جامعة بيرزيت
BIRZEIT UNIVERSITY

Faculty of Graduate Study
Program of Scientific Computing

Dynamics of Some Rational Nonlinear Difference Equations

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M.Sc. Thesis

Palestine
2007

الملخص

إن الهدف الرئيسي من هـذا البحـث هـو دراسـة بعـض المعـادلات الغيـر خطيـة النفصلة, بحيث تم التركيز على طريقـة تصـرف الحـل لهـذه المعـادلات. وقـد تـم الإعتماد في دراسة بعض هذه المعادلات علـى بعـض الإسـتراتيجيات والتقنيـات التي كانت متبعة في دراسة بعض من المعادلات المنفصلة غير الخطية المشـابهة للمعادلات التي تمت دراستها.

لقد تم التركيز في هذاالبحث على دراسة ثلاثة معادلات منفصلة.

أولاً لقد تمت دراسة المعادلة المنفصلة الغير خطية

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{B x_n + C x_{n-k}}, \qquad n = 0, 1, ...$$

وذلـك عنـدما تكـون كـل مـن المتغيـرات $\alpha, \beta, \gamma, B, C$ والقيـم الابتدائيـة $x_{-k}, ..., x_{-1}, x_0$ أعدادًا حقيقية وموجبة وأن يكون المتغيـر k عـددًا صـحيحًا موجبًا.

أما المعادلة الثانية التي تمت دراستها هي

$$x_{n+1} = \frac{a x_n + b x_{n-k}}{A + B x_{n-k}}$$

وذلك عندما تكـون كـل مـن المتغيـرات a, b, A, B هـي أعـداد حقيقيـة موجبـة ويكون المتغير k عددًا صحيحًا موجبًا, وتكون القيم الابتدائيـة $x_{-k}, ..., x_{-1}, x_0$ هي أعداد حقيقية غير سالبة.

أخيرًا, لقد تمت دراسة المعادلة

$$x_n = \frac{A}{x_{n-k}} + \frac{B}{x_{n-3k}},$$

عندما تكون القيم الابتدائية $x_{-k}, ..., x_{-1}, x_0$ وكل من A, B أعدادًا حقيقيـة موجبة و k هو عدد صحيح موجب.

Dynamics of Some Rational Nonlinear Difference Equations

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Master in Scientific Computing

Program of Scientific Computing
2007

Abstract

The main goal of this thesis is to study some methods for solving some rational difference equations. We study the solution of rational difference equations basing on some approaches and methods that were studied for some other rational equations.

We mainly study the solution of three difference equations.

The first that we study in this thesis is the rational equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \qquad n = 0, 1, ... \tag{0.0.1}$$

where the parameters $\alpha, \beta, \gamma, B, C$ and the initial conditions $x_{-k}, ..., x_{-1}, x_0$ are positive real numbers, $k = \{1, 2, 3, ...\}$.

The second equation that has been studied is

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}} \tag{0.0.2}$$

where a, b, A, B are all positive real numbers, $k \geq 1$ is a positive integer, and the initial conditions $x_{-k}, x_{-k+1}, ..., x_0$ are nonnegative real numbers.

Finally, we study the equation

$$x_n = \frac{A}{x_{n-k}} + \frac{B}{x_{n-3k}}, \tag{0.0.3}$$

where $x_{-3k+1},\ x_{-3k+2}, ...,\ x_0 \in (0, \infty),\ A, B > 0$ and $k \in \{1, 2, 3, 4, ...\}$.

DECLARATION

I certify that this thesis, submitted for the degree of Master of Science to the Department of Scientific Computing in Birzeit University, is of my own research expect where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Aseel Farhat Farhat Signature ....

July 26, 2007

DEDICATION

I dedicate this thesis to all of those who stood behind me, believed in me, supported me, taught me and contributed in any way (big or small) to enriching my life experience at any point in time. There are too many important friends and relatives to mention, but they know who they are, for without their love and support I would not have been able to reach this point. You all know who you are, for a piece of this belongs to you.

Of course some played a major role in my life, and the most important ones I cite them now.

Professor Mohammad Saleh. He was my mentor ever since I was a freshman. More than just a mentor, he was also a role model and eventually became a very good friend. He is the reason why I will become a mathematician. I will never be able to thank him enough for all the advice, guidance and words of wisdom.

The two most important people of all: my parents. Probably their best qualities are my fathers vision and wisdom and my mothers care. But it was their unconditional love and support that allowed me to start this journey and be at where I am now. Most important of all, they provided the means for me to become the person I am today and made my life the most enjoyable life anyone could ask for.

I would like to express my thanks to Assoc. Prof. Wasfi S. Kafri for his support and encouragement through my study and also his valuable comments on this thesis.

Contents

Chapter 1

Introduction

This thesis mainly consists of two parts. The first part is a theoretical background of difference equations. The second part deals with some difference equations basing on some methods that have been used in solving some worked on difference equations.

Part one includes chapters 2 and 3, where part two includes chapters 4, 5 and 6.

Chapter 2 deals with first order difference equations. We focus on equilibrium points and their stability, cobweb diagrams, periodic points and cycles.

In chapter 3, we present the theory of higher order difference equations. We mainly deals with both linear homogeneous difference equations with constant coefficients and the higher order scalar difference equations. We discuss the linearization of nonlinear difference equations and the local and global stability theorems of higher order scalar difference equations.

The equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{B x_n + C x_{n-1}}, \qquad n = 0, 1, ... \tag{1.0.1}$$

was studied by El-Afifi in [20]. In chapter 4, we study the equilibrium points and the local and global stability of the solution of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{B x_n + C x_{n-k}}, \qquad n = 0, 1, ... \tag{1.0.2}$$

where the parameters $\alpha, \beta, \gamma, B, C$ and the initial conditions $x_{-k}, ..., x_{-1}, x_0$ are positive real numbers, $k = \{1, 2, 3, ...\}$.

Chapter 5 discusses the equation

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}} \tag{1.0.3}$$

where a, b, A, B are all positive real numbers, $k \geq 1$ is a positive integer, and the initial conditions $x_{-k}, x_{-k+1}, ..., x_0$ are nonnegative real numbers taking on referee what was been worked on the equation

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_n} \tag{1.0.4}$$

in [22] by Yan, Li and Zhao.

Chapter 6 deals with the equation

$$x_n = \frac{A}{x_{n-k}} + \frac{B}{x_{n-3k}}, \tag{1.0.5}$$

where $x_{-3k+1},\ x_{-3k+2}, ...,\ x_0 \in (0, \infty),\ A, B > 0$ and $k \in \{1, 2, 3, 4, ...\}$. which was studied in [4] by Douraki, Deghghan and Razzaghi.

The last section of the chapter 3 till 6 is a numerical discussion for the solution of each of our three equations. In this section a comparison between numerical solutions and the theoretical solution that we studied for each equation. All numerical solution was hold on MATLAB 6.5. All MATLAB codes are included in the thesis.

Part I

Theory of Difference Equations

Chapter 2

Dynamics of First Order Difference Equations

2.1 Introduction

Difference equations are as differential equations in Calculus. Difference equations usually describe the evolution of certain phenomena. In difference equations the term x(n+1) is related to the term x(n) and the relation is expressed in the *difference equation*

$$x(n+1) = f(x(n)) \tag{2.1.1}$$

Starting with the initial point x_0, we can generate the sequence

$$x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), ...$$

As $f^2(x_0) = f(f(x_0)) = x_2$ and $f^3(x_0) = f(f(f(x_0))) = x_3$, then in general $f^n(x_0) = x_n$. Thus we can have,

$$x(n+1) = f^{n+1}(x_0) = f[f^n(x_0)] = f(x(n))$$

This iterative procedure is an example of a *discrete dynamical system.*

The simplest case of the linear difference equation is

$$x(n+1) = ax(n) \tag{2.1.2}$$

where $x(n_0) = x_0,\ n \geq n_0 \geq 0$.

This equation is called *the linear first-order homogeneous difference equation.*

and *the linear first-order nonhomogeneous difference equation* is given by

$$x(n+1) = ax(n) + b \tag{2.1.3}$$

where $x(n_0) = x_0,\ n \geq n_0 \geq 0$.

We assumed in Eq. 2.1.2 and Eq. 2.1.3 that $a, b \neq 0$.

2.2 The Equilibrium Points

The notion of equilibrium point is very used in the dynamics of any physical system.

Definition 2.1. *A point $\bar{x}$ is said to be an equilibrium point of (2.1.1) if it is a fixed point of f; i.e.; $f(\bar{x}) = \bar{x}$.*

Example: Take the difference equation

$$x(n+1) = x^2(n) - x(n) + 1$$

So, $f(x) = x^2 - x + 1$, and by letting $\bar{x} = \bar{x}^2 - \bar{x} + 1$, we can conclude that this equation has only one equilibrium point $\bar{x} = 1$.

Graphically, an equilibrium point is the x-coordinate of the point where the graph of f intersects the diagonal line $y = x$.

Example: The equation

$$x(n+1) = x^3(n)$$

has three equilibrium points as we can see from Figure 2.2, and they are $\bar{x} = -1, 0, 1$.

One of the one main objectives in the study of a dynamical system is to analyze the behavior of its solutions near an equilibrium point. This study is called the stability theory.

Definition 2.2. *Let $\bar{x}$ be an equilibrium point of Eq.2.1.1. Then the equilibrium point $\bar{x}$ is called*

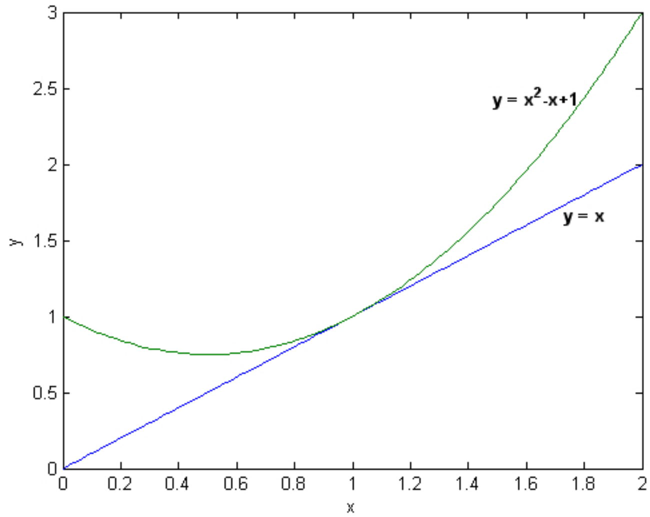

Figure 2.1: The equilibrium points of $x(n+1) = x^2(n) - x(n) + 1$

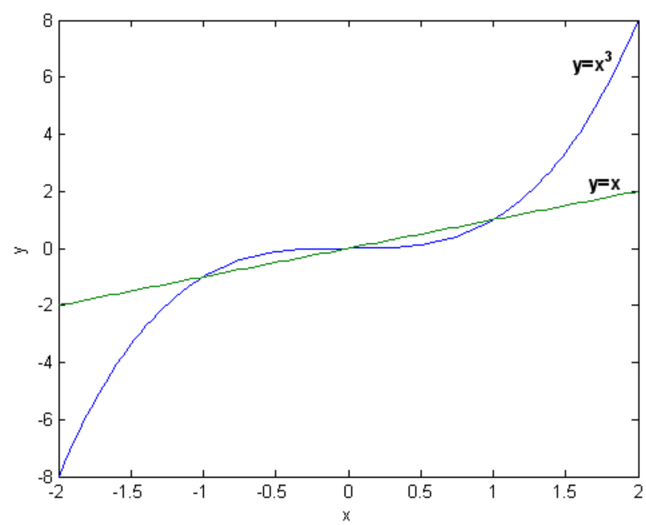

Figure 2.2: The equilibrium points of $x(n+1) = x^3(n)$

1. *locally stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that for x_0 with $|x_0 - \bar{x}| < \delta$, we have $|x_n - \bar{x}| < \epsilon$ for all $n \geq 0$,*

2. *locally asymptotically stable if it is locally stable and if there exists $\gamma > 0$ such that for x_0 with $|x_0 - \bar{x}| < \gamma$, we have $lim_{n\to\infty} x_n = \bar{x}$.*

3. *a global attractor if for any x_0, we have $lim_{n\to\infty} x_n = \bar{x}$.*

4. *globally asymptotically stable if $\bar{x}$ is locally stable and $\bar{x}$ is a global attractor.*

Example: Consider the first order difference equation

$$x(n+1) = \frac{1}{2}x(n) - 1$$

then the fixed point of the function $f(x) = \frac{1}{2}x - 1$ is the point $x = -2$, hence the equilibrium point of our difference equation is $\bar{x} = -2$. If we take our initial condition for the equation $x(0) = 1$, then as it is obvious from Figure 2.3, we can say that $\bar{x} = -2$ is a stable point, furthermore it is asymptotically stable.

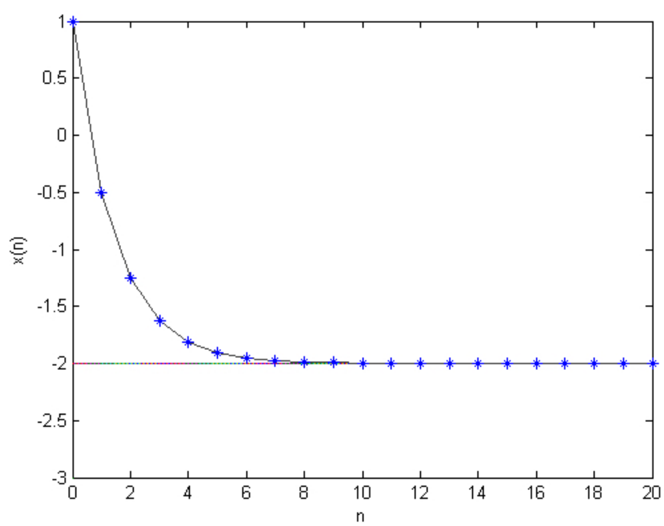

Figure 2.3: The Stability of $\bar{x} = -2$ of $x(n+1) = \frac{1}{2}x(n) - 1$

Example: Consider the difference equation

$$x(n+1) = x(n)^2$$

The equilibrium points are $\bar{x} = 0, \bar{x} = 1$. It is obvious from Figure 2.4

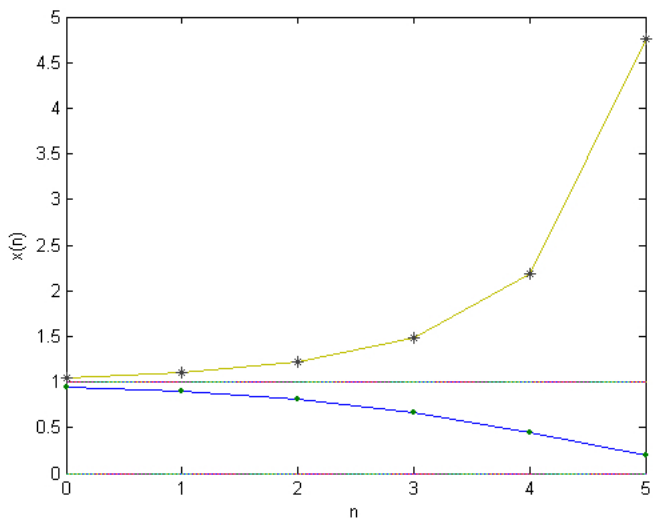

Figure 2.4: The Stability of $\bar{x} = 1$ of $x(n+1) = x(n)^2$

that $\bar{x} = 1$ is unstable equilibrium point.

2.2.1 The Cobweb Diagram

The cobweb diagram is a graphical method for analyzing the stability of the equilibrium point.We may draw the graph of f in the (x(n),x(n+1)) plane. As we choose our initial point x_0, then we can find x_1 from the graph. This could be done by drawing a vertical line from the point x_0, then find where it will intersect the graph, draw then a horizontal line now from the point (x_0, x_1) on the graph to meet the diagonal $y = x$, a vertical line from this point (x_1, x_1) will intersect the graph in the point (x_1, x_2). By continuing this procedure we can find x(n) for all $n \geq 0$.

For our first example, we will draw the cobweb diagram around the equilibrium point $\bar{x} = 0$ by taking $x_0 = 0.6$.

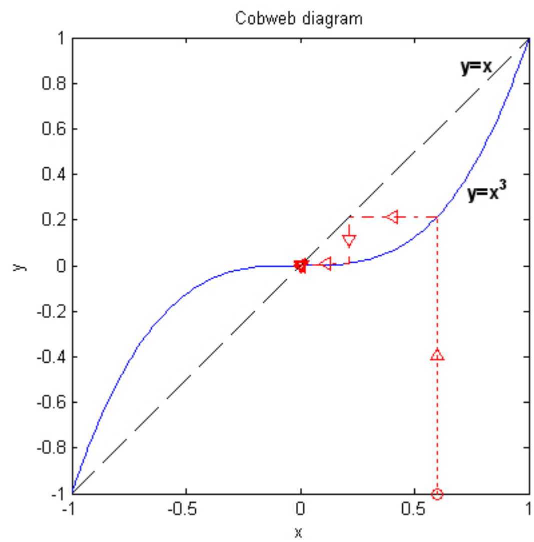

Figure 2.5: Stability of $\bar{x} = 0$ of$x(n+1) = x^3(n)$

As we can see from Figure 2.5, we can say that $\bar{x}$ is asymptotically stable.

Lets now examine the equilibrium point $\bar{x} = -1$ for the same equation using the cobweb diagram by taking $x_0 = -1.05$.

From Figure 2.6, it is obvious that $\bar{x} = -1$ is unstable.

2.3 Periodic Points

It is important for studying any dynamical system is to study its periodicity. As an example, the motion of a pendulum is periodic.

Definition 2.3. *Let b be in the domain of f. Then:*

1. *b is called a periodic point of f if for some positive integer k, $f^k(b) = b$. b is called a k-periodic point of f and it is a fixed point for the function f^k.*
   The periodic orbit of b, $O(b) = \{b, f(b), f^2(b), ..., f^{k-1}(b)\}$ and its often called a k-orbit.

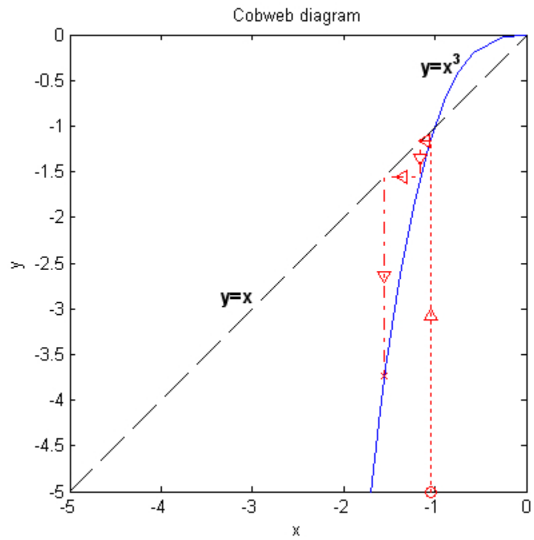

Figure 2.6: Stability of $\bar{x} = -1$ of $x(n+1) = x^3(n)$

2. *b is called eventually k-periodic if for some positive integer m, $f^m(b)$ is a k-periodic point; in other words $f^{m+k}(b) = f^m(b)$.*

We can find the k-periodic point of such a function by finding the point for which the diagonal $y = x$ intersects the graph of $f^k(x)$ and then finding the x-coordinate of such a point.

