ψ, ϕ satisfy the following system:

$$\phi = \frac{\psi}{p+\phi} \quad \text{and} \quad \phi = \frac{\phi}{p+\psi}$$

simplifying the relation in the system to get

$$(\psi - \phi)\,[p-1] = 0$$

implies that $p = 1$ as expected.

- If k is even, then $y_n = y_{n-k}$ and ψ, ϕ satisfy the following system:

$$\psi = \frac{\phi}{p+\psi} \quad \text{and} \quad \phi = \frac{\psi}{p+\phi}$$

simplifying the relation in the system to get

$$(\psi - \phi)\,[p + (\psi + \phi) + 1] = 0$$

implies that

$$\psi = \phi \quad \text{or} \quad \psi + \phi = -(p+1)$$

which contradicts that ψ and ϕ are nonnegative.

The proof is complete. □

4.2.4 Characteristics of $x_{n+1} = \frac{\beta x_n}{Bx_n + Cx_{n-k}}$

The change of variables $x_n = \frac{\beta}{Cy_n}$ reduces equation (4.2.4) to the difference equation

$$y_{n+1} = p + \frac{y_n}{y_{n-k}}, \qquad n = 0, 1, \ldots \tag{4.2.12}$$

where $p = \frac{B}{C}$.

Equation (4.2.12) investigated by Abu-Saris and DeVault in [1].

4.2.5 Characteristics of $x_{n+1} = \frac{\beta x_n}{A+Cx_{n-k}}$

The change of variables $x_n = \frac{A}{C} y_n$ reduces equation (4.2.5) to the difference equation

$$y_{n+1} = \frac{py_n}{1+y_{n-k}}, \qquad n = 0, 1, \ldots \tag{4.2.13}$$

where $p = \frac{\beta}{A}$.

To find the equilibrium points of equation (4.2.13), let

$$\begin{aligned} \bar{y} &= \frac{p\bar{y}}{1+\bar{y}} \\ \bar{y}(1+\bar{y}) &= p\bar{y} \end{aligned}$$

implies

$$\bar{y} = 0 \quad \text{or} \quad \bar{y} = p - 1.$$

By assuming the function $f(x, y) = \dfrac{px}{1+y}$, then the linearized equation associated with equation (4.2.13) about zero equilibrium point is

$$z_{n+1} - pz_n = 0$$

and the linearized equation about the positive equilibrium point is

$$z_{n+1} - z_n - \left(\frac{1}{p} - 1\right) z_{n-k} = 0$$

We can construct the following theorem easily by applying the theorems (3.3) and (3.21).

Theorem 4.14. *The following statements are true:*

(i) Assume $p < 1$, then the zero equilibrium point of equation (4.2.13) is globally asymptotically stable.

(ii) Assume $p > 1$ and the initial conditions of equation (4.2.13) are nonnegative, then the positive equilibrium point $\bar{y} = p - 1$ is globally asymptotically stable.

The proof of the theorem follows immediately by applying theorem (3.3) on the linearized equations and theorem (3.21).

Theorem 4.15. *The equation (4.2.13) has no nonnegative solution with prime period two under any condition.*

Proof. Let

$$\ldots, \psi, \phi, \psi, \phi, \ldots$$

be a period two solution of equation (4.2.13), where ψ and ϕ are two arbitrary positive and distinct real numbers.

- If k is odd, then $y_{n+1} = y_{n-k}$ and ψ, ϕ satisfy the following system:

  $$\phi = \frac{p\psi}{1+\phi} \quad \text{and} \quad \psi = \frac{p\phi}{1+\psi}$$

  simplifying the relation in the system to get

  $$(\psi - \phi)\left[1 + p + (\psi + \phi)\right] = 0$$

  implies that $(\psi + \phi) = -(1 + p)$ which is impossible since ψ and ϕ are nonnegative.

- If k is even, then $y_n = y_{n-k}$ and ψ, ϕ satisfy the following system:

$$\phi = \frac{p\psi}{1+\psi} \quad \text{and} \quad \psi = \frac{p\phi}{1+\phi}$$

simplifying the relation in the system to get

$$(\psi - \phi)(1+p) = 0$$

implies that $\psi = \phi$ since $p \neq -1$.