Example: Take the equation

$$x(n+1) = x^2(n)$$

Then $f(x) = x^2$, As we want to find the 2-periodic points, we must find $f^2(x)$.

We know that $f^2(x) = f(f(x)) = f(x^2) = x^4$. We will plot f^2 and see where will it intersect $y = x$.

As we see from the figure, the 2-periodic points of our equation are 0 and 1.

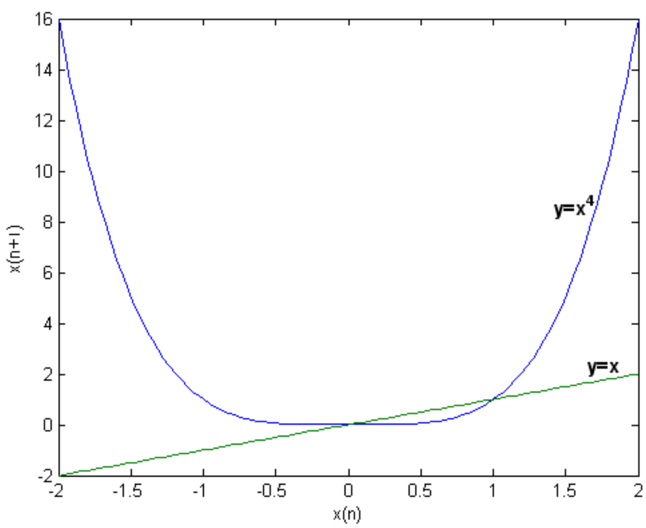

Figure 2.7: 2-periodic points of $x(n+1) = x^2(n)$

Chapter 3

Difference Equations of Higher Order

3.1 Theory of Linear Difference Equations

The general form of a kth-order *nonhomogeneous linear difference equation* is

$$y(n+k) + p_1(n)y(n+k-1) + ... + p_k(n)y(n) = g(n) \qquad (3.1.1)$$

where $p_i(n)$ and $g(n)$ are real valued functions defined for $n \geq n_0$ and $p_k(n) \neq 0$ for all $n \geq n_0$.

If g(n) is zero, then the equation is said to be kth-order *homogeneous difference equation*.

The general form of the kth-order *homogeneous difference equation* is

$$y(n+k) + p_1(n)y(n+k-1) + ... + p_k(n)y(n) = 0 \qquad (3.1.2)$$

The sequence $\{y_n\}_{n_0}^{\infty}$ is said to be a solution of Eq. 3.1.1 if it satisfies the equation.If we specify our initial conditions of the equation, that is led us to the initial value problem

$$y(n+k) + p_1(n)y(n+k-1) + ... + p_k(n)y(n) = g(n) \qquad (3.1.3)$$

$$y(n_0) = a_0, y(n_0+1) = a_1, ..., y(n_0+k-1) = a_{k-1}, \qquad (3.1.4)$$

where a_i are real numbers.

Example: Consider the 2nd order nonhomogeneous difference equation

$$y(n+2) = 2y(n+1) + 3y(n) + 5$$

where $y(0) = 1, y(1) = 2$, then we can find y(2),y(3):

$y(2) = 2y(1) + 3y(0) + 5 = 12.$
$y(3) = 2y(2) + 3y(1) + 5 = 35.$
and by the same method we can get y(4),y(5),.....

So , if we have the initial conditions , then we can find the whole solution of our difference equation.

Theorem 3.1. *[1] The initial value problem (3.1.3) and (3.1.4) have a unique solution y(n).*

Definition 3.1. *The functions $f_1(n), f_2(n), ..., f_r(n)$ are said to be linearly dependent for $n \geq n_0$ if there are nonzero constants $a_1, a_2, ..., a_r$ such that*

$$a_1 f_1(n) + a_2 f_2(n) + ... + a_r f_r(n) = 0$$

The negation of linear dependence is linear independence. Then, the set of functions $f_1(n), f_2(n), ...f_r(n)$ are said to be *linearly independent* if wherever

$$a_1 f_1(n) + a_2 f_2(n) + ... + a_r f_r(n) = 0$$

for all $n \geq n_0$, then we must have $a_1 = a_2 = ... = a_r = 0$.

Definition 3.2. *A set of k linearly independent solutions of (3.1.2) is called the fundamental set of solutions.*

Theorem 3.2. ***The Fundamental Theorem****[1]*
If $p_k(n) \neq 0$ for all $n \geq n_0$, then (3.1.2) has a fundamental set of solutions for $n \geq n_0$.

Theorem 3.3. ***Superposition Principle****[1]*
If $y_1(n), y_2(n), ..., y_r(n)$ are solutions of (3.1.2), then also

$$y(n) = a_1 y_1(n) + a_2 y_2(n) + ... + a_r y_r(n)$$

is a solution of (3.1.2), where $a_1, a_2, ..., a_r$ are real numbers.

Example: Consider the third order homogeneous difference equation

$$x(n+3) + 3x(n+2) - 4x(n+1) - 12x(n) = 0$$

where the functions $2^n, (-2)^n$ *and* $(-3)^n$ form the fundamental set of solutions of the equation.

We can verify that 2^n is a solution by substituting $x(n) = 2^n$ in the equation

$$2^{n+3} + 3(2^{n+2}) - 4(2^n) - 12(2^n) = 2^n(8 + 12 - 8 - 12) = 0$$

also, we can prove it easily for the other two functions.

From superposition principle we can say that

$$x(n) = c_1 2^n + c_2(-2)^n + c_3(-3)^n$$

where c_1, c_2 *and* c_3 are real numbers.

is also a solution to the equation, and this can be proved easily.

3.1.1 Linear Homogeneous Equations with Constant Coefficients

Consider kth-order difference equation

$$y(n+k) + p_1 y(n+k-1) + p_2 y(n+k-2) + ... + p_k y(n) = 0 \qquad (3.1.5)$$

where $p_i's$ are all real constants and $p_k \neq 0$.

Suppose that our solution is in the form of λ^n, where λ is either a real or a complex number. Substituting in (3.1.5) we get

$$\lambda^k + p_1 \lambda^{k-1} + ... + p_k = 0 \qquad (3.1.6)$$

This equation is called the *characteristic equation* of (3.1.5), and its roots λ are called the *characteristic roots*.

Example: Consider the 2nd order homogeneous difference equation

$$x(n+2) - 4x(n+1) + 3x(n) = 0$$

then its characteristic equation is

$$\lambda^2 - 4\lambda + 3 = 0$$

then the characteristic roots are $\lambda_1 = 1, \lambda_2 = 3$.

We have two cases to take into consideration.

Case One: Distinct Roots
Suppose that our characteristic roots of (3.1.5) $\lambda_1, \lambda_2, ...\lambda_k$ are distinct. The fundamental set of solutions will be $\lambda_1^n, \lambda_2^n, ..., \lambda_k^n$, and the general solution of (3.1.5) is given by

$$y(n) = \Sigma_{i=1}^{k} a_i \lambda_i^n \tag{3.1.7}$$

where $\{a_i\}$ are complex numbers.

Case Two: Repeated Roots
Suppose that the distinct characteristic roots are $\lambda_1, \lambda_2, ..., \lambda_r$ with multiplicities $m_1, m_2, ..., m_r$ with $\Sigma_{i=1}^{r} m_i = k$, respectively. In this case the general solution of (3.1.5) is given by

$$y(n) = \Sigma_{i=1}^{r} \lambda_i^n (a_{i0} + a_{i1}n + a_{i2}n^2 + ... + a_{i,m_i-1}n^{m_i-1}) \tag{3.1.8}$$

where $a_{i0}, ...a_{m_i-1}$ are complex numbers.

3.1.2 The Limiting Behavior of Solutions

Consider the ***second order homogeneous difference equation***

$$y(n+2) + p_1 y(n+1) + p_2 y(n) = 0 \tag{3.1.9}$$

then the characteristic equation of such an equation will be

$$\lambda^2 + p_1 \lambda + p_2 = 0$$

The quadratic equation has two solutions λ_1, λ_2 that break down into three cases:

Case One : Two Real Roots and Distinct:

Suppose that λ_1, λ_2 are the characteristic roots of the equation. If λ_1 and λ_2 are distinct real roots then the general solution is given by

$$y(n) = a_1 \lambda_1^n + a_2 \lambda_2^n$$

Example:Consider the equation

$$x(n+2) = x(n+1) + x(n)$$

then the characteristic equation is

$$\lambda^2 - \lambda - 1 = 0$$

The solutions to this quadratic equation are

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

then the solution will be

$$x(n) = a_1(\frac{1+\sqrt{5}}{2})^n + a_2(\frac{1-\sqrt{5}}{2})^n$$

If we assume that our initial conditions x(0)=x(1)=1 , the we can get easily that our solution is

$$x(n) = (\frac{\sqrt{5}+1}{2\sqrt{5}})(\frac{1+\sqrt{5}}{2})^n + (\frac{\sqrt{5}-1}{2\sqrt{5}})(\frac{1-\sqrt{5}}{2})^n$$

Without loss of generality we can assume that the two distinct roots of our equation satisfy

$$|\lambda_1| > |\lambda_2|$$

and by so λ_1 is called the *dominant root* and $y_1(n)$ the *dominant solution.*

The general solution could be written now as

$$y(n) = \lambda_1^n(a_1 + a_2(\frac{\lambda_2}{\lambda_1})^n)$$

It is obvious that the $\lim_{n\to\infty} y(n) = \lim_{n\to\infty} a_1\lambda_1^n$ since as $\frac{\lambda_2}{\lambda_1} < 1$, then $\lim_{n\to\infty} \frac{\lambda_2}{\lambda_1} = 0$.

It is easy to conclude that

1. If $|\lambda_1| > 1$, then the solution y(n) will diverge.
2. If $|\lambda_1| = 1$, then the solution will be a constant solution.
3. If $|\lambda_1| < 1$, then the solution will converge to zero.

Case Two: One Real Repeated Root

Suppose that λ_1, λ_2 are the two characteristic roots of the equation and suppose that $\lambda_1 = \lambda_2 = \lambda$, the the general solution will be

$$y(n) = \lambda^n (a_1 + a_2 n)$$

Example: Consider the equation

$$x(n+2) + 2x(n+1) + (n) = 0$$

then the characteristic equation is

$$\lambda^2 + 2\lambda + 1 = 0$$

which has the solution $\lambda = -1$, then the solution is of the form

$$x(n) = a_1(-1)^n + a_2 n(-1)^n$$

It is obvious that $\lim_{n\to\infty} y(n) = \lim_{n\to\infty} \lambda^n (a_1 + a_2 n)$, so we can conclude easily now that

1. If $|\lambda| \geq 1$, then the solution y(n) will diverge.
2. If $|\lambda| < 1$, then the solution will converge to zero since $\lim_{n\to\infty} n\lambda^n = 0$.

Case Three: Two Complex Roots

The last case that we will consider for our equation, is when the roots λ_1 and λ_2 are to be complex roots. Set $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. The general solution will be

$$y(n) = a_1(\alpha + i\beta)^n + a_2(\alpha - i\beta)^n$$

In polar coordinates the complex number $\alpha + i\beta$ could be written as

$$r = \sqrt{\alpha^2 + \beta^2},\ \alpha = r\cos\theta,\ \beta = r\sin\theta,\ \theta = \tan^{-1}(\frac{\beta}{\alpha})$$

Hence

$$\begin{aligned} y(n) &= a_1(r\cos\theta + ir\sin\theta)^n + a_2(r\cos\theta - i\sin\theta)^n \\ &= r^n((a_1+a_2)\cos(n\theta) + i(a_1-a_2)sin(n\theta)) \\ &= r^n(c_1\cos(n\theta) + c_2\sin(n\theta)). \end{aligned}$$

where $c_1 = a_1 + a_2$ and $c_2 = a_1 - a_2$.

Let

$$\cos\mu = \frac{c_1}{\sqrt{c_1^2+c_2^2}},\ \sin\mu = \frac{c_2}{\sqrt{c_1^2+c_2^2}},\ \mu = \tan^{-1}(\frac{c_2}{c_1})$$

Then we can write the solution as

$$y(n) = Cr^n\cos(n\theta - \mu),\ C = \sqrt{c_1^2+c_2^2} \tag{3.1.10}$$

Example: Consider the equation

$$x(n+2) + 2x(n+1) + 5x(n) = 0$$

the characteristic equation will be

$$\lambda^2 + 2\lambda + 5 = 0$$

which gives the solution

$$\lambda = 1 \pm 2i$$

then $r = \sqrt{5}$ and $\theta = \tan^{-1}(\frac{1}{2})$.

the real formed solution will be

$$x(n) = 5^{\frac{n}{2}}(c_1\cos(n\theta) + c_2\sin(n\theta))$$

The solution y(n) clearly is oscillating since the cosine function oscillates. But this oscillation can have three different cases

1. If $r < 1$, then λ_1 and $\lambda_2 = \overline{\lambda_1}$ lie inside the unitary disk and our solution will converge to zero.

2. If $r = 1$, then λ_1 and $\lambda_2 = \overline{\lambda_1}$ lie on the unitary disk and the solution will oscillate in constant magnitude.

3. If $r > 1$, then λ_1 and $\lambda_2 = \overline{\lambda_1}$ lie outside the unitary disk and the solution will diverge.

Theorem 3.4. *[1] The following statements hold :*

1. *All solutions of (3.1.9) oscillate about zero if and only if the characteristic equation has no positive real roots.*
2. *All solutions of (3.1.9) converge to zero if and only if*

$$max\{|\lambda_1|, |\lambda_2|\} < 1$$

Consider the ***second order nonhomogeneous difference equation***

$$y(n+2) + p_1 y(n+1) + p_2 y(n) = M \tag{3.1.11}$$

where M is nonzero. Suppose that y^* is an equilibrium point of such equation, then

$$y^* + p_1 y^* + p_2 y^* = M$$

then

$$y^* = \frac{M}{1 + p_1 + p_2}$$

But as we know, the general equation of nonhomogeneous equations is

$$y(n) = y_p(n) + y_c(n) \tag{3.1.12}$$

and for this equation we can take $y_p(n) = y^*$. Thus we can conclude the following theorem.

Theorem 3.5. *[1] The following statements hold :*

1. *All solutions of (3.1.11) oscillate about y^* if and only if the* ***characteristic homogeneous equation of (3.1.9)*** *has no positive real roots.*
2. *All solutions of (3.1.11) converge to y^* if and only if $max\{|\lambda_1|, |\lambda_2|\} < 1$ where λ_1 and λ_2 are the roots of the* ***homogeneous characteristic equation of (3.1.9)****.*

3.2 Higher-Order Scalar Difference Equations

In this section we will take into consideration some stated theorems that will be so useful for us in our study of higher-order difference equation. We concentrate on the the theorems that study the local and global asymptotical stability.

3.2.1 Asymptotic Stability Theorems of Linear Scalar Equations

Consider the second order difference equation

$$x(n+2) + p_1 x(n+1) + p_2 x(n) = 0 \tag{3.2.1}$$

the characteristic equation is

$$\lambda^2 + p_1\lambda + p_2 = 0 \tag{3.2.2}$$

Theorem 3.6. *[1] The conditions*

$$1 + p_1 + p_2 > 0, \quad 1 - p_1 + p_2 > 0, \quad 1 - p_2 > 0$$

are sufficient and necessary conditions for the equilibrium point of equations (3.2.1) and (3.1.11) to be asymptotically stable.

These conditions can be written as

$$|p_1| < 1 + p_2 < 2$$

Theorem 3.7. *[1] The zero solution of (3.2.1) is asymptotically stable if and only if*

$$|p_1| < 1 + p_2 < 2$$

Theorem 3.8. *[1] The zero solution of the third order homogeneous difference equation*

$$x(n+3) + p_1 x(n+2) + p_2 x(n+1) + p_3 x(n) = 0 \tag{3.2.3}$$

will be asymptotically stable if and only if

$$|p_1 + p_3| < 1 + p_2, \ \textit{and} \ |p_2 - p_1 p_3| < 1 - p_3^2$$

Consider the kth-order equation

$$x(n+1) - ax(n) + bx(n-k) \tag{3.2.4}$$

Theorem 3.9. *[1] Let a be a nonnegative real number, b an arbitrary real number, and k a positive integer. The zero solution of (3.2.4) is asymptotically stable if and only if*

$$|a| < \frac{(k+1)}{k}$$

and

1. $|a| < b < (a^2 + 1 - 2|a|\cos\phi)^{\frac{1}{2}}$ *for k odd, or*
2. $|b - a| < 1$ *and* $|b| < (a^2 + 1 - 2|a|\cos\phi)^{\frac{1}{2}}$ *for k even,* *where ϕ is the solution in $(0, \frac{\pi}{(k+1)})$ of*

$$\frac{sin(k\theta)}{\sin(k+1)\theta} = \frac{1}{|a|}$$

Lets be back to the general form of the kth-order homogeneous difference equation

$$x(n+k) + p_1x(n+k-1) + p_2x(n+k-2) + ... + p_{k-1}x(n+1) + p_kx(n) = 0 \tag{3.2.5}$$

Theorem 3.10. *[1] The zero solution of (3.2.5) is asymptotically stable if*

$$\Sigma_{i=1}^{k} |p_i| < 1 \tag{3.2.6}$$

and the zero solution of this equation is unstable if

$$|p_1| - \Sigma_{i=2}^{k} |p_i| > 1 \tag{3.2.7}$$

3.2.2 Linearization of Nonlinear Equations

Consider the k+1-order difference equation of the form

$$x(n+1) = f(x(n), x(n-1), ..., x(n-k)) \tag{3.2.8}$$

where $f : I^{k+1} \longrightarrow I$ is a continuous function, and $x_{-k}, x_{-k+1}, ..., x_0$ are the initial conditions. So there exists a unique solution $\{x(n)\}_{n=-k}^{\infty}$ such that $x(-k) = x_{-k}, x(-k+1) = x_{-k+1}, ..., x(0) = x_0$.

Definition 3.3. *A point $x^* \in I$ is an equilibrium point of (3.2.8) if*

$$f(x^*, x^*, ..., x^*) = x^*$$

Definition 3.4. *An equilibrium point x^* of (3.2.8) is stable if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that if*

$$(|x(-k) - x^*| + |x(-k+1) - x^*| + ... + |x(0) - x^*|) < \delta,$$

then

$$|x(n) - x^*| < \varepsilon \text{ for all } n \geq -k.$$

If f is continuously differentiable in some neighborhood around x^*, then we can linearize (3.2.8) around x^*. Thus, by chain rule, the linearized equation around x^* becomes

$$u(n+1) = p_0 u(n) + p_1 u(n-1) + ... + p_k u(n-k) \tag{3.2.9}$$

where

$$p_i = \frac{\partial f}{\partial u_i}(x^*, x^*, ..., x^*)$$

The characteristic equation of (3.2.9) is given by

$$\lambda^{k+1} - p_0 \lambda^k - p_1 \lambda^{k-1} - .... - p_k = 0 \tag{3.2.10}$$

Example: Consider the nonlinear difference equation

$$x(n+2) = 2x(n+1)^2 - \frac{1}{x(n)}$$

We can find the equilibrium point by

$$\bar{x} = 2\bar{x}^2 - \frac{1}{\bar{x}}$$

this can be written as

$$2\bar{x}^3 - \bar{x}^2 - 1 = 0$$

which is

$$(\bar{x} - 1)(2\bar{x}^2 + \bar{x} + 1) = 0$$

Then the equilibrium point of this equation is $\bar{x} = 1$.

Take $f(u,v) = 2u^2 - \frac{1}{v}$ then

$$\frac{\partial f}{\partial u} = 4u, \qquad \frac{\partial f}{\partial v} = \frac{1}{v^2}$$

So , our linearized equation around the equilibrium point $\bar{x} = 1$ is

$$z(n+2) = 4z(n+1) + z(n)$$

which can be written as

$$z(n+2) - 4z(n+1) - z(n) = 0$$

Theorem 3.11. ***(The Linearized Stability Result)***
Suppose that f is continuously differentiable on an open neighborhood around $(x^*, x^*, ..., x^*)$, where x^* is an equilibrium point of (3.2.8). Then the following statements are true:

1. *If all the characteristic roots of (3.2.10) lie inside the unit disk in the complex plane, then the equilibrium point x^* of (3.2.8) is locally asymptotically stable.*

2. *If at least one characteristic root of (3.2.10) is outside the unit disk in the complex plane, the the equilibrium point x^* of (3.2.8) is unstable.*

3. *If one of the characteristic roots of (3.2.10) lies on the unit disk and all the other roots lie either inside or on the unit disk in the complex plane, then the equilibrium point x^* of (3.2.8) may be stable, unstable, or asymptotically stable.*

3.2.3 The Global Stability Theorems of Nonlinear Equations

Consider the difference equation of order (k+1)

$$x(n+1) = f(x(n), x(n-1), ..., x(n-k)) \qquad (3.2.11)$$

where f is continuously differentiable on I and $x^* \in I$.

Theorem 3.12. *Let x^* be an equilibrium point of (3.2.11) and let f be satisfying the following:*

1. *f is nonincreasing in each of the arguments; that means that if $a \leq b$ then $f(..., a, ...) \leq f(..., b, ...)$.*

2. *$(u - x^*)[f(u, u, ..., u) - u] < 0$ for all $u \in I$ and $u \neq x^*$.*

then with all the initial conditions $(x(0), x(-1), ..., x(-k)) \in I$, we have all $x(n) \in I$ for all $n \geq -k$, also $\lim_{n \longrightarrow \infty} x(n) = x^*$.

Definition 3.5. *The function $f(u_1, u_2, ..., u_{k+1})$ is said to be weakly monotonic if f is nondecreasing or nonincreasing in each of its arguments; that means if $a \leq b$ then either*

$$f(..., a, ...) \leq f(..., b, ...) \text{ or } f(..., a, ...) \geq f(..., b, ...)$$

where a and b lie in the jth slot, where $1 \leq j \leq k + 1$ and all other slots are filled with some fixed numbers $z_1, z_2, ..., z_{j-1}, z_{j+1}, ..., z_{k+1}$.

The following theorems was discussed by Elaydi in [1], and they are the most useful theorems that we depend upon.

Theorem 3.13. *Consider the difference equation*

$$x_{n+1} = \sum\nolimits_{i=0}^{k} x_{n-i} F_i(x_n, x_{n-1}, ..., x_{n-k}) n = 0, 1, ... \tag{3.2.12}$$

with initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in [0, \infty)$, where

1. *$k \in \{1, 2, ...\}$;*

2. *$F_0, F_1, ..., F_k \in C\left[[0, \infty)^{k+1}, [0, 1)\right]$;*

3. *$F_0, F_1, ..., F_k$ are nonincreasing in each argument;*

4. *$\sum_{i=0}^{k} F_i(y_0, y_1, ..., y_k) < 1$ for all $(y_0, y_1, ..., y_k) \in (0, 1)^k$;*

5. *$F_0(y, y, ..., y) > 0$ for all $y \geq 0$.*

Then, $\bar{x} = 0$ is globally asymptotically stable for such equation.

Theorem 3.14. *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}); n = 0, 1, ... \tag{3.2.13}$$

where $k \in \{1, 2, ...\}$. Let $I = [a, b]$ be some interval of real numbers and assume that

$$f : [a, b] \times [a, b] \rightarrow [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(u, v)$ *is nondecreasing in* u *and nondecreasing in* v.

(b) If $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$m = f(m, m) \text{ and } M = f(M, M),$$

then $m = M$. *Then Eq.3.2.13 has a unique equilibrium* $\bar{y}$ *and every solution of Eq.3.2.13 converges to* $\bar{y}$.

Theorem 3.15. *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}); n = 0, 1, ... \tag{3.2.14}$$

where $k \in \{1, 2, ...\}$. *Let* $I = [a, b]$ *be some interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(u, v)$ *is nondecreasing in* u *and nonincreasing in* v.

(b) If $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$m = f(m, M) \text{ and } M = f(M, m),$$

then $m = M$. *Then Eq.3.2.14 has a unique equilibrium* $\bar{y}$ *and every solution of Eq.3.2.14 converges to* $\bar{y}$.

Part II

A Study of Some Nonlinear Difference Equations

Chapter 4

Dynamics of a Rational Difference Equation $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-k}}{Bx_n+Cx_{n-k}}$

4.1 Introduction

In this chapter, we will study the nonlinear rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \qquad n = 0, 1, ... \qquad (4.1.1)$$

where the parameters $\alpha, \beta, \gamma, B, C$ and the initial conditions $x_{-k}, ..., x_{-1}, x_0$ are positive real numbers, $k = \{1, 2, 3, ...\}$.

Our concentration is on invariant intervals, periodic character, the character of semi-cycles and global asymptotic stability of all positive solutions of Eq.4.1.1.
It is worth mentioning that the results in [20],[11],[12],[14],[17] are special cases of our main results.

The global stability of Eq.4.1.1 for $k = 1$ has been investigated in [20]. They showed, in respect to variation of the parameters, the positive equilibrium point is globally asymptotically stable or every solution lies eventually in an invariant interval. Kulenovic and Ladas, in addition, considered Eq.4.1.1

in their monograph [16].

Dehghan in [18] investigated the global stability, invariant intervals, the character of semi-cycles, and the boundedness of the equation

$$x_{n+1} = \frac{x_n + p}{x_n + qx_{n-k}}, \quad n = 0, 1, 2, ... \tag{4.1.2}$$

where the parameters p and q and the initial conditions $x_{-k}, ..., x_{-1}$and x_0 are positive real numbers, $k = \{1, 2, 3, ...\}$.

Li and Sun [17] investigated the periodic character, invariant intervals, oscillation and global stability of all positive solutions of the equation

$$x_{n+1} = \frac{px_n + x_{n-k}}{q + x_{n-k}}, \quad n = 0, 1, 2, ... \tag{4.1.3}$$

where the parameters p and q and the initial conditions $x_{-k}, ..., x_{-1}$and x_0 are positive real numbers, $k = \{1, 2, 3, ...\}$.

DeVault [12] investigated the global stability and the periodic character of solutions of the equation

$$x_{n+1} = \frac{p + x_{n-k}}{qx_n + x_{n-k}}, \quad n = 0, 1, 2, ... \tag{4.1.4}$$

where the parameters p and q and the initial conditions $x_{-k}, ..., x_{-1}$and x_0 are positive real numbers, $k = \{1, 2, 3, ...\}$.

In [20] , the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{Bx_n + Cx_{n-1}} \tag{4.1.5}$$

was studied by M.M. El-Afifi. He studied the local and global stability and the semi-cycles of this equation.
Our interest now to study and solve Eq.4.1.1 in the general case.

The change of variable $x_n = \frac{\beta}{B} y_n$ in [20] changes the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{Bx_n + Cx_{n-1}}, \qquad n = 0, 1, ...$$

into the equation

$$y_{n+1} = \frac{P + y_n + l y_{n-1}}{y_n + q y_{n-1}}, \qquad n = 0, 1, ... \tag{4.1.6}$$

where $P = \frac{B\alpha}{\beta^2}$, $l = \frac{\gamma}{\beta}$, $q = \frac{C}{B}$.

This change of variable works for our equation also and we will show that. Let $x_n = \frac{\beta}{B} y_n$, then the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{B x_n + C x_{n-k}}, \qquad n = 0, 1, ...$$

can be changed to become

$$\frac{\beta}{B} y_{n+1} = \frac{\alpha + \frac{\beta^2}{B} y_n + \frac{\gamma\beta}{B} y_{n-k}}{\beta y_n + \frac{C\beta}{B} y_{n-k}}$$

then

$$y_{n+1} = \frac{\alpha B^2 + B\beta^2 y_n + \gamma\beta B y_{n-k}}{\beta^2 B y_n + C\beta^2 y_{n-k}}$$

taking out the term $\beta^2 B$ from the dominator , we get

$$y_{n+1} = \frac{\frac{B\alpha}{\beta^2} + y_n + \frac{\gamma}{\beta} y_{n-k}}{y_n + \frac{C}{B} y_{n-k}}$$

then

$$y_{n+1} = \frac{D + y_n + p y_{n-k}}{y_n + q y_{n-k}}, \qquad n = 0, 1, ... \tag{4.1.7}$$

where $D = \frac{B\alpha}{\beta^2}$, $p = \frac{\gamma}{\beta}$, $q = \frac{C}{B}$.

Before studying the behavior of solutions of this rational difference equation, we will review some definitions and basic results that will be used throughout this chapter.

Lemma 4.1. [*1*][*15*] *Let I be some interval of real numbers and let*

$$f : I^{k+1} \to I$$

be a continuously differentiable function .Then for every initial conditions $x_{-k}, ..., x_{-1}, x_0 \in I$, $k = \{1, 2, 3, ...\}$. , the difference equation

$$x_{n+1} = f(x_n, x_{n-k}), \quad n = 0, 1, ... \tag{4.1.8}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 4.1. *We say that a solution* $\{y_n\}_{n=-k}^{\infty}$ *of a difference equation* $y_{n+1} = f(y_n, y_{n-1}, ..., y_{n-k})$ *is periodic if there exists a positive integer p such that* $y_{n+p} = y_n$. *The smallest such positive integer p is called the prime period of the solution of the difference equation.*

Definition 4.2. *The equilibrium point* $\bar{y}$ *of the equation*

$$y_{n+1} = f(y_n, y_{n-1}, ..., y_{n-k}), n = 0, 1, ... \tag{4.1.9}$$

is the point that satisfies the condition

$$\bar{y} = f(\bar{y}, \bar{y}, ..., \bar{y}).$$

Definition 4.3. *[1] Let* $\bar{y}$ *be an equilibrium point of Eq.4.1.9. Then the equilibrium point* $\bar{y}$ *is called*

1. *locally stable if for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all* $y_{-k}, y_{-k+1}, ..., y_0 \in I$ *with* $|y_{-k} - \bar{y}| + |y_{-k+1} - \bar{y}| + ... + |y_0 - \bar{y}| < \delta$, *we have* $|y_n - \bar{y}| < \epsilon$ *for all* $n \geq -k$,
2. *locally asymptotically stable if it is locally stable and if there exists* $\gamma > 0$ *such that for all* $y_{-k}, y_{-k+1}, ..., y_0 \in I$ *with* $|y_{-k} - \bar{y}| + |y_{-k+1} - \bar{y}| + ... + |y_0 - \bar{y}| < \gamma$, *we have* $lim_{n\to\infty} y_n = \bar{y}$.
3. *a global attractor if for all* $y_{-k}, y_{-k+1}, ..., y_0 \in I$, *we have* $lim_{n\to\infty} y_n = \bar{y}$.
4. *globally asymptotically stable if* $\bar{y}$ *is locally stable and* $\bar{y}$ *is a global attractor.*

Let

$$p = \frac{\partial f}{\partial u}(\bar{x}, \bar{x}) \quad and \quad q = \frac{\partial f}{\partial v}(\bar{x}, \bar{x})$$

where $f(u, v)$ is the function in Eq.4.1.8 and $\bar{x}$ is an equilibrium of the equation. Then the equation

$$y_{n+1} = p y_n + q y_{n-k}, \qquad n = 0, 1, ... \tag{4.1.10}$$

is called the linearized equation associated with Eq.4.1.8 about the equilibrium point $\bar{x}$. Its characteristic equation is

$$\lambda^{k+1} - p\lambda^{k} - q = 0 \tag{4.1.11}$$

Theorem 4.1. *[1][11][14][15][16]* ***Linearized Stability****. Consider the difference equation*

$$y_{n+1} = py_n + qy_{n-k}, \qquad n = 0, 1, ...$$

(a) If both roots of the equation have absolute values less than one, then the equilibrium $\bar{y}$ of the equation is locally asymptotically stable.

(b)If at least one of the roots of the equation has an absolute value greater than one, then $\bar{y}$ is unstable .

(c)Both roots of the equation have absolute values less than one if and only if

$$|p| < 1 - q < 2$$

in this case, $\bar{y}$ is a locally asymptotically stable.

(d)Both roots of the equation have absolute values greater than one if and only if

$$|q| > 1 \qquad \text{and} \qquad |p| > |1 - q|$$

in this case, $\bar{y}$ is a repeller.

(e)One root of the equation has an absolute value greater than one while the other root has an absolute value less than one if and only if

$$p^2 + 4p > 0 \qquad \text{and} \qquad |p| > |1 - q| \, .$$

in this case, $\bar{y}$ is unstable and is called a saddle point.

4.2 The Equilibrium Points

Next we investigate the equilibrium points of the nonlinear rational difference equation

$$y_{n+1} = \frac{D + y_n + py_{n-k}}{y_n + qy_{n-k}}, \qquad n = 0, 1, ... \tag{4.2.1}$$

where the parameters p, q, D and the initial conditions $y_{-k}, ..., \; y_{-1}, y_0$ are positive real numbers, $k = \{1, 2, 3, ...\}$.

The equilibrium points of Eq.4.2.1 are the positive solutions of the equation

$$\begin{aligned}\bar{y} &= \frac{D+\bar{y}+p\bar{y}}{\bar{y}+q\bar{y}} \\ &= \frac{D+\bar{y}(p+1)}{\bar{y}(q+1)}\end{aligned}$$

hence the equilibrium point is given by

$$\bar{y} = \frac{1+p+\sqrt{(1+p)^2+4D(1+q)}}{2(q+1)}$$

To find the linearization of our problem, consider

$$f\left(u,v\right) = \frac{D+u+pv}{u+qv}$$

now,

$$\frac{\partial f}{\partial u} = \frac{(u+qv)-(D+u+pv)}{(u+qv)^2}$$

so

$$\frac{\partial f}{\partial u} = \frac{v(q-p)-D}{(u+qv)^2}$$

hence

$$\frac{\partial f}{\partial u}(\bar{y},\bar{y}) = \frac{\bar{y}(q-p)-D}{(q\bar{y}+\bar{y})^2}$$

also

$$\frac{\partial f}{\partial v} = \frac{u(p-q)-qD}{(u+qv)^2}$$

hence

$$\frac{\partial f}{\partial v}(\bar{y},\bar{y}) = \frac{\bar{y}(p-q)-qD}{(q\bar{y}+\bar{y})^2}$$

so, the linearized equation is

$$z_{n+1} = \frac{\bar{y}(q-p)-D}{(q\bar{y}+\bar{y})^2}z_n + \frac{\bar{y}(p-q)-qD}{(q\bar{y}+\bar{y})^2}z_{n-k} \tag{4.2.2}$$

and its characteristic equation is

$$\lambda^{k+1} - \frac{\bar{y}(q-p) - D}{(q\bar{y}+\bar{y})^2}\lambda^k + \frac{\bar{y}(q-p) - qD}{(q\bar{y}+\bar{y})^2} = 0 \tag{4.2.3}$$

4.3 The Local Stability

The following facts are important to study the local stability

Lemma 4.2. *[1] Assume that a,b are real numbers and $k \in \{1, 2, ...\}$. Then*

$$|a| + |b| < 1 \tag{4.3.1}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+1} + ay_n + by_{n-k} = 0,\ n = 0, 1, ... \tag{4.3.2}$$

Suppose in addition that one of the following two cases holds.

1. *k odd and $b < 0$.*
2. *k even and $ab < 0$.*

Then 4.3.1 is also a necessary condition for the asymptotic stability of Eq.4.3.3.

Lemma 4.3. *[1] Assume that a,b are real numbers. Then*

$$|a| < b + 1 < 2$$

is a necessary and sufficient condition for the asymptotic stability of the difference equation

$$y_{n+k} + ay_n + by_{n-k} = 0,\ n = 0, 1, ... \tag{4.3.3}$$

Lemma 4.4. *Assume that all the roots of the characteristic equation of the above equation lie inside the unit circle, then the positive equilibrium is locally asymptotically stable.*

Theorem 4.2. *Let $\bar{y}$ be an equilibrium point of Eq.4.2.1, then $\bar{y}$ is locally asymptotically stable.*

Proof. The equilibrium point $\bar{y}$ of Eq.4.2.1 is

$$\bar{y} = \frac{1 + p + \sqrt{(1+p)^2 + 4D(1+q)}}{2(q+1)}$$

and the linearized equation about it is

$$z_{n+1} = \frac{\bar{y}(q-p) - D}{(q\bar{y} + \bar{y})^2} z_n + \frac{\bar{y}(p-q) - qD}{(q\bar{y} + \bar{y})^2} z_{n-k}$$

We will use Lemma 4.3 to show that $\bar{y}$ is asymptotically stable. From our linearized equation we have

$$a = -\frac{\bar{y}(q-p) - D}{(q\bar{y} + \bar{y})^2}$$

$$b = \frac{\bar{y}(q-p) + qD}{(q\bar{y} + \bar{y})^2}$$

But as

$$(\bar{y} + q\bar{y})^2 = (1+q)(D + (1+p)\bar{y})$$

Then

$$a = -\frac{\bar{y}(q-p) - D}{(1+q)(D + (1+p)\bar{y})}$$

$$b = \frac{\bar{y}(q-p) + qD}{(1+q)(D + (1+p)\bar{y})}$$

Its easy to show that

$$|a| < b < b + 1$$

and as

$$b < 1$$

Then we proved that

$$|a| < b + 1 < 2$$

Then by Lemma 4.3 we can say that the equilibrium point $\bar{y}$ is asymptotically stable.