The proof is complete. □

4.2.6 Characteristics of $x_{n+1} = \frac{\beta x_n}{A+Bx_n}$

The change of variables $x_n = \frac{1}{y_n}$ reduces equation (4.2.6) to the difference equation

$$y_{n+1} = \frac{A}{\beta} y_n + \frac{B}{\beta}, \qquad n = 0, 1, \ldots \tag{4.2.14}$$

and this equation is linear first order difference equation.

4.2.7 Characteristics of $x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{C x_{n-k}}$

The change of variables $x_n = \frac{\beta}{C} y_n$ reduces equation (4.2.7) to the difference equation

$$y_{n+1} = p + \frac{y_n}{y_{n-k}}, \qquad n = 0, 1, \ldots \tag{4.2.15}$$

where $p = \frac{\gamma}{\beta}$.

Equation (4.2.15) is the same as equation (4.2.12) which is investigated by Abu-Saris and DeVault in [1] as mentioned previously.

4.2.8 Characteristics of $x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Bx_n}$

The change of variables $x_n = \frac{\gamma}{B} y_n$ reduces equation (4.2.8) to the difference equation

$$y_{n+1} = p + \frac{y_{n-k}}{y_n}, \qquad n = 0, 1, \ldots \tag{4.2.16}$$

where $p = \frac{\beta}{\gamma}$.

Equation (4.2.16) is the same as equation (4.2.9) which is investigated in [9, 21] independently as mentioned previously.

CHAPTER 5

Numerical Approach

In order to support our theoretical discussions and to illustrate the main results of chapter (3), we consider several interesting numerical examples in this chapter. These examples represent different types of qualitative behavior of solutions to nonlinear difference equation

$$y_{n+1} = \frac{py_n + y_{n-k}}{r + qy_n + y_{n-k}}, \qquad n = 0, 1, \ldots \tag{5.0.1}$$

In this chapter, to observe this numerical results clearly, we present tables of solutions that were carried out using MATLAB code in Appendix (A.1). We choose different values for the parameters p, q and r. It should be noted that $y_k, y_{k+1}, \ldots, y_1, y_0$ are also different initial values.

To simplify our exposition we restrict our discussion to the following sections.

5.1 Numerical Approach of Local and Global Stability

Here in this section, we will study the global stability of our equation numerically based on some data and figures that we can get using MATLAB.

Example 5.1. Assume that equation (5.0.1) holds, take $k = 4$, $p = 0.5$, $q = 3$ and $r = 1$. So the equation will be reduced to the following:

$$y_{n+1} = \frac{0.5\, y_n + y_{n-4}}{1 + 3\, y_n + y_{n-4}}$$

In this case, the values of p, q, and r satisfy the conditions of local stability (3.9.5) and (3.9.6), so by theory the value of $\bar{y}$ is

$$\bar{y} = \frac{p+1-r}{q+1} = \frac{0.5+1-1}{3+1} = \frac{1}{8} = 0.125$$

We assume the initial points $y_0 = 1$, $y_1 = 8$, $y_2 = 5$, $y_3 = 8$ and $y_4 = 3$, and the output of the numerical illustrated in the following table (Table 5.1) and figure (Fig. 5.1), as expected as the theoretical results.

Example 5.2. In this example, let the parameters $p = 2$, $q = 5$ and $r = 4$, and $k = 3$. So the equation (5.0.1) will be reduced to the following:

$$y_{n+1} = \frac{2\, y_n + y_{n-3}}{4 + 5\, y_n + y_{n-3}}$$

We assume the initial points $y_0 = 3$, $y_1 = 1$, $y_2 = 2$ and $y_3 = 0.5$.

In this case, the values of p, q, and r satisfy the condition of global stability of the zero equilibrium point ($p+1 \leqslant r$), and this is clear numerically as illustrated in table (5.2) and figure (5.2).