This completes the proof. □

4.4 Analysis of Semi-Cycles

Definition 4.4. *We say that a solution $\{y_n\}$ of a difference equation*

$$y_{n+1} = f(y_n, y_{n-1}, ..., y_{n-k}) \tag{4.4.1}$$

is bounded and persists if there exist positive constants P and Q such that

$$P \le x_n \le Q \text{ for } n = -1, 0, \dots .$$

Definition 4.5. *A positive semi-cycle of a solution $\{y_n\}$ of Eq.4.4.1 consists of a "string" of terms $\{y_l, y_{l+1}, \dots, y_m\}$, all greater than or equal to the equilibrium $\bar{y}$, with $l \ge -k$ and $m \le \infty$ and such that*

$$\text{either } l = -k, \text{ or } l > -k \text{ and } y_{l-1} < \bar{y}$$

and

$$\text{either } m = \infty, \text{ or } m < \infty \text{ and } y_{m+1} < \bar{y}.$$

Definition 4.6. *A negative semi-cycle of a solution $\{y_n\}$ of Eq.4.4.1 consists of a "string" of terms $\{y_l, y_{l+1}, \dots, y_m\}$, all less than the equilibrium $\bar{y}$, with $l \ge -k$ and $m \le \infty$ and such that*

$$\text{either } l = -k, \text{ or } l > -k \text{ and } y_{l-1} \ge \bar{y}$$

and

$$\text{either } m = \infty, \text{ or } m < \infty \text{ and } y_{m+1} \ge \bar{y}.$$

The first semi-cycle of a solution starts with the term y_{-k} and is positive if $y_{-k} \ge \bar{y}$ and negative if $y_{-k} < \bar{y}$.

Definition 4.7. *A solution $\{y_n\}$ of Eq.4.4.1 is called nonoscillatory if there exists $N \ge -k$ such that $y_n > \bar{y}$ for all $n \ge N$ or $y_n < \bar{y}$ for all $n \ge N$.*

And a solution $\{y_n\}$ is called oscillatory if it is not nonoscillatory.

Now, we will list some theorems which will be useful in our investigation.

Theorem 4.3. *[15] Assume that $f \in [(0,\infty) \times (0,\infty), (0,\infty)]$ is such that: $f(x,y)$ is increasing in x for each fixed y, and $f(x,y)$ is decreasing in y for each fixed x. Let $\bar{x}$ be a positive equilibrium of equation*

$$x_{n+1} = f(x_n, x_{n-k}) \tag{4.4.2}$$

Then except possibly for the first semi-cycle, every oscillatory solution of Eq.4.4.2 has semi-cycle of length at least k. Furthermore, if we assume that

$$f(u,u) = \bar{x} \tag{4.4.3}$$

and

$$f(x,y) < x \quad \text{for every } \bar{x} < y < x \tag{4.4.4}$$

then $\{x_n\}$ *oscillates about the equilibrium* $\bar{x}$ *with semi-cycles of length* $k+1$ *or* $k+2$*, except possibly for the first semi-cycle which may have length* k*. The extreme in each semi-cycle occurs in the first term if the semi-cycle has two terms and in the second term if the semi-cycle has three terms, and in the* $k+1$ *term if the semi-cycle has* $k+2$ *terms.*

Theorem 4.4. *[15] Assume that* $f \in [(0,\infty) \times (0,\infty), (0,\infty)]$ *is such that:* $f(x,y)$ *is increasing in* x *for each fixed* y*, and* $f(x,y)$ *is increasing in* y *for each fixed* x*. Let* $\bar{x}$ *be a positive equilibrium of Eq.4.4.2. Then except possibly for the first semi-cycle, every oscillatory solution of Eq.4.4.2 has semi-cycle of length* k*.*

Now, we give necessary and sufficient condition for Eq.4.2.1 to have a prime period-two solution and we exhibit all prime period-two solutions of the equation.

Theorem 4.5. *If* k *is even, then Eq. 4.2.1 has no nonnegative prime period-two solution.*

Proof. Let k is even. Assume for the sake of contradiction that there exist distinctive nonnegative real number Φ and Ψ, such that

$$..., \Phi, \Psi, \Phi, \Psi, ...$$

is a prime period two solution of Eq.4.2.1, then Φ and Ψ satisfy

$$\Phi = \frac{D + p\Psi + \Psi}{q\Psi + \Psi},$$

and

$$\Psi = \frac{D + p\Phi + \Phi}{q\Phi + \Phi}.$$

so

$$\Phi = \frac{\Psi(p+1) + D}{\Psi(q+1)}$$

and

$$\Psi \quad = \quad \frac{D+\Phi(p+1)}{\Phi(q+1)}$$

By substituting Φ into the equation of Ψ we get easily that

$$\Psi(q+1)(D+(p+1)\Psi) = D\Psi(q+1) + (D+\Psi(p+1))(p+1)$$

then we get that

$$\Psi(\Psi-1) = \frac{D}{q+1}$$

After solving the quadratic equation

$$\Psi^2 - \Psi - \frac{D}{q+1} = 0$$

we get that

$$\Psi = \frac{1 \mp \sqrt{1+\frac{4D}{q+1}}}{2}$$

but as $\sqrt{1+\frac{4D}{q+1}} > 1$ and Ψ is nonnegative , then

$$\Psi = \frac{1 + \sqrt{1+\frac{4D}{q+1}}}{2}$$

The same could be done for Φ, and then

$$\Phi = \frac{1 + \sqrt{1+\frac{4D}{q+1}}}{2}$$

so $\Phi = \Psi$. This contradicts the hypothesis that Φ and Ψ distinct nonnegative real number.
Thus there exists no prime periodic-two solution for the Eq.4.2.1.

$\square$

Theorem 4.6. *If k is odd, then Eq. 4.2.1 has no nonnegative prime period-two solution.*

Proof. Let k is odd. Assume for the sake of contradiction that there exist distinctive nonnegative real number Φ and Ψ, such that

$$..., \Phi, \Psi, \Phi, \Psi, ...$$

is a prime period two solution of Eq.4.2.1, then Φ and Ψ satisfy

$$\Phi = \frac{D + \Psi + p\Phi}{\Psi + q\Phi}$$

and

$$\Psi = \frac{D + \Phi + p\Psi}{\Phi + q\Psi}$$

Solving such two equations using MATLAB will show that the two solutions are identical, then as a result

$$\Phi = \Psi$$

and that is contradicts the fact that Ψ and Ψ must be different.
Then Eq. 4.2.1 has no prime two-periodic solution if k is odd.
This completes the proof. □

Then we can get out this results.

Corollary 4.1. *Eq. 4.2.1 possess no prime periodic-two solution.*

Semi-cycle analysis of the solution of Eq.4.2.1 is a powerful tool for a detailed understanding of the entire character of solutions.

Next, we present some results about the semi-cycle character of solutions of Eq.4.2.1.

Theorem 4.7. *Let $\{y_n\}$ be a nontrivial solution of Eq.4.2.1, then the following statements are true:*

1. *Assume $D + p > q$, then $\{y_n\}$ oscillates about the equilibrium $\bar{y}$ with semi-cycles of length $k + 1$ or $k + 2$, except possibly for the first semi-cycle which may have length k. The extreme in each semi-cycle occurs in the first term if the semi-cycle has two terms and in the second term if the semi-cycle has three terms, and in the $k+1$ term if the semi-cycle has $k + 2$ terms.*

2. *Assume $D+p<q$, then either $\{y_n\}$ oscillates about the equilibrium $\bar{y}$ with semi-cycles of length k after the first semi-cycle, or $\{y_n\}$ converges monotonically to $\bar{y}$.*

Proof. 1. The proof follows from Theorem 4.3 by observing that the condition $D+p>q$ implies that the function

$$f(x,y)=\frac{D+x+py}{x+qy}$$

is increasing in x and decreasing in y. This function also satisfies conditions 4.4.3, 4.4.4.

2. The proof follows from Theorem 4.4 by observing that the condition $D+p<q$ implies that the function

$$f(x,y)=\frac{D+x+py}{x+qy}$$

is decreasing in x and increasing in y.

The proof is complete. $\square$

We now examine the existence of intervals which attract all solution of Eq.4.2.1.

Theorem 4.8. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.4.2.1. Then the following statements are true :*

(1) Suppose $D+p<q$ and assume that for some $N\geq 0$.

$$y_{N-k},...,y_{N-1},y_N\in\left[\frac{D+p}{q},1\right],$$

then

$$y_n\in\left[\frac{D+p}{q},1\right],\ \textit{for all } n>N.$$

(2) Suppose $D+p>q$ and assume that for some $N\geq 0$.

$$y_{N-k},...,y_{N-1},y_N\in\left[1,\frac{D+p}{q}\right],$$

then

$$y_n\in\left[1,\frac{D+p}{q}\right],\ \textit{for all } n>N.$$

Proof. (1) If for some $N > 0$,

$$y_{N-k}, ..., y_{N-1}, y_N \in \left[\frac{D+p}{q}, 1\right],$$

then $\frac{D+p}{q} \leq y_{N-k} \leq 1$, then

$$\begin{aligned} y_{N+1} &= \frac{D + y_N + py_{N-k}}{y_N + qy_{N-k}} \\ &\leq \frac{y_N + D + p}{y_N + q\frac{D+p}{q}} \\ &\leq 1. \end{aligned}$$

Now take into consideration that the function

$$f(u, v) = \frac{D + u + pv}{u + qv}$$

is increasing in u and decreasing in v. Then for

$$y_{N+1} = \frac{D + y_N + py_{N-k}}{y_N + qy_{N-k}}$$

y_{N+1} is increasing in y_N for some fixed value for y_{N-k}.

We can take this fixed value for y_{N-k} to be 1.and since $\frac{D+p}{q} \leq y_N \leq 1$ then

$$\begin{aligned} y_{N+1} &= \frac{D + y_N + py_{N-k}}{y_N + qy_{N-k}} \\ &\geq \frac{D + \frac{D+p}{q} + p}{\frac{D+p}{q} + q} \\ &\geq \frac{(D+p)(1 + \frac{1}{q})}{q(1 + \frac{1}{q}\frac{D+p}{q})} \\ &\geq \frac{D+p}{q} \end{aligned}$$

By Mathematical Induction we can prove that

$$\frac{D+p}{q} \leq y_n \leq 1 \; for \; all \; n \geq N$$

Then

$$y_n \in \left[\frac{D+p}{q}, 1\right], \text{ for all } n > N.$$

(2) This proof is similar to the above one. If for some $N > 0$,

$$y_{N-k}, ..., y_{N-1}, y_N \in \left[1, \frac{D+p}{q}\right],$$

then $1 \leq y_{N-k} \leq \frac{D+p}{q}$,then

$$\begin{aligned} y_{N+1} &= \frac{D + y_N + p y_{N-k}}{y_N + q y_{N-k}} \\ &\geq \frac{y_N + D + p}{y_N + q\frac{D+p}{q}} \\ &\geq 1. \end{aligned}$$

As we saw, y_{N+1} is increasing in y_{N-k} for some fixed value for y_N.

We can take this fixed value for y_N to be 1.and since since $1 \leq y_{N-k} \leq \frac{D+p}{q}$ then

$$\begin{aligned} y_{N+1} &= \frac{D + y_N + p y_{N-k}}{y_N + q y_{N-k}} \\ &\leq \frac{D + 1 + p\frac{D+p}{q}}{1 + q\frac{D+p}{q}} \\ &\leq \frac{D+p}{q}\frac{p + \frac{q(D+1)}{D+p}}{1 + D + p} \\ &\leq \frac{D+p}{q}\frac{1 + D + p}{1 + D + p} \\ &\leq \frac{D+p}{q} \end{aligned}$$

By Mathematical Induction we can prove that

$$1 \leq y_n \leq \frac{D+p}{q} \ for\ all\ n \geq N$$

Then

$$y_n \in \left[1, \frac{D+p}{q}\right], \text{ for all } n > N.$$

The proof is complete. $\square$

Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.4.2.1. Then the following identities are hold :

$$y_{n+1} - 1 = (q-p)\frac{\frac{D}{(q-p)} - y_{n-k}}{y_n + qy_{n-k}} \tag{4.4.5}$$

$$y_{n+1} - (\frac{D+p}{q}) = \frac{(1-(\frac{D+p}{q}))y_n + D(1-y_{n-k})}{y_n + qy_{n-k}} \tag{4.4.6}$$

First we will analyze the semi-cycles of the solution $\{y_n\}_{n=-k}^{\infty}$ under the assumption that

$$D + p > q,\ q > p \tag{4.4.7}$$

The following result is a direct consequences of 4.4.5-4.4.6.

Lemma 4.5. *Assume that 4.4.6 holds and let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.4.2.1. Then the following statements are true:*

1. *If for some $N \geq 0$, $y_{N-k} < (D+p)/q$, then $y_{N+1} > 1$;*
2. *If for some $N \geq 0$, $y_{N-k} = D/(q-p)$, then $y_{N+1} = 1$;*
3. *If for some $N \geq 0$, $y_{N-k} > D/(q-p)$, then $y_{N+1} < 1$;*
4. *If for some $N \geq 0$, $y_{N-k} \geq 1$, then $y_{N+1} < (D+p)/q$;*
5. *If for some $N \geq 0$, $y_{N-k} \leq 1$, then $y_{N+1} \geq 1$;*
6. *If for some $N \geq 0$, $1 \leq y_{N-k} \leq (D+p)/q$, then $1 \leq y_{N+1} \leq (D+p)/q$;*
7. *If for some $N \geq 0$, $1 \leq y_{N-k}, y_N \leq (D+p)/q$, then $1 \leq y_n \leq (D+p)/q$, for all $n \geq N$; Thats $[1, (D+p)/q]$ is an invariant interval of Eq.4.2.1.*
8. $1 < \bar{y} < (D+p)/q$,

   Indeed: when $D + p > q$

$$Dq + Dp + p^2 > pq + Dp$$

$$Dq > pq + Dq - Dp - p^2$$

$$Dq > (D+p)(q-p)$$

$$\frac{D}{q-p} > \frac{D+p}{q}$$

Theorem 4.9. *Assume that Eq.4.2.1 holds. Then every nontrivial and oscillatory solution of Eq.4.2.1 which lies in the interval $[1, (D+p)/q]$, oscillates about $\bar{y}$ with semi-cycle of length k or $k+1$.*

Now we will analyze the semi-cycles of the solution $\{y_n\}_{n=-k}^{\infty}$ under the assumption that

$$D + p < q, \ q > p \tag{4.4.8}$$

The following is a direct consequences of 4.4.5-4.4.6 and 4.4.8.

Lemma 4.6. *Assume that 4.4.8 holds and let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.4.2.1. Then the following statements are true:*

1. *If for some $N \geq 0$, $y_{N-k} > (D+p)/q$, then $y_{N+1} < 1$;*
2. *If for some $N \geq 0$, $y_{N-k} = D/(q-p)$, then $y_{N+1} = 1$;*
3. *If for some $N \geq 0$, $y_{N-k} < D/(q-p)$, then $y_{N+1} > 1$;*
4. *If for some $N \geq 0$, $y_{N-k} \leq 1$, then $y_{N+1} > (D+p)/q$;*
5. *If for some $N \geq 0$, $y_{N-k} \leq (D+p)/q$, then $y_{N+1} > (D+p)/q$;*
6. *If for some $N \geq 0$, $(D+p)/q \leq y_{N-k} \leq 1$, then $(D+p)/q \leq y_{N+1} \leq 1$;*
7. *If for some $N \geq 0$, $(D+p)/q \leq y_{N-k}, y_N \leq 1$, then $(D+p)/q \leq y_n \leq 1$, for all $n \geq N$; Thats $[1, (D+p)/q]$ is an invariant interval of Eq.4.2.1.*
8. *$(D+p)/q < \bar{y} < 1$,*

   Indeed: when $D + p < q$

$$Dq + Dp + p^2 < pq + Dp$$

$$Dq < pq + Dq - Dp - p^2$$

$$Dq < (D+p)(q-p)$$
$$\frac{D}{q-p} < \frac{D+p}{q}$$

Theorem 4.10. *Assume that Eq.4.2.1 holds. Then every nontrivial and oscillatory solution of Eq.4.2.1 which lies in the interval* $[(D+p)/q, 1]$*, oscillates about* $\bar{y}$ *with semi-cycle of length at least* $k+1$*.*

Next we will analyze the semi-cycles of the solution $\{y_n\}_{n=-k}^{\infty}$ under the assumption that

$$D + p = q \tag{4.4.9}$$

In this case Eq.4.2.1 reduces to

$$y_{n+1} = \frac{D + y_n + py_{n-k}}{y_n + (D+p)y_{n-k}} \tag{4.4.10}$$

with the unique equilibrium point $\bar{y} = 1$. Also Eqs.4.4.5-4.4.6 reduce to

$$y_{n+1} - 1 = \frac{D(1 - y_{n-k})}{y_n + (D+p)y_{n-k}} \tag{4.4.11}$$

and so the following results follow immediately.

Lemma 4.7. *Let* $\{y_n\}_{n=-k}^{\infty}$ *be a solution of Eq.4.4.10. Then the following statements are true:*

1. *If for some* $N \geq 0$*,* $y_{N-k} < 1$*, then* $y_{N+1} > 1$*;*
2. *If for some* $N \geq 0$*,* $y_{N-k} = 1$*, then* $y_{N+1} = 1$*;*
3. *If for some* $N \geq 0$*,* $y_{N-k} > 1$*, then* $y_{N+1} < 1$*;*

Corollary 4.2. *Let* $\{y_n\}_{n=-k}^{\infty}$ *be a nontrivial solution of Eq.4.4.10. Then* $\{y_n\}_{n=-k}^{\infty}$ *oscillates about the equilibrium 1.*

Now assume that $D + p > q$ and let $\{y_n\}_{n=-k}^{\infty}$ be a solution which does not eventually lie in the interval $I = [1, (D+p)/q]$. Then one can see that the solution oscillates relative to the interval $I = [1, (D+p)/q]$, essentially in the following two ways:

1. k+1 consecutive terms in $((D+p)/q, \infty)$ are followed by k+1 consecutive terms in $((D+p)/q, \infty)$ and so on. The solution never visits the interval $(1, (D+p)/q)$.

2. There exists exactly k terms in $((D+p)/q, \infty)$, which is followed by exactly k terms in $(1, (D+p)/q)$, which is followed by exactly k terms in $(0, 1)$, which is followed by exactly k terms in $(1, (D+p)/q)$, which is followed by exactly k terms in $((D+p)/q, \infty)$, and so on. The solution visits consecutively the intervals
$..., ((D+p)/q, \infty), (1, (D+p)/q), (0, 1), (1, (D+p)/q), ((D+p)/q, \infty), ...$
in order with k terms per interval.

The situation is essentially the same relative to the interval $[(D+p)/q, 1]$ when $D+p<q$.

4.5 The Global Stability

The next results are about the global stability for the positive equilibrium of Eq.4.2.1.

Here are the theorems that we need.

Theorem 4.11. *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), n = 0, 1... \tag{4.5.1}$$

where $k \in \{1, 2, ...\}$. Let $I = [a, b]$ be some interval of real numbers and assume that

$$f : [a, b] \times [a, b] \longrightarrow [a, b]$$

is a continuous function satisfying the following properties :

(a) $f(x, y)$ is non-increasing in each of its arguments;

(b) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system $f(M, M) = m$ and $f(m, m) = M$ then $m = M$.

Then Eq.4.5.1 has a unique equilibrium $\bar{y}$ and every solution of Eq.4.5.1 converges to $\bar{y}$.