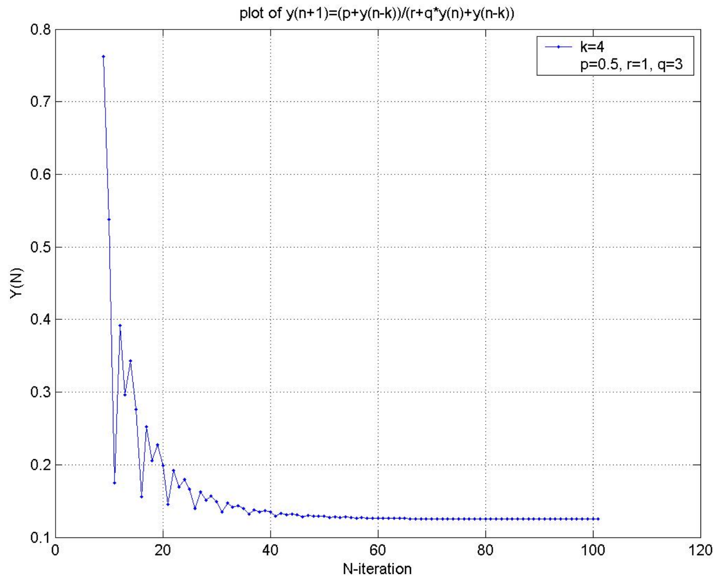

Figure 5.1: The Behavior of the equilibrium point of equation $y_{n+1} = \frac{0.5\,y_n + y_{n-4}}{1+3\,y_n + y_{n-4}}$

5.2 Numerical Approach of Unstable Case

Example 5.3. Assume that equation (5.0.1) holds, take $k = 4$, $p = 0.25$, $q = 20$ and $r = 0.1$. So the equation will be reduced to the following:

$$y_{n+1} = \frac{0.25\,y_n + y_{n-4}}{0.1 + 20\,y_n + y_{n-4}}$$

We assume the initial points $y_0 = 2$, $y_1 = 3$, $y_2 = 1$, $y_3 = 4$ and $y_4 = 3$.

In this case, the values of p, q, and r contradict the condition of local stability (3.9.6) such that

$$q + r > 3p + 1 + qr + pq$$

$$20 + 0.1 > 3 \times 0.25 + 1 + 20 \times 0.1 + 0.25 \times 20$$

Table 5.1: The solution of equation $y_{n+1} = \frac{0.5\, y_n + y_{n-4}}{1+3\, y_n + y_{n-4}}$

n	y(n)	n	y(n)	n	y(n)	n	y(n)
1	1.0000	26	0.1392	51	0.1271	76	0.1253
2	8.0000	27	0.1628	52	0.1279	77	0.1253
3	5.0000	28	0.1512	53	0.1276	78	0.1253
4	8.0000	29	0.1564	54	0.1278	79	0.1253
5	3.0000	30	0.1495	55	0.1274	80	0.1253
6	0.2273	31	0.1348	56	0.1264	81	0.1252
7	0.8380	32	0.1469	57	0.1268	82	0.1252
8	0.6365	33	0.1411	58	0.1267	83	0.1252
9	0.7625	34	0.1437	59	0.1267	84	0.1252
10	0.5378	35	0.1400	60	0.1265	85	0.1252
11	0.1747	36	0.1317	61	0.1260	86	0.1251
12	0.3918	37	0.1380	62	0.1261	87	0.1251
13	0.2960	38	0.1351	63	0.1261	88	0.1251
14	0.3435	39	0.1364	64	0.1261	89	0.1251
15	0.2763	40	0.1344	65	0.1260	90	0.1251
16	0.1561	41	0.1296	66	0.1256	91	0.1251
17	0.2526	42	0.1328	67	0.1257	92	0.1251
18	0.2056	43	0.1314	68	0.1257	93	0.1251
19	0.2277	44	0.1320	69	0.1257	94	0.1251
20	0.1991	45	0.1309	70	0.1256	95	0.1251
21	0.1458	46	0.1281	71	0.1254	96	0.1251
22	0.1926	47	0.1298	72	0.1255	97	0.1250
23	0.1693	48	0.1291	73	0.1254	98	0.1250
24	0.1800	49	0.1294	74	0.1254	99	0.1250
25	0.1662	50	0.1288	75	0.1254	100	0.1250

So we see from the figure (5.3) and table (5.3) that there is no any stable solution for this case.