Theorem 4.12. *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), n = 0, 1 \tag{4.5.2}$$

where $k \in \{1, 2, ...\}$. Let $I = [a, b]$ be some interval of real numbers and assume that

$$f : [a, b] \times [a, b] \longrightarrow [a, b]$$

is a continuous function satisfying the following properties :

(a) $f(x, y)$ is non-decreasing in $x \in [a, b]$ for each $y \in [a, b]$, and $f(x, y)$ is non-increasing in $y \in [a, b]$ for each $x \in [a, b]$;

(b) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system $f(m, M) = m$ and $f(M, m) = M$ then $m = M$.

Then Eq.4.5.2 has a unique equilibrium $\bar{y} \in [a, b]$ and every solution of Eq.4.5.2 converges to $\bar{y}$.

Now we will apply these theorems on our equation.

Theorem 4.13. *Assume that $D + p > q$, then the positive equilibrium of Eq.4.2.1 on the interval $[1, \frac{D+p}{q}]$ is globally asymptotically stable.*

Proof. This proof can be done easily depending on Theorem 4.11.

Assume that $D + p > q$ and consider the function

$$f(x, y) = \frac{D + x + py}{x + qy} \tag{4.5.3}$$

First, note that f(x,y) on the interval $[1, \frac{D+p}{q}]$ is nonincreasing in both of its arguments x, y.

Second, let $(m, M) \in [a, b] \times [a, b]$ is a solution of the system $f(M, M) = m$ and $f(m, m) = M$, then

$$m = \frac{D + M + pM}{M + qM}$$

and

$$M = \frac{D + m + pm}{m + qm}$$

but as we showed that our equation has no periodic-two solutions, then the only solution is $m = M$.

So, the both conditions of Theorem 4.11 hold, so, If $\bar{y}$ is an equilibrium point of Eq. 4.2.1, then every solution of Eq. 4.2.1 converges to $\bar{y}$ in the interval $[1, \frac{D+p}{q}]$. As $\bar{y}$ is asymptotically stable, then it is globally asymptotically stable on $[1, \frac{D+p}{q}]$.

$\square$

Theorem 4.14. *Assume that $D + p < q$, then the positive equilibrium of Eq.4.2.1 on the interval $[\frac{D+p}{q}, 1]$ is globally asymptotically stable.*

Proof. This proof can be done easily depending on Theorem 4.12.

Assume that $D + p < q$ and consider the function

$$f(x, y) = \frac{D + x + py}{x + qy} \tag{4.5.4}$$

First, note that f(x,y) on the interval $[\frac{D+p}{q}], 1$ is nondecreasing in x, and nonincreasing in y.

Second, let $(m, M) \in [a, b] \times [a, b]$ is a solution of the system $f(m, M) = m$ and $f(M, m) = M$, then

$$m = \frac{D + m + pM}{m + qM}$$

and

$$M = \frac{D + M + pm}{M + qm}$$

but as we showed that our equation has no periodic-two solutions, then the only solution is $m = M$.

So, the both conditions of Theorem 4.12 hold, so, If $\bar{y}$ is an equilibrium point of Eq. 4.2.1, then every solution of Eq. 4.2.1 converges to $\bar{y}$ in the interval $[\frac{D+p}{q}, 1]$. As $\bar{y}$ is asymptotically stable, then it is globally asymptotically stable on $[\frac{D+p}{q}, 1]$.

□

4.6 Numerical Discussion

Here in this section, we will study the global stability of our equation numerically based on some data and figures that we can get using MATLAB 6.5.

Example1:

Assume that Equation 4.1.1 holds, take $k = 2$, $\alpha = 1$, $\beta = 2$, $\gamma = 1$, $B = 2$, $C = 3$. So the equation will be reduced to the following:

$$x_{n+1} = \frac{1 + 2x_n + x_{n-2}}{2x_n + 3x_{n-2}}, \qquad n = 0, 1, ... \tag{4.6.1}$$

We assumed that the initial points $\{x_{-2},\ x_{-1}, x_0\}$ *all to be* $\in (0,\infty)$ and are $\{0.2, 0.5, 1\}$.

The change of variable $x_n = \frac{\beta}{B} y_n$ changes the equation into the equation

$$y_{n+1} = \frac{D + y_n + p y_{n-2}}{y_n + q y_{n-2}}, \qquad n = 0, 1, ... \tag{4.6.2}$$

where $D = \frac{B\alpha}{\beta^2} = 0.5$, $p = \frac{\gamma}{\beta} = 0.5$, $q = \frac{C}{B} = 1.5$.

then the theoretical positive equilibrium point will be $\bar{y} = 0.83851648071345$.

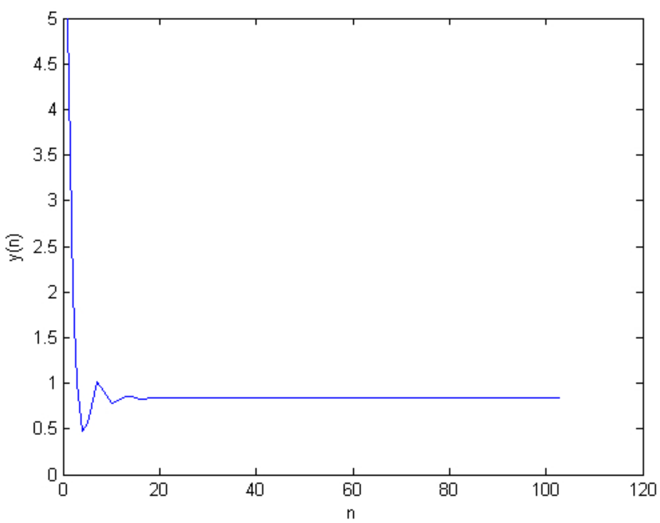

Figure 4.1: The Behavior of the Equilibrium point of the Equation $y_{n+1} = \frac{0.5 + y_n + 0.5 y_{n-2}}{y_n + 1.5 y_{n-2}}$

By theory, the equilibrium point $\bar{y}$ is globally asymptotically stable, and it is obvious from Figure4.1 that it is globally asymptotically stable, as we have shown theoretically.Lets take another example now.

Example2:
Assume that Equation 4.1.1 holds, take $k = 4$, $\alpha = 3$, $\beta = 1$, $\gamma = 2$, $B = 4$, $C = 5$. So the equation will be reduced to the following:

$$x_{n+1} = \frac{3 + x_n + 2x_{n-4}}{4x_n + 5x_{n-4}}, \qquad n = 0, 1, ... \tag{4.6.3}$$

We assumed that the initial points $\{x_{-4},\ x_{-3}, ..., x_0\}$ *all to be* $\in (0, \infty)$ and are $\{2, 1.4, 1.3, 0.9, 1.5\}$.

The change of variable $x_n = \frac{\beta}{B} y_n$ changes the equation into the equation

$$y_{n+1} = \frac{D + y_n + py_{n-4}}{y_n + qy_{n-4}}, \qquad n = 0, 1, ... \tag{4.6.4}$$

where $D = \frac{B\alpha}{\beta^2} = 3$, $p = \frac{\gamma}{\beta} = 2$, $q = \frac{C}{B} = 1.25$.

then the theoretical positive equilibrium point will be $\bar{y} = 2$.

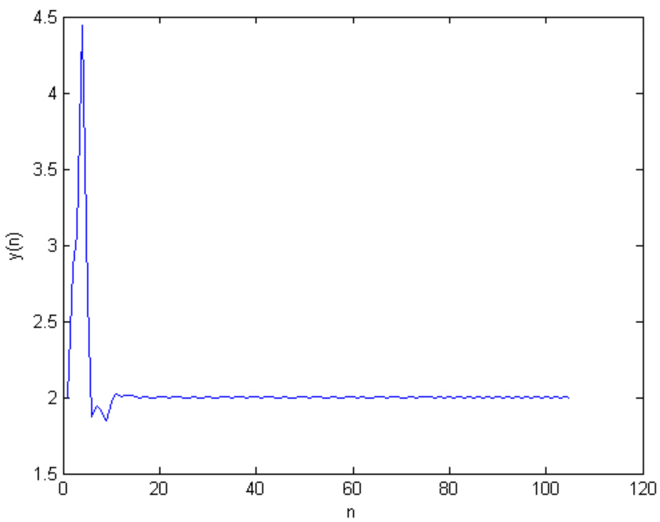

Figure 4.2: The Behavior of the Equilibrium point of the Equation $y_{n+1} = \frac{3+y_n+2y_{n-4}}{y_n+1.25y_{n-4}}$

By theory, the equilibrium point $\bar{y} = 2$ is globally asymptotically stable, and it is obvious from Figure4.2 that it is globally asymptotically stable, as we have shown theoretically.

Here, it is obvious that from Figure4.2 that our equilibrium point is around the point 2.

Example3:
Assume that Equation 4.1.1 holds, take $k = 2$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $B = 1$, $C = 2$. So the equation will be reduced to the following:

$$x_{n+1} = \frac{1 + x_n + x_{n-2}}{x_n + 2x_{n-2}}, \qquad n = 0, 1, ... \tag{4.6.5}$$

We assumed that the initial points $\{x_{-2},\ x_{-1}, x_0\}$ *all to be* $\in (0, \infty)$ and are $\{2, 8, 3\}$.

The change of variable $x_n = \frac{\beta}{B} y_n$ changes the equation into the equation

$$y_{n+1} = \frac{D + y_n + p y_{n-2}}{y_n + q y_{n-2}}, \qquad n = 0, 1, ... \tag{4.6.6}$$

where $D = \frac{B\alpha}{\beta^2} = 1$, $p = \frac{\gamma}{\beta} = 1$, $q = \frac{C}{B} = 2$.

then the theoretical positive equilibrium point will be $\bar{y} = 1$.

Its obvious from figure that the equilibrium point $\bar{y} = 1$ is globally asymptotically stable.

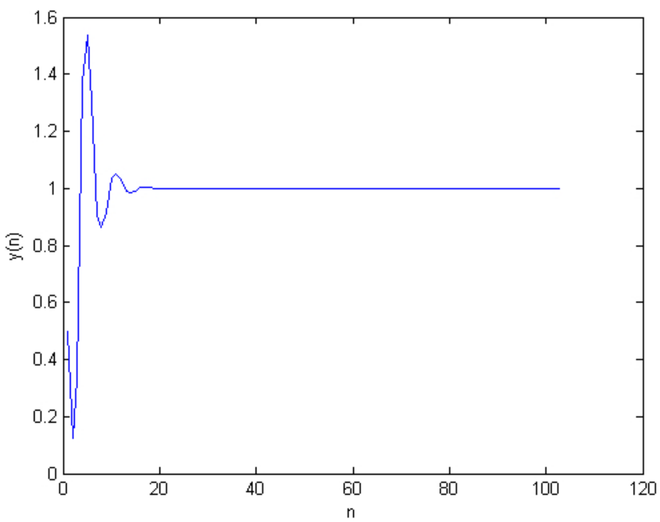

Figure 4.3: The Behavior of the Equilibrium point of the Equation $y_{n+1} = \frac{1+y_n+y_{n-2}}{y_n+2y_{n-2}}$

Chapter 5

Global Asymptotic Stability of the Higher Order Equation $x_{n+1} = \frac{ax_n+bx_{n-k}}{A+Bx_{n-k}}$

5.1 Introduction

Our goal in this chapter is to study the rational higher order difference equation

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}} \tag{5.1.1}$$

where a, b, A, B are all positive real numbers, $k \geq 1$ is a positive integer, and the initial conditions $x_{-k}, x_{-k+1}, ..., x_0$ are nonnegative real numbers.

Here, we recall some basic notations and definitions and results that was discussed in Part One.

Definition 5.1. *The equilibrium point $\bar{y}$ of the equation*

$$y_{n+1} = f(y_n, y_{n-1}, ..., y_{n-k}), n = 0, 1, ... \tag{5.1.2}$$

is the point that satisfies the condition

$$\bar{y} = f(\bar{y}, \bar{y}, ..., \bar{y}).$$

Definition 5.2. *Let $\bar{y}$ be an equilibrium point of Eq.5.1.2. Then the equilibrium point $\bar{y}$ is called*

1. *locally stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $y_{-k}, y_{-k+1}, ..., y_0 \in I$ with $|y_{-k} - \bar{y}| + |y_{-k+1} - \bar{y}| + ... + |y_0 - \bar{y}| < \delta$, we have $|y_n - \bar{y}| < \epsilon$ for all $n \geq -k$,*

2. *locally asymptotically stable if it is locally stable and if there exists $\gamma > 0$ such that for all $y_{-k}, y_{-k+1}, ..., y_0 \in I$ with $|y_{-k} - \bar{y}| + |y_{-k+1} - \bar{y}| + ... + |y_0 - \bar{y}| < \gamma$, we have $lim_{n\to\infty} y_n = \bar{y}$.*

3. *a global attractor if for all $y_{-k}, y_{-k+1}, ..., y_0 \in I$, we have $lim_{n\to\infty} y_n = \bar{y}$.*

4. *globally asymptotically stable if $\bar{y}$ is locally stable and $\bar{y}$ is a global attractor.*

Theorem 5.1. *(**Linearized Stability**).*
Consider the difference equation

$$y_{n+1} = py_n + qy_{n-k}, \qquad n = 0, 1, ...$$

(a) If both roots of the equation have absolute values less than one, then the equilibrium $\bar{y}$ of the equation is locally asymptotically stable.

(b)If at least one of the roots of the equation has an absolute value greater than one, then $\bar{y}$ is unstable .

Theorem 5.2. *Assume that a,b are real numbers and $k \in \{1, 2, ...\}$. Then*

$$|a| + |b| < 1 \tag{5.1.3}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_n + ay_{n-k} + by_{n-l} = 0, \; n = 0, 1, ... \tag{5.1.4}$$

Suppose in addition that one of the following two cases holds.

(a) k odd and $b < 0$.

(b) k even and $ab < 0$.

Then 5.1.3 is also a necessary condition for the asymptotic stability of Eq.5.1.4.

Theorem 5.3. *Assume that a,b are real numbers. Then $|a| < b + 1 < 2$ is a necessary and sufficient condition for the asymptotic stability of the difference equation*

$$y_n + ay_{n-k} + by_{n-l} = 0, \; n = 0, 1, ... \tag{5.1.5}$$

Theorem 5.4. *Consider the difference equation*

$$x_{n+1} = \sum_{i=0}^{k} x_{n-i} F_i(x_n, x_{n-1}, ..., x_{n-k}) n = 0, 1, ... \qquad (5.1.6)$$

with initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in [0, \infty)$, *where*

1. $k \in \{1, 2, ...\}$;
2. $F_0, F_1, ..., F_k \in C\left[[0, \infty)^{k+1}, [0, 1)\right]$;
3. $F_0, F_1, ..., F_k$ *are nonincreasing in each argument;*
4. $\sum_{i=0}^{k} F_i(y_0, y_1, ..., y_k) < 1$ *for all* $(y_0, y_1, ..., y_k) \in (0, 1)^k$;
5. $F_0(y, y, ..., y) > 0$ *for all* $y \geq 0$.

Then, $\bar{x} = 0$ *is globally asymptotically stable for such equation.*

Theorem 5.5. *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}); n = 0, 1, ... \qquad (5.1.7)$$

where $k \in \{1, 2, ...\}$. *Let* $I = [a, b]$ *be some interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(u, v)$ *is nondecreasing in* u *and nondecreasing in* v.

(b) If $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$m = f(m, m) \text{ and } M = f(M, M),$$

then $m = M$. *Then Eq.5.1.7 has a unique equilibrium* $\bar{y}$ *and every solution of Eq.5.1.7 converges to* $\bar{y}$.

Theorem 5.6. *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}); n = 0, 1, ... \qquad (5.1.8)$$

where $k \in \{1, 2, ...\}$. *Let* $I = [a, b]$ *be some interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(u,v)$ is nondecreasing in u and nonincreasing in v.

(b) If $(m,M) \in [a,b] \times [a,b]$ is a solution of the system

$$m = f(m,M) \text{ and } M = f(M,m),$$

then $m = M$. Then Eq.5.1.8 has a unique equilibrium $\bar{y}$ and every solution of Eq.5.1.8 converges to $\bar{y}$.

5.2 The Equilibrium Points

Next, we investigate the equilibrium points of our rational difference equation,

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}} \tag{5.2.1}$$

where a, b, A, B are all positive real numbers, $k \geq 1$ is a positive integer, and the initial conditions $x_{-k}, x_{-k+1}, ..., x_0$ are nonnegative real numbers.

The equilibrium points of Eq.5.2.1 are the positive solutions of the equation

$$\begin{aligned} \bar{x} &= \frac{a\bar{x} + b\bar{x}}{A + B\bar{x}} \\ &= \frac{(a+b)\bar{x}}{A + B\bar{x}} \end{aligned}$$

Then,

$$\bar{x}(A + B\bar{x}) = (a+b)\bar{x}$$

$$\bar{x}(A + B\bar{x} - a - b) = 0$$

So, $\bar{x} = 0$ is always an equilibrium point of Eq.5.2.1, and when $a+b > A$, Eq.5.2.1 has another positive equilibrium point , $\bar{x} = \frac{a+b-A}{B}$.

To find the linearization of our problem, consider

$$f(u,v) = \frac{au + bv}{A + Bv}$$

now,

$$\frac{\partial f}{\partial u} = \frac{a}{A + Bv}$$

$$\frac{\partial f}{\partial v} = \frac{(A + Bv)b - (au + bv)B)}{(A + Bv)^2}$$

so

$$\frac{\partial f}{\partial v} = \frac{Ab - au}{(A + Bv)^2}$$

Hence, for $\bar{x} = 0$,

$$\frac{\partial f}{\partial u}(\bar{x}, \bar{x}) = \frac{a}{A}$$

also

$$\frac{\partial f}{\partial v(\bar{x}, \bar{x})} = \frac{b}{A}$$

So, the linearized equation about the zero equilibria is

$$z_{n+1} = \frac{a}{A} z_n + \frac{b}{A} z_{n-k} \tag{5.2.2}$$

and its characteristic equation is

$$\lambda^{k+1} - \frac{a}{A}\lambda^k - \frac{b}{A} = 0 \tag{5.2.3}$$

For the positive equilibria $\bar{x} = \frac{a+b-A}{B}$,

$$\frac{\partial f}{\partial u}(\bar{x}, \bar{x}) = \frac{a}{a+b}$$

also

$$\frac{\partial f}{\partial v(\bar{x}, \bar{x})} = \frac{A - a}{a+b}$$

So, the linearized equation about the positive equilibria is

$$z_{n+1} = \frac{a}{a+b} z_n + \frac{A-a}{a+b} z_{n-k} \tag{5.2.4}$$

and its characteristic equation is

$$\lambda^{k+1} - \frac{a}{a+b}\lambda^k - \frac{A-a}{a+b} = 0 \tag{5.2.5}$$

5.3 The Local Stability of the Equilibrium Points

5.3.1 The Local Stability of the Zero Equilibrium Point

Our equation posses the equilibrium point $\bar{x} = 0$ always in all of the following cases:

1. $a + b < A$
2. $a + b = A$
3. $a + b > A$

We will study the stability of $\bar{x} = 0$ in all of the cases.