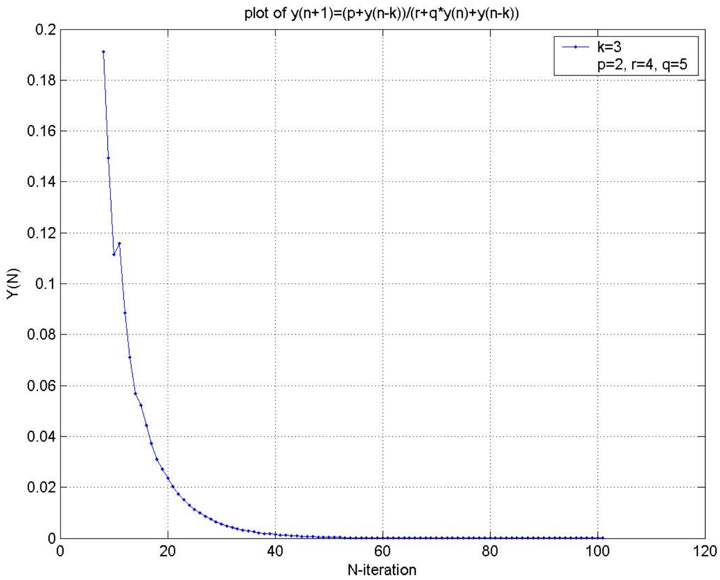

Figure 5.2: The Behavior of the zero equilibrium point of equation $y_{n+1} = \frac{2\,y_n + y_{n-3}}{4+5\,y_n + y_{n-3}}$

So, all what we have to say now is that our theoretical discussion was satisfied with the data we get from our numerical discussion. So we have correctly illustrated our study for the equation

$$y_{n+1} = \frac{py_n + y_{n-k}}{r + qy_n + y_{n-k}}, \qquad n = 0, 1, \ldots$$

Table 5.2: The solution of equation $y_{n+1} = \frac{2\,y_n + y_{n-3}}{4 + 5\,y_n + y_{n-3}}$

n	y(n)	n	y(n)	n	y(n)	n	y(n)
1	3.0000	26	0.0098	51	0.0003	76	0.0000
2	1.0000	27	0.0085	52	0.0003	77	0.0000
3	2.0000	28	0.0074	53	0.0002	78	0.0000
4	0.5000	29	0.0065	54	0.0002	79	0.0000
5	0.4211	30	0.0056	55	0.0002	80	0.0000
6	0.2593	31	0.0049	56	0.0002	81	0.0000
7	0.3452	32	0.0043	57	0.0001	82	0.0000
8	0.1912	33	0.0037	58	0.0001	83	0.0000
9	0.1494	34	0.0033	59	0.0001	84	0.0000
10	0.1115	35	0.0028	60	0.0001	85	0.0000
11	0.1159	36	0.0025	61	0.0001	86	0.0000
12	0.0887	37	0.0022	62	0.0001	87	0.0000
13	0.0711	38	0.0019	63	0.0001	88	0.0000
14	0.0568	39	0.0016	64	0.0001	89	0.0000
15	0.0522	40	0.0014	65	0.0000	90	0.0000
16	0.0444	41	0.0013	66	0.0000	91	0.0000
17	0.0372	42	0.0011	67	0.0000	92	0.0000
18	0.0309	43	0.0010	68	0.0000	93	0.0000
19	0.0271	44	0.0008	69	0.0000	94	0.0000
20	0.0236	45	0.0007	70	0.0000	95	0.0000
21	0.0203	46	0.0006	71	0.0000	96	0.0000
22	0.0173	47	0.0006	72	0.0000	97	0.0000
23	0.0150	48	0.0005	73	0.0000	98	0.0000
24	0.0131	49	0.0004	74	0.0000	99	0.0000
25	0.0114	50	0.0004	75	0.0000	100	0.0000

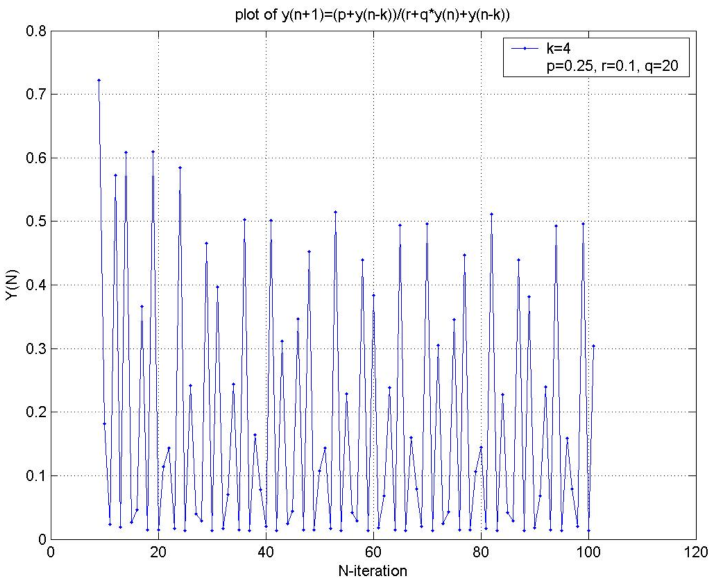

Figure 5.3: The Behavior of unstable solution of equation $y_{n+1} = \frac{0.25\,y_n+y_{n-4}}{0.1+20\,y_n+y_{n-4}}$

Table 5.3: The solution of equation $y_{n+1} = \frac{0.25\, y_n + y_{n-4}}{0.1 + 20\, y_n + y_{n-4}}$

n	y(n)	n	y(n)	n	y(n)	n	y(n)
1	2.0000	26	0.2414	51	0.1440	76	0.0143
2	3.0000	27	0.0402	52	0.0168	77	0.4474
3	1.0000	28	0.0293	53	0.5142	78	0.0150
4	4.0000	29	0.4657	54	0.0138	79	0.1064
5	3.0000	30	0.0138	55	0.2291	80	0.1447
6	0.0443	31	0.3967	56	0.0417	81	0.0168
7	0.7555	32	0.0173	57	0.0286	82	0.5116
8	0.0733	33	0.0709	58	0.4394	83	0.0138
9	0.7218	34	0.2437	59	0.0139	84	0.2275
10	0.1814	35	0.0150	60	0.3832	85	0.0420
11	0.0238	36	0.5029	61	0.0176	86	0.0285
12	0.5722	37	0.0141	62	0.0686	87	0.4392
13	0.0186	38	0.1647	63	0.2388	88	0.0139
14	0.6083	39	0.0783	64	0.0150	89	0.3816
15	0.0268	40	0.0206	65	0.4934	90	0.0177
16	0.0462	41	0.5010	66	0.0141	91	0.0682
17	0.3658	42	0.0137	67	0.1600	92	0.2396
18	0.0148	43	0.3115	68	0.0788	93	0.0150
19	0.6093	44	0.0244	69	0.0206	94	0.4925
20	0.0145	45	0.0438	70	0.4964	95	0.0141
21	0.1140	46	0.3465	71	0.0138	96	0.1592
22	0.1436	47	0.0142	72	0.3054	97	0.0793
23	0.0170	48	0.4524	73	0.0247	98	0.0205
24	0.5850	49	0.0150	74	0.0435	99	0.4964
25	0.0136	50	0.1073	75	0.3458	100	0.0138