The Case $a + b < A$

Theorem 5.7. *The zero equilibrium point $\bar{x} = 0$ will be locally asymptotically stable when $a + b < A$.*

Proof. The characteristic equation of the zero equilibrium point is

$$\lambda^{k+1} - \frac{a}{A}\lambda^k - \frac{b}{A} = 0 \tag{5.3.1}$$

where a,b,A,B are all positive real numbers.

By applying Theorem 5.2, and as we assume $a + b < A$, then it is easy then to show that $\bar{x} = 0$ is asymptotically equilibrium point.
This completes the proof. □

The case $a+b=A$

Theorem 5.8. *The zero equilibrium point* $\bar{x}=0$ *will be locally stable when* $a+b=A$.

Proof. Our difference equation is

$$x_{n+1}=\frac{ax_n+bx_{n-k}}{A+Bx_{n-k}} \tag{5.3.2}$$

Let $\{x_n\}_{n=-k}^{\infty}$ be a nonnegative solution of our equation with the initial points $x_{-k},......,x_0$ are to be nonnegative.
Let $\epsilon>0$ and

$$|x_{-k}-\bar{x}|<\epsilon,......,|x_0-\bar{x}|<\epsilon$$

So,and since $\bar{x}=0$,

$$0\le x_{-k}<\epsilon,......,0\le x_0<\epsilon$$

Then, when $a+b=A$

$$\begin{aligned} x_1 &= \frac{ax_0+bx_{-k}}{A+Bx_{-k}} \\ &< \frac{ax_0+bx_{-k}}{A} \\ &< \frac{(a+b)\epsilon}{A} \\ &< \epsilon \end{aligned}$$

So, $0\le x_1<\epsilon$, and by Mathematical Induction we get

$$0\le x_n<\epsilon, \forall n\ge -k$$

Then,

$$|x_n-\bar{x}|<\epsilon, \forall n\ge -k$$

By definition, then $\bar{x}=0$ is a locally stable equilibrium point when $a+b=A$. This completes the proof. □

The Case $a+b>A$

Theorem 5.9. *The zero equilibrium point* $\bar{x}=0$ *is unstable when* $a+b>A$

Proof. The characteristic polynomial of the zero equilibrium point

$$f(\lambda) = \lambda^{k+1} - \frac{a}{A}\lambda^k - \frac{b}{A} \tag{5.3.3}$$

is a continuous function for all λ.

For $\lambda = 1$,

$$f(1) = 1 - \frac{a}{A} - \frac{b}{A} = \frac{A - (a+b)}{A}$$

and since $a + b > A$, $f(1) < 0$.

But also, as $\lambda \to \infty$

$$\lim_{\lambda \to \infty} f(\lambda) = \infty > 0$$

By Roll's Theorem, we can say that $f(\lambda)$ has a zero solution λ_0, where $f(\lambda_0) = 0$ and $\lambda_0 \in (1, \infty)$.

Then there exists a solution λ_0, where λ_0 lies outside the unitary disk.

By Theorem 5.1, $\bar{x} = 0$ is unstable equilibrium point when $a + b > A$. This completes the proof. □

5.3.2 The Local Stability of the Positive Equilibria

Our equation posses a positive equilibrium $\bar{x} = \frac{a+b-A}{B}$ under the condition $a + b > A$. The linearized equation of this equilibrium point is

$$z_{n+1} = \frac{a}{a+b} z_n + \frac{A-a}{a+b} z_{n-k} \tag{5.3.4}$$

and its characteristic equation is

$$\lambda^{k+1} - \frac{a}{a+b}\lambda^k - \frac{A-a}{a+b} = 0 \tag{5.3.5}$$

Lets apply Theorem 5.2 under the both cases $A \geq a$ and $A < a$.

(1) **The case $A \geq a$:**

$$\begin{aligned} \left|\frac{a}{a+b}\right| + \left|\frac{A-a}{a+b}\right| &= \frac{a+|A-a|}{a+b} \\ &= \frac{a+A-a}{a+b} \\ &= \frac{A}{a+b} \end{aligned}$$

Then the necessary condition

$$\frac{A}{a+b} < 1$$

can be written as

$$a \leq A < a+b$$

(2) **The case $A < a$:**

$$\begin{aligned} \left|\frac{a}{a+b}\right| + \left|\frac{A-a}{a+b}\right| &= \frac{a+|A-a|}{a+b} \\ &= \frac{a+a-A}{a+b} \\ &= \frac{2a-A}{a+b} \end{aligned}$$

Then the necessary condition

$$\frac{2a-A}{a+b} < 1$$

can be written as

$$a-b < A < a$$

If we combine the both cases, we get the following theorem.

Theorem 5.10. *If k is odd or if k is even, the unique positive equilibrium point $\bar{x} = \frac{a+b-A}{B}$ will be locally asymptotically stable if and only if $a-b < A < a+b$.*

5.4 The periodic Two Solution

We study here the periodic solution of our equation,

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}} \tag{5.4.1}$$

Lets assume that the two periodic nonnegative solution of our equation will be in the form

$$...., \psi, \phi, \psi, \phi, ...$$

If k is odd then,

$$x_{n+1} = x_{n-k} \tag{5.4.2}$$

By so we get,

$$\psi = \frac{a\phi + b\psi}{A + B\psi} \tag{5.4.3}$$

$$\phi = \frac{a\psi + b\phi}{A + B\phi} \tag{5.4.4}$$

This yields to

$$\psi(A + b + B\psi) = a\phi$$

$$\phi(A + b + B\phi) = a\psi$$

By subtract the second equation from the first we get the equation

$$(A + a + b)(\psi - \phi) + B(\psi^2 - \phi^2) = 0 \tag{5.4.5}$$

$$(\psi - \phi)(A + a + b + B(\psi + \phi)) = 0 \tag{5.4.6}$$

Then, either $\psi = \phi$ or $(\psi + \phi) = -\frac{A+a+b}{B}$ which is impossible since both ϕ and ψ are nonnegative. Then in this case there is no two periodic nonnegative solution for our equation.

Lets now take k to be even and see what we will get.

If k is even then,

$$x_n = x_{n-k} \tag{5.4.7}$$

So,

$$\psi = \frac{a\phi + b\phi}{A + B\phi} \tag{5.4.8}$$

$$\phi = \frac{a\psi + b\psi}{A + B\psi} \tag{5.4.9}$$

We get that

$$\phi(A + B\psi) = (a + b)\psi$$
$$\psi(A + B\phi) = (a + b)\phi$$

By subtract both equations we get

$$(\phi - \psi)(A + a + b) = 0 \tag{5.4.10}$$

Then either $\phi = \psi$ or $A = -(a + b)$ which i also impossible. Then also in this case there exists no two periodic nonnegative solution for our equation. We can conclude the following thus.

Theorem 5.11. *There exists no two periodic nonnegative solution for the equation*

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}}$$

under any condition.

5.5 The Global Stability

Here also we will consider the two equilibrium points separately.

5.5.1 The Global Stability of the Zero Equilibria

We study the global stability of the zero equilibrium point under the condition $a + b \leq A$.

Our equation

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}} \tag{5.5.1}$$

can be written as

$$x_{n+1} = \frac{a}{A + Bx_{n-k}} x_n + \frac{b}{A + Bx_{n-k}} x_{n-k} \tag{5.5.2}$$

Lets apply Theorem 5.4 now. We can consider $F_0 = \frac{a}{A+Bx_{n-k}}$ and $F_1, ...F_{k-1} = 0$ and $F_k = \frac{b}{A+Bx_{n-k}}$. Then it is obvious that that theorem could be applied so easily since:

1. $k \in \{1, 2, ...\}$;

2. $F_0, F_k \in \mathrm{C}\ [[0, \infty)^{k+1}, [0, 1)]$;

3. F_0, F_k are nonincreasing in each argument;

4. $F_0 + F_k = \frac{a+b}{A+Bx_{n-k}}$, and since $a + b < A$,

$$\begin{aligned} F_0 + F_k &= \frac{a+b}{A+Bx_{n-k}} \\ &< \frac{a+b}{A} \leq 1 \\ &< 1 \end{aligned}$$

5. $F_0(y, y, ..., y) = \frac{a}{A+By} > 0$ for all $y \geq 0$.

We can conclude now

Theorem 5.12. *The zero equilibrium point $\bar{x} = 0$ is globally asymptotically stable for the equation*

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}}$$

under the condition $a + b \leq A$.

5.5.2 The Global Stability of the Positive Equilibria

We study the global stability of the positive equilibria under the condition $a + b > A$.

Our equation as we said before could be written as

$$x_{n+1} = \frac{a}{A + Bx_{n-k}} x_n + \frac{b}{A + Bx_{n-k}} x_{n-k} \tag{5.5.3}$$

So, take

$$f(u, v) = \frac{a}{A + Bv} u + \frac{b}{A + Bv} v \tag{5.5.4}$$

f(u,v) is always increasing in the argument u. For v, f(u,v) can either being increasing or decreasing in such argument.

First, consider the case when f(u,v) is increasing in the argument v. We will apply now Theorem 5.5, we will show that m=M where,

$$m = f(m,m) = \frac{(a+b)m}{A+Bm} \tag{5.5.5}$$

$$M = f(M,M) = \frac{(a+b)M}{A+BM} \tag{5.5.6}$$

So we get that,

$$m(A+Bm-(a+b)) = 0$$

$$M(A+BM-(a+b)) = 0$$

and since we are looking for the positive solution we get that,

$$m = M = \frac{a+b-A}{B} \tag{5.5.7}$$

Then, $m = \frac{a+b-A}{B}$ is a global attractor. For this case we can say that the positive equilibrium point $\bar{x} = \frac{a+b-A}{B}$ is globally asymptotically stable.

Second, consider the case when f(u,v) is decreasing in the argument v. We will apply Theorem 5.6, we will show that m=M where,

$$m = f(m,M) = \frac{am+bM}{A+BM} \tag{5.5.8}$$

$$M = f(M,m) = \frac{aM+bm}{A+Bm} \tag{5.5.9}$$

Suppose that $m = \frac{a+b-A}{B}$ and show that M=m.

As we proved , our equation possess no periodic two solution under any condition, by so we can conclude easily that the only solution foe such system is that

$$M = m = \frac{a+b-A}{B}$$

So positive equilibrium point $\bar{x} = \frac{a+b-A}{B}$ is globally asymptotically stable.

Theorem 5.13. *The positive equilibria* $\bar{x} = \frac{a+b-A}{B}$ *is a globally asymptotically stable point of the equation*

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_{n-k}}$$

under the condition $a + b > A$.

5.6 Numerical Discussion

In this section, we investigate some examples that include all the case that the two equilibrium points are globally asymptotically stable by theory in order to illustrate the results we got. The examples were carried on MATLAB 6.5.

Example1:
Assume that Equation 5.1.1 holds, take $k = 2$, $A = 4$, $B = 3$, $a = 1$, $b = 2$. So the equation will be reduced to the following:

$$x_{n+1} = \frac{x_n + 2x_{n-2}}{4 + 3x_{n-2}} \tag{5.6.1}$$

In this case, $a + b = 3 < A = 4$ We assumed that the initial points $x_{-2}, x_{-1}, x_0 \in [0, \infty)$ are to be respectively $\{0.7, 0, 1.2\}$.

By theory, the zero equilibrium point under the condition $a + b < A$ is globally asymptotically stable as it is also obvious from Figure5.1.

Example2:
Assume that Equation 5.1.1 holds, take $k = 3$, $A = 6$, $B = 2$, $a = 4$, $b = 2$. So the equation will be reduced to the following:

$$x_{n+1} = \frac{4x_n + 2x_{n-3}}{6 + 2x_{n-3}} \tag{5.6.2}$$

In this case, $a+b = 6 = A$ We assumed that the initial points $x_{-3}, x_{-2}, x_{-1}, x_0 \in [0, \infty)$ are to be respectively $\{0.5, 1.2, 1.9, 2.4\}$.

By theory, the zero equilibrium point under the condition $a + b = A$ is globally asymptotically stable as it is also obvious from Figure5.2.

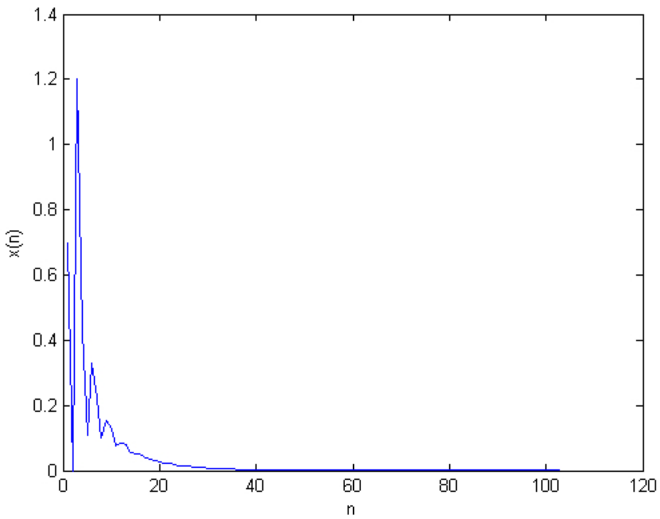

Figure 5.1: The Behavior of the Zero Equilibrium Point of $x_{n+1} = \frac{x_n+2x_{n-2}}{4+3x_{n-2}}$

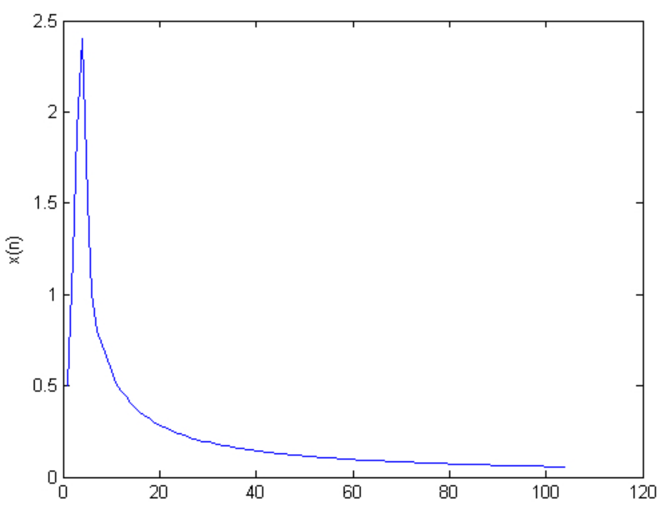

Figure 5.2: The Behavior of the Zero Equilibrium Point of $x_{n+1} = \frac{4x_n+2x_{n-3}}{6+2x_{n-3}}$

Example3:

Assume that Equation 5.1.1 holds, take $k = 4$, $A = 3$, $B = 5$, $a = 1$, $b = 5$. So the equation will be reduced to the following:

$$x_{n+1} = \frac{x_n + 5x_{n-4}}{3 + 5x_{n-4}} \tag{5.6.3}$$

In this case, $a + b = 6 > A = 3$ We assumed that the initial points $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 \in [0, \infty)$ are to be respectively $\{0, 0.4, 1, 0.8, 1.3\}$. Here the positive equilibrium point will be

$$\bar{x} = \frac{a + b - A}{B} = \frac{3}{5} = 0.6$$

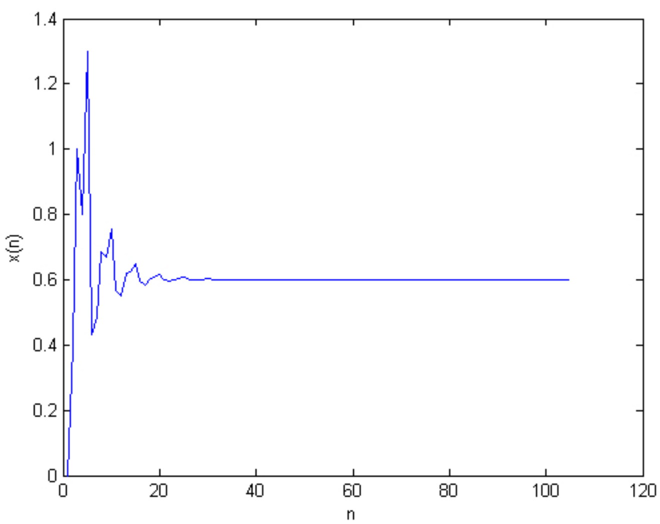

Figure 5.3: The Behavior of the Positive Equilibrium Point of $x_{n+1} = \frac{x_n+5x_{n-4}}{3+5x_{n-4}}$

By theory, the positive equilibrium point $\bar{x} = 0.6$ under the condition $a + b > A$ should be globally asymptotically stable as it is also obvious from Figure5.3.

So, all what we have to say now is that our theoretical discussion was satisfied with the data we get from our numerical discussion. So we have correctly illustrated our study for the equation $x_{n+1} = \frac{ax_n+bx_{n-k}}{A+Bx_{n-k}}$.

Chapter 6

On the Dynamics of the Rational Difference Equation $x_n = \dfrac{A}{x_{n-k}} + \dfrac{B}{x_{n-3k}}$

6.1 Introduction

In [2] the periodicity of the difference equation

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad n = 0, 1, \ldots \tag{6.1.1}$$

where $y_{-k}, ..., y_{-1}, y_0, A \in (0, \infty)$ and $k \in \{2, 3, 4, ...\}$ was studied.

It was shown in [3] that for the case $k = 1$ the positive equilibrium $\bar{y} = 1 + C$ of Eq.6.1.1 is globally asymptotically stable for $C > 1$. In [2], the periodicity of Eq.6.1.1 was investigated.

In [4] the equation

$$x_n = \frac{A}{x_{n-k}} + \frac{B}{x_{n-3k}}, \tag{6.1.2}$$

was studied. The local and global stability was investigated for such an equation, also the semi-cycles analysis was done. It was proved that the equilibrium point of such and equation $\bar{y} = 1 + C$ is globally asymptotically stable.

In this chapter, other related results of asymptotic, periodicity, and semi-cycles of a more general formula are investigated.

We list below some definitions and basic results that will be needed in this chapter.

Definition 6.1. *We say that a solution* $\{y_n\}_{n=-k}^{\infty}$ *of a difference equation* $y_{n+1} = f(y_n, y_{n-1}, ..., y_{n-k})$ *is periodic if there exists a positive integer* p *such that* $y_{n+p} = y_n$. *The smallest such positive integer* p *is called the prime period of the solution of the difference equation.*

Definition 6.2. *The equilibrium point* $\bar{y}$ *of the equation*

$$y_{n+1} = f(y_n, y_{n-1}, ..., y_{n-k}), n = 0, 1, ... \tag{6.1.3}$$

is the point that satisfies the condition

$$\bar{y} = f(\bar{y}, \bar{y}, ..., \bar{y}).$$

Definition 6.3. *Let* $\bar{y}$ *be an equilibrium point of Eq.6.1.3. Then the equilibrium point* $\bar{y}$ *is called*

1. *locally stable if for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all* $y_{-k}, y_{-k+1}, ..., y_0 \in I$ *with* $|y_{-k} - \bar{y}| + |y_{-k+1} - \bar{y}| + ... + |y_0 - \bar{y}| < \delta$, *we have* $|y_n - \bar{y}| < \epsilon$ *for all* $n \geq -k$,

2. *locally asymptotically stable if it is locally stable and if there exists* $\gamma > 0$ *such that for all* $y_{-k}, y_{-k+1}, ..., y_0 \in I$ *with* $|y_{-k} - \bar{y}| + |y_{-k+1} - \bar{y}| + ... + |y_0 - \bar{y}| < \gamma$, *we have* $lim_{n\to\infty} y_n = \bar{y}$.

3. *a global attractor if for all* $y_{-k}, y_{-k+1}, ..., y_0 \in I$, *we have* $lim_{n\to\infty} y_n = \bar{y}$.

4. *globally asymptotically stable if* $\bar{y}$ *is locally stable and* $\bar{y}$ *is a global attractor.*

In [4] the change of variables $y_n = \frac{x_n x_{n-k}}{B}$ changes the equation

$$x_n = \frac{A}{x_{n-k}} + \frac{B}{x_{n-3k}},$$

into the form

$$y_n = C + \frac{y_{n-k}}{y_{n-2k}}$$

We will try now to show this.

Let $y_n = \frac{x_n x_{n-k}}{B}$, then we can get easily that $x_n = \frac{By_n}{x_{n-k}}$. Substituting this in our equation we get

$$\frac{By_n}{x_{n-k}} = \frac{A}{x_{n-k}} + \frac{B}{x_{n-3k}}$$

Multiplying the equation by x_{n-k} we get

$$By_n = A + B\frac{x_{n-k}}{x_{n-3k}}$$

But from our assumption we can get that

$$y_{n-k} = \frac{x_{n-k}x_{n-2k}}{B} \quad so \quad x_{n-k} = \frac{By_{n-k}}{x_{n-2k}}$$

Also

$$y_{n-2k} = \frac{x_{n-2k}x_{n-3k}}{B} \quad so \quad x_{n-3k} = \frac{By_{n-2k}}{x_{n-2k}}$$

From so we can conclude that

$$\frac{x_{n-k}}{x_{n-3k}} = \frac{y_{n-k}}{y_{n-2k}}$$

Substituting back in our equation to get

$$y_n = \frac{A}{B} + \frac{y_{n-k}}{y_{n-2k}}$$

which can be written

$$y_n = C + \frac{y_{n-k}}{y_{n-2k}}, \tag{6.1.4}$$

where $C = \frac{A}{B} > 0$,$y_{-2k+1},\ y_{-2k+2}, ...,\ y_0 \in (0, \infty)$ and $k \in \{1, 2, 3, 4, ...\}$.

Let $\bar{y}$ be the equilibrium point of (6.1.4), then

$$\bar{y} = C + \frac{\bar{y}}{\bar{y}} \tag{6.1.5}$$

so we can conclude that

$$\bar{y} = 1 + C$$

6.2 The Local Stability of the Equilibrium Point

We want now to find the linearized equation of Eq.6.1.4 about the positive equilibrium $\bar{y} = 1 + C$.

Let $f(u, v) = C + \frac{u}{v}$, then

$$\frac{\partial f}{\partial u} = \frac{1}{v}$$

$$\frac{\partial f}{\partial u} = \frac{-u}{v^2}$$

then

$$\frac{\partial f}{\partial y_{n-k}}(\bar{y}, \bar{y}) = \frac{1}{\bar{y}} = \frac{1}{1+C}$$

$$\frac{\partial f}{\partial y_{n-2k}}(\bar{y}, \bar{y}) = \frac{-\bar{y}}{\bar{y}^2} = \frac{-1}{\bar{y}} = \frac{-1}{1+C}$$

The linearized difference equation will become

$$z_n = \frac{1}{1+C} z_{n-k} - \frac{1}{1+C} z_{n-2k} \tag{6.2.1}$$

which can be written as

$$z_n - \frac{1}{1+C} z_{n-k} + \frac{1}{1+C} z_{n-2k} = 0. \tag{6.2.2}$$

Lemma 6.1. *[6][5][7] Assume that a,b are real numbers and $k \in \{1, 2, ...\}$. Then*

$$|a| + |b| < 1 \tag{6.2.3}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_n + a y_{n-k} + b y_{n-l} = 0, \ n = 0, 1, ... \ . \tag{6.2.4}$$

Suppose in addition that one of the following two cases holds.

(a) k odd and $b < 0$.

(b) k even and $ab < 0$.

Then 6.2.3 is also a necessary condition for the asymptotic stability of Eq.6.2.4.

Lemma 6.2. *[6][5][7] Assume that a,b are real numbers. Then*

$|a| < b + 1 < 2$

is a necessary and sufficient condition for the asymptotic stability of the difference equation

$$y_n + ay_{n-k} + by_{n-l} = 0,\ n = 0, 1, \dots . \tag{6.2.5}$$

The above two lemmas prove that for all values of $C > 1$, the equilibrium point of 6.1.4 is locally asymptotically stable.

Lemma 6.3. *([1],[8]) The difference equation*

$$y_n - by_{n-k} + by_{n-l} = 0,\ n = 0, 1, \dots . \tag{6.2.6}$$

is asymptotically stable iff $0 < |b| < 1/2\cos(\frac{k\pi}{k+2})$

We can conclude now this lemma.

Lemma 6.4. *Consider Eq.6.1.4. If $C > 2\cos(\frac{k\pi}{k+2}) - 1$ then the unique positive equilibrium $\bar{y} = 1 + C$ of Eq.6.1.4 is locally asymptotically stable, while if $C < 2\cos(\frac{k\pi}{k+2}) - 1$ then the positive equilibrium is unstable.*

Proof. The proof is a direct consequence of the conditions in Lemma 6.3.

For the Linearized equation

$$z_n - \frac{1}{1+C}z_{n-k} + \frac{1}{1+C}z_{n-2k} = 0$$

If $C > 2\cos(\frac{k\pi}{k+2}) - 1$ then

$$C + 1 > 2\cos(\frac{k\pi}{k+2})$$

So

$$0 < \frac{1}{1+C} < \frac{1}{2\cos(\frac{k\pi}{k+2})}$$

Now from Lemma 6.3 we can conclude easily that the equilibrium point $\bar{y} = 1 + C$ is asymptotically stable.

By the same technique we can show that if $C < 2\cos(\frac{k\pi}{k+2}) - 1$ then $\bar{y} = 1 + C$ is unstable. □

In [4], it was based on the following theorem in order to proof that the positive equilibrium oint $\bar{y} = 1 + C$ is asymptotically stable.

Theorem 6.1. *[4] Consider the equation*

$$\theta^2 - \frac{1}{1+C}\theta + \frac{1}{1+C} = 0$$

1. *If both of the quadratic roots of the above equation lie in open unit disk, then the equilibrium point $\bar{y}$ of Eq 6.1.4 is locally asymptotically stable.*

2. *If at least one root of the above equation has absolute value greater than one, then the equilibrium point $\bar{y}$ of Eq 6.1.4 is unstable.*

3. *A necessary and sufficient condition for both roots of the above equation to lie inside the open unit disk is*

$$|p| < 1 - q \leq 2.$$

   In this case the locally asymptotically stable equilibrium $\bar{y}$ of Eq 6.1.4is also called a sink.

4. *A necessary and sufficient condition for both roots of the above equation to have absolute value greater than one is*

$$|q| > 1 \quad \textit{and} \quad |p| < |1 - q|.$$

   In this case the equilibrium point $\bar{y}$ of Eq 6.1.4is called a repeller.

5. *A necessary and sufficient condition for one root of the above equation to have absolute value less than one and the other root to have absolute value greater than one is*

$$p^2 + 4q > 0 \quad \textit{and} \quad |p| > |1 - q|.$$

   In this case the unstable equilibrium point $\bar{y}$ of Eq 6.1.4 is called a saddle point.

The following result is one of the results in [4] but not proved, here we will prove it.

Theorem 6.2. *The equilibrium point $\bar{y} = 1+C$ of Eq 6.1.4 is asymptotically stable iff C >1.*

Proof. Lets be back to our Linearized equation around $\bar{y} = 1 + C$

$$z_n - \frac{1}{1+C} z_{n-k} + \frac{1}{1+C} z_{n-2k} = 0$$

The characteristic equation here will be

$$\lambda^{2k} - \frac{1}{1+C}\lambda^k + \frac{1}{1+C} = 0 \tag{6.2.7}$$

taking $\theta = \lambda^k$ will transform this equation into

$$\theta^2 - \frac{1}{1+C}\theta + \frac{1}{1+C} = 0 \tag{6.2.8}$$

for this equation we have $p = -\frac{1}{1+C}$ and $q = \frac{1}{1+C}$. If we assume that $C > 1$ then we get that

$$\frac{1}{1+C} < \frac{C}{1+C}$$

but after doing our computations we get that

$$1 - q = \frac{C}{1+C} \quad and \quad |p| = \frac{1}{1+C}.$$

then putting all together to get that

$$|p| = \frac{1}{1+C} < \frac{C}{1+C} < 1 \leq 2$$

So by the above theorem, we get that the two roots lie inside the open unitary disk, for so $|\theta| < 1$.
But since $\theta = \lambda^k$,this means that $|\theta| < 1$ implies that $|\lambda| < 1$.

And from so we can conclude easily that $C > 1$ is both necessary and sufficient condition for the positive equilibrium point $\bar{y} = 1+C$ to be asymptotically stable.
The proof is complete.

□

6.3 Analysis Of The Global Stability, And The Semi-Cycles Of Solutions

In this section we will show that every positive solution of Eq.6.1.4 is globally asymptotically stable and thus we get as a corollary the boundedness and persistence of solutions.

Definition 6.4. *We say that a solution* $\{y_n\}$ *of a difference equation* $y_{n+1} = f(y_n, y_{n-1}, ..., y_{n-k})$ *is bounded and persists if there exist positive constants* P *and* Q *such that*

$$P \leq x_n \leq Q \ \text{ for } \ n = -1, 0, \dots .$$

Definition 6.5. *A positive semi-cycle of a solution* $\{y_n\}$ *of Eq.6.1.3 consists of a "string" of terms* $\{y_l, y_{l+1}, \dots, y_m\}$*, all greater than or equal to the equilibrium* $\bar{y}$*, with* $l \geq -k$ *and* $m \leq \infty$ *and such that*

$$\text{either } l = -k, \text{ or } l > -k \text{ and } y_{l-1} < \bar{y}$$

and

$$\text{either } m = \infty, \text{ or } m < \infty \text{ and } y_{m+1} < \bar{y}.$$

Definition 6.6. *A negative semi-cycle of a solution* $\{y_n\}$ *of Eq.6.1.3 consists of a "string" of terms* $\{y_l, y_{l+1}, \dots, y_m\}$*, all less than the equilibrium* $\bar{y}$*, with* $l \geq -k$ *and* $m \leq \infty$ *and such that*

$$\text{either } l = -k, \text{ or } l > -k \text{ and } y_{l-1} \geq \bar{y}$$

and

$$\text{either } m = \infty, \text{ or } m < \infty \text{ and } y_{m+1} \geq \bar{y}.$$

The first semi-cycle of a solution starts with the term y_{-k} and is positive if $y_{-k} \geq \bar{y}$ and negative if $y_{-k} < \bar{y}$.

Definition 6.7. *A solution* $\{y_n\}$ *of Eq.6.1.3 is called nonoscillatory if there exists* $N \geq -k$ *such that* $y_n > \bar{y}$ *for all* $n \geq N$ *or* $y_n < \bar{y}$ *for all* $n \geq N$*.*

And a solution $\{y_n\}$ *is called oscillatory if it is not nonoscillatory.*

Now we will present one of our results on the Eq.6.1.4.

Theorem 6.3. *Eq.6.1.4 has no solution of prime period 2 if* $C = 1$ *or* k *is even.*

Proof. Assume that our solution is of prime period $p = 2$ and takes the form $..., \Phi, \Psi, \Phi, \Psi, ...$

If k is even then $\Phi = \Psi = C + 1$, this leads to a contradiction with our assumption.So, in this case $p \neq 2$.

If k is odd then $\Phi = C + \frac{\Phi}{\Psi}$ and $\Psi = C + \frac{\Psi}{\Phi}$.
It follows that

$$\frac{\Phi}{\Psi} = \Phi - C$$

and

$$\frac{\Psi}{\Phi} = \Psi - C$$

Multiplying the last two equations, we get

$$(\Psi - C)(\Phi - C) = 1$$

Thus,

$$\Phi \neq C \ and \ \Psi \neq C$$

Moreover, we conclude that

$$\Psi = \frac{1}{\Phi - C} + C$$

But on the other hand, we have

$$\frac{1}{\Psi} - \frac{1}{\Phi} = \frac{1}{\Psi^2} - \frac{1}{\Phi^2}$$

Therefore, we get

$$\frac{1}{\Psi} + \frac{1}{\Phi} = 1$$

Solving for Ψ, we get

$$\Psi = \frac{\Phi}{1 + \Phi}$$

but as

$$\frac{1}{\Psi} = 1 - \frac{C}{\Phi} \quad and \quad \frac{1}{\Phi} = 1 - \frac{C}{\Psi}$$

we get that

$$1 - \frac{C}{\Phi} + 1 - \frac{C}{\Psi} = 1$$

which leads to

$$2 - C(\frac{1}{\Psi} + \frac{1}{\Phi}) = 1$$

but as $\frac{1}{\Psi} + \frac{1}{\Phi} = 1$ we get that $C = 1$. By so, in order to have a solution of period $p = 2$, C must equal 1.

We can conclude that the periodic solution of prime period $p = 2$ takes the form ..., $\Phi, \frac{\Phi}{1+\Phi}, \Phi, \frac{\Phi}{1+\Phi}, ...$
This completes the proof.

□

Now we will study the global asymptotic stability for the general case $k \in \{1, 2, 3, 4, ...\}$.

Theorem 6.4. *[9] Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}); n = 0, 1, ... \tag{6.3.1}$$

where $k \in \{1, 2, ...\}$. Let $I = [a, b]$ be some interval of real numbers and assume that

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(u, v)$ is nonincreasing in u and nondecreasing in v.

(b) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$m = f(M, m) \text{ and } M = f(m, M),$$

then $m = M$. Then Eq.6.3.1 has a unique equilibrium $\bar{y}$ and every solution of Eq.6.3.1 converges to $\bar{y}$.

Theorem 6.5. *[1] Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}); n = 0, 1, ... \tag{6.3.2}$$

where $k \in \{1, 2, ...\}$. Let $I = [a, b]$ be some interval of real numbers and assume that

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(u,v)$ is nonincreasing in u and nonincreaing in v.

(b) If $(m,M) \in [a,b] \times [a,b]$ is a solution of the system

$$m = f(M,M) \text{ and } M = f(m,m),$$

then $m = M$. Then Eq.6.3.2 has a unique equilibrium $\bar{y}$ and every solution of Eq.6.3.2 converges to $\bar{y}$.

Here is another result from [4], but we will present it in a different way.

Lemma 6.5. *Let $C > 1$. Then every solution of Eq.6.1.4 is bounded and persists.*

Proof. For $n > 3k$, the following inequality hold

$$C < y_n < C + \frac{1+C}{C}$$

since, our difference equation was become

$$y_n = C + \frac{y_{n-k}}{y_{n-2k}}$$

so

$$C = y_n - \frac{y_{n-k}}{y_{n-2k}}$$

so we can conclude

$$C < y_n \quad for\ all\ n > 0$$

Now, from the equation we get

$$y_{n+k} = C + \frac{y_n}{y_{n-k}}$$

but also from our equation we get

$$\frac{y_n}{y_{n-k}} = \frac{C}{y_{n-k}} + \frac{1}{y_{n-2k}}$$

But since $y_n > C$ for all $n > 0$,then for all $n \geq 2k$ $y_{n-k} > C$ and $y_{n-2k} > C$, by so

$$\frac{y_n}{y_{n-k}} < \frac{C}{C} + \frac{1}{C} = 1 + \frac{1}{C} = \frac{1+C}{C}$$

then

$$y_{n+k} = C + \frac{y_n}{y_{n-k}} < C + \frac{1+C}{C}$$

we get by the end that

$$C < y_{n+k} < C + \frac{1+C}{C} \quad for\ all\ n \geq 2k$$

So

$$C < y_n < C + \frac{1+C}{C} \quad for\ all\ n \geq 3k$$

This completes the proof. □

Here is another own result for the Eq.6.1.4.

Theorem 6.6. *Let $C > 0$. Then the unique positive equilibrium $\bar{y} = 1+C$ of Eq.6.1.1 is a global attractor.*

Proof. Define $f(u, v) = C + u/v$ on the interval $[C, C + \frac{(C+1)}{C}]$. Then the result follows directly from Theorem 6.5.

Since f is nonincreasing in both u and v then

$$m = f(M, M) = C + \frac{M}{M} = C + 1$$

and

$$M = f(m, m) = C + \frac{m}{m} = C + 1$$

Then it is shown that $m = M$, so we can conclude from Theorem 6.5 that $\bar{y} = 1 + C$ is a global attractor. □

Theorem 6.7. *Let $C > 1$. Then the unique positive equilibrium $\bar{y} = C + 1$ of Eq.6.1.4 is globally asymptotically stable.*

Proof. Let

$$I = \lim_{n\to\infty} \inf \{y_n\} > 0 \text{ and } S = \lim_{n\to\infty} \sup \{y_n\} < \infty.$$

Then it is easy to see from Eq.(6.1.4) that

$$S \leq C + \frac{S}{I} \text{ and } I \geq C + \frac{I}{S}.$$

Thus

$$IS \leq CI + S \text{ and } IS \geq CS + I.$$

From here it follows that

$$CS + I \leq CI + S,$$

or

$$(C-1)S \leq (C-1)I.$$

Thus if $C > 1$ then $S \leq I$, and the result follows.

Another proof can be held easily, We have shown that If $C > 1$ then the equilibrium of point $\bar{y} = 1 + C$ of Eq 6.1.4 is asymptotically stable, and we also showed in the last theorem that for all $C > 0$, $\bar{y} = 1 + C$ is a global attractor, then we get that for $C > 1$, then $\bar{y} = 1 + C$ is a globally asymptotically stable equilibrium point of Eq 6.1.4. □

In [4], it was showed that the solution $\{y_n\}$ of Eq 6.1.2 oscillates about the equilibrium $\bar{y}$ with a semi-cycle of length wt most 3k.

Theorem 6.8. *[4] Let $\{y_n\}$ be a nontrivial solution of Eq.6.1.4, $C > 1$, $k \geq 1$. Then every semi-cycle has at most $3k$ terms.*

Proof. Assume that we have a positive semicycle with $y_N \geq \bar{y}$ to be its first term.
If $y_{N+k} < y_N$ then

$$y_{N+2k} = C + \frac{y_{N+k}}{y_N} < C + \frac{y_N}{y_N} = C + 1 = \bar{y}$$

We conclude that the theorem holds true in this case.