APPENDIX A

Appendix

A.1 Matlab Code-Plot Difference Equation

This code of MATLAB is to plot the difference equation

$$y_{n+1} = \frac{py_n + y_{n-k}}{r + qy_n + y_{n-k}}$$

with the entered parameters p, q, r and the initial values $y_{-k}, \ldots, y_0$.

```
%Dynamical of Nonlinear Difference Equation
%Amer Jafar
%Master of Sci Computing Program
clear all;
format short;
%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
disp('------------------------------------ ');
disp('This matlab program to plot the difference equation:');
disp('------------------------------------ ');
fprintf('\n First: Input The Constants Values of
the Difference Equation \n')
disp(' ')
p=input('The value of the positive parameter p= ');
q=input('The value of the positive parameter q= ');
r=input('The value of the positive parameter r= ');
disp('------------------------------------ ');
k=input('Second: Input The value of k \n k= ');
disp('------------------------------------ ');
fprintf('\n Now Enter the initial conditions of the Diff.Equation\n ')
%-----------------------
for i=1:k+1;
    fprintf('Enter the value of y%0.0f \n ',i-1);
y(i)=input (' ');
end
for n=k+1:100;
y(n+1)=(p*y(n)+y(n-k))/(r+q*y(n)+y(n-k));
y(n+1);
end
t=1:101;
ans=[t;y]';
plot(t(k+5:101),y(k+5:101),'b.-')
```

```
xlabel('N-iteration'); ylabel('Y(N)');
title('plot of y(n+1)=(p+y(n-k))/(r+q*y(n)+y(n-k))')
hold on
grid on
p1=strcat('k=',num2str(k));
p2=strcat('p= ',num2str(p),', r=' ,num2str(r),', q= ',num2str(q));
legend(p1,p2)
%--------------------------------
disp('The results are: ')
disp('__________________________________________________')
disp(' n y(n) n y(n) n y(n) n y(n)')
disp('__________________________________________________')
D=[ans(1:25,:),ans(26:50,:),ans(51:75,:),ans(76:100,:)]; disp(D)
```

A.2 Matlab Code-Simplify the Linearized Equation

This code is to find $f_x(x,y)$ and $f_y(x,y)$ of the function $f(x,y) = \frac{px+y}{r+qx+y}$, then simplify the values of $f_x(\bar{y},\bar{y})$ and $f_y(\bar{y},\bar{y})$ to find the linearized equation.

```
%This code simplify the linearized equation
clear all;
syms x y  F0  q p r fx fy
F0=(p*x+y)/(r+q*x+y); %define the fuction
```

```
ybar=(p+1-r)/(q+1);   %set the equilibrium point value
Fx = diff(F0, x);  % differeniate F0 with respect to x
latex(Fx)          %write the differenial of F0_x in latex
Fy = diff(F0, y);   % differeniate F0 with respect to y
latex(Fy)           %write the differenial of F0_y in latex
%substitite ybar and simplify
SFx=simplify(subs(Fx,{x,y},{ybar,ybar}))
%substitite ybar and simplify
SFy=simplify(subs(Fy,{x,y},{ybar,ybar}))
latex(SFx)  %write the simplification in latex
latex(SFy)  %write the simplification in latex
```

References

[1] R. M. Abu-Saris and R. DeVault. Global stability of $y_{n+1} = A + \frac{y_n}{y_{n-k}}$. *Applied Mathematics Letters*, 16(2):173–178, 2003.

[2] K. Cunningham, M. R. S. Kulenović, G. Ladas, and S. V. Valicenti. On the recursive sequence $x_{n+1} = \frac{\alpha+\beta x_n}{Bx_n+Cx_{n-1}}$. *Nonlinear Analysis: Theory, Methods & Applications*, 47(7):4603–4614, 2001. Proceedings of the Third World Congress of Nonlinear Analysts.

[3] S. Ebru Das and Mustafa Bayram. Dynamics of a higher - order nonlinear rational difference equation. *International Journal of the Physical Sciences*, 6(12):2950–2957, 2011.

[4] Mehdi Dehghan, Majid Jaberi Douraki, and Marjan Jaberi Douraki. Dynamics of a rational difference equation using both theoretical and computational approaches. *Applied Mathematics and Computation*, l68(2):756–775, 2005.

[5] R. DeVault, C. Kent, and W. Kosmala. On the recursive sequence $x_{n+1} = p + \frac{x_{n-k}}{x_n}$. *Journal of Difference Equations and Applications*, 9(8):721–730, 2003.

[6] R. DeVault, W. Kosmala, G. Ladas, and S.W. Schultz. Global behavior of $y_{n+1} = \frac{p+y_{n-k}}{qy_n+y_{n-k}}$. *Nonlinear Analysis: Theory, Methods & Applications*, 47(7):4743–4751, 2001.

[7] M. Jaberi Douraki, Mehdi Dehghan, and Javad Mashreghi. Dynamics of the difference equation $x_{n+1} = \frac{x_n+px_{n-k}}{x_n+q}$. *Computers and Mathematics with Applications*, 56(1):186–198, 2008.

[8] M. El-Afifi. On the recursive sequence $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{Bx_n+Cx_{n-1}}$. *Applied Mathematics and Computation*, 147(3):617–628, 2004.

[9] H.M. El-Owaidy, A.M. Ahmed, and M.S. Mousa. On asymptotic behaviour of the difference equation $x_{n+1} = \alpha + \frac{x_{n-k}}{x_n}$. *Applied Mathematics and Computation*, 147(1):163–167, 2004.

[10] Saber N. Elaydi. *an Introduction to Difference Equations.* Springer, New York, third edition, 2005.

[11] Aseel Farhat. Dynamic of some rational nonlinear difference equations. Master's thesis, Birzeit University, 2007.

[12] E. A. Grove and G. Ladas. *Periodicities in Nonlinear Difference Equations*, volume Four of *Advances in Discrete Mathematics and Applications.* CHAPMAN & HALL/CRC, Boca Raton, first edition, 2005.

[13] Lin-Xia Hu, Wan-Tong Li, and Stevo Stević. Global asymptotic stability of a second order rational difference equation. *Journal of Difference Equations and Applications*, 14(8):779–797, 2008.

[14] Lin-Xia Hu, Wan-Tong Li, and Hong-Wu Xu. Global asymptotical stability of a second order rational difference equation. *Computers and Mathematics with Applications*, 54(9-10):1260–1266, 2007.

[15] V.L. Kocic and G. Ladas. *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, volume 256 of *Mathematics and Its Applications*. Kluwer Academic Publishers, Netherlands, 1993.

[16] M. R. S. Kulenović and G. Ladas. *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures*. Chapman and Hall/CRC, Florida, 2002.

[17] M. R. S. Kulenović, G. Ladas, L. F. Martins, and I. W. Rodrigues. The dynamics of $x_{n+1} = \frac{\alpha+\beta x_n}{A+Bx_n+Cx_{n-1}}$ facts and conjectures. *Computers and Mathematics with Applications*, 45(6–9):1087–1099, 2003.

[18] Wan-Tong Li and Hong-Rui Sun. Dynamics of a rational difference equation. *Applied Mathematics and Computation*, 163(2):577–591, 2005.

[19] Reza Mazrooei-Sebdani and Mehdi Dehghan. Dynamics of a non-linear difference equation. *Applied Mathematics and Computation*, 178(2):250–261, 2006.

[20] M. Saleh and S. Abu-Baha. Dynamics of a higher order rational difference equation. *Applied Mathematics and Computation*, 181(1):84–102, 2006.

[21] M. Saleh and M. Aloqeili. On the rational difference equation $y_{n+1} = A + \frac{y_{n-k}}{y_n}$. *Applied Mathematics and Computation*, 171(2):862–869, 2005.

[22] Kardi Teknomo. Difference equation tutorial. `http://people.revoledu.com/kardi/tutorial/DifferenceEquation/index.html`.

[23] Xing-Xue Yan, Wan-Tong Li, and Zhu Zhao. Global asymptotic stability for a higher order nonlinear rational difference equations. *Applied Mathematics and Computation*, 182(2):1819–1831, 2006.

[24] E. M. E. Zayed. Dynamics of the nonlinear rational difference equation $x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n+x_{n-k}}{q+x_{n-k}}$. *European Journal of Pure and Applied Mathematics*, 3(2):254–268, 2010.

[25] E.M.E. Zayed and M.A. El-Moneam. On the rational recursive sequence $x_{n+1} = \gamma x_{n-k} + \frac{Ax_n+Bx_{n-k}}{Cx_n-Dx_{n-k}}$. *Bulletin of the Iranian Mathematical Society*, 36(1):103–115, 2010.

Index