If $y_{N+k} \geq y_N$ then

$$y_{N+2k} = C + \frac{y_{N+k}}{y_N} \geq C + \frac{y_N}{y_N} \geq \bar{y}$$

But as

$$\begin{aligned} y_{N+2k} = C + \frac{y_{N+k}}{y_N} &\leq C + \frac{y_{N+k}}{\bar{y}} \\ &\leq C + \frac{y_{N+k}}{1+C} \\ &\leq y_{N+k} \end{aligned}$$

then

$$y_{N+3k} = C + \frac{y_{N+2k}}{y_{N+k}} \leq C + \frac{y_{N+k}}{y_{N=k}}$$
$$\leq C + 1$$
$$\leq \bar{y}$$

Also in this case the theorem hold true.
This completes the proof. □

The following result was not studied in [4].

Theorem 6.9. *Let k be odd and let*

$$y_{-2k+1}, y_{-2k+3}, ..., y_{-1} \leq C + 1, \ 0 > y_{-2k+2}, y_{-2k+4}, ..., y_0 > C + 1$$

Then, the solution $\{y_n\}_{n=-2k+1}^{\infty}$ is oscillatory and every semi-cycle has length at most $2k$. Moreover, every term of $\{y_n\}_{n=-2k+1}^{\infty}$ is strictly greater than $C+1$ with the possible exception of the first $2k$ semi-cycles, no term of $\{y_n\}_{n=1}^{\infty}$ is ever equal to $C + 1$.

Proof. Just notice that, for any $n \geq 1$, we can get from Eq.6.1.4 that

$$y_{2n} = C + \frac{y_{2n-k}}{y_{2n-2k}}$$

but since k is odd, then $2n - k$ is also odd and $2n - 2k$ is even. So by the assumption we get

$$y_{2n} = C + \frac{y_{2n-k}}{y_{2n-2k}} < C + 1$$

Also,

$$y_{2n+1} = C + \frac{y_{2n-k+1}}{y_{2n-2k+1}}$$

but since k is odd, then $2n - k + 1$ is even and $2n - 2k + 1$ is odd. So by the assumption we get

$$y_{2n+1} = C + \frac{y_{2n-k+1}}{y_{2n-2k+1}} > C + 1$$

The result then follows. □

6.4 The Periodic Behavior of The Solution

Here we will show that the equilibrium point of the solution of Eq.6.1.2 is periodic of period k. From Eq.6.1.2, we obtain

$$x_{2n} = \frac{A}{x_{2n-k}} + \frac{B}{x_{2n-3k}} \tag{6.4.1}$$

and

$$x_{2n-k} = \frac{A}{x_{2n-2k}} + \frac{B}{x_{2n-4k}} \tag{6.4.2}$$

and

$$x_{2n-3k} = \frac{A}{x_{2n-4k}} + \frac{B}{x_{2n-6k}} \tag{6.4.3}$$

So it follows that

$$x_{2n} = \frac{A}{\frac{A}{x_{2n-2k}} + \frac{B}{x_{2n-4k}}} + \frac{B}{\frac{A}{x_{2n-4k}} + \frac{B}{x_{2n-6k}}} \tag{6.4.4}$$

Based on what was done in [4], we obtain the following statements which outline properties of Eq.6.1.2. Their proofs are based on Eq.6.4.4.

Lemma 6.6. *[4] Let $\{x_n\}$ be a positive solution of Eq.6.1.2. Then the following statements hold:*

1. *For $N > 0$, let*

$$m_N = Min(x_{2N-4k}, x_{2N-2k}, x_{2N})$$

$$M_N = Max(x_{2N-4k}, x_{2N-2k}, x_{2N})$$

Then $m_N \leq x_{2N+2lk} \leq M_N \qquad$ *for* $l \geq 1$

2. $\lim_{n\to\infty} \frac{x_{2nk+i}}{x_{(2n-2)k+i}} = 1 \qquad$ *for* $i = 0, 1, ..., k-1$

Proof. 1. Define the function f

$$f(x, y, z) = \frac{A}{\frac{A}{x} + \frac{B}{y}} + \frac{B}{\frac{A}{y} + \frac{B}{z}}$$

It is obvious that f is increasing in x, y, and z. So

$$x_{2N+2k} = \frac{A}{\frac{A}{x_{2N}} + \frac{B}{x_{2N-2k}}} + \frac{B}{\frac{A}{x_{2N-2k}} + \frac{B}{x_{2N-4k}}}$$

but from the definition of M_N, we get

$$x_{2N+2k} \leq \frac{A}{\frac{A}{M_N} + \frac{B}{M_N}} + \frac{B}{\frac{A}{M_N} + \frac{B}{M_N}} = M_N$$

Now by Mathematical Induction we can prove easily that

$$x_{2N+2lk} \leq M_N$$

We will do the same but for m_N,

$$x_{2N+2k} = \frac{A}{\frac{A}{x_{2N}} + \frac{B}{x_{2N-2k}}} + \frac{B}{\frac{A}{x_{2N-2k}} + \frac{B}{x_{2N-4k}}}$$

but from the definition of m_N, we get

$$x_{2N+2k} \geq \frac{A}{\frac{A}{m_N} + \frac{B}{m_N}} + \frac{B}{\frac{A}{m_N} + \frac{B}{m_N}} = m_N$$

Now by Mathematical Induction we can prove easily that

$$x_{2N+2lk} \geq m_N$$

It is proved now that

$$m_N \leq x_{2N+2lk} \leq M_N \quad for\ all\ l \geq 1$$

2. It was shown that the sequence $\{y_n\}$ converges to the equilibrium point,so

$$y_{2n} = \frac{x_{2n}x_{2n-k}}{B}$$

converges to the equilibrium.
We can conclude that $\{y_{2nk+i}\}$ and $\{y_{2nk-k+i}\}$ also converges to the equilibrium. But

$$\frac{y_{2nk+i}}{y_{2nk-k+i}} = \frac{x_{2nk+i}}{x_{(2n-2)k+i}}$$

As

$$\lim_{n\to\infty} \frac{y_{2nk+i}}{y_{2nk-k+i}} = 1$$

Then

$$\lim_{n\to\infty} \frac{x_{2nk+i}}{x_{(2n-2)k+i}} = 1$$

This completes the proof. □

Lemma 6.7. *[4] There exist positive numbers S and I such that $I < x_n < S$ for all $n \geq 0$.*

Proof. This proof can be done so easily as similar as the previous proof of Lemma 6.6(1).
From Eq.6.1.2, we obtain

$$x_n = \frac{A}{x_{n-k}} + \frac{B}{x_{n-3k}} \tag{6.4.5}$$

and so

$$x_{n-k} = \frac{A}{x_{n-2k}} + \frac{B}{x_{n-4k}} \tag{6.4.6}$$

So it follows that

$$x_n = \frac{A}{\frac{A}{x_{n-2k}} + \frac{B}{x_{n-4k}}} + \frac{B}{\frac{A}{x_{n-4k}} + \frac{B}{x_{n-6k}}} \tag{6.4.7}$$

For $n > 0$, let

$$m_n = Min(x_{n-2k}, x_{n-4k}, x_{n-6k})$$
$$M_n = Max(x_{n-2k}, x_{n-4k}, x_{n-6k})$$

and take

$$S = \sup M_n$$

and

$$I = \inf m_n$$

Then we can show easily that

$$I \le m(n) < x_n < M(n) \le S$$

This completes the proof. $\square$

Theorem 6.10. *[25] Let $\{x_n\}$ be a positive solution of Eq.6.1.2. Then there exist positive constants $l_0, l_1, ...l_{k-1}$ such that $l_i l_{k+i} = A+B$ and $\lim_{n\to\infty} x_{2nk+i} = l_i$ for $i = 0, 1, ..., k-1$*

Proof. We show that that the sequence $\{x_{2nk+i}\}$ for $i = 0, 1, ..., k-1$ are Cauchy. By Lemma 6.6 and Lemma 6.7 and Eq.6.4.2, We have

$$\left|\frac{x_{2nk+i}}{x_{2nk-2k+i}} - 1\right| = \left|\frac{x_{2nk+i} - x_{2nk-2k+i}}{x_{2nk-2k+i}}\right| \ge \left|\frac{x_{2nk+i} - x_{2nk-2k+i}}{S}\right|$$

Since

$$\lim_{n\to\infty}\left|\frac{x_{2nk+i}}{x_{2nk-2k+i}} - 1\right| = 0$$

Hence,

$$\lim_{n\to\infty} |x_{2nk+i} - x_{2nk-2k+i}| = 0,\ i = 0, 1, ..., k-1$$

Since $\{x_n\}$ is bounded by Lemma 6.6 , $\lim_{n\to\infty} x_{2nk+i}$ exists and is a positive number l_i.

We can conclude from above that

$$l_i = l_{i-2k} = l_{i+2k}$$

Finally, from Eq.6.4.2 we see

$$x_{2nk-k+i} = \frac{A}{x_{2nk-2k+i}} + \frac{B}{x_{2nk-4k+i}}$$

we observe that

$$l_{i-k} = \frac{A}{l_{i-2k}} + \frac{B}{l_{i-4k}}$$

Since $l_i = l_{i-2k} = l_{i-4k}$ and $l_{i-k} = l_{i+2k-k} = l_{i+k}$ then

$$l_{i+k} = \frac{A+B}{l_i}$$

So we have

$$l_{i+k} l_i = A + B$$

This completes the proof. $\square$

6.5 Numerical Discussion

In this section, to illustrate the result of this study, two numerical examples are given, which were carried on MATLAB 6.5.

Example1:

Assume that Equation 6.1.2 holds, take $k = 4$, $A = 5$, $B = 3$. So the equation will be reduced to the following:

$$x_n = \frac{5}{x_{n-4}} + \frac{3}{x_{n-12}}, \tag{6.5.1}$$

We assumed that the initial points $x_{-11},\ x_{-10}, ...,\ x_0 \in (0, \infty)$ are all to be equal 1.

Using the change of variables:$y_n = \frac{x_n x_{n-4}}{3}$. The corresponding equation will be as follows:

$$y_n = C + \frac{y_{n-4}}{y_{n-8}}, \tag{6.5.2}$$

where $C = \frac{5}{3} > 1$.

By theory, the equilibrium point $\bar{y} = 1 + C = 2.666667$, and it is obvious from Figure 6.2 that it is globally asymptotically stable, as we have shown theoretically.Lets take another example now.

Example2:

Assume that Equation 6.1.2 holds, take $k = 2$, $A = 9$, $B = 4$. So the equation will be reduced to the following:

$$x_n = \frac{9}{x_{n-2}} + \frac{4}{x_{n-6}}, \tag{6.5.3}$$

Taking the initial points $x_{-5},\ x_{-4}, ...,\ x_0 \in (0, \infty)$ are respectively to be 0.6, 0.82, 1.7, 2.7, 1.2, 0.5 .

Using the change of variables:$y_n = \frac{x_n x_{n-2}}{4}$. The corresponding equation will be as follows:

$$y_n = C + \frac{y_{n-2}}{y_{n-4}}, \tag{6.5.4}$$

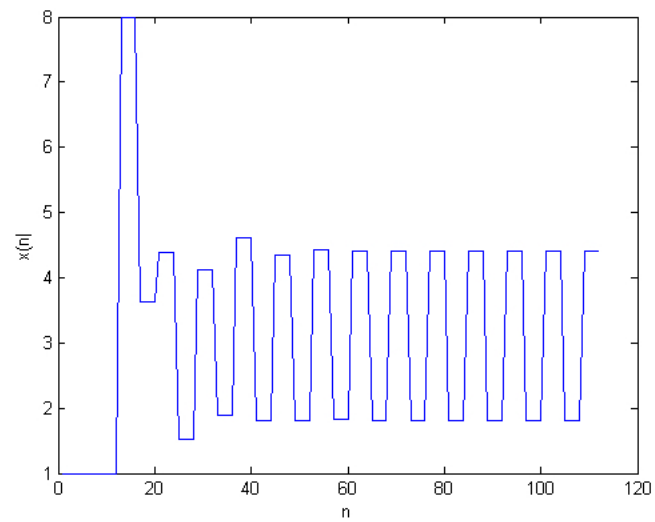

Figure 6.1: The Behavior of the Equilibrium point of the Equation $x_n = \frac{5}{x_{n-4}} + \frac{3}{x_{n-12}}$

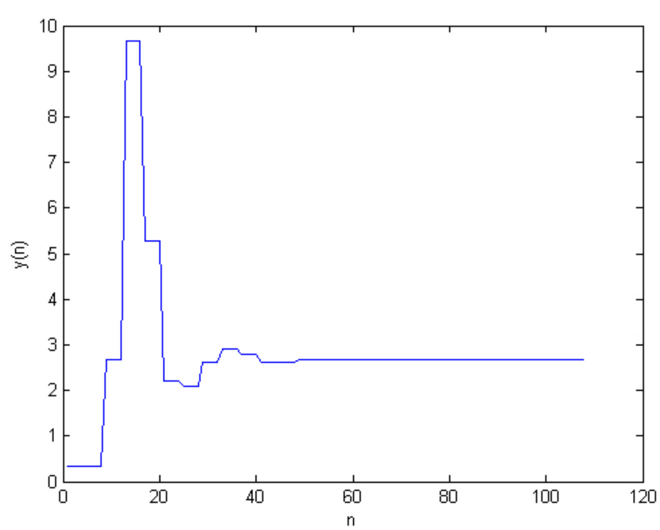

Figure 6.2: The Behavior of the Equilibrium point of the Equation $y_n = C + \frac{y_{n-4}}{y_{n-8}}$

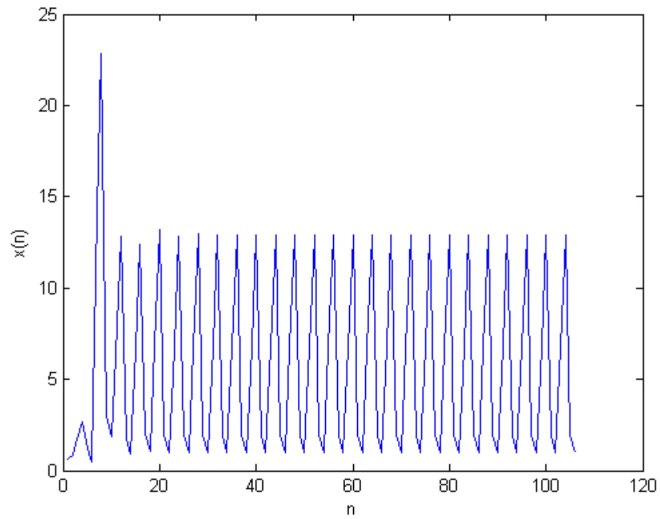

Figure 6.3: The Behavior of the Equilibrium point of the Equation $x_n = \frac{9}{x_{n-2}} + \frac{4}{x_{n-6}}$

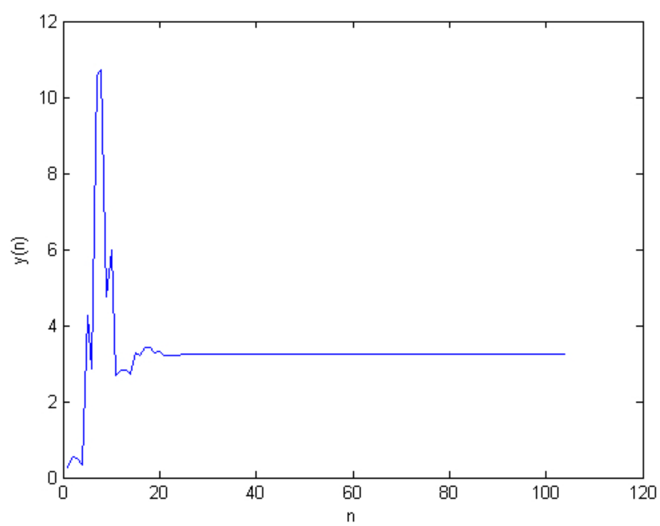

Figure 6.4: The Behavior of the Equilibrium point of the Equation $y_n = C + \frac{y_{n-2}}{y_{n-4}}$

where $C = \frac{9}{4} > 1$.

Here, it is obvious that from Figure 6.4 that our equilibrium point is around the point 1.4. Lets calculate it theoretically, $\bar{y} = 1 + C = 3.25$. We have gotten it.

Part III

The MATLAB 6.5 Codes

```
% On The Dynamics Of A Higher Order Rational Difference Equations
% Program 1

% Aseel Farhat
% 1055334

clear all
format long

fprintf(1,' \n Input the constants of your difference Equation \n');

fprintf(1,'A = ');
A = input(' ');
fprintf(1,'B = ');
B = input(' ');
fprintf(1,'m = ');
m = input(' ');
fprintf(1,'k = ');
k = input(' ');

fprintf(1,'\n Input the initial conditions of your Equation \n');
x = input(' ');                              % n= 1,2,.....3mk

for n=-3*m*k+1 : 100

   if (n>0)
       x(n+3*m*k)= A/x(n+2*m*k) + B/x(n);
   end
end

fprintf(1,'The equilibrium of the sequence x(n) is');
x_bar = sqrt(A+B)

plot(x);
fprintf(1,'\n Input some integer in order to continue ');
e = input(' ');

C = A/B;
fprintf(1,'\n The equilibrium of the y(n) sequence is');
y_bar = C+1

for n=-2*m*k+1 : 100

       y(n+2*m*k)= x(n+3*m*k)*x(n+2*m*k)/B;
end
plot(y);
```

```
% On The Dynamics Of A Higher Order Rational Difference Equations
% Program 2

% Aseel Farhat
% 1055334

clear all
format long

fprintf(1,' \n Input the constants of your difference Equation \n');

fprintf(1,'alfa = ');
alfa = input(' ');
fprintf(1,'beta = ');
beta = input(' ');
fprintf(1,'gamma = ');
gamma = input(' ');
fprintf(1,'B = ');
B = input(' ');
fprintf(1,'C = ');
C = input(' ');
fprintf(1,'k = ');
k = input(' ');

fprintf(1,'\n Input the initial conditions of your Equation \n');
x = input(' ');                     % n= 1,2,.....k+1

D = (alfa*beta)/(beta*beta)
p = gamma/beta
q = C/B

fprintf(1,'\n The equilibrium of the y(n) sequence is');
y_bar = ((1+p)+sqrt((1+p)^2+4*D*(1+q)))/(2*(1+q))

for n= -k : 100

   if (n>0)
      y(n+k+1) = (D+y(n+k)+p*y(n))/(y(n+k)+q*y(n));
   else
      y(n+k+1) = B/(beta*x(n+k+1));
   end
end
plot(y);
```

```
% On The Dynamics Of A Higher Order Rational Difference Equations
% Program 3

% Aseel Farhat
% 1055334

clear all
format long

fprintf(1,' \n Input the constants of your difference Equation \n');

fprintf(1,'A = ');
A = input(' ');
fprintf(1,'B = ');
B = input(' ');
fprintf(1,'a = ');
a = input(' ');
fprintf(1,'b = ');
b = input(' ');
fprintf(1,'k = ');
k = input(' ');

fprintf(1,'\n Input the initial conditions of your Equation \n');
x = input(' ');                              % n= 1,2,.....k+1

for n=1 : 100
   x(n+k+1) = ( a*x(n+k) + b*x(n) )/( A + B*x(n) );
end

x_bar= (a+b-A)/(B)

plot(x)
```

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